Distribution of photons under the control of the classical equipartition and Boltzmann distribution laws—Resolution of the ultraviolet catastrophe paradox

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Abstract

Since Planck's explanation of blackbody radiation over one century ago, researchers have considered that the law of equipartition does not apply to photons. Furthermore, since the publication of Bose distribution theory, photons have been presumed to follow the Bose distribution (quantum statistics) rather than the Boltzmann distribution. I first show that the mean energy of photons based on Wien's radiation law, which slightly differs from that of Planck's law, equals the mean energy $3kT$ of the lattice vibrations of a solid and hence satisfies the idealized classical law of equipartition (where k is Boltzmann's constant and T is the equilibrium temperature). However, zero-frequency photons are logically nonexistent because all of their components are zero; accordingly, they must be excluded from the distribution. Using this fact, I demonstrate that Planck's law is merely Wien's radiation law with the zero-frequency photons excluded from the distribution, which follows the classical law of equipartition and the Boltzmann distribution law. Additionally, I show that the accepted theory of photons cannot exclude zero-frequency photons from the distribution. The present study might help to elucidate the mechanism of the specific heat of solids mediated by phonons, which (like photons in a cavity) form standing waves.

Keywords: Boltzmann distribution law; Planck's law; Wien's radiation law; Law of equipartition; Rayleigh–Jeans law; Classical physics

1. Introduction

The equipartition and Boltzmann distribution laws are the core concepts of classical thermodynamics and are inextricably linked [1, 2]. More than a century ago, it was thought that the energy in a cavity diverges if the law of equipartition is applied to electromagnetic fields. In blackbody radiation terms, this divergence is called the ultraviolet catastrophe. Similarly, it was believed that Planck dismissed the law of equipartition in his initial explanation of blackbody radiation, thus creating quantum physics [3-5]. Furthermore, photons are thought to follow the later-published Bose distribution (quantum statistics) rather than the Boltzmann distribution [6], although some studies still adopt the classical interpretation of blackbody radiation [7-11]. Researchers who recognize the importance of the classical interpretation of blackbody radiation for photons have dwindled in number, but their works point in an important direction. This study is an extension of their works.

We first show that the mean energy of photons in a cavity derived from Wien's radiation law equals the mean energy $3kT$ of lattice vibrations of a solid, and hence satisfies the idealized law of equipartition in classical physics. Here, k is Boltzmann's constant and T is the equilibrium temperature. Meanwhile, the mean energy of photons derived from Planck's law is $2.701kT$, which slightly differs from that based on Wien's radiation law [12]. I elucidate the cause of the difference between these two laws.

Zero-frequency photons are logically nonexistent because all their components are zero. Therefore, they must be excluded in a correct derivation of the distributive law for photons[1]. Based on this fact, I demonstrate that the Wien's radiation law follows not only the idealized law of equipartition and Boltzmann distribution law but also that the Planck's law is merely the Wien's radiation with zero-frequency photons removed from the distribution, which follows the law of equipartition and Boltzmann distribution law (our new normalization constant added to Planck's law does not change the relative energy distribution of the photons.)

In this interpretation, the term $1/(e^{h\nu/kT} - 1)$ in Planck's law (where h is Planck's constant and ν is the photon frequency) is merely a modified Boltzmann factor that excludes the zero-frequency photons from the distribution.

We also show that the accepted theory of photons cannot exclude zero-frequency photons from the distribution. My demonstrations confirm that classical physics can explain the behaviors of photons, thus challenging the quantum-physical understanding of photons.

Finally, I suggest that this interpretation can help to elucidate the mechanism of the specific heat of solids mediated by phonons, which (like photons in a cavity) form standing waves.

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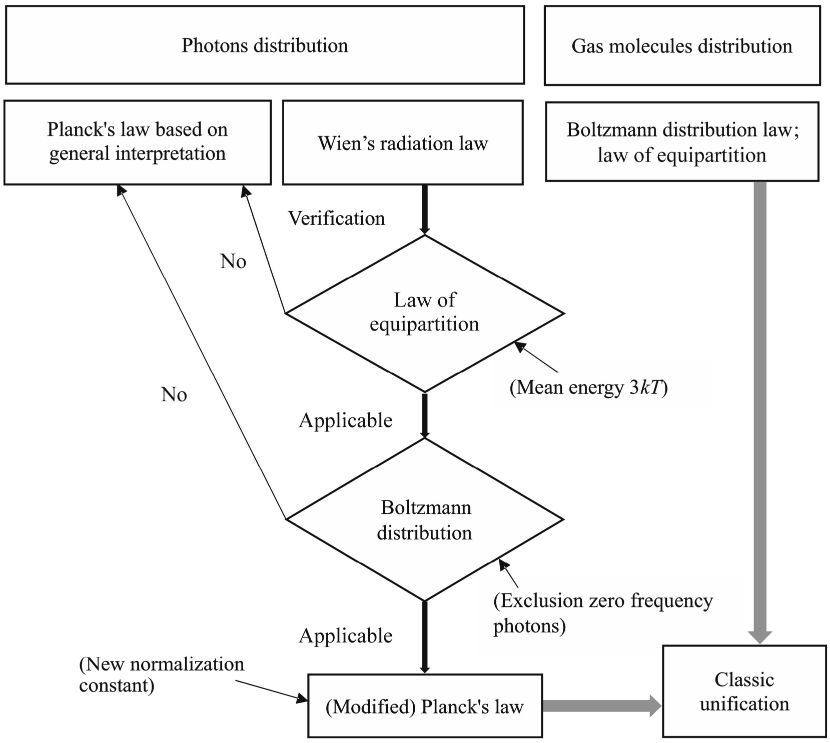
[1] The zero point energy is thought to be irrelevant in an electromagnetic field, and is not considered in Planck's law. Therefore, this study accepts a stochastic interpretation without considering the zero point energy.

