A proof questioning principles of series and sums in Analysis by using a formula of Poisson

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Abstract:
This is a mathematical example and a logical opportunity for those who doubt the summation of Poisson or the related definitions of Riemann. By using a formula of Poisson, this article makes a mathematical proof by considering that a limit of a sum is rational. Finally, you will find a logical explanation based on mathematical principles that may be helpful for all students of Analysis who will find a new opportunity to demonstrate.

Keywords: Series, sum, Poisson, Riemann, rational, irrational, limit, lgics, mathematics, analysis.

Introduction:
It is a fact that all scientists and engineers need mathematicians. However, many principles of discrete mathematics are not well accepted among people who only apply mathematical tools. I made a previous short article giving the example of a runner trying to run in a field that can be integrated by respecting Riemann's definition of the integrals [1]. Here is a new mathematical example and a logical opportunity for those who doubt the summation of Poisson or hate the hypothesis related to the Zeta Function of Riemann [2].

This article makes a mathematical proof by making a consideration that a limit of a sum is rational. Of course, most of the readers of this article will agree about this initial consideration since in this example, we are using the sum of the series \( \sum_{n \in \mathbb{N}} \left( \frac{1}{2} \right)^n \) and it seems that there is logically a last digit in the decimal part of this sum since

\[
\lim_{n \to \infty} \left( \frac{1}{2} \right)^n = 0.
\]

You will find the mathematical proof followed by a logical explanation based on Analysis principles in mathematics.

1. The example of the approach:

For Riemann Zeta Function, Riemann needed this formula of Poisson:
\[
\sum_{n \in \mathbb{Z}} \exp(-\pi \times n^2 \times s) = \frac{1}{\sqrt{s}} \times \sum_{n \in \mathbb{Z}} \exp(-\pi \times n^2) \quad \text{(for any: } s > 0) \quad (1)
\]

Let's consider that \( t > 1 \) and let's use \( s = \ln(t) \) in this formula. \hspace{1cm} (2)

We have:

\[
\sum_{n \in \mathbb{Z}} \exp(-\pi \times n^2 \times \ln(t)) = \frac{1}{\sqrt{\ln(t)}} \times \sum_{n \in \mathbb{Z}} \exp(-\pi \times n^2) \quad (3)
\]

\[
\Leftrightarrow \sum_{n \in \mathbb{Z}} t^{-\pi \times n^2} = \frac{1}{\sqrt{\ln(t)}} \times \sum_{n \in \mathbb{Z}} \exp(-\pi \times n^2) \quad (4)
\]

Now let's consider that: \( u = \frac{1}{t} \) and we have consequently: \( 0 < u < 1 \). \hspace{1cm} (5)

And we have:

\[
2 \times \sum_{n \in \mathbb{N}} u^{\pi \times n^2} - 1 = \frac{1}{\sqrt{-\ln(u)}} \times \sum_{n \in \mathbb{Z}} \exp\left(\frac{\pi \times n^2}{\ln(u)}\right) \quad (6)
\]

Hence:

\[
\sum_{n \in \mathbb{N}} u^{\pi \times n^2} = \frac{1}{2 \times \sqrt{-\ln(u)}} \times \left( \sum_{n \in \mathbb{Z}} \exp\left(\frac{\pi \times n^2}{\ln(u)}\right) + \frac{1}{2} \right) \quad (7)
\]

Now let's consider that: \( u = \left(\frac{1}{2}\right)^{\pi} \) \hspace{1cm} (8)

We can verify that: \( 0 < u < 1 \).

Consequently, we get this formula:

\[
\sum_{n \in \mathbb{N}} \left(\frac{1}{2}\right)^{n^2} = \frac{1}{2 \times \sqrt{\ln(2)}} \times \left( \sum_{n \in \mathbb{Z}} \exp\left(-\frac{\pi^2 \times n^2}{\ln(2)}\right) + \frac{1}{2} \right) \quad (9)
\]

2. The development and the considerations:

The series \( \sum_{n \in \mathbb{N}} \left(\frac{1}{2}\right)^{n^2} \) is composed of some terms of the series \( \sum_{n \in \mathbb{N}} \left(\frac{1}{2}\right)^{n} \) and since these two series are increasing monotonic series then:
\[ \sum_{n \in \mathbb{N}} \left( \frac{1}{2} \right)^{n^2} < \sum_{n \in \mathbb{N}} \left( \frac{1}{2} \right)^n = 2 \]  

