The Yang Algebra, Born Reciprocal Relativity Theory and Curved Phase Spaces

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September 2022

Abstract

We begin with a review of the basics of the Yang algebra of non-commutative phase spaces and Born Reciprocal Relativity. A solution is provided for the exact analytical mapping of the non-commuting x^μ, p^μ operator variables (associated to an $8D$ curved phase space) to the canonical Y^A, Π^A operator variables of a flat $12D$ phase space. We explore the geometrical implications of this mapping which provides, in the *classical* limit, with the embedding functions $Y^A(x,p), \Pi^A(x,p)$ of an $8D$ curved phase space into a flat $12D$ phase space background. The latter embedding functions determine the functional forms of the base spacetime metric $g_{\mu\nu}(x,p)$, the fiber metric of the vertical space $h^{ab}(x,p)$, and the nonlinear connection $N_{a\mu}(x,p)$ associated with the $8D$ cotangent space of the $4D$ spacetime. A review of the mathematical tools behind curved phase spaces, Lagrange-Finsler, and Hamilton-Cartan geometries follows. This is necessary in order to answer the key question of whether or not the solutions found for $g_{\mu\nu}, h^{ab}, N_{a\mu}$ as a result of the embedding, also solve the generalized gravitational vacuum field equations in the $8D$ cotangent space. We finalize with an Appendix with the key calculations involved in solving the exact analytical mapping of the x^μ, p^μ operator variables to the canonical Y^A, Π^A operator ones.

Keywords : Yang algebra; Curved Phase Space; Born Reciprocal Relativity Theory ; Finsler Geometry; Noncommutative Geometry.

1 Introduction : The Yang Algebra and Born Reciprocal Relativity Theory

The idea of a Quantum Spacetime where the spacetime coordinates do not commute was proposed early on by Heisenberg and Ivanenko as a way to eliminate infinities from Quantum Field Theory. Snyder published the first concrete example [1] of a noncommutative algebra involving the spacetime coordinates, and it was generalized shortly after by Yang [2], to include noncommuting momentum variables as well. We learnt from General Relativity that the Poincare algebra cannot be implemented on a curved spacetime, but only on its flat tangent space (Minkowski spacetime). The momentum operators don't commute on a curved spacetime. And vice versa, by Born's principle of reciprocity [3], the coordinate operators do not commute on a curved *momentum* space. This prompted the formulation of Quantum Mechanics and Quantum Field Theory in Noncommutative spacetimes (also called Noncommutative QFT), and which might cast some light in the formulation of Quantum Gravity by encoding both key aspects of a curved and a noncommuting spacetime (a curved noncommuting spacetime).

In [12] we suggested that Born's Reciprocal Relativity Theory in Phase spaces is the arena to implement a space-time-matter unification. More precisely : quantum matter curves noncommuting spacetime, and vice versa, noncommuting spacetime curves quantum matter (quantum momentum space) as a result of the back-reaction of quantum spacetime on quantum matter. We believe that it is this Born's reciprocity principle that holds important clues to quantize gravity (geometry) in curved phase spaces within the context of Finsler geometry.

Given a flat $6D$ spacetime with coordinates $Y^A = \{Y^1, Y^2, Y^3, Y^4, Y^5, Y^6\}$, and a metric $\eta_{AB} = diag(-1, +1, +1, \ldots, +1)$, the Yang algebra [2], which is an extension of the Snyder algebra [1], can be derived in terms of the $so(5,1)$ Lorentz algebra generators described by the angular momentum/boost operators

$$J^{AB} = -(Y^A\ \Pi^B\ -\ Y^B\ \Pi^A)\ =\ i\ Y^A \frac{\partial}{\partial Y_B}\ -i\ Y^B \frac{\partial}{\partial Y_A} \qquad (1.1)$$

where $\Pi^A = -i(\partial/\partial Y_A)$ is the canonical conjugate momentum variable to Y^A. Their commutators are

$$[Y^A, Y^B]\ =\ 0,\ [\Pi^A, \Pi^B]\ =\ 0,\ \ [Y^A, \Pi^B]\ =\ i\,\eta^{AB},\ \ A, B = 1,2,3,4,5,6 \quad (1.2)$$

The coordinates Y^A commute. The momenta Π^A also commute, and Y^A, Π^B obey the Weyl-Heisenberg algebra in $6D$.

Adopting the units $\hbar = c = 1$, the correspondence among the noncommuting $4D$ spacetime coordinates x^μ, the noncommuting momenta p^μ, and the Lorentz $so(5,1)$ algebra generators leading to the Yang algebra [2] is given by

$$x^\mu \leftrightarrow L_P\ J^{\mu 5} = -L_P\ (Y^\mu\ \Pi^5 - Y^5\ \Pi^\mu) \quad (1.3a)$$

$$p^\mu \leftrightarrow \frac{1}{\mathcal{L}}\ J^{\mu 6} = -\frac{1}{\mathcal{L}}\ (Y^\mu\ \Pi^6 - Y^6\ \Pi^\mu), \quad \mu,\nu = 1,2,3,4 \quad (1.3b)$$

and which requires the introduction of an ultra-violet cutoff scale L_P given by the Planck scale, and an infra-red cutoff scale $\mathcal{L}$ that can be set equal to the Hubble scale R_H (which determines the cosmological constant). It is very important to emphasize that despite the introduction of two length scales $L_P, \mathcal{L}$ the Lorentz symmetry is not lost. This is one of the most salient features of the Snyder [1] and Yang [2] algebras.

The other generators are

$$\mathcal{N} \equiv J^{56} = -(Y^5\ \Pi^6 - Y^6\ \Pi^5), \quad J^{\mu\nu} = -(Y^\mu\ \Pi^\nu - Y^\nu\ \Pi^\mu), \quad \mu,\nu = 1,2,3,4 \quad (1.4)$$

One can then verify that the Yang algebra is recovered after imposing the above correspondence (1.3)

$$[x^\mu,\ x^\nu] = -i\ L_P^2\ J^{\mu\nu}, \quad [p^\mu,\ p^\nu] = -i\ (\frac{1}{\mathcal{L}})^2\ J^{\mu\nu}, \quad \eta^{55} = \eta^{66} = 1 \quad (1.5)$$

$$[x^\mu,\ J^{\nu\rho}] = i\ (\eta^{\mu\rho}\ x^\nu - \eta^{\mu\nu}\ x^\rho) \quad (1.6)$$

$$[p^\mu,\ J^{\nu\rho}] = i\ (\eta^{\mu\rho}\ p^\nu - \eta^{\mu\nu}\ p^\rho\ ) \quad (1.7)$$

$$[x^\mu,\ p^\nu] = -i\ \eta^{\mu\nu}\ \frac{L_P}{\mathcal{L}}\ \mathcal{N}, \quad [J^{\mu\nu}, \mathcal{N}] = 0 \quad (1.8)$$

$$[x^\mu,\ \mathcal{N}] = i\ L_P \mathcal{L}\ p^\mu, \quad [p^\mu,\ \mathcal{N}] = -i\ \frac{1}{L_P \mathcal{L}}\ x^\mu \quad (1.9)$$

and where the $[J^{\mu\nu}, J^{\rho\sigma}]$ commutators are the same as in the $so(3,1)$ Lorentz algebra in $4D$. They are of the form

$$[\ J^{\mu_1\mu_2},\ J^{\nu_1\nu_2}\ ] = -i\ \eta^{\mu_1\nu_1}\ J^{\mu_2\nu_2} + i\ \eta^{\mu_1\nu_2}\ J^{\mu_2\nu_1} + i\ \eta^{\mu_2\nu_1}\ J^{\mu_1\nu_2} - i\ \eta^{\mu_2\nu_2}\ J^{\mu_1\nu_1}, \quad \hbar = c = 1 \quad (1.10)$$

The generators are assigned to be Hermitian so there are i factors in the right-hand side of eq-(1.10) since the commutator of two Hermitian operators is anti-Hermitian. The $4D$ spacetime metric is $\eta_{\mu\nu} = diag(-1,1,1,1)$.

Given the above correspondence (1.3), we were able to *extend* it further to the higher grade polyvector-valued coordinates and momenta operators in noncommutative Clifford phase spaces [11]. We found the corresponding commutators among the antisymmetric tensors $x^{\mu_1\mu_2}, x^{\mu_1\mu_2\mu_3}, \cdots; p^{\mu_1\mu_2}, p^{\mu_1\mu_2\mu_3}, \cdots$ of different ranks. In addition, we found the spectrum of the quantum harmonic oscillator in noncommutative spaces in terms of the eigenvalues of the generalized angular momentum operators in higher dimensions, and discussed how to extend these results to higher grade polyvector-valued coordinates and momenta.

Instead of working with the above *canonical* coordinates Y^A and momenta Π^A in a flat $12D$ phase space ($A = 1, 2, \cdots, 5, 6$), the authors [5] were interested in finding Hermitian realizations of the above Yang algebra in an $8D$ phase space, and given in terms of the *canonical* variables $\tilde{x}_\mu, \tilde{p}_\mu$ satisfying $[\tilde{x}_\mu, \tilde{x}_\nu] = [\tilde{p}_\mu, \tilde{p}_\nu] = 0$, and $[\tilde{x}_\mu, \tilde{p}_\nu] = i\eta_{\mu\nu}$, with $\mu, \nu = 1, 2, 3, 4$.