The procedure for this study is shown in Fig. 1.

Fig.1: Flowchart showing the procedure of this study



2. Comparison of Wien's radiation law and Planck's law, and derivation of the law of equipartition for photons

***2.1.* Wien's radiation law follows the classical law of equipartition**

If the energy $h\nu$ of a photon with frequency ν follows Wien's radiation law, its energy $U_w(\nu)$ at the equilibrium temperature T is given by [13]

$$U_w(\nu) = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/kT}}V \,, \tag{1}$$

where V is the cavity volume and c is the speed of light. The energy density U_{Dw} at that frequency is

$$U_{Dw} = \int_0^\infty U_w(\nu)d\nu / V = \frac{8\pi h}{c^3}\int_0^\infty \frac{\nu^3}{e^{h\nu/kT}}d\nu \,.$$

Letting $x = h\nu/kT$ and rearranging for $\nu = kTx/h$, this expression becomes

$$= \frac{8\pi h}{c^3}\int_0^\infty \frac{(kTx/h)^3}{e^x}(kT/h)dx = \frac{8\pi k^4T^4}{c^3h^3}\int_0^\infty \frac{x^3}{e^x}dx \,.$$

As $\int_0^\infty \frac{x^3}{e^x}dx = \Gamma(4) = 6$, the right hand side of this expression reduces to

$$=\frac{48\pi k^4 T^4}{c^3 h^3} \tag{2}$$

The number density N_{Dw} of photons is given by

$$N_{Dw} = \int_0^\infty [U_w(\nu)/h\nu] d\nu / V = \frac{8\pi}{c^3}\int_0^\infty \frac{\nu^2}{e^{h\nu/kT}} d\nu,$$

Again, letting $x = h\nu/kT$ and rearranging as $\nu = kTx/h$, I get

$$= \frac{8\pi}{c^3}\int_0^\infty \frac{(kTx/h)^2}{e^x}(kT/h)dx = \frac{8\pi k^3 T^3}{c^3 h^3}\int_0^\infty \frac{x^2}{e^x}dx.$$

As $\int_0^\infty \frac{x^2}{e^x}dx = \Gamma(3) = 2$, the right hand side of this expression reduces to

$$=\frac{16\pi k^3 T^3}{c^3 h^3} \tag{3}$$

Therefore, the mean energy E_{Mw} of photons following Wien's radiation law is

$$E_{Mw} = U_{Dw}/N_{Dw} = 3kT, \tag{4}$$

which is exactly the mean energy of the lattice vibrations of a solid.

As a photon is simply a vibration of an electromagnetic field, it can be approximated as a three-dimensional (3D) harmonic oscillator with six degrees of freedom [14-16]. Therefore, I can write

$$E_{Mw} = 3kT = kT/2 \times 6, \tag{5}$$

where I have allocated an energy of $kT/2$ to each of the six degrees of freedom. Therefore, if photons follow the Wien's radiation law, they also follow the idealized law of equipartition, and fundamentally behave analogously to lattice vibrations in a solid [8].

2.2. Law of equipartition for photons

2.2.1. Mean energy of photons based on Planck's law

Although a modified version of Planck's law will be presented later in this report, the original Planck's law can be used in this section, as described in Footnote 9.

According to Planck's law, the energy $U(\nu)$ per unit frequency of photons with frequency ν and temperature T is [17]

$$U(\nu) = \frac{8\pi h \nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1} V. \tag{6}$$

The energy density U_D of the photons in the cavity is then given by [18]

$$U_D = \int_0^\infty U(\nu) d\nu \, /V = \frac{8\pi h}{c^3} \int_0^\infty \frac{\nu^3}{e^{h\nu/kT} - 1} d\nu \, .$$

