Abstract

To help fill the need for examples of introductory-level problems that have been solved via Geometric Algebra (GA), we derive the equation for a plane that is tangent to three given planes. The approach that we use determines the unit bivector of the tangent plane from the interior and exterior products of the vectors that connect the centers of the given spheres. A more-general version of this approach is presented in an appendix.
1 Introduction

To specify a plane, all we need to know is the unit bivector \( \hat{B} \) parallel to the plane, and one point \((p)\) within the plane. We can then write the equation of the tangent plane as

\[
(x - p) \wedge \hat{B} = 0.
\]  

(1.1)

Our challenge is to identify the required points and unit bivectors for the two planes that are tangent to a given set of three spheres.

2 Some of the Ideas that We Will Find Useful

1. For any two vectors \( \mathbf{a} \) and \( \mathbf{b} \), \( \| \mathbf{a} \wedge \mathbf{b} \|^2 = a^2 b^2 - (\mathbf{a} \cdot \mathbf{b})^2 \).
2. The relationship between a vector $v$ and its projection and “rejection” with respect to a bivector $B$ is as shown in Fig. 1.

3. The unit vector from the center of a sphere to a point of tangency is the dual of the unit bivector of the tangent plane. Note that we need to choose the correct sense of rotation of the bivector, according to the right-hand rule, so that $BI_3^{-1}$ is directed away from the center (Fig. 2). See also [1], pp. 105-108.

4. Let $v$ be a vector, and $\alpha$, $\beta$ be scalars. Then, the multiplicative inverse of the multivector $\alpha I_3 + \beta v$ is $\frac{-\alpha I_3 + \beta v}{\alpha^2 + \beta^2} = \frac{\alpha I_3^{-1} + \beta v}{\alpha^2 + \beta^2}$.

5. Whenever an expression contains the outer product of two perpendicular vectors (call them $a$ and $b$), we should consider substituting the geometric product $ab$ for $a \wedge b$.

### 3 Formulation of the Problem in GA Terms

For any given set of three spheres, there are two planes that are tangent to all three of them (Fig. 3). We will formulate the problem for only one of the two tangent planes (Fig. 4), leaving the formulation for the other plane as an exercise to the reader. We call the smallest sphere $S_1$, and its radius $R_1$. The radii of $S_2$ and $S_3$ are (respectively) $R_2$ and $R_3$.

We begin by reducing the problem from “three spheres” to “one point and two spheres”, as in the solution of the classic Apollonius problem. The center reducing the problem to one of “two spheres and one point” is not necessary, but does help us to identify the essential elements of the problem.
Figure 2: The unit vector from the center of a sphere to a point of tangency is
the dual of the unit bivector of the tangent plane. (Note that we need to choose
the correct sense of rotation of the bivector, according to the right-hand rule, so
that $B I_3^{-1}$ is directed away from the center.)

Figure 3: For any given set of three spheres, there are two planes that are
tangent to three all of them.
Figure 4: We will identify the tangent plane that is shown here. We denote the radii of the three spheres as (respectively) \( R_1, R_2, \) and \( R_3 \), with \( R_1 \) being the smallest.

of the “reduced” sphere \( S'_2 \) is the same as for \( S_2 \); \( S'_2 \)’s radius is \( r_2 = R_2 - R_1 \) (Fig. 5). Similarly, center of the “reduced” sphere \( S'_3 \) is the same as for \( S_3 \); and \( S'_3 \)’s radius is \( r_3 = R_3 - R_1 \).

To complete the formulation, we use the center of \( S_1 \) as the origin, and define the vectors as in Fig. 6. According to this formulation,

\[
\begin{align*}
\mathbf{s}_2 &= P_{\hat{B}}(s_2) + (s_2 \wedge \hat{B}) \hat{B}^{-1} \\
&= P_{\hat{B}}(s_2) - r_2 \hat{B} I_3^{-1} \\
&= P_{\hat{B}}(s_2) + r_2 \hat{B} I_3 .
\end{align*}
\tag{3.1}
\]

\[
\begin{align*}
\mathbf{s}_3 &= P_{\hat{B}}(s_3) + (s_3 \wedge \hat{B}) \hat{B}^{-1} \\
&= P_{\hat{B}}(s_3) - r_3 \hat{B} I_3^{-1} \\
&= P_{\hat{B}}(s_3) + r_2 \hat{B} I_3 .
\end{align*}
\tag{3.2}
\]