The Yang model studied by [5] was characterized by the choice of the commutator $[x_\mu, p_\nu] = i\gamma_{\mu\nu}(x, p)$, and where the rank-2 tensor $\gamma_{\mu\nu}(x, p)$ is of the form

$$\gamma_{\mu\nu} \ = \ h(x^2, p^2, x \cdot p + p \cdot x) \ \eta_{\mu\nu} \tag{1.11}$$

with h a judicious function of the Lorentz scalars $x^2, p^2, x \cdot p + p \cdot x$, and that is determined from solving the Jacobi identities. The rank-2 tensor $\gamma_{\mu\nu}(x, p)$ is what leads to the generalized uncertainty relations. The Triple Special Relativity model [7] which was an extension of [6] was characterized by a different choice of $\gamma_{\mu\nu}(x, p)$. The Lorentz generators are represented as

$$\mathcal{J}_{\mu\nu} \ = \ \frac{1}{2} \ (x_\mu \ p_\nu \ - \ x_\nu \ p_\mu \ + \ p_\nu \ x_\mu \ - \ p_\mu \ x_\nu) \tag{1.12}$$

In particular, the authors [5] looked for representations where the generators $\mathcal{J}_{\mu\nu}$ and the tensor $\gamma_{\mu\nu}$ can be written in terms of the canonical variables $\tilde{x}_\mu$ and $\tilde{p}_\nu$. This required the arduous task of finding the nontrivial map among the *noncanonical* variables x_μ, p_μ, and the canonical ones $\tilde{x}_\mu, \tilde{p}_\mu : x_\mu = x_\mu(\tilde{x}, \tilde{p}); p_\mu = p_\mu(\tilde{x}, \tilde{p})$. The map was found iteratively in powers of $\tilde{x}, \tilde{p}$. The explicit technical details of this map can be found in [5].

In this work we shall follow a very $different$ approach and such that one is able to find *exact* analytical solutions. Instead of finding the maps $x_\mu = x_\mu(\tilde{x}, \tilde{p}); p_\mu = p_\mu(\tilde{x}, \tilde{p})$, *iteratively* and consistent with the Yang algebra and the Jacobi identities, we shall follow a much *simpler* procedure by finding the *exact* analytical expression for the embedding maps $Y^A = Y^A(x, p), \Pi^A = \Pi^A(x, p)$ expressing the $12D$ flat phase space coordinates Y^A and momenta Π^A in terms of the $8D$ *curved* phase space variables x_μ, p_μ. Not unlike in string theory where one has the embedding of the string's world sheet into a D-dim target spacetime background $X^\mu(\sigma^1, \sigma^2)$, where σ^1, σ^2 are the world sheet string coordinates.

Most of the work devoted to Quantum Gravity has been focused on the geometry of spacetime rather than phase space per se. The first indication that phase space should play a role in Quantum Gravity was raised by [3]. The principle behind Born's reciprocal relativity theory [8], [9] was based on the idea proposed long ago by [3] that coordinates and momenta should be unified on the same footing. Consequently, if there is a limiting speed (temporal derivative of the position coordinates) in Nature there should be a maximal force as well, since force is the temporal derivative of the momentum. The principle of maximal acceleration was advocated earlier on by [4]. A *maximal* speed limit (speed of light) must be accompanied with a *maximal* proper force (which is also compatible with a *maximal* and *minimal* length duality) [9].

We explored in [9] some novel consequences of Born's reciprocal Relativity theory in flat phase-space and generalized the theory to the curved spacetime scenario. We provided, in particular, some specific results resulting from Born's reciprocal Relativity and which are *not* present in Special Relativity. These are : momentum-dependent time delay in the emission and detection of photons; relativity of chronology; energy-dependent notion of locality; superluminal behavior; relative rotation of photon trajectories due to the aberration of light; invariance of areas-cells in phase-space and modified dispersion relations.

The generalized velocity and force (acceleration) boosts (rotations) transformations of the $flat$ $8D$ Phase space coordinates , where $X^i, T, E, P^i; i = 1,2,3$ are **c** -valued (classical) variables which are *all* boosted (rotated) into each-other, were given by [8] based on the group $U(1,3)$ and which is the Born version of the Lorentz group $SO(1,3)$. The $U(1,3) = SU(1,3) \times U(1)$ group transformations leave invariant the symplectic 2-form $\Omega = -\, dT \wedge dE + \delta_{ij} dX^i \wedge dP^j; i,j = 1,2,3$ and also the following Born-Green line interval in the $flat$ $8D$ phase-space

$$(d\omega)^2 \;=\; c^2 (dT)^2 \;-\; (dX)^2 \;-\; (dY)^2 \;-\; (dZ)^2 \;+$$

$$\frac{1}{b^2}\; \left((dE)^2 \;-\; c^2 (dP_x)^2 \;-\; c^2 (dP_y)^2 \;-\; c^2 (dP_z)^2 \right) \qquad (1.13)$$

The maximal proper force is set to be given by b. The rotations, velocity and force (acceleration) boosts leaving invariant the symplectic 2-form and the line interval in the $8D$ phase-space are rather elaborate, see [8] for details.

These transformations can be simplified drastically when the velocity and force (acceleration) boosts are both parallel to the x-direction and leave the transverse directions Y, Z, P_y, P_z intact. There is now a subgroup $U(1,1) = SU(1,1) \times U(1) \subset U(1,3)$ which leaves invariant the following line interval

$$(d\omega)^2 \;=\; c^2 (dT)^2 \;-\; (dX)^2 \;+\; \frac{(dE)^2 \;-\; c^2 (dP)^2}{b^2} \;=$$

$$(d\tau)^2 \;\left(\;1 \;+\; \frac{(dE/d\tau)^2 \;-\; c^2 (dP/d\tau)^2}{b^2} \;\right) = (d\tau)^2 \;\left(\;1 \;-\; \frac{F^2}{F^2_{max}} \;\right), \quad P = P_x \qquad (1.14)$$

where one has factored out the proper time infinitesimal $(d\tau)^2 = c^2 dT^2 - dX^2$ in (1.14). The proper force interval $(dE/d\tau)^2 - c^2 (dP/d\tau)^2 = -F^2 < 0$ is "spacelike" when the proper velocity interval $c^2 (dT/d\tau)^2 - (dX/d\tau)^2 > 0$ is timelike. The analog of the Lorentz relativistic factor in eq-(1.14) involves the ratios of two proper *forces*.

One may set the maximal proper-force acting on a fundamental particle of Planck mass to be given by $F_{max} = b \equiv m_P c^2 / L_P$, where m_P is the Planck mass and L_P is the postulated minimal Planck length. Invoking a minimal/maximal length duality one can also set $b = M_U c^2 / R_H$, where R_H is the Hubble scale and M_U is the observable mass of the universe. Equating both expressions for b leads to $M_U / m_P = R_H / L_P \sim 10^{60}$. The value of b may also be interpreted as the maximal string tension. In the most general case there are four scales of time, energy, momentum and length that can be constructed from the three constants $b, c, \hbar$ as follows

$$\lambda_t \;=\; \sqrt{\frac{\hbar}{bc}};\;\; \lambda_l \;=\; \sqrt{\frac{\hbar\; c}{b}};\;\; \lambda_p \;=\; \sqrt{\frac{\hbar\; b}{c}};\;\; \lambda_e \;=\; \sqrt{\hbar\; b\; c} \tag{1.15}$$

The gravitational constant can be written as $G = \alpha_G \; c^4/b$ where α_G is a dimensionless parameter to be determined experimentally. If $\alpha_G = 1$, then the four scales in eq-(1.15) coincide with the *Planck* time, length, momentum and energy, respectively.

The $U(1,1)$ group transformation laws of the phase-space coordinates X, T, P, E which leave the interval (1.14) invariant are [8]

$$T' \;=\; T\; cosh\xi \;+\; (\frac{\xi_v\; X}{c^2} \;+\; \frac{\xi_a\; P}{b^2})\; \frac{sinh\xi}{\xi} \tag{1.16a}$$

$$E' \;=\; E\; cosh\xi \;+\; (-\xi_a\; X \;+\; \xi_v P)\; \frac{sinh\xi}{\xi} \tag{1.16b}$$

$$X' \;=\; X\; cosh\xi \;+\; (\xi_v\; T \;-\; \frac{\xi_a\; E}{b^2})\; \frac{sinh\xi}{\xi} \tag{1.16c}$$

$$P' \;=\; P\; cosh\xi \;+\; (\frac{\xi_v\; E}{c^2} \;+\; \xi_a\; T)\; \frac{sinh\xi}{\xi} \tag{1.16d}$$

ξ_v is the velocity-boost rapidity parameter; ξ_a is the force (acceleration) boost rapidity parameter, and ξ is the net effective rapidity parameter of the primed-reference frame. These parameters ξ_a, ξ_v, ξ are defined respectively in terms of the velocity $v = dX/dT$ and force $f = dP/dT$ (related to acceleration) as

$$tanh(\frac{\xi_v}{c}) = \frac{v}{c};\quad tanh(\frac{\xi_a}{b}) \;=\; \frac{F}{F_{max}},\;\; \xi \;=\; \sqrt{(\frac{\xi_v}{c})^2 + (\frac{\xi_a}{b})^2} \tag{1.17}$$

The $U(3,1)$ generators $Z_{ab} \;=\; J_{[ab]} \,+\, M_{(ab)}$ are comprised of the 6 ordinary Lorentz generators $J_{[ab]}$, and 10 force (acceleration) boost/rotation generators $M_{(ab)}$ giving a total of 16 generators. The commutation relations are of the form $[J, J] \sim J; [J, M] \sim M; [M, M] \sim J$ [8], [15].