Letting $x = h\nu/kT$ and rearranging to give $\nu = kTx/h$, the right hand side of this expression becomes

$$= \frac{8\pi h}{c^3} \int_0^\infty \frac{(kTx/h)^3}{e^x - 1} (kT/h) dx = \frac{8\pi k^4 T^4}{c^3 h^3} \int_0^\infty \frac{x^3}{e^x - 1} dx \, ,$$

which, given $\int_0^\infty \frac{x^3}{e^x - 1} dx = \frac{\pi^4}{15}$, reduces to

$$= \frac{8\pi^5 k^4 T^4}{15 c^3 h^3} \tag{7}$$

By (6), the number density N_D of the photons can be expressed as follows, where N_P is the number of photons in the cavity:

$$N_D = \frac{N_P}{V} = \int_0^\infty \left[U(\nu) / h\nu \right] d\nu / V$$

$$= \int_0^\infty \left(\frac{8\pi h \nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1} \div h\nu \right) d\nu = \frac{8\pi}{c^3} \int_0^\infty \frac{\nu^2}{e^{h\nu/kT} - 1} d\nu \, .$$

Again, letting $x = h\nu/kT$ and rearranging for $\nu = kTx/h$, I have

$$= \frac{8\pi}{c^3} \int_0^\infty \frac{(kTx/h)^2}{e^x - 1} (kT/h) dx = \frac{8\pi k^3 T^3}{c^3 h^3} \int_0^\infty \frac{x^2}{e^x - 1} dx \, .$$

As $\int_0^\infty \frac{x^2}{e^x - 1} dx = \Gamma(3)\, \zeta(3)$, where $\zeta(x)$ is the Riemann zeta function and $\Gamma(3) = 2$, the right hand side of this expression reduces to

$$= \frac{8\pi k^3 T^3 \times 2\varsigma(3)}{c^3 h^3} \tag{8}$$

Therefore, the mean energy E_M of photons based on Planck's law is [12]

$$E_M = U_D / N_D$$

$$= \frac{\pi^4}{30\varsigma(3)} \times kT = \frac{\varsigma(4)}{\varsigma(3)} \times 3kT \approx 2.701kT, \tag{9}$$

where $\zeta(3) \approx 1.202$ and $\zeta(4) = \pi^4/90$.

The mean energy $3kT$ based on Wien's radiation law slightly differs from $2.701kT$ based on Planck's law. The relationship between this difference and the law of equipartition will be clarified in the next subsection.

2.2.2. Derivation of the law of equipartition for photons and confirmation with state equations

Here, I derive the law of equipartition for photons by comparing the behaviors of gas molecules and photons.

According to the ideal gas state equation, $PV = m_oRT = m_oN_AkT = NkT$ (where P is pressure, V is volume of a vessel, m_o is the number of moles, R is the universal gas constant, N_A is Avogadro's number, and N is the number of gas molecules contained in the vessel) [1, 2]. The Boltzmann's constant k is the following function of N, T, V, and P:

$$k = \frac{PV}{TN}.$$

Denoting the mean and total energies of the gas molecules as E_{Mg} and $E_{Mg}N$, respectively, and setting P equal to 2/3 of the energy density ($2E_{Mg}N/3V$) [1, 2], I have

$$k = \frac{(2E_{Mg}N/3V)V}{TN} = \frac{2E_{Mg}/3}{T} \quad \text{(thus, } E_{Mg} = \frac{3kT}{2}\text{).} \tag{10}$$

Equation (10) is a direct manifestation of the law of equipartition, which confirms that the mean energy of each degree of freedom is $kT/2$, as one gas molecule has three degrees of freedom [1, 2]. Thus, (10) results from an exquisite balance among N, T, V, and P under the law of equipartition.

We first examine whether the following equation, which is equivalent to (10), holds for photons following Wien's radiation law, where N_{Pw} is the number of photons and P_{Rw} *is* the radiation pressure based on Wien's radiation law[2]:

$$k = \frac{P_{Rw}V}{TN_{Pw}}.$$

Adopting Wien's radiation law, the total energy of the photons is $E_{Mw}N_{Pw}$ and P_{Rw} is one-third of the energy density ($E_{Mw}N_{Pw}/3V$) [19]. I thus have

$$k = \frac{(E_{Mw}N_{Pw}/3V)V}{TN_{Pw}} = \frac{E_{Mw}}{3T} \quad \text{(by which } E_{Mw} = 3kT\text{).} \tag{11}$$

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[2] Radiation pressure and the pressure of gas molecules both act on the walls of a cavity or vessel. Therefore, they are regarded as identical in this discussion.

If (11) holds, then $E_{Mw} = 3kT$ as shown in (4). Therefore, (11) directly results from the law of equipartition, confirming that the mean energy of each degree of freedom is $kT/2$ and that a photon has six degrees of freedom as shown in (5).

