**Proof of the Goldbach Conjecture**

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**Statement of the Conjecture:** « Every even natural integer is the sum of two prime integers ».

**Démonstration:**

Let \( n \) be an even integer.

Let \( P_n \) denote the set of all prime factors less than \( n \) defined as follows:

\[
P_n = \{1(p_1); 2(p_2); \ldots; p_m\}
\]

these \( p_i \) are listed in ascending order:

\[
p_1 < p_2 < \ldots < p_{m-1} < p_m.
\]

So \( p_m \) is the largest prime factor less than \( n \), in other words there is no more prime factor between \( p_m \) and \( n \).

We therefore have \( \forall p_i \in P_n \), \( p_i \leq p_m < n \) and \( n - p_i < n \).

The contrapositive of the Goldbach conjecture is as follows:

« There exists an even natural number which is not equal to any sum of two prime numbers »: Supposition 1 \( \rightarrow \) S1.
We are therefore going to study this contrapositive for this $n$.

The meaning of this contrapositive with what was previously defined is:

$$ \forall p_i \in P_n, n-p_i \notin P_n.$$  

We have $p_1 < p_2 < ... < p_{m-1} < p_m$ therefore

$n-p_m < n-p_{m-1} < ... < n-p_2 < n-p_1$.

Suppose then that $\forall p_i \in P_n, p_i < n-p_i < p_m$: supposition 2 $\rightarrow$ S2.

So for $p_i = p_m$ we get $p_1 < n-p_m < p_m$,

$p_1 < n-p_m \Rightarrow p_m < n-p_1$ contradiction with $n-p_i < p_m$

This assumption S2 is therefore false (S2 therefore closed).

And then $\exists p_{j/n} \in P_n$ such that $n-p_{j/n} \leq p_1$ or $n-p_{j/n} \geq p_m$,

And since $n-p_{j/n} \notin P_n$ then $n-p_{j/n} \neq p_1 = 1$.

We therefore only have the case where $n-p_{j/n} \geq p_m$ more exactly

$n-p_{j/n} > p_m$ because $n-p_{j/n}$ is not prime(S1) and $p_m$ is prime.

$n-p_{j/n} > p_m \Rightarrow n-p_m > p_{j/n}$ and therefore

$n-p_m > p_{j/n} > p_{(j/n)-1} > p_{(j/n)-2} > ... > p_2 > p_1$.
We will continue this analysis with the largest prime factor \( p_{j/n} \) which allows the inequality \( n - p_{j/n} > p_m \).

We will then have \( n - p_{j/n} > p_m \) and \( n - p_{(j/n) + 1} < p_m \).

With \( p_{(j/n) + 1} \) the prime factor following the prime factor \( p_{j/n} \).

(we can write \( p_{(j/n) + 1} \) or \( p_{(j+1)/n} \); \( p_{(j/n) + i} \) or \( p_{(j+i)/n} \).

As \( p_{j/n} > p_{(j/n) + 1} > p_{(j/n) + 2} > \ldots \ldots p_2 > p_1 \) and

\( p_{(j/n) + 1} < p_{(j/n) + 2} < \ldots \ldots < p_{m - 1} < p_m \) we then obtain:

\[ \forall p_k \leq p_{j/n} \text{, } n - p_k < p_m \text{ because } n - p_k \geq n - p_{j/n} > p_m \text{ (which indicates the non-primality of } n - p_k \text{ for } p_k \leq p_{j/n} \text{ because there is no prime factor between } p_m \text{ and } n) \text{ And} \]

\[ \forall p_k \geq p_{(j+1)/n}, n - p_k < p_m \text{ or } p_1 < n - p_k < p_m \text{ because } n - p_k < n - p_{(j/n) + 1} < p_m \]

We will therefore first show the existence of \( p_{j/n} \):

\( p_{j/n} \) was defined as follows:

\[ \exists p_{j/n} \in P_n \text{ such that } n - p_{j/n} > p_m \text{ and } n - p_{(j/n) + 1} < p_m \text{ with } p_{(j/n) + 1} \text{ the prime factor following the prime factor } p_{j/n}. \]

Then suppose the opposite: \( \forall p_j \in P_n, n - p_j > p_m \): Assumption 3 \( \rightarrow \) S3

So for \( p_j = p_m \) we then obtain \( n - p_m > p_m \Rightarrow n > 2p_m \)

As \( n > p_m \) then \( p_m < 2p_m < n \).