(10)

consequently:  \[ \sum_{n \in \mathbb{N}} \left( \frac{1}{2} \right)^{n^2} - \frac{1}{2} < \frac{3}{2} \]  

(Remark 1)  

(11)

Let's consider that \( l \) is the limit of:  \[ \sum_{n \in \mathbb{N}} \left( \frac{1}{2} \right)^{n^2} \], and let's consider that \( l \in \mathbb{Q} \) because it seems that there is logically a last digit in the decimal part of this sum since  \[ \lim_{n \to \infty} \left( \frac{1}{2} \right)^n = 0 \] .

Hence:  \[ l = \frac{p}{q} \in \mathbb{Q} \quad \text{with} \quad l \neq 0 \] .

Consequently:  \[ B = \frac{1}{2 \times \sqrt{\frac{\ln(2)}{\pi}}} \times \left( \sum_{n \in \mathbb{Z}} \exp\left( -\frac{\pi^2 \times n^2}{\ln(2)} \right) \right) + \frac{1}{2} = \frac{p}{q} \]  

(Formula 1)  

(12)

Hence:  \[ A = \frac{q}{\sqrt{\frac{\ln(2)}{\pi}}} \times \left( \sum_{n \in \mathbb{Z}} \exp\left( -\frac{\pi^2 \times n^2}{\ln(2)} \right) \right) = (2 \times p - q) \in \mathbb{Z} \setminus \{0\} \]  

(Remark 2)  

(13)

since:  \( 2p \neq q \) because:  \( p \in \mathbb{N} \setminus \{0\} \) and  \( q \in \mathbb{N} \setminus \{0\} \) and  \( p \land q = 1 \) .

**A useful remark:**

From Remark 1 and Remark 2, we deduce that:  \( A < 3q \) . Hence:  \( 2p - q < 3q \)  

(14)

Consequently:  \[ B = \frac{p}{q} < 2 \]  

(Remark 3)  

(15)

**3. The contradiction and the conclusion:**

A simple calculation gives this approximative value for \( C \) the coefficient of Formula 1:
\[ C = \frac{1}{2 \times \sqrt{\frac{\ln(2)}{\pi}}} \approx 1.064467 \]  

(16)

And we know that the biggest term \( D \) of  \[ \sum_{n \in \mathbb{Z}} \exp\left( -\frac{\pi^2 \times n^2}{\ln(2)} \right) \] is  \( D = \exp\left( -\frac{\pi^2 \times 0}{\ln(2)} \right) = 1 \)

With  \[ \sum_{n \in \mathbb{Z}} \exp\left( -\frac{\pi^2 \times n^2}{\ln(2)} \right) \] is an increasing monotonic series.
Hence: \( B > C \times D + \frac{1}{2} \approx 1.564467 \). \hfill (17)

However, we proved that: \( B = \frac{p}{q} < 2 \)

Consequently, we proved that: \( 1.564467 < B < 2 \) \hfill (18)

This means that: \( 1.564467 < \sum_{n \in \mathbb{N}} \left( \frac{1}{2} \right)^n < 2 \) \hfill (19)

Also, we can easily prove from Remark 2 that: \( 2q \times \left( \sum_{n \in \mathbb{N}} \left( \frac{1}{2} \right)^n - \frac{1}{2} \right) \in \mathbb{N} \setminus \{0\} \) \hfill (20)

Hence: \( 2q \times \left( \sum_{n \in \mathbb{N}} \left( \frac{1}{2} \right)^n - \frac{1}{2} \right) = 2q \times \left( \sum_{n \in \mathbb{N}} \left( \frac{1}{2} \right)^n \right) - q \in \mathbb{N} \setminus \{0\} \) \hfill (21)

Consequently: \( \sum_{n \in \mathbb{N}} \left( \frac{1}{2} \right)^n \in \mathbb{N} \setminus \{0\} \) since: \( q \in \mathbb{N} \setminus \{0\} \). \hfill (22)

However, we proved that: \( 1.564467 < \sum_{n \in \mathbb{N}} \left( \frac{1}{2} \right)^n < 2 \)

And this is a contradiction since no natural number exists between 2 and 1.564467.