Substituting the number of photons N_P and the radiation pressure P_R based on Planck's law into (11) and letting the obtained value be k_P

$$k_P = \frac{P_R V}{T N_P}.$$

Letting the total energy of the photons be $E_M N_P$ and P_R be one-third of the energy density ($E_M N_P/3V$) [19], I have

$$k_P = \frac{(E_M N_P / 3V)V}{T N_P} = \frac{E_M}{3T} \quad \text{(that is, } E_M = 3k_P T\text{).} \tag{12-a}$$

If (12-a) holds, $E_M = 3k_P T$ must hold. Using $N_P = V \times N_D$ from (8), $P_R = U_D/3$ from (7) [19], and $\zeta(3) = 1.202$, (12-a) can be rewritten as

$$k_P = \frac{P_R V}{T N_P} = \frac{(8\pi^5 k^4 T^4 / 15c^3 h^3 \div 3) \times V}{T \times (V \times 8\pi k^3 T^3 \times 2\varsigma(3) / c^3 h^3)} \approx 0.9004k \text{ (constant).} \tag{12-b}$$

From (12-a) and (12-b), I obtain

$$E_M = 3k_P T \approx 3 \times 0.9004kT = 2.701kT. \tag{13}$$

This result perfectly agrees with $E_M = 2.701kT$ in (9), implying that $E_M = 3k_P T$ holds and that (12-a) holds for photons in a cavity. Therefore, like (10) and (11), (12-a) should confirm to the law of equipartition.

In fact, given that a photon (a vibration of an electromagnetic field) can be approximated as a 3D harmonic oscillator with six degrees of freedom [14-16], I can write

$$E_M = 3k_P T = k_P T/2 \times 6 \; (\approx 0.9004k\, T/2 \times 6 \approx 2.701kT). \tag{14}$$

The term $3k_P T$ in (13) is the result of allocating $k_P T/2$ to each of the six degrees of freedom of photons, as shown for the *law of equipartition of photons* based on Planck's law. Therefore, the classical law of equipartition (or a law based on the same mechanism) can be applied to photons[3]. Like (10), (12-a) is the product of an exquisite balance among N_P, T, V, and P_R under the law of equipartition.

We now clarify that the reexpression of k as k_P in (12-a)–(14) is the result of applying the Boltzmann distribution law to photons, and that the distribution law must be expressed in terms of k (not k_P).

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[3] Equations (12-a)–(14) also hold by (23), the modified version of Planck's Law (see Footnote 9).

3. Distribution of all photons determined by applying the Boltzmann distribution law

3.1. Wien's radiation law following the Boltzmann distribution law and its error: Physical meaning of $8\pi v^2/c^3$

As discussed above, (12-a) reflects the exquisite balance among N_P, T, V, and P_R. Therefore, N_P and V are strongly correlated. When a cavity is divided, the equilibrium temperature and pressure in each partition are constant and equal to T and P_R, respectively, to preserve (12-a). The ratio of the volume to the number of photons in each part must also be constant and equal to V/N_P. This is realized by the following mechanism:

If the mean volume per photon is $s_m = V/N_P$ and n is an integer, then when the volume ns_m is excluded from V (i.e., $V - ns_m$), N_P will inevitably reduce to $N_P - n$, and thus, V/N_P is always constant as follows.

$$\frac{V \pm ns_m}{N_P \pm n} = \frac{V \pm nV/N_P}{N_P \pm n} = \frac{V}{N_P} \quad \text{(constant).} \tag{15}$$

According to (15), one photon and one volume s_m (on average) must always be added or subtracted in pairs, and (15) and (12-a) cannot hold without this pair relationship. That is, each photon is enclosed by its own space with volume s_m (on average) and cannot overlap with other photons (see Fig. 2a)[4, 5]. The space enclosing one photon can be regarded as a balloon or an exclusive cell. Consequently, only the spaces containing photons with frequency v can be gathered in theory (see Fig. 2b, c). As each photon is separated from all other photons, the number density ρ_v of photons in the space of photons with v necessarily equals the number density of standing waves based on the Rayleigh–Jeans law (see Section 4 for a comparison with the accepted theory) [20, 21]:

$$\rho_v = \frac{8\pi}{c^3} v^2 . \tag{16}$$

Therefore, the particular volume (not s_m) paired with a photon of frequency v is $c^3/8\pi v^2$ (the reciprocal of ρ_v), which contains only one photon. The average of all particular volumes is s_m (later, I will show that the particular volumes sum to V.)

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[4] This overlap differs from wave superposition in that the amplitude increases additively.

[5] The particular volume paired with a photon in (16) is not s_m because (12-a) and (15) are statistical expressions based on the average value of each element, as shown in (12-a).