It is straightforwad to verify that the transformations (1.16) leave invariant the phase space interval $c^2(dT)^2 - (dX)^2 + ((dE)^2 - c^2(dP)^2)/b^2$ but *do not* leave separately invariant the proper time interval $(d\tau)^2 \;=\; c^2 dT^2 - dX^2$, nor the interval in energy-momentum space $\frac{1}{b^2}[(dE)^2 - (dP)^2]$. Only the *combination*

$$(d\omega)^2 \;=\; (d\tau)^2 \left(1 \;-\; \frac{F^2}{F^2_{max}} \right) \tag{1.17}$$

is truly left invariant under force (acceleration) boosts (1.16). They also leave invariant the symplectic 2-form (phase space areas) $\Omega = -\; dT \wedge dE + dX \wedge dP$.

Having reviewed the basics of the Yang algebra and Born Reciprocal Relativity, we proceed with the outline of this work. Section **2** is devoted to solving the exact analytical mapping of the (non-canonical) non-commuting x^μ, p^μ operator variables (associated to an $8D$ curved phase space) into the canonical Y^A, Π^A operator variables of a flat $12D$ phase space.

In section **3** we explore the geometrical implications of the classical limit of this mapping which provides the embedding functions $Y^A(x,p), \Pi^A(x,p)$ of an $8D$ (classical) curved phase space into a flat $12D$ (classical) phase space background. The latter embedding functions determine the functional forms of the base spacetime metric $g_{\mu\nu}(x,p)$, the fiber metric of the vertical space $h^{ab}(x,p)$, and the nonlinear connection $N_{a\mu}(x,p)$ associated with the $8D$ cotangent space of the $4D$ spacetime.

In section **4** we review the mathematical tools behind curved phase spaces, Lagrange-Finsler, and Hamilton-Cartan geometries. This is necessary in order to answer the key question of whether or not the solutions found for $g_{\mu\nu}, h^{ab}, N_{a\mu}$ as a result of the embedding of the $8D$ curved phase space into the $12D$ flat phase space, also solve the generalized gravitational vacuum field equations in the $8D$ cotangent space. We finalize with an Appendix with the calculations involved in solving the the exact analytical mapping of the x^μ, p^μ operator variables to the canonical Y^A, Π^A operator ones.

2 Mapping of x^μ, p^μ to the Y^A, Π^A variables in Flat Phase Space

The Y^5, Y^6, Π^5, Π^6 canonical coordinates and momenta (operators) in the flat 12-dim phase space are scalars from the point of view of the 8-dim curved phase space parametrized by the non-canonical coordinates x^μ and momenta p^μ. Therefore, Y^5, Y^6, Π^5, Π^6 must be functions of the Lorentz scalars

$$x^2 = \eta_{\mu\nu} x^\mu x^\nu, \;\; p^2 = \eta_{\mu\nu} p^\mu p^\nu, \;\; x \cdot p = \eta_{\mu\nu} x^\mu p^\mu, \;\; p \cdot x = \eta_{\mu\nu} p^\mu x^\nu, \;\; \mu, \nu = 1,2,3,4 \tag{2.1}$$

Setting $\alpha = \mathcal{L}^{-1}, \beta = L_P$, due to the Born reciprocity principle, one must have functions $f(z_1, z_2, z_3)$ of the arguments z_1, z_2, z_3 given by the following combination of Hermitian variables (operators)

$$z_1 \equiv (\alpha^2 x^2 + \beta^2 p^2), \;\; z_2 \equiv (x \cdot p + p \cdot x), \;\; z_3 \equiv i\,(x \cdot p - p \cdot x), \;\; \alpha = \mathcal{L}^{-1}, \; \beta = L_P \tag{2.2}$$

The arguments z_1, z_2, z_3 are invariant under $\alpha \leftrightarrow \beta$, $x \leftrightarrow p$, and $i \leftrightarrow -i$ if one wishes to implement Born's reciprocity symmetry. Therefore, one must have functions of the form

$$Y^5 = Y^5(z_1, z_2, z_3), \;\; Y^6 = Y^6(z_1, z_2, z_3), \;\; \Pi^5 = \Pi^5(z_1, z_2, z_3), \;\; \Pi^6 = \Pi^6(z_1, z_2, z_3) \tag{2.3}$$

For instance, one could have functions linear in z_1, z_2, z_3 defined as follows

$$Y^5(x,p) \;=\; a_1(\alpha^2 x^2 \;+\; \beta^2 p^2) \;+\; b_1(x\cdot p) \;+\; b_1^*(p\cdot x) \;+\; c_1 \qquad (2.4a)$$

$$Y^6(x,p) \;=\; a_2(\alpha^2 x^2 \;+\; \beta^2 p^2) \;+\; b_2(x\cdot p) \;+\; b_2^*(p\cdot x) \;+\; c_2 \qquad (2.4b)$$

$$\Pi^5(x,p) \;=\; a_3(\alpha^2 x^2 \;+\; \beta^2 p^2) \;+\; b_3(x\cdot p) \;+\; b_3^*(p\cdot x) \;+\; c_3 \qquad (2.4c)$$

$$\Pi^6(x,p) \;=\; a_4(\alpha^2 x^2 \;+\; \beta^2 p^2) \;+\; b_4(x\cdot p) \;+\; b_4^*(p\cdot x) \;+\; c_4. \qquad (2.4d)$$

where a_i, b_i, c_i $(i = 1,2,3,4)$ are judicious numerical (dimensionful) coefficients. The units of the coefficients in eqs-(2.4a,2b) are those of length, while those in eqs-(2.4c,2.4d) are those of mass. Note that the b_i coefficients in eqs-(2.4) are complex-valued $b_i = \gamma_i + i\delta_i$. The reason is that the combination

$$b_i\,(x{\cdot}p) + b_i^*\,(p{\cdot}x) \;=\; \gamma_i\,(x{\cdot}p{+}p{\cdot}x) + i\,\delta_i\,(x{\cdot}p{-}p{\cdot}x) \;=\; \gamma_i\,z_2 + \delta_i\,z_3,\;\; i = 1,2,3,4 \qquad (2.4e)$$

ensures that eq-(2.4e) is Hermitian by construction. Eq-(2.4e) is also invariant under Born's reciprocity $x \leftrightarrow p$ and $i \leftrightarrow -i$. We shall show that eqs-(2.4) should, in principle, provide satisfactory solutions to the embedding problem defined below.

The $[x^\mu, p^\nu]$ commutator is defined as

$$[x^\mu, p^\nu] \;=\; x^\mu\; p^\nu \;-\; p^\nu\; x^\mu \;=\; i\;\gamma^{\mu\nu}(x,p) \qquad (2.5)$$

where $\gamma^{\mu\nu}(x,p)$ is a second rank tensor, not necessarily symmetric, that we refrain from identifying it to a metric tensor.

The above commutator can also be expressed in terms of the $6D$ angular momenta variables (see eq-(1.8)) as

$$[x^\mu, p^\nu] = \; i\;\gamma^{\mu\nu}(x,p) \;=\; -\;i\;\alpha\beta\;J^{56}(x,p)\;\eta^{\mu\nu} \;=$$

$$i\;\alpha\beta\;[\;Y^5(x,p)\;\Pi^6(x,p) \;-\; Y^6(x,p)\;\Pi^5(x,p)\;]\;\eta^{\mu\nu}, \quad \alpha = \mathcal{L}^{-1}, \beta = L_P \qquad (2.6)$$

Therefore, from eqs-(2.5,2.6) one arrives at the following relation, after contracting both equations with $\eta_{\mu\nu}$,

$$\frac{1}{4i}\;\eta_{\mu\nu}\;(x^\mu\; p^\nu \;-\; p^\nu\; x^\mu) \;=\; \frac{1}{4i}\;(x\cdot p \;-\; p\cdot x) \;=$$

$$\alpha\beta\;(\;Y^5(x,p)\;\Pi^6(x,p) \;-\; Y^6(x,p)\;\Pi^5(x,p))\;) \;=\; -\;\alpha\beta\;\mathcal{N} \qquad (2.7)$$

Therefore, in this particular case, one finds that the tensor is symmetric $\gamma^{\mu\nu}(x,p) = \Phi(x,p)\eta^{\mu\nu}$ and such that the conformal factor $\Phi(x,p)$ is Hermitian and given by the left hand side of eq-(2.7). The r.h.s of (2.7) is Hermitian because J^{56} is Hermitian due to the canonical and Hermiticity nature of the $6D$ variables : $(Y^5\Pi^6)^\dagger = \Pi^6 Y^5 = Y^5\Pi^6$, and $(Y^6\Pi^5)^\dagger = \Pi^5 Y^6 = Y^6\Pi^5$ resulting from the commutators of the $6D$ canonical variables given by eq-(1.2).