4 Our Strategy

We need to identify the unit bivector \( \hat{B} \), and one point within the tangent plane. We will identify \( \hat{B} \) via the outer product of the projections of \( s_2 \) and \( s_3 \). As the required point, we will use \( R_1 \hat{B} I_3^{-1} \) (the point of tangency with \( S_1 \)). Then, we will write the equation of the plane as

\[
\left( \mathbf{x} - R_1 \hat{B} I_3^{-1} \right) \wedge \hat{B} = 0.
\tag{4.1}
\]
Figure 5: Reduction of the problem to “two spheres and one point”. The origin is at the center of \( S_1 \). The centers and radii of radius center of spheres \( S'_2 \) and \( S'_3 \) are as follows:

The center of \( S'_2 \) is the same as for \( S_2 \); \( S'_2 \)’s radius is \( r_2 = R_2 - R_1 \).

The center of \( S'_3 \) is the same as for \( S_3 \); \( S'_3 \)’s radius is \( r_3 = R_3 - R_1 \).

Figure 6: Our formulation of the problem in GA terms. Note that the unit vector of the direction from the ends of the projections to the centers of the spheres is \( \hat{\mathbf{B}}I_3^{-1} = \hat{\mathbf{B}}I_3 \). \( \hat{\mathbf{M}} \) is the unit bivector of the plane that contains the vectors \( s_2 \) and \( s_3 \). \( \hat{\mathbf{B}} \) (the bivector that we wish to identify) contains their projections.
5 Solution

5.1 Preliminary Work

Here, we will write a few results that we will need later. First, \( \hat{\mathbf{B}} \) is the unit bivector of the plane that contains (or to be more correct, is parallel to) the bivector \( \mathbf{P}_B (s_2) \land \mathbf{P}_B (s_3) \). Therefore,

\[
\mathbf{P}_B (s_2) \land \mathbf{P}_B (s_3) = \| \mathbf{P}_B (s_2) \land \mathbf{P}_B (s_3) \| \hat{\mathbf{B}}. \tag{5.1}
\]

From Item 1 in Section 2, we might infer that to find \( \| \mathbf{P}_B (s_2) \land \mathbf{P}_B (s_3) \| \), we will need to know the lengths of the projections of \( \mathbf{P}_B (s_2) \) and \( \mathbf{P}_B (s_3) \). Therefore, we will write expressions for those lengths now:

\[
\| \mathbf{P}_B (s_2) \| = \sqrt{s_2^2 - r_2^2}, \quad \text{and} \quad \| \mathbf{P}_B (s_3) \| = \sqrt{s_3^2 - r_3^2}. \tag{5.2}
\]

5.2 Identify \( \| \mathbf{P}_B (s_2) \land \mathbf{P}_B (s_3) \| \)

We will deal with \( \| \mathbf{P}_B (s_2) \land \mathbf{P}_B (s_3) \| \) first, so that when we have obtained an expression for \( \mathbf{P}_B (s_2) \land \mathbf{P}_B (s_3) \), we may replace that expression with \( \| \mathbf{P}_B (s_2) \land \mathbf{P}_B (s_3) \| \hat{\mathbf{B}} \).

Item 1 in Section 2 noted that for any two vectors \( \mathbf{a} \) and \( \mathbf{b} \), \( \| \mathbf{a} \land \mathbf{b} \|^2 = a^2 b^2 - (\mathbf{a} \cdot \mathbf{b})^2 \). Thus,

\[
\| \mathbf{P}_B (s_2) \land \mathbf{P}_B (s_3) \|^2 = \| \mathbf{P}_B (s_2) \|^2 \| \mathbf{P}_B (s_3) \|^2 - [\mathbf{P}_B (s_2) \cdot \mathbf{P}_B (s_3)]^2.
\]

We have already found expressions for \( \| \mathbf{P}_B (s_2) \| \) and \( \| \mathbf{P}_B (s_3) \| \) (Eqs. (5.2)). To find \( \mathbf{P}_B (s_2) \cdot \mathbf{P}_B (s_3) \), we start from the relations (Eqs. (3.1) and (3.2)) between each “s” vector and its projection. Specifically,

\[
\begin{align*}
\mathbf{P}_B (s_2) + r_2 \hat{\mathbf{B}} L_3 & = s_2 \cdot s_3 \\
\mathbf{P}_B (s_3) + r_3 \hat{\mathbf{B}} L_3 & = s_2 \cdot s_3
\end{align*}
\]

\( \mathbf{P}_B (s_2) \) and \( \mathbf{P}_B (s_3) \) are perpendicular to \( \hat{\mathbf{B}} L_3 \); therefore, the left-hand side reduces to

\[
\mathbf{P}_B (s_2) \cdot \mathbf{P}_B (s_3) + r_2 r_3 (\hat{\mathbf{B}} L_3)^2 = s_2 \cdot s_3.
\]

\( \hat{\mathbf{B}} L_3 \) is a unit vector, so \( \mathbf{P}_B (s_2) \cdot \mathbf{P}_B (s_3) + r_2 r_3 = s_2 \cdot s_3 \), from which

\[
\mathbf{P}_B (s_2) \cdot \mathbf{P}_B (s_3) = s_2 \cdot s_3 - r_2 r_3.
\]

Putting the previous ideas together,

\[
\| \mathbf{P}_B (s_2) \land \mathbf{P}_B (s_3) \| = \sqrt{\| \mathbf{P}_B (s_2) \|^2 \| \mathbf{P}_B (s_3) \|^2 - [\mathbf{P}_B (s_2) \cdot \mathbf{P}_B (s_3)]^2} \\
= \sqrt{(s_2^2 - r_2^2)(s_3^2 - r_3^2) - [s_2 \cdot s_3 - r_2 r_3]^2}.
\]
We’ll see more of the vector \((r_3 s_2 - r_2 s_3)\) shortly.

Expanding the products and simplifying,

\[
\|P_B(s_2) \wedge P_B(s_3)\| = \sqrt{s_2^2 s_3^2 - (s_2 \cdot s_3)^2 - (r_3^2 s_2^2 - 2 r_3 r_2 s_2 \cdot s_3 + r_2^2 s_3^2)}
\]

\[
= \sqrt{\|s_2 \wedge s_3\|^2 - \|r_3 s_2 - r_2 s_3\|^2}. \tag{5.3}
\]

### 5.3 Obtain an expression for \(P_B(s_2) \wedge P_B(s_3)\)

We begin by writing \[
\left[ P_B(s_2) + r_2 \mathbf{B}_3 \right] \wedge \left[ P_B(s_3) + r_3 \mathbf{B}_3 \right] = s_2 \wedge s_3.
\]

Because \([\mathbf{B}_3] \wedge [\mathbf{B}_3] = 0\),

\[
P_B(s_2) \wedge P_B(s_3) = \left[ P_B(s_2) \right] \wedge \left[ P_B(s_3) \right] + \left[ r_2 \mathbf{B}_3 \right] \wedge \left[ P_B(s_3) \right] = s_2 \wedge s_3,
\]

and

\[
P_B(s_2) \wedge P_B(s_3) = \left[ r_3 P_B(s_2) - r_2 P_B(s_3) \right] \wedge \left[ \mathbf{B}_3 \right] = s_2 \wedge s_3.
\]

We wish to find \(\mathbf{B}\), so let’s introduce it in the first term on the left-hand side, via the substitution indicated by Eq. (5.1). Our idea is that we may be able to transform the left-hand side into a product of \(\mathbf{B}\) and some multivector that has an inverse.

\[
\frac{\|P_B(s_2) \wedge P_B(s_3)\| \mathbf{B} + \left[ r_3 P_B(s_2) - r_2 P_B(s_3) \right] \wedge [\mathbf{B}_3]}{= P_B(s_2) \wedge P_B(s_3)} = s_2 \wedge s_3.
\]

With the same idea in mind, we note that we can write

\[
\left[ r_3 P_B(s_2) - r_2 P_B(s_3) \right] \wedge \left[ \mathbf{B}_3 \right]
\]

as

\[
\left[ r_3 P_B(s_2) - r_2 P_B(s_3) \right] [\mathbf{B}_3]\] .

Thus,

\[
\frac{\|P_B(s_2) \wedge P_B(s_3)\| \mathbf{B} + \left[ r_3 P_B(s_2) - r_2 P_B(s_3) \right] [\mathbf{B}_3]}{= P_B(s_2) \wedge P_B(s_3)} = s_2 \wedge s_3. \tag{5.4}
\]

Although that result could be transformed readily into the form of a product of \(\mathbf{B}\) and some multivector, we would still not be able to solve for \(\mathbf{B}\), because we cannot as yet identify the vector \([ r_3 P_B(s_2) - r_2 P_B(s_3) ]\). To get an idea, let’s “see” Eq. (5.4) as an addition of bivectors, examine a geometric representation of the addition of bivectors in general (Fig. 7). The key element in that representation is the “hinge” vector that is parallel to both of the addends. Could \([ r_3 P_B(s_2) - r_2 P_B(s_3) ]\) be that “hinge” in our case? More specifically,
we showed (above) that that vector is parallel to $\hat{B}$. Is that vector also parallel to $s_2 \wedge s_3$? Yes, it is. We can show this by writing $P_{\hat{B}}(s_2)$ as $s_2 - r_2\hat{B}I_3$, and $P_{\hat{B}}(s_3)$ as $s_3 - r_3\hat{B}I_3$:

$$r_3 \left[ s_2 - r_2\hat{B}I_3 \right] - r_2 \left[ s_3 - r_3\hat{B}I_3 \right] = r_3s_2 - r_2s_3.$$  

Therefore, Eq. (5.4) becomes

$$\|P_{\hat{B}}(s_2) \wedge P_{\hat{B}}(s_3)||\hat{B} + |r_3s_2 - r_2s_3| \hat{B}I_3 = s_2 \wedge s_3. \quad (5.5)$$

Two comments are in order. First, a study of Eq. (5.5) shows us why Eq. (5.3) makes sense: the bivector $s_2 \wedge s_3$ is the sum of the bivectors $P_{\hat{B}}(s_2) \wedge P_{\hat{B}}(s_3)$ and $|r_3s_2 - r_2s_3| \hat{B}I_3$, which are perpendicular to each other. Therefore, the relation between the norms of the three vectors has the form of the Pythagorean Theorem.

The second comment is that although we could now proceed with our idea of identifying $\hat{B}$, we can also change our goal, with benefit.

5.4 Changing Our Goal to Take Advantage of an Unforeseen Opportunity

Our goal has been to identify $\hat{B}$, after which we would write the equation of the tangent plane as $\left( x - R_1\hat{B}I_3^{-1} \right) \wedge \hat{B} = 0$. (See Eq. (4.1).) However, an examination of Eq. (5.3) suggests that we could instead find the vector $\hat{B}I_3^{-1}$ directly. One advantage of this change of plans is that we can then write the
Another advantage (as we shall see) is that the expression for \( x \cdot \hat{B}I_3^{-1} = R_1 \).
and thus as
\[
x \cdot \hat{B}I_3^{-1} = R_1.
\] (5.6)

Another advantage (as we shall see) is that the expression for \( \hat{B}I_3^{-1} \) will be both readily interpretable, and easy to check via programs such as GeoGebra.

### 5.5 Continuing in Our New Direction: Find \( \hat{B}I_3^{-1} \) Rather than \( B \)

To make the necessary transformation of Eq. (5.5), we will use the identities \( I_3I_3^{-1} = 1 \) and \( I_3^{-1} = -I_3 \). We begin by writing Eq. (5.5) as
\[
\left[ \left\| P_B(s_2) \wedge P_B(s_3) \right\| I_3 - (r_3s_2 - r_2s_3) \right] \hat{B}I_3^{-1} = s_2 \wedge s_3
\]

Now, we use \( I_3^{-1} \hat{B} = \hat{B}I_3^{-1} \), and the fact that GA products possess the associative property, to write
\[
\left[ \left\| P_B(s_2) \wedge P_B(s_3) \right\| I_3 - (r_3s_2 - r_2s_3) \right] \hat{B}I_3^{-1} = s_2 \wedge s_3
\]

Factoring, we obtain
\[
\left\{ \left\| P_B(s_2) \wedge P_B(s_3) \right\| I_3 - (r_3s_2 - r_2s_3) \right\} \hat{B}I_3^{-1} = s_2 \wedge s_3
\]

Left-multiplying both sides by the inverse of the factor in \( \{ \} \)’s,
\[
\hat{B}I_3^{-1} = \left[ \left\| P_B(s_2) \wedge P_B(s_3) \right\| I_3 - (r_3s_2 - r_2s_3) \right]^{-1} \left[ s_2 \wedge s_3 \right].
\]