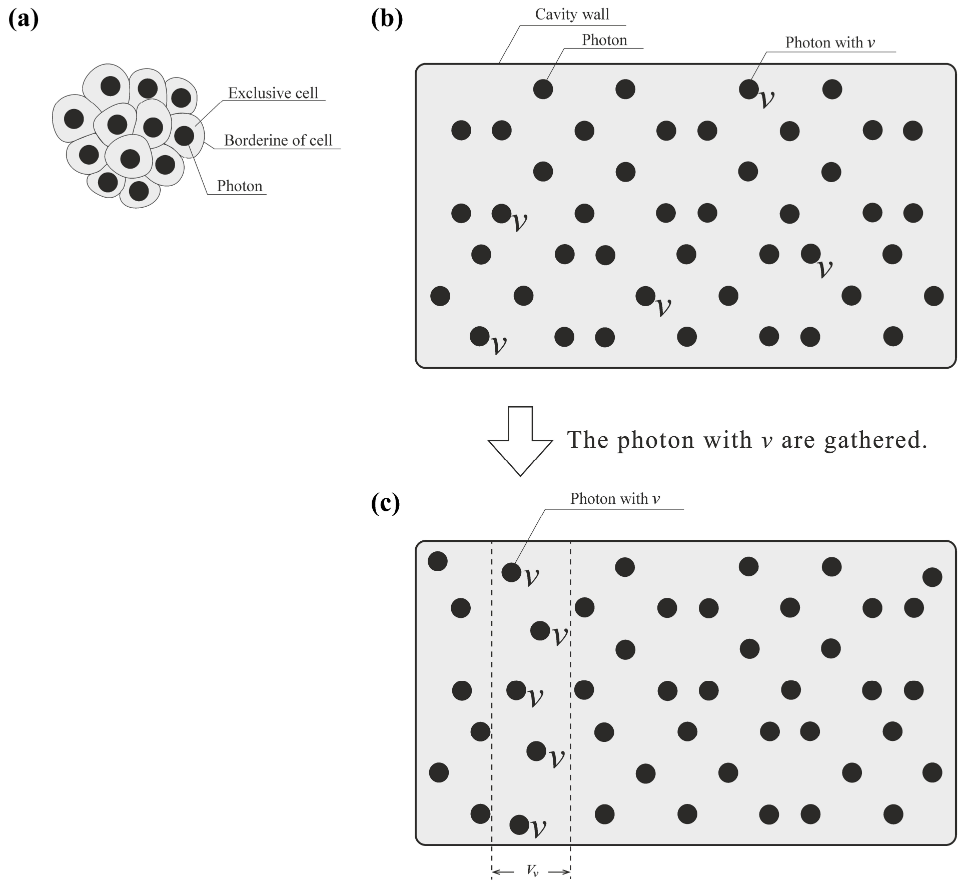

Fig. 2 Relationship between the volume of a cavity and photons. (a) Single photons in exclusive cells, where each point (•) is a photon and each circle is an exclusive cell. (b) Conditions in a cavity, where each point ($\bullet_\nu$) is a photon with frequency ν. (c) The gathering of photons of frequency ν, where V_ν is the distributed volume of this group of photons. Panels (b) and (c) exclude the borders of the exclusive cells.

The above derivation of ρ_ν is nonstochastic so a statistical distribution function is not required. In fact, ρ_ν is only the number density. The number of photons distributed in each energy level can be determined only be determining the volume occupied by ρ_ν. Therefore, to provide a distribution law for ρ_ν, I must establish a distribution function for the volumes in the cavity.

Dividing V into N_P cells of equal volume s_m for convenience[6], and assuming that the cell distribution follows

[6] However, s_m defines the mean volume per photon as mentioned above, so the number of photons in the cell is not determined by this division alone.

the Boltzmann distribution law, the volume Vv and energy $U(v)$ of energy level hv (omitting the normalization constant) are respectively given by

$$Vv = s_m \times N_P \times \frac{1}{e^{h\nu/kT}}, \tag{17}$$

and

$$\begin{aligned} U(v) &= \frac{8\pi}{c^3}\nu^2 \times V\nu \times h\nu \\ &= \frac{8\pi}{c^3}\nu^2 \times s_m \times N_P \times \frac{1}{e^{h\nu/kT}} \times h\nu \\ &= \frac{8\pi}{c^3}\nu^2 \times h\nu \times V \times \frac{1}{e^{h\nu/kT}} \quad \text{(because } s_m \times N_P = V\text{).} \end{aligned} \tag{18}$$

Equation (18) is consistent with Wien's radiation law given by (1).

As the cavity exists in thermal equilibrium, the correct distribution law for photons should satisfy the second law of thermodynamics; that is, that entropy is always increasing [17, 22, 23][7]. Planck rigorously proved that (18) satisfies the second law of thermodynamics before deriving (6) [22, 23]. This fact supports that (17) and (18) are based on the correct theory, meaning that the distribution of photons follows the Boltzmann distribution law.

However, (18) is slightly erroneous in the low-frequency region. The following subsection describes a method for resolving these errors without violating the theory in (17) and (18). (Note: Planck considered these errors as a manifestation of theoretical errors in Wien's radiation law and derived (6) by a new approach using Boltzmann's principle [17, 23]. However, this approach is problematic as discussed in Section 4.)

3.2. Interpretation of Planck's law as a modified Wien's radiation law and its normalization: Physical meaning of $1/(e^{hv/kT} - 1)$

In a cubic cavity of side length L, photons exist as standing waves with the wavelength components of $2L/n_x$, $2L/n_y$, and $2L/n_z$ parallel to the x-, y-, and z-axes, respectively, where n_x, n_y, and n_z are integers [20, 21]. According to Maxwell's equations, $n_x^2 + n_y^2 + n_z^2 = 4L^2/\lambda^2$, $\lambda = 2L/\sqrt{n_x^2 + n_y^2 + n_z^2}$, and $v = c\sqrt{n_x^2 + n_y^2 + n_z^2}/2L$ (where λ and v are the photon wavelength and frequency, respectively) [20, 21].