From eqs-(1.3) one learnt that the $4D$ operators x^μ, p^μ admitted a $6D$ angular momentum realization of the form

$$x^\mu = \beta \; J^{\mu 5} \;\; = \;\; - \; \beta \; (Y^\mu \; \Pi^5 \; - \; Y^5 \; \Pi^\mu), \quad \beta = L_P \tag{2.8}$$

$$p^\mu \;\; = \; \alpha \; J^{\mu 6} \;\; = \;\; - \; \alpha \; (Y^\mu \; \Pi^6 \; - \; Y^6 \; \Pi^\mu), \quad \alpha = \mathcal{L}^{-1} \tag{2.9}$$

From eqs-(2.8, 2.9) one can deduce the relation

$$\mathcal{J}^{\mu\nu} \;\; = \;\; x^\mu \; p^\nu \; - \; x^\nu \; p^\mu \;\; = \;\; \alpha\beta \; J^{56} \; (Y^\mu \; \Pi^\nu \; - \; Y^\nu \; \Pi^\mu) \tag{2.10}$$

where $J^{56} \equiv \mathcal{N}$ and $J^{\mu\nu}$ are given by eq-(1.4) explicitly in terms of the $6D$ canonical variables Y^A, Π^B.

One can *invert* the relations in eqs-(2.8,2.9) as follows. After multiplying eqs-(2.8 2.9) on the *right* by Π^6, Π^5, respectively, and subtracting the top equation from the bottom one, it yields

$$\beta^{-1} \; x^\mu \; \Pi^6 \; - \; \alpha^{-1} \; p^\mu \; \Pi^5 \;\; = \;\; \Pi^\mu \; \mathcal{N} \;\; = \;\; \mathcal{N} \; \Pi^\mu \tag{2.11a}$$

due to the canonical nature of the $6D$ variables Y^A, Π^A described by the commutators in eqs-(1.2) and which allows us to re-order the relevant factors due to the commutativity.

And multiplying eqs-(2.8, 2.9) on the *right* by Y^6, Y^5, respectively, and subtracting the top equation from the bottom one, it yields

$$\beta^{-1} \; x^\mu \; Y^6 \; - \; \alpha^{-1} \; p^\mu \; Y^5 \;\; = \;\; Y^\mu \; \mathcal{N} \;\; = \;\; \mathcal{N} \; Y^\mu \tag{2.11b}$$

We shall see next that the functional forms of $Y^5(x,p), Y^6(x,p), \Pi^5(x,p)$, $\Pi^6(x,p)$ provided eqs-(2.4) lead to solutions to eq-(2.7), and which in turn, yield automatically the solutions to eqs-(2.11a, 2.11b). And, in doing so, one has found solutions to the embedding problem : $Y^\mu = Y^\mu(x,p); \Pi^\mu = \Pi^\mu(x,p)$, with $\mathcal{N}(x,p) \equiv J^{56}(x,p) = -(Y^5\Pi^6 - Y^6\Pi^5)(x,p)$, and where $[\mathcal{N}, Y^\mu] = [\mathcal{N}, \Pi^\mu] = 0$.

Thus, from eqs-(2.7,2.11) one can construct the maps from the x^μ, p^μ non-canonical (operator) variables in $4D$ to the canonical (operator) variables Y^A, Π^A in $6D$. The next step is to take the classical limit in order to find the embeddings from the $8D$ (classical) curved phase space into the $12D$ (classical) flat phase space given by $Y^A(x,p), \Pi^A(x,p)$ given in terms of the corresponding *classical* coordinates (**c**-numbers) associated with the operator-valued variables.

One can also exploit the fact that the operator variables x^μ, p^μ, and Y^A, Π^A are Hermitian and *rewrite* eqs-(211a,2.12b) in commutator form. Taking the Hermitian conjugates of eqs-(2.11a,2.11b), and then subtracting these conjugates from eqs-(2.11a,2.11b), yields the relations

$$\beta^{-1} \; [\; \Pi^6(x,p), \; x^\mu \;] \; + \alpha^{-1} \; [\; p^\mu, \; \Pi^5(x,p) \;] \;\; = \;\; 0 \tag{2.12a}$$

$$\beta^{-1} \; [\; Y^6(x,p), \; x^\mu \;] \; + \alpha^{-1} \; [\; p^\mu, \; Y^5(x,p) \;] \;\; = \;\; 0 \tag{2.12b}$$

Eqs-(2.12) combined with eq-(2.7) define three equations in commutator form where Y^μ and Π^μ do *not* appear explicitly, like they do in eqs-(2.11).

Let us show how eqs-(2.4) do, in principle, solve the key eq-(2.7). Upon inserting the expressions provided by eqs-(2.4) directly into eq-(2.7) one ends up with 13 terms (involving the Lorentz scalars z_1, z_2, z_3 in eq-(2.2)) of the form

$$constant; \ z_1, \ z_2, \ z_3, \ z_1^2, \ z_2^2, \ z_3^2,$$
$$z_1 z_2, \ z_2 z_1, \ z_1 z_3, \ z_3 z_1, \ z_2 z_3, \ z_3 z_2 \tag{2.13a}$$

As a reminder, because we are dealing with coordinate and momentum operators, the products do not commute. The left hand side of eq-(2.7) is just given by $-\frac{z_3}{4}$. Thus, the only non-zero entry containing the terms (monomials) described by eq-(2.13a) is the one involving the z_3 term with an overall coefficient of $-\frac{1}{4}$. The remaining 12 terms (monomials) must all cancel and appear with zero coefficients. Therefore, one finds 13 equations to be obeyed by the $4 \times 4 = 16$ numerical coefficients which appear in eqs-(2.4).

Since the system of equations is underdetermined $13 < 16$ one may fix 3 of the coefficients. Let us impose the following relations

$$a \ = \ a_1 \ = \ a_2 \ = \ \beta^2 \ a_3 \ = \ \beta^2 \ a_4 \tag{2.13b}$$

such that one ends with a total of $1+12 = 13$ coefficients to be determined given by $a; \gamma_i, \delta_i, c_i; i = 1, 2, 3, 4$. In the Appendix we explicitly write down all the 13 equations in terms of 13 coefficients. After a laborious but straightforward procedure we find the following family of solutions

$$Y^5(x,p) \ = \ \kappa_1 \ \beta \ z_1 \ + \ \kappa_2 \ \beta \ z_2 \ + \ \kappa_3 \ \beta \ z_3 \ + \ \kappa_4 \ \beta \tag{2.14a}$$

$$Y^6(x,p) \ = \ \kappa_1 \ \beta \ z_1 \ + \ \kappa_2 \ \beta \ z_2 \ + \ \kappa_3 \ \beta \ z_3 \ + \ (\kappa_4 + 1) \ \beta \tag{2.14b}$$

$$\Pi^5(x,p) \ = \ \kappa_1 \ \beta^{-1} \ z_1 \ + \ \kappa_2 \ \beta^{-1} \ z_2 \ + \ \frac{5}{4} \ \kappa_3 \ \beta^{-1} \ z_3 \ + \ \kappa_4 \ \beta^{-1} \tag{2.14c}$$

$$\Pi^6(x,p) \ = \ \kappa_1 \ \beta^{-1} \ z_1 \ + \ \kappa_2 \ \beta^{-1} \ z_2 \ + \ \frac{5}{4} \ \kappa_3 \ \beta^{-1} \ z_3 \ + \ (\kappa_4 + 1) \ \beta^{-1} \tag{2.14d}$$

where $\kappa_3 = (\alpha\beta)^{-1}$ and $\kappa_1, \kappa_2, \kappa_4$ are three arbitrary parameters. This is due to the nonlinearity of the equations that one is solving. The solutions (2.14) have the form $Y^6 = Y^5 + \beta; \Pi^5 = \Pi^6 - \beta^{-1}$ such that $\alpha\beta \ Y^{[5} \ \Pi^{6]} = -\frac{z_3}{4}$ as required by eq-(2.7).