To find that inverse, we use Item 4 from Section 2
\[
\left[ \left\| P_B(s_2) \wedge P_B(s_3) \right\| I_3 - (r_3s_2 - r_2s_3) \right]^{-1} = -\frac{\left\| P_B(s_2) \wedge P_B(s_3) \right\| I_3 - (r_3s_2 - r_2s_3)}{\left\| P_B(s_2) \wedge P_B(s_3) \right\|^2 + \left( r_3s_2 - r_2s_3 \right)^2}.
\] (5.7)

From Eq. (5.3), \( \left\| P_B(s_2) \wedge P_B(s_3) \right\|^2 = \left\| s_2 \wedge s_3 \right\|^2 - \left( r_3s_2 - r_2s_3 \right)^2 \). Making this substitution in the denominator of Eq. (5.7), then simplifying (and recalling that \( I_3^{-1} = -I_3 \)), we obtain
\[
\left[ \left\| P_B(s_2) \wedge P_B(s_3) \right\| I_3 - (r_3s_2 - r_2s_3) \right]^{-1} = -\frac{\left\| P_B(s_2) \wedge P_B(s_3) \right\| I_3 - (r_3s_2 - r_2s_3)}{\left\| s_2 \wedge s_3 \right\|^2 + \left( r_3s_2 - r_2s_3 \right)^2}.
\] (5.8)
Therefore,
\[
\hat{B}_3^{-1} = \left[ \frac{\|P_B(s_2) \wedge P_B(s_3)\| I_3^{-1} + (r_2s_3 - r_3s_2)}{\|s_2 \wedge s_3\|^2} \right] [s_2 \wedge s_3]
\]
\[
= \left[ \frac{\|P_B(s_2) \wedge P_B(s_3)\| I_3^{-1} + (r_2s_3 - r_3s_2)}{\|s_2 \wedge s_3\|} \right] \hat{M},
\]
(5.9)

where \(\hat{M} = [s_2 \wedge s_3] / \|s_2 \wedge s_3\|\).

We can put that result in a more readily-interpreted form as follows:
\[
\hat{B}_3^{-1} = \left[ \frac{\|P_B(s_2) \wedge P_B(s_3)\|}{\|s_2 \wedge s_3\|} \right] [I_3^{-1} \hat{M}] + \left[ \frac{1}{\|s_2 \wedge s_3\|} \right] [(r_2s_3 - r_3s_2) \hat{M}]
\]
\[
= \left[ \frac{\sqrt{\|s_2 \wedge s_3\|^2 - \|r_3s_2 - r_2s_3\|^2}}{\|s_2 \wedge s_3\|} \right] [\hat{M} I_3^{-1}] + \left[ \frac{1}{\|s_2 \wedge s_3\|} \right] [(r_2s_3 - r_3s_2) \hat{M}]
\]
\[
= \left\{ \sqrt{1 - \left[ \frac{\|r_3s_2 - r_2s_3\|}{\|s_2 \wedge s_3\|} \right]^2} \right\} [\hat{M} I_3^{-1}] + \left[ \frac{1}{\|s_2 \wedge s_3\|} \right] [(r_2s_3 - r_3s_2) \hat{M}].
\]
(5.10)

The first term on the right-hand side is a scalar multiple of the unit normal vector (in the right-hand sense) of the bivector \(s_2 \wedge s_3\) — and thus of the plane that contains the centers of the three given spheres. The second term is a scalar multiple of the 90° rotation (in the right-hand sense) of the vector \((r_2s_3 - r_3s_2)\) in the plane of \(s_2 \wedge s_3\). The result is shown in Fig. 8.

6 For the Reader

In what way does this procedure need to be changed in order to identify the unit bivector of the second tangent plane?

References

Figure 8: The result: All vectors \( \mathbf{x} \) whose endpoints lie in the tangent plane fulfill the condition \( \mathbf{x} \cdot \left[ \hat{B}I_3^{-1} \right] = R_1 \).

\section{A General Case}

Here, we will develop an equation analogous to Eq. (5.10), but for the general case in which

\[ \mathbf{v}_1 = P_B (\mathbf{v}_1) + \lambda_1 \hat{B}I_3^{-1} \]  

(A.1)

and

\[ \mathbf{v}_2 = P_B (\mathbf{v}_2) + \lambda_2 \hat{B}I_3^{-1}, \]  

(A.2)

where either both of the \( \lambda \)'s are positive, or both are negative. As in the main text, we find \( \|P_B (\mathbf{v}_1) \wedge P_B (\mathbf{v}_2)\| \) by first finding \( \|P_B (\mathbf{v}_1)\|,\|P_B (\mathbf{v}_2)\| \), and \( P_B (\mathbf{v}_1) \cdot P_B (\mathbf{v}_2) \), so that we may use the relation

\[ \|P_B (\mathbf{v}_1) \wedge P_B (\mathbf{v}_2)\|^2 = \left[ P_B (\mathbf{v}_1) \right]^2 \left[ P_B (\mathbf{v}_2) \right]^2 - \left[ P_B (\mathbf{v}_1) \cdot P_B (\mathbf{v}_2) \right]^2. \]  

(A.3)

From Eqs. (A.1) and (A.2),

\[ \|P_B (\mathbf{x}_1)\|^2 = \nu_1^2 - \lambda_1^2, \]  

and

\[ \|P_B (\mathbf{v}_2)\|^2 = \nu_2^2 - \lambda_2^2. \]  

(A.4)

To obtain an expression for \( P_B (\mathbf{v}_1) \cdot P_B (\mathbf{v}_2) \), we write

\[ \left[ P_B (\mathbf{v}_1) + \lambda_1 \hat{B}I_3^{-1} \right] \cdot \left[ P_B (\mathbf{v}_2) + \lambda_2 \hat{B}I_3^{-1} \right] = \mathbf{v}_1 \cdot \mathbf{v}_2, \]

from which

\[ P_B (\mathbf{v}_1) \cdot P_B (\mathbf{v}_2) = \mathbf{v}_1 \cdot \mathbf{v}_2 - \lambda_1 \lambda_2. \]  

(A.5)
Therefore, Eq. (A.3) becomes
\[
\|P_B(v_1) \wedge P_B(v_2)\|^2 = [\lambda_1^2 - \lambda_2^2][v_1^2 - v_2^2] - [v_1 \cdot v_2 - \lambda_1 \lambda_2]^2 \\
= v_1^2 v_2^2 - (v_1 \cdot v_2)^2 - \left[(\lambda_2 v_1)^2 - 2\lambda_1 \lambda_2 v_1 \cdot v_2 + (\lambda_1 v_2)^2\right] \\
= \|v_1 \wedge v_2\|^2 - \|\lambda_2 v_1 - \lambda_1 v_2\|^2. \\
\text{(A.6)}
\]

Next, we form the outer product from Eq. (A.1) and (A.2):
\[
\mathbf{B} = \left[ P_B(v_1) + \lambda_1 \mathbf{B} I_3^{-1} \right] \wedge \left[ P_B(v_2) + \lambda_2 \mathbf{B} I_3^{-1} \right] = v_1 \wedge v_2,
\]
from which
\[
P_B(v_1) \wedge P_B(v_2) = P_B(v_1) \wedge \left[\lambda_2 P_B(v_1) - \lambda_1 P_B(v_2)\right] \wedge \left[I_3^{-1}\right] = v_1 \wedge v_2, \\
\|P_B(v_1) \wedge P_B(v_2)\| \mathbf{B} = \left[\lambda_2 P_B(v_1) - \lambda_1 P_B(v_2)\right] \wedge \left[I_3^{-1}\right] = v_1 \wedge v_2,
\]
and (because \(\mathbf{B} I_3^{-1}\) is perpendicular to the vector \(\left[\lambda_2 P_B(v_1) - \lambda_1 P_B(v_2)\right]\))
\[
\|P_B(v_1) \wedge P_B(v_2)\| \mathbf{B} = \left[\lambda_2 P_B(v_1) - \lambda_1 P_B(v_2)\right] \wedge \left[I_3^{-1}\right] = \|v_1 \wedge v_2\| \mathbf{M},
\]
where \(\mathbf{M} = [v_1 \wedge v_2] / \|v_1 \wedge v_2\|\). Writing \(P_B(v_1)\) as \(v_1 - \lambda_1 \mathbf{B} I_3^{-1}\) and \(P_B(v_2)\) as \(v_2 - \lambda_2 \mathbf{B} I_3^{-1}\), then simplifying, the factor \(\left[\lambda_2 P_B(v_1) - \lambda_1 P_B(v_2)\right]\) becomes \(\lambda_2 v_1 - \lambda_1 v_2\).
Thus,
\[
\|P_B(v_1) \wedge P_B(v_2)\| \mathbf{B} = \left[\lambda_2 v_1 - \lambda_1 v_2\right] \wedge \left[I_3^{-1}\right] = \|v_1 \wedge v_2\| \mathbf{M}.
\]

