ON THE GENERAL ERDŐS-MOSER EQUATION VIA THE NOTION OF OLLOIDS

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Abstract. We introduce and develop the notion of the olloid. We apply this notion to study a variant and a generalized version of the Erdős-Moser equation under some special local condition.

1. Introduction

The Erdős-Moser equation is an equation of the form
\[ 1^k + 2^k + \cdots + m^k = (m + 1)^k \]
where \( m \) and \( k \) are positive integers. The only known solution to the equation is \( 1^1 + 2^1 = 3^1 \) and Paul Erdős is known to have conjectured that the equation has no further solution. The exponent \( k \) and the arguments in the Erdős-Moser equation has also been studied quite extensively. In other words, several contraints on the exponent \( k \) and the argument \( m \) of the Erdős-Moser equation have been studied under a presumption that other solutions - if any - exists. In particular, it has been shown that \( k \) must be divisible by 2 and that there is no solution with \( m < 10^{1000000} \) \[1\]. The methods introduced by Moser were later refined and adapted to show that \( m > 2.7139 \times 10^{1.667,658,416} \) in \[2\]. This was improved to the lower bound \( m > 200 \times 10^{1,667,658,416} \) in \[5\] via large scale computation of \( \ln(2) \). It is also shown (see \[3\]) that \( 6 \leq k + 2 < m < 2k \). It is also known that \( \text{lcm}(1, 2, \cdots, 200) \) must divide \( k \) and that any prime factor of \( m + 1 \) must be irregular and > 1000 \[4\]. In 2002, it was shown that all primes \( 200 < p < 1000 \) must divide the exponent \( k \) in the Erdős-Moser equation
\[ 1^k + 2^k + \cdots + m^k = (m + 1)^k \]
where \( m \) and \( k \) are positive integers.

In this paper we introduce and study the notion of the olloid and develop a technique for extending the solution of the generalized Erdős-Moser equation upto exponents \( k \) under some special local conditions of the underlying generator. In particular, we obtain the following result

**Theorem 1.1** (The generalized extension method). Let \( h : \mathbb{N} \rightarrow \mathbb{R}^+ \) have continuous derivative on \([1, s]\) and decreasing on \( \mathbb{R}^+ \) with \( f : \mathbb{N} \rightarrow (0, \infty) \) such that \( f(s + 1) > h(i) \) for all \( 1 \leq i \leq s \) for \( s \in \mathbb{N} \). If the equation
\[ \sum_{i=1}^{s} h(i)^k = f(s + 1)^k \]
for $k > 1$ has a solution and there exist some $r \in \mathbb{N}$ such that
\[ 1 - \frac{1}{g(s)^r} > \int_1^s \frac{g'(t)}{g(t)^2} \, dt + \frac{1}{g(s)} \int_1^s \frac{g'(t)}{g(t)^2} \, dt + \cdots + \frac{1}{g(s)^{r-1}} \int_1^s \frac{g'(t)}{g(t)^2} \, dt \]
with
\[ g(i) := \frac{h(i)}{f(s + 1)} \]
for $1 \leq i \leq s$. Then the equation
\[ \sum_{i=1}^s h(i)^{k+r} = f(s + 1)^{k+r} \]
also has a solution.

This result is a consequence of the more fundamental result using the notion of the olloid.

**Lemma 1.2** (Expansion principle). Let $\mathbb{F}^k_s$ be an s-dimensional olloid of degree $k$ for a fixed $k \in \mathbb{N}$ with $k > 1$. If $g : \mathbb{N} \rightarrow \mathbb{R}^+$ is a generator with continuous derivative on $[1, s]$ and decreasing on $\mathbb{R}^+$ such that
\[ 1 - \frac{1}{g(s)^r} > \int_1^s \frac{g'(t)}{g(t)^2} \, dt + \frac{1}{g(s)} \int_1^s \frac{g'(t)}{g(t)^2} \, dt + \cdots + \frac{1}{g(s)^{r-1}} \int_1^s \frac{g'(t)}{g(t)^2} \, dt \]
for $r \in \mathbb{N}$ then $g : \mathbb{N} \rightarrow \mathbb{R}^+$ is also a generator of the olloid $\mathbb{F}^{k+r}_s$ of degree $k + r$.

2. The notion of the olloid

In this section we launch the notion of the olloid and prove a fundamental lemma, which will be relevant for our studies in the sequel.

**Definition 2.1.** Let $\mathbb{F}^k_s := \{ (u_1, u_2, \ldots, u_s) \in \mathbb{R}^s \mid \sum_{i=1}^s u_i^k = 1, \ k > 1 \}$. Then we call $\mathbb{F}^k_s$ an s-dimensional olloid of degree $k > 1$. We say $g : \mathbb{N} \rightarrow \mathbb{R}$ is a generator of the s-dimensional olloid of degree $k$ if there exists some vector $(v_1, v_2, \ldots, v_s) \in \mathbb{F}^k_s$ such that $v_i = g(i)$ for each $1 \leq i \leq s$.

**Question 2.2.** Does there exists a fixed generator $g : \mathbb{N} \rightarrow \mathbb{R}$ with infinitely many olloids?

**Remark 2.3.** While it may be difficult to provide a general answer to question 2.2, we can in fact provide an answer by imposition certain conditions for which the generator of the olloid must satisfy. In particular, we launch a basic and a fundamental principle relevant for our studies in the sequel.