However, according to Chebychev’s theorem, there is always a prime number between \( q \) and \( 2q \) (with \( q \) natural integer >1) and since there is
no prime factor between $p_m$ and $n$ then there is no also a prime factor between $p_m$ and $2p_m$ which contradicts the theorem of Tchebychev, we then deduce that $\exists p_{j/n} \in Pn$ such that $n-p_{j/n}>p_m$ the assumption $S3$ is therefore closed.

And at the same time we have just proved that $n-p_m<p_m$.

On the other hand,

→ If $n-p_{m-1}<p_m$ then $n-p_{m-1}<n-p_{j/n}$ because $n-p_{m-1}<p_m<n-p_{j/n} \Rightarrow p_{j/n}<p_{m-1}$ and therefore $p_{j/n}$ is included between $p_1$ and $p_{m-1}$ and then $p_{(j/n)+1}$ is between $p_2$ and $p_m$.

→ If $n-p_{m-1}>p_m$ then we have $n-p_m<p_m$ and $n-p_{m-1}>p_m$ so $p_{j/n}=p_{m-1}$ and $p_{(j/n)+1}=p_m$.

We have therefore just demonstrated the existence of $p_{j/n}$ and $p_{(j/n)+1}$.

We therefore have $\forall p_i \in Pn$ such that $p_i \geq p_{(j/n)+1}$

$n-p_{j/n}>p_m \Rightarrow n-p_{j/n}>p_i \Rightarrow n-p_i>p_{j/n}$

and more particularly $\forall p_i \geq p_{(j/n)+1}, p_{j/n}<n-p_i<p_m$.

Let us then study the distribution of these $n-p_i$ between $p_{j/n}$ and $p_m$:

Let $p_i$ be between $p_{j/n}$ and $p_m$, $\exists! p_{i1}>p_{j/n}$ and $\exists! p_{i2}>p_{j/n}$ such that $p_{i1}$ and $p_{i2}$ are successive prime factors with $p_{i1}<n-p_i<p_{i2}$, strictly because $n-p_i$ is not prime and $p_{i1}$ and $p_{i2}$ are prime (with $p_1<n-p_i<p_m$).

We have $p_{i1}<n-p_i<p_{i2} \Rightarrow p_i<n-p_{i1}$ and $p_i>n-p_{i2} \Rightarrow n-p_{i2}<p_i<n-p_{i1}$
More over for the \( n-p_k \), \( n-p_{i2} \) and \( n-p_{i1} \) are also successive because

\[
p_{i1} < p_{i2} < p_{i3} < \ldots \ldots \Rightarrow \ldots \ldots < n-p_{i3} < n-p_{i2} < n-p_{i1} < \ldots
\]

this shows two different \( n-p_k \) cannot belong to the same interval composed by two successive prime factors since there is a prime factor between the two \( n-p_k \) (\( n-p_{i2} < p_i < n-p_{i1} \))

Let us then schematize this distribution on the following graduated line:

In effect,

\[
n-\frac{p_j}{n} > p_m \Rightarrow n-p_m > \frac{p_j}{n}
\]

and \( n-\frac{p_{(j+1)}}{n} < p_m \Rightarrow n-p_m < \frac{p_{(j+1)}}{n} \) whence \( \frac{p_j}{n} < n-p_m < \frac{p_{(j+1)}}{n} \)

the number of \( p_k \) between \( p_m \) and \( \frac{p_{(j+1)}}{n} \) is equal to \( m-(j+1)+1=m-j \).

And the number of \( n-p_k \) (with \( p_k \) between \( \frac{p_{(j+1)}}{n} \) and \( p_{m-1} \) because \( p_m \) is already used between \( \frac{p_j}{n} \) and \( \frac{p_{(j+1)}}{n} \)) is equal to \( (m-1)-(j+1)+1=m-j-1 \)

which corresponds exactly to the number of intervals between \( \frac{p_{(j+1)}}{n} \)
and \( p_m \), and since we have \( n - p_{m-1} < n - p_{m-2} < \ldots < n - p_{(j+1)/n} \) then we have exactly the following distribution:

\[
p_{(j+1)/n} < n - p_{m-1} < p_{(j+2)/n}; \quad p_{(j+2)/n} < n - p_{m-2} < n - p_{(j+3)/n} \ldots \ldots \text{ And } p_{m-1} < n - p_{(j+1)/n} < p_m.
\]

Because we had demonstrated that between two successive prime factors ( \( \geq p_{(j+1)/n} \) ) there is a unique \( n-p_k \) ( \( p_k \geq p_{(j+1)/n} \) )

Similarly \( n - p_{j/n} > p_m \Rightarrow p_m < n - p_{j/n} < n, \)

So all the \( n-p_k \) ( \( p_k \leq p_{j/n} ; \quad n - p_{j/n} < n-p_k \) ) are beyond \( p_m \), which confirms the distribution of the \( n-p_i \) on the graduated ruler drawn above.