This is a clear contradiction and since we made two considerations:

1) \( l \) the limit of: \( \sum_{n \in \mathbb{N}} \left( \frac{1}{2} \right)^n \) is a rational number.

2) Poisson Formula is correct.

We conclude that:

Either \( l \) the limit of: \( \sum_{n \in \mathbb{N}} \left( \frac{1}{2} \right)^n \) is not a rational number, or the formula of Poisson is false. If the formula of Poisson is false, then the proof of Riemann Zeta Function is false since it needs this formula of Poisson. Hence, there is no need of proving Riemann's Hypothesis in this case. This would be interesting for scientists who hate this famous Hypothesis [2].

However, these are some important explanations based on principles of Analysis since there is no proof that limit \( l \) is rational:

In general, we don't know easily if the sum of a series of « not rational » terms is rational or not. This is also the case for some series with rational terms.
In this special case of the series \( \sum_{n \in \mathbb{N}} \left( \frac{1}{2} \right)^n \) there is logically a last digit in the decimal part of this sum since \( \lim_{n \to \infty} \left( \frac{1}{2} \right)^n = 0 \). The last digit in the decimal part of the sum \( \sum_{n \in \mathbb{N}} \left( \frac{1}{2} \right)^n \) is 5 since there are only terms divided by 2. However, when \( n \) reaches the infinite, all the new digits of the decimal part of the sum \( \sum_{n \in \mathbb{N}} \left( \frac{1}{2} \right)^n \) are zeros. We can define the last digit « 5 » of the decimal part of the limit « \( l \) » of the sum \( \sum_{n \in \mathbb{N}} \left( \frac{1}{2} \right)^n \) at the biggest finite natural number « \( n-m \) » when « \( n \) » reaches the infinite. Hence, all the new digits of the decimal part of « \( l \) » are 0 starting from the term « \( n-m+1 \) » which is infinite.

However, since the series \( \sum_{n \in \mathbb{N}} \left( \frac{1}{2} \right)^n \) reaches the limit « \( l \) » at the term of « \( n-m \) » which is finite, and since \( (n-m)^2 \) is the number of digits of the decimal part of the limit « \( l \) » of the series \( \sum_{n \in \mathbb{N}} \left( \frac{1}{2} \right)^n \) and since \( (n-m)^2 > (n-m) \) then \( (n-m)^2 \) should be the term of the last digit 5. Otherwise \( (n-m)^2 \) is infinite and the number of digits is infinite.

We will stay repeating the process of substituting \( (n-m) \) by \( (n-m)^2 \) as the term of the last digit 5 until we reach the infinite. This means that the number of digits of the decimal part of the sum \( \sum_{n \in \mathbb{N}} \left( \frac{1}{2} \right)^n \) is infinite. Hence « \( l \) » the limit of \( \sum_{n \in \mathbb{N}} \left( \frac{1}{2} \right)^n \) is « not a rational number ».

This is what mathematics professors can say: even if \( \lim_{n \to \infty} \left( \frac{1}{2} \right)^n = 0 \), the limit of the sum \( \sum_{n \in \mathbb{N}} \left( \frac{1}{2} \right)^n \) is « not rational », and

\[
\sum_{n \in \mathbb{N}} \left( \frac{1}{2} \right)^n \in \mathbb{R} \setminus \mathbb{Q} \iff \sum_{n \in \mathbb{N}} \left( \frac{1}{2} \right)^n - \sum_{n \in \mathbb{N}} \left( \frac{1}{2} \right)^n \in \mathbb{R} \setminus \mathbb{Q} \quad \text{since} \quad \sum_{n \in \mathbb{N}} \left( \frac{1}{2} \right)^n = 2.
\]

But, wait !!! This should be valid even for the similar series \( \sum_{n \in \mathbb{N}} \left( \frac{1}{2} \right)^n \) since it is made only of terms divided by 2. However, everybody knows that \( \sum_{n \in \mathbb{N}} \left( \frac{1}{2} \right)^n = 2 \).

*Now, can you prove that the limit of the sum \( \sum_{n \in \mathbb{N}} \left( \frac{1}{2} \right)^n \) is « rational » ????

References:
