On Legendre's conjecture

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Abstract: In this paper, we interesting in most conjecture problem relies with the prime number which is Legendre's conjecture. We also introduced polynomials that check this conjecture with algebraic proof. Also, we reinforced the conjecture with some rules.

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1. Introduction

In 1912, during the International Conference on Mathematics, Edmund Landau posed four basic problems about prime numbers, among them the Legendre’s conjecture. Which states:

Is there always at least one prime number between consecutive square number \( n^2 \) and \((n+1)^2\)?

After algebraic operations, we found that the numbers that fulfill Legendre's conjecture are written in the following form

\[ n^2 + n - 1 = (n + 1)^2 - (n + 2) \quad (1) \]

for \( n \in \mathbb{N} \) where \( n \geq 2 \) with \( n \neq 5k + 2 \) for \( k \geq 1 \) is a natural number. And of course, we found that the numbers written in the form (1) are prime numbers if \( n \neq 5k + 2 \) where \( k \geq 1 \).

It is easy to prove that the numbers written in the form (1) satisfy the relationship

\[ n^2 < n^2 + n - 1 < (n + 1)^2 \]

by proof the correctness of the following:

\[ n^2 - (n^2 + n - 1) = 0 \quad \text{and} \quad (n + 1)^2 - (n^2 + n - 1) > 0 \]

for \( n \geq 2 \).

In the second step we will prove that \( \text{PGCD} \left( n^2, n^2 + n - 1 \right) = 1 \) it is true for every natural number \( n \in \mathbb{N} \). We have \( n^2 + n - 1 = n^2 + 1 \cdot n - 1 \), then

\[ \text{PGCD} \left( n^2, n^2 + n - 1 \right) = \text{PGCD} \left( n^2, n - 1 \right) \]

Also, we get \( 1 \cdot n^2 + (n - 1)(n + 1) = 1 \). That is, there are two integers such that \( \alpha = 1 \) and \( \beta = n + 1 \). According to Puzo's theorem we have \( n^2 \) and \( n - 1 \) are relatively prime. Then \( n^2 \) and \( n^2 + n - 1 \) are relatively prime also.

On the other hand, we find for \( \text{PGCD} \left( n, n - 1 \right) = 1 \) that \( n \) and \( n - 1 \) are relatively prime for every natural number \( n \in \mathbb{N} \). Because, according to Puzo's theory we have \( n - (n - 1) = 1 \). We conclude under Dirichlet's theorem that for \( \text{PGCD} \left( n, n - 1 \right) = 1 \) there is infinitely many primes given by

\[ p(n) = n^2 + n - 1 \quad (2) \]

Let \( \Delta \) be the discriminant of (2). \( \Delta = 5 = 1 \ [4] \). It can be said that the polynomial (2) overlaps the definition of Rabinowitsch polynomial for instant see [4].

Next, we consider the natural number \( n \geq 1 \) where \( n = 5k + r \) with \( r = \{0, 1, 2, 3, 4\} \) and \( k \in \mathbb{N} \). And we will write the polynomial (2) in terms of \( k \) for the following values of \( n = \{5k, 5k + 1, 5k + 2, 5k + 3, 5k + 4\} \) respectively. Then

First, for \( n = 5k \) we find

\[ n^2 + n - 1 = 25k^2 + 5k - 1 = 5k (5k + 1) - 1 = 5m - 1 \]

And for \( n = 5k + 1 \) we find

\[ n^2 + n - 1 = 25k^2 + 15k + 1 = 5k (5k + 3) + 1 = 5m + 1 \]

Next, for \( n = 5k + 2 \) we find

\[ n^2 + n - 1 = 25k^2 + 25k + 5 = 5(5k^2 + 5k + 1) = 5m \]

And for \( n = 5k + 3 \) we find

\[ n^2 + n - 1 = 25k^2 + 45k + 19 = 5(5k^2 + 9k + 4) - 1 = 5m - 1 \]

According to the results obtained previously, we conclude that the polynomial given in (2) became an odd number for \( n = \{5k, 5k + 1, 5k + 2, 5k + 3, 5k + 4\} \). It is written in one of the two forms \( 5m + 1 \) or \( 5m - 1 \).

For \( n = 5k + 2 \), the polynomial (2) is a multiple of 5 and is written in the form \( 5m \) with \( m \in \mathbb{N} \). Of course, except for number 5 for \( k = 0 \).

Also, we give more than one polynomial that give a prime number. Which define as the following:

\[ 25n^2 + 5n - 1, \ 25n^2 + 15n + 1 \text{ and } 25n^2 + 35n + 11 \]

with \( 5n^2 + 5n + 1 \). Finally, \( 25n^2 + 45n + 19 \).

There more, we reduced for the numerical results of the polynomial (2) shown in table (1), this relation as following: for consecutive \( p_j(n) \) and \( p_{j+1}(n) \) we have

\[ p_{j+1}(n) = p_j(n) - 2 + 2n \quad (3) \]

Since, if \( n = 5k + 2 \) for \( k \geq 1 \). We get

\[ p_{j+1}(n) = p_j(n - 2) + 2(2n - 1) \quad (4) \]

And that through the two relations (3) and (4) we can write an algorithm for this formula for \( 2 \leq n < 22 \), see the following table.

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There are infinite prime numbers. And after \( n = 5k + 2 \) we found that the Legendre's conjecture is not true because our polynomial gives us a multiple of 5 and therefore the number is not prime number.

For this we will put a new conjecture, it is as follows:

For the natural number \( n \geq 1 \) where \( n = 5k + r \) with \( r = \{0, 1, 3, 4\} \) and \( k \in \mathbb{N} \). There infinite prime numbers written in the form \( p(n) = n^2 + n - 1 \) and checked

\[
 n^2 < p(n) < (n + 1)^2.
\]

It’s clear from Table (1) this conjecture is often true for \( 2 \leq n < 22 \) without \( n = 5k + 2 \) for \( k \in \mathbb{N}^{+} \). And after \( n = 22 \) we note that, in addition to the numbers are written as (2) for \( n = 5k + 2 \) with \( k \in \mathbb{N}^{+} \) from Table (1), we find other numbers are not prime. But it can be expressed by multiplying prime numbers

### REFERENCES


### AUTHORS’ BIOGRAPHIES

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