However, photons with $v = c\sqrt{n_x^2 + n_y^2 + n_z^2}/2L = 0$ (i.e., at $n_x = n_y = n_z = 0$) are logically nonexistent

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[7] Planck proved that (18) satisfies the second law of thermodynamics by considering the entropy of oscillators.

because all their components are zero. This situation contrasts with that of gas molecules, which can be motionless. As shown in (15), a photon must always be paired with a particular volume (averaging $V/N_P = s_m$), and thus, the volume distributed to the photons with $v = 0$ cannot exist. Therefore, the volume allocated to photons with $\nu = 0$ must be excluded from the distribution (see Footnote 1)[8]. These volumes are removed from the distribution by the following method.

As shown in Fig. 3, the reciprocal of probability (the total number of elements per target element) statistically equals the *population size per target element* (Popu.S/TE). The probability can thus be expressed as Popu.S/TE.

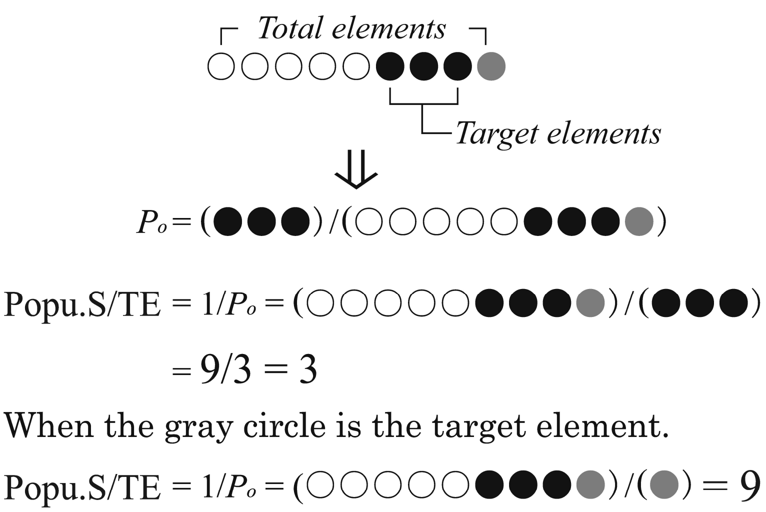

Fig. 3 Conceptual illustration of *population size per target element* (Popu.S/TE). Among a population of nine elements (represented by circles), the number of target elements is three (black circles) or one (gray circle). The probability P_o of drawing a black or gray target element from the total pool of elements is 3/9 (= 1/3) and 1/9, respectively. The corresponding Popu.S/TE is $1/P_o = 3$ and 9, respectively.

Popu.S/TE in the Boltzmann distribution is the reciprocal ($f(x) = e^x$) of the Boltzmann factor (where $x = h\nu/kT$ and the normalization constant is omitted). Exploiting the peculiar property of the Napier number (i.e.,$\int e^x\,dx = e^x + C$, where C is an integration constant), we have $\int_0^j e^x dx = e^j - e^0$ and thus $e^j = \int_0^j e^x dx + 1$.

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[8] Analogously, if fictional spot (e.g., 0) is added to the population of elements on a die, I cannot derive the correct probability.

That is, Popu.S/TE at $x = j$ can be obtained by sequentially adding the Popu.S/TE values from $x = 0$ to j (see **Fig. 4A**), and thus each Popu.S/TE always includes Popu.S/TE at $x = 0$ (i.e., $f(0) = e^0 = 1$) as a part of it. Note that "Popu.S/TE at $x = 0$ (i.e., $f(0) = e^0 = 1$)" means that that zero-energy elements exist and are the only elements occupying the population (when not considering normalization constants). Therefore, applying the Boltzmann distribution to elements, the existence of element with $v = 0$ is essential. On the other hand, in the case of photons, modifying the boundary condition to be $f(0) = 0$ (i.e., $f(x) = e^x - 1$), as shown in **Fig.4B**, it is possible to exclude only the part of Popu.S/TE at $x = 0$ (i.e., $f(0) = e^0 = 1$) from each Popu.S/TE in the Boltzmann distribution. As a result, Popu.S/TE at $x = 0$ changes to $f(0) = e^0 - 1 = 0$, which means that there is no population when $x = 0$ ($v = 0$), and thus the photon with $v = 0$ does not exist. Therefore, the reciprocal (W_B) of the correct distribution function for photons becomes (omitting the normalized constant)

$$W_B = e^x - 1\,. \tag{19}$$

This approach alone can exclude the volume allocated to photons with $x = 0$ ($v = 0$) from the population. (As elaborated in Section 4, the accepted theory of photons cannot exclude the photons with $v = 0$ from the population.)