In order to define *classical* $\hbar \to 0$ limit, one must reinstate $\hbar$ which was set to unity. Eq-(2.7) in the classical limit becomes

$$lim_{\hbar \to 0} \ \frac{x \cdot p - p \cdot x}{4i\hbar} \ \to \ 1 \ \Rightarrow \ \mathcal{N} \ \to \ - \ \frac{1}{\alpha\beta} \tag{2.15}$$

as a result of Dirac's prescription turning commutators into Poisson brackets

$$lim_{\hbar \to 0} \ \frac{[\ \hat{x}^\mu, \ \hat{p}^\nu \ ]}{i\hbar} \ \to \ \{ \ x^\mu, \ p^\nu \ \}_{PB}, \ \ \{ \ x^0, \ p^0 \ \}_{PB} = -1, \ \ \{ \ x^i, \ p^j \ \} \ = \ \delta^{ij} \tag{2.16}$$

with $\eta_{\mu\nu} = diag(-1,+1,+1,+1)$. Inserting the classical limit of $\mathcal{N}$ into eqs-(2.11) (where x, p are now **c**-valued) yield

$$\alpha\ x^\mu\ \Pi^6\ -\ \beta\ p^\mu\ \Pi^5\ =\ -\ \Pi^\mu \tag{2.17a}$$

$$\alpha\ x^\mu\ Y^6\ -\ \beta\ p^\mu\ Y^5\ =\ -\ Y^\mu \tag{2.17b}$$

From eq-(2.15) one learns that in the classical limit one obtains

$$\mathcal{N}\ \rightarrow\ -\frac{1}{\alpha\beta}\ \Rightarrow\ Y^5\ \Pi^6\ -\ Y^6\ \Pi^5\ \rightarrow\ \frac{1}{\alpha\beta} \tag{2.18}$$

The four embedding functions Y^5, Y^6, Π^5, Π^6, in the *classical* limit, are functions of the two Lorentz scalars because

$$z_1\ \rightarrow\ w_1\ =\ \alpha^2 x^2\ +\ \beta^2 p^2;\ \ z_2\ \rightarrow\ w_2\ =\ 2x\cdot p,\ z_3\ \rightarrow\ 0 \tag{2.19}$$

and obey the quadratic constraint equation

$$Y^5(w_1,w_2)\ \Pi^6(w_1,w_2)\ -\ Y^6(w_1,w_2)\ \Pi^5(w_1,w_2)\ =\ \frac{1}{\alpha\beta}\ =\ \frac{\mathcal{L}}{L_P} \tag{2.20}$$

The classical limit of the equations involving the commutators (2.12) is

$$\beta^{-1}\ \{\ \Pi^6(w_1,w_2),\ x^\mu\ \}_{PB}\ +\alpha^{-1}\ [\{\ p^\mu,\ \Pi^5(w_1,w_2)\ \}_{PB}\ =\ 0 \tag{2.21a}$$

$$\beta^{-1}\ \{\ Y^6(w_1,w_2),\ x^\mu\ \}_{PB}\ +\alpha^{-1}\ \{\ p^\mu,\ Y^5(w_1,w_2)\ \}_{PB}\ =\ 0 \tag{2.21b}$$

There are some subtleties with these eqs-(2.17) and eqs-(2.21) worth mentioning. Note that in the quantum case, eqs-(2.11) are "equivalent" to eqs-(2.12) because the latter were obtained from the former via the process of taking the Hermitian conjugation. Whereas, in the classical case, the *algebraic* eqs-(2.17) do not lead to the *differential* eqs-(2.21) involving the Poisson brackets.

Because the variable $z_3 \rightarrow 0$ decouples in the classical limit, there are only $1+4+4 = 9$ parameters left over in eqs-(2.4) given by $a; \gamma_i; c_i; i = 1,2,3,4$. The monomials appearing in eq-(2.20) are now of the form

$$constant,\ w_1,\ w_2,\ w_1^2,\ w_2^2,\ w_1 w_2 \tag{2.22}$$

leading to 6 equations. Thus one has 9 parameters to obey 6 equations. Once again, one ends with an under determined systems of equations.

On the other hand, because eqs-(2.17) do not lead to eqs-(2.21), there is no guarantee that the solutions of the type given by eqs-(2.4) are going to solve the algebraic eq-(2.20), *and* the two extra *differential* eqs-(2.21a,2.21b) involving the Poisson brackets.

Furthermore, there is also a caveat worth mentioning when one takes the classical limit. Upon restoring $\hbar$ which was set to unity in the terms $\gamma_i z_2 \rightarrow \frac{\gamma_i}{\hbar} z_2$

of eqs-(2.4e), in order to match units, one can see that these terms are *singular* in the $\hbar \to 0$ limit. Whereas the terms $\frac{\delta_i}{\hbar} z_3 \to -4\delta_i$ are well behaved and yield constants.

For these reasons we shall just adhere to the following prescription in finding the *classical* limit of the embedding functions $Y^A(x,p), \Pi^A(x,p)$. Having found the solutions for the 13 coefficients $a = a_1 = a_2 = \beta^2 a_3 = \beta^2 a_4; \gamma_i, \delta_i, c_i; i = 1,2,3,4$ in the quantum case, we simply *drop* the *singular* $\frac{1}{\hbar} z_2$ terms, and set the $\frac{1}{\hbar} z_3$ terms to constants in the explicit solutions for Y^5, Y^6, Π^5, Π^6 given by eqs-(2.14).

Or one could set the arbitrary constant κ_2 to zero $\kappa_2 = 0$ so that $\frac{1}{\hbar}\kappa_2 z_2$ is *no* longer singular. In this case there is no need to drop the z_2 terms. And finally, one can obtain the explicit solutions for Y^μ, Π^μ of eqs-(2.17), and given in terms of the functions $Y^5(w_1,w_2), Y^6(w_1,w_2), \Pi^5(w_1,w_2), \Pi^6(w_1,w_2)$ found in eqs-(2.14), and x^μ, p^μ.

Consequently, this latter prescription will render eqs-(2.17) in the following form

$$\alpha\ x^\mu\ \Pi^6(w_1,w_2)\ -\ \beta\ p^\mu\ \Pi^5(w_1,w_2)\ =\ -\ \Pi^\mu \qquad (2.23a)$$

$$\alpha\ x^\mu\ Y^6(w_1,w_2)\ -\ \beta\ p^\mu\ Y^5(w_1,w_2)\ =\ -\ Y^\mu \qquad (2.23b)$$

with $w_1 = \alpha^2 x^2 + \beta^2 p^2$, $w_2 = 2x \cdot p$, and Y^5, Y^6, Π^5, Π^6 given by eqs-(2.14) after setting the $\frac{z_3}{\hbar}$ terms to constants. Next we shall study the implications of the (classical) embedding solutions found in this section.

3 Embedding a $8D$ curved phase space into a $12D$ flat phase space

The previous section involved the use of coordinates and momenta *operators*. In this section we shall deal with *classical* variables (**c**-numbers) x, p. A more rigorous notation in the previous section would have been to assign "hats" to operators $\hat{x}^\mu, \hat{p}^\mu; \hat{Y}^A, \hat{\Pi}^A$. For the sake of simplicity we avoided it in order to follow the notation used by the authors [5].

The geometry of the cotangent bundle of spacetime (phase space) can be best explored within the context of Lagrange-Finsler, Hamilton-Cartan geometry [16], [17]. The line element in the $8D$ curved phase space is

$$(ds)^2\ =\ g_{\mu\nu}(x,p)\ dx^\mu dx^\nu\ +\ h^{ab}(x,p)\ (dp_a + N_{a\mu}(x,p)dx^\mu)\ (dp_b + N_{b\nu}(x,p)dx^\nu) \qquad (3.1)$$

where $g_{\mu\nu}(x,p), h^{ab}(x,p)$ are the base spacetime and internal space metrics, respectively, with $a,b = 1,2,3,4$, $\mu,\nu = 1,2,3,4$, and $N_{a\mu}(x,p)$ is the nonlinear connection.

One should note that the metric tensor $g_{\mu\nu}$ is *not* the vertical Hessian of the square of a Finsler function, and h^{ab} is *not* the inverse of $g_{\mu\nu}$. h^{ab} represents, physically, the cotangent bundle's internal-space metric tensor which is independent from the base-spacetime metric tensor $g_{\mu\nu}$. The number of total components of $g_{\mu\nu}, h^{ab}, N_{a\mu}$ is $10 + 10 + 16 = 36 = (8 \times 9)/2)$.

The generalized (vacuum) gravitational field equations associated with the geometry of the $8D$ cotangent bundle [18] differ considerably from the the standard (vacuum) Einstein field equations in $8D$ based on Riemannian geometry. Thus, for instance, by using a base-spacetime $g_{\mu\nu}$ metric to be *independent* from the internal-space metric h_{ab}, and a nonlinear connection $N_{\mu a}$, it might avoid the reduction of the solutions of the generalized gravitational field equations to the standard Schwarzschild (Tangherlini) solutions when radial symmetry is imposed.

For example, in [12] we studied a scalar-gravity model in curved phase spaces. After a very laborious procedure, the variation of the action S with respect to the fundamental fields

$$\frac{\delta \mathcal{S}}{\delta g_{\mu\nu}} = 0, \quad \frac{\delta \mathcal{S}}{\delta h_{ab}} = 0, \quad \frac{\delta \mathcal{S}}{\delta N_{\mu a}} = 0, \quad \frac{\delta \mathcal{S}}{\delta \Phi} = 0 \tag{3.2}$$

leads to the very *complicated* field equations which differ considerably from the Einstein field equations. Exact nontrivial analytical solutions for the base-spacetime $g_{\mu\nu}$, the internal-space metric h_{ab} components, the nonlinear connection N_{ia}, and the scalar field Φ were found that obey the generalized gravitational field equations, in addition to satisfying the *zero* torsion conditions for *all* of the torsion components. See [12] for details.