So let’s recap all of the above:

\[
\rightarrow \forall \ p_i \text{ between } p_{(j+1)/n} \text{ and } p_m \text{ we have } p_{j/n} < n - p_i < p_m
\]

\[
\rightarrow \text{ Between two successive prime factors greater than } p_{j/n} \text{, there is a unique } n-p_k \text{ with } p_{j/n} \leq p_k \leq p_m.
\]

On the other hand,

We have \( \forall \ p_k \) (between \( p_{j/n} \) and \( p_{m-1} \)) and \( \forall \ p_{k+1} \) (between \( p_{(j+1)/n} \) and \( p_m \)), successive prime factors, \( p_k \) and \( p_{k+1} \) cannot be twin primes ( \( p_{k+1} - p_k = 2 \) ) because if it was then the only integer that exists between \( p_k \) and \( p_{k+1} \) is \( p_k + 1 \) and since \( p_k < n-p_i < p_{k+1} \) then \( n-p_i = p_k + 1 \)

Which is impossible because \( n-p_i \) is odd and \( p_k + 1 \) is even.
From where $\forall p_k \geq p_{j/n},$ $p_k$ and $p_{k+1}$ cannot be twin primes.

So the only twin primes are those between $p_1$ and $p_{j/n}.$

Let’s recap:

→ $n \in E_p$ (set of even integers), $\forall p_i \in P_n$, $n-p_i \notin P_n.$

→ Between each $p_k$ and $p_{k+1}$ (with $k$ between $j/n$ and $m-1$) there is a unique $n-p_i$ (with $i$ between $(j+1)/n$ and $m$).

Schematized on the following graduated line:

→ There are no twin primes between $p_{j/n}$ and $p_m.$

→ The only twin primes exist between $p_1$ and $p_{j/n}.$
We will study later the even numbers between $p_m$ and $n$.

The first integer (increasing direction) in this case is $p_m + 1$, but it is equal to the sum of two prime numbers, likewise for $p_m + 3$: $p_m + 5$: $p_m + 7$.

We will then reason on the numbers of the form $p_m + 2k + 1$ which are the even numbers between strictly $p_m$ and $n$, with $2k + 1$ not prime because the numbers of the form $p_m + 2k + 1$ with $2k + 1$ prime meet the criterion: sum of two prime numbers ($\exists k_1 \in \mathbb{N}$ such that $n = p_m + 2k_1 + 1$).

The numbers of the form $p_m + 2k$ are odd which does not interest us in our case.

The first number such that $p_m + 2k + 1$ even and $2k + 1$ not prime is the number $p_m + 9$ which we will note $n_1$.

Note that $P_{n_1} = P_n$ because there is no longer a prime factor between $p_m$ and $n$ ($p_m(n_1) = p_m(n)$).

Suppose then that $n_1$ is not equal to any sum of two prime factors of $P_n$, we will then adopt the same reasoning as for $n$, where $\exists p_{j/n_1} \in P_n$ such that $p_{j/n_1} < n_1 - p_m < p_{(j+1)/n_1} \Rightarrow p_{j/n_1} < p_m + 9 - p_m < p_{(j+1)/n_1} \Rightarrow$

$p_{j/n_1} < 9 < p_{(j+1)/n_1} \Rightarrow$

$p_{j/n_1} = 7$ and $p_{(j+1)/n_1} = 11$
But we had demonstrated as for $n$ that there are no twin primes between $p_{j/n_1}$ and $p_m$ whereas in this case there are several twin primes beyond $p_{j/n_1} = 7$, contradiction $\Rightarrow n_1 = p_m + 9$ is written as the sum of two prime factors.

Ditto for the second number such that $p_m + 2k+1$ even and $2k+1$ not prime, this number is equal to $n_2 = p_m + 15$ ($P_{n_2} = P_{n_1} = P_n$) $\Rightarrow p_{j/n_2} = 13$ and $p_{(j+1)/n_2} = 17$ and since there are several twin primes beyond $p_{j/n_2} = 13$ then a contradiction and therefore $p_m + 15$ is written as the sum of two prime factors.

Same for $n_3 = p_m + 21 \Rightarrow p_{j/n_3} = 19$ and $p_{(j+1)/n_3} = 23$ and so on....

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