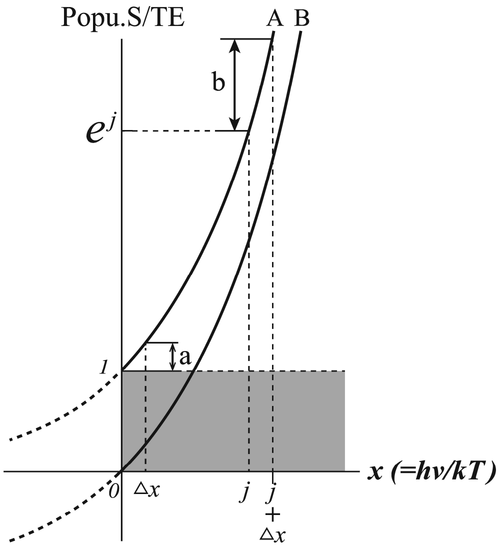

Fig. 4 Method for excluding the volume of photons with frequency ν = 0 from the distribution. (A) Plot of $f(x) = \int_0^x e^x dx + 1$ (which coincides with that of $f(x) = e^x$), where $x = hv/kT$. The vertical axis represents the *population size per target element* (Popu.S/TE). The length (a) is the increment of Popu.S/TE when x changes from 0 to $\triangle x$: (a) $= e^{\triangle x} - e^0 = \int_0^{\Delta x} e^x dx$. The length

(b) is the increment of Popu.S/TE when x changes from j to $j+\triangle x$: (b) $= e^{j+\triangle x} - e^{j} = \int_{j}^{j+\Delta x} e^{x}\,dx$.

The value of Popu.S/TE is the value obtained by sequentially adding each Popu.S/TE, including $e^0 = 1$ (gray area). (B) Plot of $f(x) = \int_0^x e^x\,dx = e^x - 1$, which excludes $e^0 = 1$ from each Popu.S/TE.

According to (19), the relative probability (P_B) based on the Boltzmann distribution law is modified as follows:

$$P_B = \frac{1}{W_B} = \frac{1}{e^x - 1} = \frac{1}{e^{h\nu/kT} - 1} \tag{20}$$

Thus, the Bose distribution function is merely the Boltzmann function excluding the volume allocated to photons with $\nu = 0$. Equation (20) requires the Boltzmann constant k rather than k_P from the process of derivation, and thus, k_P only appears to be a constant. The probability P_{BD} is expressed in terms of the normalization constant D_P as

$$P_{BD} = \frac{1}{D_P W_B} = \frac{1}{D_P(e^{h\nu/kT} - 1)}, \tag{21}$$

$$\int_0^\infty \frac{1}{D_P(e^{h\nu/kT} - 1)}d\nu = 1. \tag{22}$$

According to (21), Wein's radiation law (18) is modified as follows:

$$U(\nu) = \frac{8\pi h\nu^3}{c^3}\frac{1}{D_P(e^{h\nu/kT} - 1)}V. \tag{23}$$

Equation (23) is Wien's radiation law modified by excluding the zero-frequency photons from the distribution. It is based on the Boltzmann distribution law and differs from the original Planck's law (6) only by inclusion of the term D_P. Here, D_P is essential because the distributed volumes must sum to V, meaning that the distribution of all photons is determined by one application of the Boltzmann distribution law[9]:

[9] When (6) is replaced with (23), the D_P terms in the denominator and numerator of (9) and (12-b) cancel so the final result ($2.701kT$ and $0.9004k$) is unchanged.

$$V \times \int_0^{\infty} \frac{1}{D_P(e^{h\nu/kT}-1)} d\nu = V. \tag{24}$$

Relationship between Wien's radiation law and (23): Wien's radiation law follows the idealized Boltzmann distribution and equipartition laws (without D_P) before excluding the zero-frequency photons from the distribution, which is responsible for slight errors in the low-frequency region. As (23) is merely Wien's radiation law modified by excluding the zero-frequency photons from the distribution, it naturally follows the Boltzmann distribution law and the law of equipartition. Thus, the Wien's radiation law and (23) (or Planck's law) differ by a very simple mechanism.

Relationship between Rayleigh–Jeans law and (23) (Resolution of the Ultraviolet Catastrophe): The law of equipartition has been considered inapplicable to photons because the Rayleigh–Jeans law $U(\nu) = 8\pi\nu^2/c^3 \times kT$ based on the law of equipartition diverges at high frequencies. In contrast, although (23) is also based on the law of equipartition as shown in (14), it avoids the ultraviolet catastrophe because the volume distribution is smaller at high frequencies than at lower frequencies; this relationship reduces the number of high-frequency photons.

4. Inability of the accepted theory to rationalize the law of equipartition for photons

As shown in (12-a)-(14) and Footnote 9, *the law of equipartition for photons* [i.e., (12-a)–(14)] can be derived based on either the original Planck's law (6) or (23). However, the accepted theory of photons has interpreted various phenomena related to the Planck's law, denying the application of the law of equipartition (i.e., classical physics) to photons. Therefore, whether or not the law of equipartition (i.e., classical physics) is applied to photons should have a significant influence on the interpretation of these various phenomena.