The embedding of the $8D$ curved phase space into the 12-dim flat phase space is described by equating the $8D$ line interval ds^2 in (3.1) with the $12D$ one $ds^2 = \eta_{AB} dZ^A dZ^B$. After doing so, given $Z^A \equiv (Y^A, \Pi^A)$ one learns that

$$g_{\mu\nu} + h^{ab} N_{a\mu} N_{b\nu} = \eta_{AB} \frac{\partial Z^A}{\partial x^\mu} \frac{\partial Z^A}{\partial x^\nu} \tag{3.3}$$

$$h^{ab} = \eta_{AB} \frac{\partial Z^A}{\partial p_a} \frac{\partial Z^A}{\partial p_b} \tag{3.4}$$

$$h^{ab} N_{b\mu} = \eta_{AB} \frac{\partial Z^A}{\partial p_a} \frac{\partial Z^A}{\partial x^\mu} \quad A, B = 1, 2, \cdots, 5, 6 \tag{3.5}$$

Eqs-(3.3-3.5) *determine* the functional form of $g_{\mu\nu}, h^{ab}, N_{a\mu}$ after one inserts the functional forms of the embedding functions $Z^A(x,p) = Y^A(x,p), \Pi^A(x,p)$ found in the previous section, and by making the following replacement $p^\mu \rightarrow p_a$. We explained at the end of section **2** how the $x{\cdot}p, p{\cdot}x$ terms could *decouple* in the *classical* limit, by removing the singular terms $\frac{z_2}{\hbar}$, the $\frac{z_3}{\hbar}$ become constants, and leaving only the terms $w_1 = \alpha^2 x^2 + \beta^2 p^2$. Thus, after making the replacement $p^\mu \rightarrow p_a$ one has $\eta_{\mu\nu} p^\mu p^\nu \rightarrow \eta^{ab} p_a p_b$, and such that the indices will now match those appearing in eqs-(3.3-3.5).

Also, it is very important to emphasize that the base spacetime metric $g_{\mu\nu}(x,p)$ in eqs-(3.1,3.2,3.3) is *not* the *same* as the tensor $\gamma_{\mu\nu}(x,p)$ appearing in the definition of the commutator $[x_\mu, p_\nu] = i\gamma_{\mu\nu}(x,p) = i\Phi(x,p)\eta_{\mu\nu}$. It would be a *remarkable* coincidence if their functional form turned out to be the same.

The (classical) embedding functions $Z^A(x,p) = Y^A(x,p), \Pi^A(x,p)$ obtained in the previous section determine the functional form of $g_{\mu\nu}, h^{ab}, N_{a\mu}$ in eqs-(3.3-3.5). The key question is whether or not the solutions found in eqs-(3.3-3.5) for $g_{\mu\nu}, h^{ab}, N_{a\mu}$ *also* solve the vacuum field equations. And if not, can one find the appropriate field/matter sources which are consistent with these solutions ?. Namely, can one find the matter/field configurations which source the fields $g_{\mu\nu}, h^{ab}, N_{a\mu}$, and which in turn, originated from solving the embedding eqs-(2.7-2.11) that emerged from the Yang noncommutative algebra in phase space, and then, by inserting these solutions into eqs-(3.3-3.5).

It is natural to assume that quantum matter/fields could be the source of the noncommutativity of the spacetime coordinates and momenta. After all, quantum fields live in spacetime. If this were not the case, what then is the source of this phase space noncommutativity ? Is it space-time foam, dark matter, dark energy ? .... If one expects to have a space-time-matter unification then one has that matter curves space-time, and space-time back-reacts on matter curving momentum space, "curving matter". To tackle these questions we devote the whole of next section.

4 Curved Phase Space and Lagrange-Finsler, Hamilton-Cartan Geometry

In this section we begin with a review of our work in [12]. We deem it to be very useful for the non-expert reader in order to grasp the essentials of the ample literature on the geometry of Lagrange-Finsler, Hamilton-Cartan spaces and higher order (jet bundles) generalizations, see [16], [17], and references therein. Let us begin with the Sasaki-Finsler metric of the cotangent space of a d-dim manifold T^*M_d, and which is given by the following metric in *block diagonal* form

$$(d\omega)^2 \;=\; g_{ij}(x^k, p_a)\; dx^i d\, x^j \;+\; h^{ab}(x^k, p_c)\; \delta p_a\; \delta p_b \;= \\ g_{ij}(x^k, p_a)\; dx^i d\, x^j \;+\; h_{ab}(x^k, p_c)\; \delta p^a\; \delta p^b \tag{4.1}$$

The range of the base manifold indices is $i, j, k = 0, 1, 2, 3, ...d-1$; whereas the range of the fiber indices is $a, b, c = 0, 1, 2, 3, ...d-1$. The standard coordinate basis frame has been replaced by the following anholonomic non-coordinate basis frame comprised of the following elongated and ordinary derivatives, respectively,

$$\delta_i \;=\; \delta/\delta x^i \;=\; \partial_{x^i} \;+\; N_{ia}\, \partial^a \;=\; \partial_{x^i} \;+\; N_{ia}\, \partial_{p_a}; \;\; \partial^a \;\equiv\; \partial_{p_a} \;=\; \frac{\partial}{\partial p_a} \tag{4.2}$$

The signature is chosen to be Lorentzian $(-,+,+,+,\cdots,+)$ for both g_{ij} and h_{ab}. It is important to emphasize that one does *not* have a theory with two times because the energy coordinate is not time. One should note the *key* position of the indices that allows us to distinguish between derivatives with respect to x^i and those with respect to p_a. The dual basis of $(\delta_i = \delta/\delta x^i; \partial^a = \partial/\partial p_a)$ is

$$dx^i, \ \ \delta p_a \ = \ dp_a \ - \ N_{ja} \ dx^j, \ \ \delta p^a \ = \ dp^a \ - \ N^a_j \ dx^j \qquad (4.3)$$

where the N-coefficients define a nonlinear connection, N–connection structure.

An N-linear connection D on T^*M allows to construct covariant derivatives which are compatible with the structure induced by the nonlinear connection and that preserve the horizontal-vertical split of the cotangent bundle. Thus, an N-linear connection D on T^*M can be uniquely represented in the adapted basis in the following form

$$D_{\delta_j}(\delta_i) \ = \ H^k_{ij} \ \delta_k; \ \ D_{\delta_j}(\partial^a) \ = \ - \ H^a_{bj} \ \partial^b; \qquad (4.4a)$$

$$D_{\partial^a}(\delta_i) \ = \ C^{ka}_i \ \delta_k; \ \ D_{\partial^a}(\partial^b) \ = \ - \ C^{ba}_c \ \partial^c \qquad (4.4b)$$

where $H^k_{ij}(x,p), H^a_{bj}(x,p), C^{ka}_i(x,p), C^{ba}_c(x,p)$ are the connection coefficients. Our notation for the derivatives is

$$\partial^a = \partial/\partial p_a, \ \ \partial_i \ = \ \partial_{x^i}, \ \ \delta_i \ = \ \delta/\delta x^i \ = \ \partial_{x^i} \ + \ N_{ia} \ \partial^a \qquad (4.4c)$$

The N-connection structures can be naturally defined on (pseudo) Riemannian spacetimes and one can relate them with some anholonomic frame fields (vielbeins) satisfying the relations $\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha = W^\gamma_{\alpha\beta}\delta_\gamma$. The only nontrivial (nonvanishing) nonholonomy coefficients are

$$W_{ija} \ = \ R_{ija}; \ \ W^a_{\ jb} \ = \ \partial^a N_{jb} \ = \ - \ W_j{}^a{}_b \qquad (4.5a)$$

and

$$R_{ija} \ = \ \delta_j N_{ia} \ - \ \delta_i N_{ja} \qquad (4.5b)$$

is the nonlinear connection curvature (N-curvature).

Imposing a zero nonmetricity condition of $g_{ij}(x,p), h^{ab}(x,p)$ along the horizontal and vertical directions, respectively, gives

$$D_i g_{jk} \ = \ \delta_i \ g_{gk} \ - H^l_{ij} \ g_{lk} \ - H^l_{ik} \ g_{jl} \ = \ 0, \qquad (4.6a)$$

$$D^a h^{bc} \ = \ \partial^a \ h^{bc} \ + \ C^{ab}_d \ h^{dc} \ + \ C^{ac}_d \ h^{bd} \ = \ 0 \qquad (4.6b)$$

Performig a cyclic permutation of the indices in eqs-(6a,6b), followed by linear combination of the equations obtained yields the irreducible (horizontal, vertical) h-v-components for the connection coefficients

$$H^i{}_{jk} \ = \ \frac{1}{2} g^{in} \left(\delta_k g_{nj} + \delta_j g_{nk} - \delta_n g_{jk}\right) \qquad (4.7)$$

$$C^{ab}_c \ = \ - \ \frac{1}{2} \ h_{cd} \ \left(\ \partial^b h^{ad} \ + \ \partial^a h^{bd} \ - \ \partial^d h^{ab} \ \right) \qquad (4.8)$$