Continuing as in the main text, we use \(I_3^{-1} \mathbf{B} = \hat{\mathbf{B}} I_3^{-1}\), and the fact that GA products possess the associative property, to obtain
\[
\left[\|P_B(v_1) \wedge P_B(v_2)\| I_3 \right] \wedge \left[\hat{\mathbf{B}} I_3^{-1}\right] = \|v_1 \wedge v_2\| \mathbf{M},
\]
\[
\left[\|P_B(v_1) \wedge P_B(v_2)\| I_3 + (\lambda_2 v_1 - \lambda_1 v_2)\right] \wedge \left[I_3^{-1}\right] = \|v_1 \wedge v_2\| \mathbf{M},
\]
and
\[
\hat{\mathbf{B}} I_3^{-1} = \left\{\left[\|P_B(v_1) \wedge P_B(v_2)\| I_3 + (\lambda_2 v_1 - \lambda_1 v_2)\right]^{-1}\right\} \|v_1 \wedge v_2\| \mathbf{M}. \\
\text{(A.7)}
\]

The inverse on the right-hand side is
\[
\frac{\|P_B(v_1) \wedge P_B(v_2)\| I_3^{-1} + (\lambda_2 v_1 - \lambda_1 v_2)}{\|P_B(v_1) \wedge P_B(v_2)\|^2 + (\lambda_2 v_1 - \lambda_1 v_2)^2}.
\]
Recalling that \(\|P_B(v_1) \wedge P_B(v_2)\|^2 = \|v_1 \wedge v_2\|^2 - \|\lambda_2 v_1 - \lambda_1 v_2\|^2\), the denominator reduces to \(\|v_1 \wedge v_2\|^2\), and Eq. (A.7) becomes
\[
\hat{\mathbf{B}} I_3^{-1} = \left\{\frac{\|P_B(v_1) \wedge P_B(v_2)\| I_3^{-1} + (\lambda_2 v_1 - \lambda_1 v_2)}{\|v_1 \wedge v_2\|}\right\} \mathbf{M} \\
= \left[\|P_B(v_1) \wedge P_B(v_2)\|\right] \left[I_3^{-1} \mathbf{M}\right] + \frac{1}{\|v_1 \wedge v_2\|} \left[\lambda_2 v_1 - \lambda_1 v_2\right] \mathbf{M}.
\]

13
Now, we use $\|P_B(v_1) \wedge P_B(v_2)\| = \sqrt{\|v_1 \wedge v_2\|^2 - \|\lambda_2 v_1 - \lambda_1 v_2\|^2}$ and $I_3^{-1} \hat{M} = \hat{M} I_3^{-1}$:

$$\hat{B}_3^{-1} = \left[ \sqrt{\frac{\|v_1 \wedge v_2\|^2 - \|\lambda_2 v_1 - \lambda_1 v_2\|^2}{\|v_1 \wedge v_2\|}} \right] \left[ \hat{M} I_3^{-1} \right]$$

$$+ \frac{1}{\|v_1 \wedge v_2\|} \left[ (\lambda_2 v_1 - \lambda_1 v_2) \hat{M} \right].$$

Simplifying,

$$\hat{B}_3^{-1} = \left\{ \left[ 1 - \left( \frac{\|\lambda_2 v_1 - \lambda_1 v_2\|}{\|v_1 \wedge v_2\|} \right)^2 \right] \right\} \left[ \hat{M} I_3^{-1} \right]$$

$$+ \frac{1}{\|v_1 \wedge v_2\|} \left[ (\lambda_2 v_1 - \lambda_1 v_2) \hat{M} \right]. \quad (A.8)$$

The first term on the right-hand side is a scalar multiple of the unit normal vector (in the right-hand sense) of the bivector $v_1 \wedge v_2$. The second term is a scalar multiple of the 90 deg rotation (in the right-hand sense) of the vector $(\lambda_2 v_1 - \lambda_1 v_2)$ in the plane of $v_1 \wedge v_2$. 

14