In the generally accepted theory of photons, $8\pi\nu^2/c^3$ in Planck's law is the number density of the modes containing photons of frequency ν, and $h\nu/(e^{h\nu/kT} - 1)$ is the mean energy E_m of the modes. E_m is thus derived as the following geometric series (where n is an integer) [6, 20, 21, 24]:

$$E_m = \frac{\sum_{n=0}^{\infty} nh\nu e^{-nh\nu/kT}}{\sum_{n=0}^{\infty} e^{-nh\nu/kT}} = \frac{h\nu}{e^{h\nu/kT}-1}. \tag{25}$$

According to (25), the number of photons that can enter the volume per mode, $c^3/8\pi\nu^2$ (the reciprocal of $8\pi\nu^2/c^3$), is unlimited, and empty modes (or cells) are allowed.

However, in the interpretation of (15) and (16), $8\pi\nu^2/c^3$ is the number density of photons (rather than modes), and the particular volume (not the average volume s_m) paired with a single photon of frequency ν is $c^3/8\pi\nu^2$.

Therefore, only a single photon can enter $c^3/8\pi v^2$, and empty modes (or cells) are not allowed. This result is a natural consequence of (12-a) and (15), which I earlier proved to be correct. Therefore, in theory, the interpretation of (6) based on (25) cannot rationalize *the law of equipartition for photons* [i.e., (12-a)–(14)].

Furthermore, as shown in (19)–(22), photons with frequency $v = 0$ must be excluded from the distribution; thereby, the term −1 was added to $1/e^{hv/kT}$, yielding $1/(e^{hv/kT} - 1)$. The same term $1/(e^{hv/kT} - 1)$ in the accepted theory given by (25), which includes Bose–Einstein statistics, is derived assuming that particles are indistinguishable. Consequently, the accepted theory cannot exclude the photons with frequency $v = 0$ from the distribution[10]. The *law of equipartition for photons* must instead be understood through (23) based on the Boltzmann distribution law.

5. Possible application to the theory of specific heat of solids

A solid can be regarded as a set of 3D harmonic oscillators. The classical law of equipartition assigns an energy of $kT/2$ to each degree of freedom of each oscillator. As the mean energy is $3kT$ per oscillator, the total energy U_H of the Avogadro's number (N_A) of oscillators (i.e., 1 mole of atoms) is $U_H = 3N_AkT$. Therefore, the specific heat Cv of the solid is $Cv = dU_H/dT = 3N_Ak = 3R$, where R is the universal gas constant [8].

As revealed in this study, a cavity (like a solid) can be regarded as a set of 3D harmonic oscillators. If I accept Wien's radiation law (4), the mean energy of photons is again $3kT$. The modified Wien radiation law (23) also follows the law of equipartition (or a law based on the same mechanism) (i.e., $E_M = k_PT/2 \times 6 = 3k_PT$). Therefore, (23) should be applicable to solids after being modified to comply with the properties of solids [25]. Applying the Planck's law (or similar a theory), the Einstein and Debye models approximately explain specific heat in the low-temperature region [28, 29], but both models assume that Planck's law denies the law of equipartition. I believe that specific heat theory should be reconsidered by presuming that Planck's law follows the law of equipartition and that D_P is required.

By (17)–(22), if the volume of the solid is divided per frequency of oscillators or phonons and the energy distribution is determined accordingly, the specific heat will be contributed by oscillations or phonons with various frequencies rather than a single frequency [26, 27].

Intervention of Planck's constant *h* in specific heat theory: If the frequency of a photon with mean energy $3k_PT$ in (14) is v_m, I can define h in terms of a constant T/v_m, namely, $h = 3k_PT/v_m$. This expression relates h to the

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[10] In this study, the concept of indistinguishable particles is not necessary.

law of equipartition and lattice vibrations. Therefore, h can intervene in the theory of specific heat of solids, which also follows the law of equipartition and is explained by lattice vibrations [28, 29]. However, T/ν_{max} (where ν_{max} is the peak frequency) is constant in Wien's displacement law whereas T/ν_m depends on the mean energy.

Of course, I should account for the properties of solids and the differences between photons and phonons, but I believe that the present study will help to elucidate the specific heat of solids.

6. Conclusion

I demonstrated that Wien's radiation law follows the idealized law of equipartition and Boltzmann distribution law, and that Planck's law is merely Wien's radiation law modified by excluding the zero-frequency photons from the distribution. After excluding the zero-frequency photons from the distribution, the mean energy allocated to each degree of freedom of the photons is $k_P T/2$ (not $kT/2$). The modified Planck's law (i.e., the modified Wien's radiation law) based on the Boltzmann distribution law must also include the normalization constant D_P, by which the radiant energies yielded by the original Planck's law differ from those of the modified Planck's law (but the relative distributions of the two laws are identical). These findings, which confirm the applicability of classical physics to photons, are expected to change our quantum-physical understanding of photons. This study should help elucidate the mechanism of the specific heat of solids mediated by phonons, which (like photons) form standing waves.

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