The additional conditions $D_i h^{ab} = 0, D^a g_{ij} = 0$, yield the *mixed* components of the connection coefficients

$$H^b_{ja} = \partial^b N_{ja} + \frac{1}{2} h^{bc} \left(\delta_j h_{ac} - h_{ad} \partial^d N_{jc} - h_{cd} \partial^d N_{ja} \right) \tag{4.9}$$

and

$$C^{ja}_i = \frac{1}{2} g^{jk} \partial^a g_{ik} \tag{4.10}$$

For any N-linear connection D with the above coefficients the torsion 2-forms are

$$\Omega^i = \frac{1}{2} T^i_{jk} \, dx^j \wedge dx^k + C^{ia}_j \, dx^j \wedge \delta p_a \tag{4.11a}$$

$$\Omega_a = \frac{1}{2} R_{jka} \, dx^j \wedge dx^k + P^b_{aj} \, dx^j \wedge \delta p_b + \frac{1}{2} S^{bc}_a \, \delta p_b \wedge \delta p_c \tag{4.11b}$$

and the curvature 2-forms are

$$\Omega^i_j = \frac{1}{2} R^i_{jkm} \, dx^k \wedge dx^m + P^{ia}_{jk} \, dx^k \wedge \delta p_a + \frac{1}{2} S^{iab}_j \, \delta p_a \wedge \delta p_b \tag{4.12}$$

$$\Omega^a_b = \frac{1}{2} R^a_{bkm} \, dx^k \wedge dx^m + P^{ac}_{bk} \, dx^k \wedge \delta p_c + \frac{1}{2} S^{acd}_b \, \delta p_c \wedge \delta p_d \tag{4.13}$$

where one must recall that the dual basis of $\delta_i = \delta/\delta x^i, \ \partial^a = \partial/\partial p_a$ is given by $dx^i, \ \delta p_a = dp_a - N_{ja} dx^j$.

The distinguished torsion tensors are given by

$$T^i_{jk} = H^i_{jk} - H^i_{kj}; \ S^{ab}_c = C^{ab}_c - C^{ba}_c; \ T^{ia}_j = C^{ia}_j = - T^{ia}{}_j$$

$$P_b{}^a{}_j = H^a_{bj} - \partial^a N_{jb}, \ P_b{}^a{}_j = - P_{bj}{}^a$$

$$R_{ija} = \frac{\delta N_{ja}}{\delta x^i} - \frac{\delta N_{ia}}{\delta x^j} \tag{4.14}$$

And the distinguished tensors of the curvature are

$$R^i_{kjh} = \delta_h H^i_{kj} - \delta_j H^i_{kh} + H^l_{kj} H^i_{lh} - H^l_{kh} H^i_{lj} - C^{ia}_k R_{jha} \tag{4.15}$$

$$P^{ab}_{cj} = \partial^a H^b_{cj} + C^{ad}_c P^b_{dj} - \left(\delta_j C^{ab}_c + H^b_{dj} C^{da}_c + H^a_{dj} C^{bd}_c - H^d_{cj} C^{ab}_d \right) \tag{4.16}$$

$$P^{ak}_{ij} = \partial^a H^k_{ij} + C^{al}_i T^k_{lj} - \left(\delta_j C^{ak}_i + H^a_{bj} C^{bk}_i + H^k_{lj} C^{al}_i - H^l_{ij} C^{ak}_l \right) \tag{4.17}$$

$$S_d^{abc} \ = \ \partial^c \ C_d^{ab} \ - \ \partial^b \ C_d^{ac} \ + \ C_d^{eb} \ C_e^{ac} \ - \ C_d^{ec} \ C_e^{ab}; \qquad (4.18)$$

$$S_j^{ibc} \ = \ \partial^c C_j^{bi} \ - \ \partial^b C_j^{ci} \ + \ C_j^{bh} \ C_h^{ci} \ - \ C_j^{ch} \ C_h^{bi} \qquad (4.19)$$

$$R_{bjk}^a \ = \ \delta_k H_{bj}^a \ - \ \delta_j H_{bk}^a \ + \ H_{bj}^c \ H_{ck}^a \ - \ H_{bk}^c \ H_{cj}^a \ - \ C_b^{ca} \ R_{jkc} \qquad (4.20)$$

Adopting the units where $\hbar = c = G = 1$ such that the Planck mass and length squared are respectively $M_P^2 = 1, L_P^2 = 1$; given $g^{AB} \equiv g^{ij}, h^{ab}$, and the definitions $\partial_A \Phi(x,p) \equiv \delta_i \Phi(x,p), \partial_a \Phi(x,p)$, where the ordinary ∂_a and elongated derivatives δ_i defined by eq-(4.2) act on $\Phi(x,p)$, one may construct the simplest gravity-scalar field action of the form[1]

$$\mathcal{S} \ = \ \mathcal{S}_G \ + \ \mathcal{S}_M \ = \ \frac{1}{2\kappa} \int d^4x \ d^4p \ \sqrt{|\det g_{AB}|} \ \left(g^{ij} \ R_{(ij)} \ + \ h_{ab} \ S^{(ab)} \right) \ -$$

$$\int d^4x \ d^4p \ \sqrt{|\det g_{AB}|} \ \left(\frac{1}{2} \ g^{AB} \ \partial_A \Phi \ \partial_B \Phi \ + \ V(\Phi) \right) \qquad (4.21)$$

The determinant factorizes $det(g_{AB}) = det(g_{ij}) det(h_{ab})$ in an anhololomic basis adapted to the nonlinear connection (the metric assumes the block diagonal form (1)). κ is the gravitational coupling constant. If the phase space action action (4.21) is dimensionless, after reintroducing the physical constants that were set to unity, gives $\kappa = 8\pi \to (8\pi G/c^4)(M_p c)^4$.

After a very laborious procedure the authors [18] have shown that variation of the action (4.21)

$$\frac{\delta \mathcal{S}}{\delta g_{ij}} \ = \ 0, \ \ \frac{\delta \mathcal{S}}{\delta h_{ab}} \ = \ 0, \ \ \frac{\delta \mathcal{S}}{\delta N_{ia}} \ = \ 0, \ \ \frac{\delta \mathcal{S}}{\delta \Phi} \ = \ 0 \qquad (4.22)$$

leads to the following field equations

$$R_{(ij)}(x,p) \ - \frac{1}{2} \ g_{ij}(x,p) \ (R+S) \ + \ R_{k(ia} \ C_{j)}^{ka} \ = \ T_{ij} \qquad (4.23)$$

$$S_{(ab)}(x,p) \ - \frac{1}{2} \ h_{ab}(x,p) \ (R+S) \ = \ T_{ab} \qquad (4.24)$$

$$g^{ik} \ \partial^a H_{kj}^j \ - \ g^{kl} \ \partial^a H_{kl}^i \ = \ T^{ia} \qquad (4.25)$$

where

$$R_{kh} \ = \ R_{kjh}^i \ \delta_i^j, \ \ R \ = \ g^{kh} \ R_{(kh)} \ \ S^{ac} \ = \ S_d^{abc} \ \delta_b^d, \ \ S \ = \ h_{ac} \ S^{(ac)} \qquad (4.26)$$

after symmetrizing the indices accordingly and denoted by (). The components of the stress energy tensor are defined as

[1] $d^4x \ d^4p = dx^0 \wedge dx^1 \wedge \cdots \wedge \delta p_0 \wedge \delta p_1 \wedge \cdots = dx^0 \wedge dx^1 \wedge \cdots \wedge dp_0 \wedge dp_1 \wedge \cdots$

$$T_{ij} = - \frac{2}{\sqrt{|detG_{AB}|}} \frac{\delta(\sqrt{|detG_{AB}|}L_M)}{\delta g^{ij}}, \; T_{ab} = - \frac{2}{\sqrt{|detG_{AB}|}} \frac{\delta(\sqrt{|detG_{AB}|}L_M)}{\delta h^{ab}} \tag{4.27}$$

$$T^{ia} = - \frac{2}{\sqrt{|detG_{AB}|}} \frac{\delta(\sqrt{|detG_{AB}|}L_M)}{\delta N_{ia}} \tag{4.28}$$

and given by

$$T_{ij} = (\delta_i\Phi(x,p))\,(\delta_j\Phi(x,p)) - g_{ij}\left(\frac{1}{2}\, g^{AB}(\partial_A\Phi(x,p))\,(\partial_B\Phi(x,p)) + V(\Phi)\right) \tag{4.29}$$

$$T_{ab} = (\partial_a\Phi(x,p))\,(\partial_b\Phi(x,p)) - h_{ab}\left(\frac{1}{2}\, g^{AB}(\partial_A\Phi(x,p))\,(\partial_B\Phi(x,p)) + V(\Phi)\right) \tag{4.30}$$

$$T^{ia} = g^{ij}\; \delta_j\Phi(x,p)\; \partial^a\Phi(x,p) \tag{4.31}$$

One must include also the equation of motion for the scalar field $\Phi(x,p)$, which is a generalization of the d'Alambert equation,

$$g^{ij}\; D_iD_j\Phi + h^{ab}\; D_aD_b\Phi - \frac{\partial V(\Phi)}{\partial\Phi} = 0 \tag{4.32}$$

$$D_iD_j\Phi = \delta_i\delta_j\Phi - H^k_{ij}\;\delta_k\Phi, \;\; D_aD_b\Phi = \partial_a\partial_b\Phi - C^c_{ab}\;\partial_c\Phi \tag{4.33}$$

The system of *coupled* nonlinear differential equations (4.23,4.24,4.25,4.32) leading to the solutions for $g_{ij}(x,p), h_{ab}(x,p), N_{ai}(x,p), \Phi(x,p)$ are highly non-trivial. The scalar field $\Phi(x,p)$ curves both spacetime and momentum space. The equations have almost a similar form to the Einstein gravitational field equation with the difference of the extra term $R_{k(ia}\; C^{ka}_{j)}$ in eq-(4.23).

We found exact solutions to eqs-(4.23-4.25) in [12] when $\Phi = constant$, yielding the following $8D$ line interval, after reinstating the Planck length $\beta = L_P$ (that were set to unity) in order to match units,

$$(ds)^2 = - (dt)^2 + e^{2H_ot}\,((dx)^2 + (dy)^2 + (dz)^2) -$$
$$\beta^4\;[\;(dE)^2 + e^{2\beta^2H_oE}\,((dp_x)^2 + (dp_y)^2 + (dp_z)^2\;] +$$
$$\beta^2\; e^{-2\beta^2H_oE}\; N(t)^2\;(dt)^2 - 2\beta^3\; e^{-2\beta^2H_oE}\; N(t)\; dt\; dp_x, \;\; \beta^2 = L_P^2 \tag{4.34}$$

Note the off-diagonal piece $-2\beta^3e^{-2\beta^2H_oE}N(t)dtdp_x$ above implying a mixing of coordinates and momenta. $N(t)$ was an arbitrary function chosen judiciously so that $N(t=0) \neq \infty,\; N(t=\infty) = 0$. $N(t)$ has units of mass.

One of the salient features of the phase space metric (4.34) is that it captures both the very early inflationary ($t \sim 0, E \sim \infty$), and very-late-times ($t \sim \infty, E \sim 0$) de Sitter phases of the four-dim Universe. More general solutions are required to explain the evolution of the Universe. For example, solutions of the form $\Phi(t) \neq$ constant, and $V(\Phi) = V(t)$ such that the variable exponent $2\int H(t)dt$ associated to the Hubble function $H(t)$ leads to an early inflationary phase with a large exponential factor, and a constant factor H_o at very late times. Ideally one would hope to recover the inflationary, radiation, matter and dark energy eras of the evolution of the Universe.

Inspired by the de Sitter like solutions in (4.34), one may search for conformally flat solutions of the form

$$g_{ij}(x^i, p_a) \ = \ G(x^i, p_a)\ \eta_{ij}, \quad h^{ab}(x^i, p_a) \ = \ H(x^i, p_a)\ \eta^{ab} \qquad (4.35a)$$

and where the ansatz for the nonlinear connection is chosen to be of the form

$$N_{ai}(x^i, p_a) \ = \ \frac{\partial N_a(x^i, p_a)}{\partial x^i} \qquad (4.35b)$$

Even further, given $x^2 \equiv \eta_{ij}x^i x^j$; $p^2 \equiv \eta^{ab}p_a p_b$, a Killing-symmetry reduction of eqs-(4.35) can be chosen to be given by

$$g_{ij} \ = \ G(x^2, p^2)\ \eta_{ij}, \quad h^{ab} \ = \ H(x^2, p^2)\ \eta^{ab}, \quad N_{ai} \ = \ \frac{\partial N_a(x^2, p^2)}{\partial x^i} \qquad (4.36)$$

and it will simplify matters even further. We gave already in the prior sections the reasons why the $x \cdot p, p \cdot x$ terms in eqs-(2.4) could decouple in the classical limit leaving only the $\alpha^2 x^2, \beta^2 p^2$ ones which are compatible with the Killing-symmetry reduction.

To conclude this final section, first of all, before focusing in Killing-symmetry reductions, it remains to be seen whether or not the ansatz in eqs-(4.35) yields solutions to the generalized gravitational field equations (4.23-4.25), with the stress energy tensors given by eqs-(4.29-4.31), and the equation of motion for the scalar field given by (4.32). Namely, whether or not one can find consistent solutions in terms of the functions $G(x^i, p_a), H(x^i, p_a), N_a(x^i, p_a)$. This particular $8D$ curved phase space geometry requires the above $1+1+4 = 6$ functions to satisfy a large number of equations. This is left for future calculations, at the moment is beyond our computational capabilities.

APPENDIX

In this Appendix we provide solutions to eq-(2.7). Inserting the ansatz provided by eqs-(2.4a-2.4e) into eq-(2.7), assembling the monomials accordingly to eq-(2.13a), and after imposing the relations (2.13b), one arrives at the 13 equations

$$-\ \frac{1}{4} \ = \ \delta_1\ c_4 \ + \ \delta_4\ c_1 \ - \ \delta_2\ c_3 \ - \ \delta_3\ c_2 \qquad (A.1)$$

$$a\ c_4 \ + \ \beta^{-2}\ a\ c_1 \ - \ a\ c_3 \ - \ \beta^{-2}\ a\ c_2 \ = \ 0 \qquad (A.2)$$

$$\gamma_1\ c_4\ +\ \gamma_4\ c_1\ -\ \gamma_2\ c_3\ -\ \gamma_3\ c_2\ =\ 0 \qquad (A.3)$$

$$a^2\beta^{-2}\ -\ a^2\beta^{-2}\ =\ 0,\ \ a\ \gamma_4\ -\ a\ \gamma_3\ =\ 0 \qquad (A.4)$$

$$a\ \delta_4\ -\ a\ \delta_3\ =\ 0,\ \ \gamma_1\ \delta_4\ -\ \gamma_2\ \delta_3\ =\ 0 \qquad (A.5)$$

$$\beta^{-2}\ a\ \gamma_1\ -\ \beta^{-2}\ a\ \gamma_2\ =\ 0,\ \ \gamma_1\ \gamma_4\ -\ \gamma_2\ \gamma_3\ =\ 0. \qquad (A.6)$$

$$\beta^{-2}\ a\ \delta_1\ -\ \beta^{-2}\ a\ \delta_2\ =\ 0,\ \ \gamma_4\ \delta_1\ -\ \gamma_3\ \delta_2\ =\ 0 \qquad (A.7)$$

$$\delta_1\ \delta_4\ -\ \delta_2\ \delta_3\ =\ 0,\ \ c_1\ c_4\ -\ c_2\ c_3\ =\ 0 \qquad (A.8)$$

After some algebra one finds the following relations among the coefficients

$$\gamma_1\ =\ \gamma_2,\ \ \gamma_3\ =\ \gamma_4,\ \ \delta_1\ =\ \delta_2,\ \ \delta_3\ =\ \delta_4. \qquad (A.9)$$

$$\gamma_4\ =\ \beta^{-2}\ \gamma_1,\ \ \gamma_3\ =\ \beta^{-2}\ \gamma_2,\ \ c_4\ =\ \beta^{-2}\ c_2,\ \ c_3\ =\ \beta^{-2}\ c_1 \qquad (A.10)$$

such that

$$c_1\ -\ c_2\ \neq\ 0,\ \ c_3\ -\ c_4\ \neq\ 0 \qquad (A.11)$$

and

$$-\frac{1}{4}\ =\ (c_1 - c_2)\ (\delta_3\ -\ \beta^{-2}\ \delta_1) \qquad (A.12)$$

Finally one finds the family of solutions in eqs-(2.14)

$$Y^5(x,p)\ =\ \kappa_1\ \beta\ z_1\ +\ \kappa_2\ \beta\ z_2\ +\ \kappa_3\ \beta\ z_3\ +\ \kappa_4\ \beta \qquad (2.14a)$$

$$Y^6(x,p)\ =\ \kappa_1\ \beta\ z_1\ +\ \kappa_2\ \beta\ z_2\ +\ \kappa_3\ \beta\ z_3\ +\ (\kappa_4 + 1)\ \beta \qquad (2.14b)$$

$$\Pi^5(x,p)\ =\ \kappa_1\ \beta^{-1}\ z_1\ +\ \kappa_2\ \beta^{-1}\ z_2\ +\ \frac{5}{4}\ \kappa_3\ \beta^{-1}\ z_3\ +\ \kappa_4\ \beta^{-1} \qquad (2.14c)$$

$$\Pi^6(x,p)\ =\ \kappa_1\ \beta^{-1}\ z_1\ +\ \kappa_2\ \beta^{-1}\ z_2\ +\ \frac{5}{4}\ \kappa_3\ \beta^{-1}\ z_3\ +\ (\kappa_4 + 1)\ \beta^{-1} \qquad (2.14d)$$

where $\kappa_3 = (\alpha\beta)^{-1}$ and $\kappa_1, \kappa_2, \kappa_4$ are three arbitrary parameters. The solutions (2.14) have the form $Y^6 = Y^5 + \beta; \Pi^5 = \Pi^6 - \beta^{-1}$ such that $\alpha\beta\ Y^{[5}\ \Pi^{6]} = -\frac{z_3}{4}$ as required by eq-(2.7).

Acknowledgments

We thank M. Bowers and J.C Zambrano for friendship and computer assistance.

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