**Lemma 2.4** (Expansion principle). Let $\mathbb{F}^k_s$ be an s-dimensional olloid of degree $k > 1$ for a fixed $k \in \mathbb{N}$. If $g : \mathbb{N} \rightarrow \mathbb{R}^+$ is a generator with continuous derivative on $[1, s]$ and decreasing on $\mathbb{R}^+$ such that
\[ 1 - \frac{1}{g(s)^r} > \int_1^s \frac{g'(t)}{g(t)^2} \, dt + \frac{1}{g(s)} \int_1^s \frac{g'(t)}{g(t)^2} \, dt + \cdots + \frac{1}{g(s)^{r-1}} \int_1^s \frac{g'(t)}{g(t)^2} \, dt \]
for $r \in \mathbb{N}$ then $g : \mathbb{N} \rightarrow \mathbb{R}^+$ is also a generator of the olloid $\mathbb{F}^{k+r}_s$ of degree $k + r$. 
Proof. Suppose \( g : \mathbb{N} \to \mathbb{R}^+ \) is a generator of the olloid \( F^k \) with continuous derivative on \([1, s]\). Then there exists a vector \((v_1, v_2, \ldots, v_s) \in F^k_s\) such that \( v_i = g(i) \) for each \(1 \leq i \leq s\), so that we can write

\[
\sum_{i=1}^{s} g(i)^k = 1.
\]

Let us assume to the contrary that there exists no \( r \in \mathbb{N} \) such that \( g : \mathbb{N} \to \mathbb{R}^+ \) is a generator of the olloid \( F^k_{s+r} \). By applying the summation by parts, we obtain the inequality

\[
\frac{1}{g(s)} \sum_{i=1}^{s} g(i)^{k+1} \geq 1 - \int_1^s \frac{g'(t)}{g(t)^2} dt \tag{2.1}
\]

by using the inequality

\[
\sum_{i=1}^{s} g(i)^{k+1} < \sum_{i=1}^{s} g(i)^k = 1.
\]

By applying summation by parts on the left side of (2.1) and using the contrary assumption, we obtain further the inequality

\[
\frac{1}{g(s)^2} \sum_{i=1}^{s} g(i)^{k+2} \geq 1 - \int_1^s \frac{g'(t)}{g(t)^2} dt - \frac{1}{g(s)} \int_1^s \frac{g'(t)}{g(t)^2} dt. \tag{2.2}
\]

By induction we can write the inequality as

\[
\frac{1}{g(s)^r} \sum_{i=1}^{s} g(i)^{k+r} \geq 1 - \int_1^s \frac{g'(t)}{g(t)^2} dt - \frac{1}{g(s)} \int_1^s \frac{g'(t)}{g(t)^2} dt - \cdots - \frac{1}{g(s)^{r-1}} \int_1^s \frac{g'(t)}{g(t)^2} dt
\]

for any \( r \geq 2 \) with \( r \in \mathbb{N} \). Since \( g : \mathbb{N} \to \mathbb{R}^+ \) is decreasing, it follows that

\[
1 - \int_1^s \frac{g'(t)}{g(t)^2} dt - \frac{1}{g(s)} \int_1^s \frac{g'(t)}{g(t)^2} dt - \cdots - \frac{1}{g(s)^{r-1}} \int_1^s \frac{g'(t)}{g(t)^2} dt > 1
\]

and using the requirement

\[
1 - \frac{1}{g(s)^r} > \int_1^s \frac{g'(t)}{g(t)^2} dt + \frac{1}{g(s)} \int_1^s \frac{g'(t)}{g(t)^2} dt + \cdots + \frac{1}{g(s)^{r-1}} \int_1^s \frac{g'(t)}{g(t)^2} dt
\]

for \( r \in \mathbb{N} \), we have the inequality

\[
1 = \sum_{i=1}^{s} g(i)^k
\]

\[
\geq \sum_{i=1}^{s} g(i)^{k+r} > 1
\]

which is absurd. This completes the proof of the Lemma. \( \square \)
Lemma 2.4 - albeit fundamental - is ultimately useful for our study of variants and possibly extensions of the Erdős-Moser equation. It can be seen as a tool for extending the solution of equations of the form
\[ \sum_{i=1}^{s} g(i)^k = 1 \]
for \( k > 1 \) - under the presumption that it exists - to the solution of equations of the form
\[ \sum_{i=1}^{s} g(i)^{k+r} = 1 \]
for a fixed \( r \in \mathbb{N} \) under some special requirements of the generator \( g : \mathbb{N} \to \mathbb{R} \).

3. Application to solutions of the generalized Erdős-Moser equation

In this section we apply the notion of theolloid to study solutions of the Erdős-Moser equation. We launch the following method as an outgrowth of Lemma 2.4.

**Theorem 3.1** (The generalized extension method). Let \( h : \mathbb{N} \to \mathbb{R}^+ \) have continuous derivative on \([1, s]\) and decreasing on \(\mathbb{R}^+\) with \( f : \mathbb{N} \to (0, \infty) \) such that \( f(s+1) > h(i) \) for all \( 1 \leq i \leq s \) for \( s \in \mathbb{N} \). If the equation
\[ \sum_{i=1}^{s} h(i)^k = f(s+1)^k \]
for \( k > 1 \) has a solution and there exist some \( r \in \mathbb{N} \) such that
\[ 1 - \frac{1}{g(s)} > \int_{1}^{s} \frac{g'(t)}{g(t)^2} dt + \frac{1}{g(s)} \int_{1}^{s} \frac{g'(t)}{g(t)^2} dt + \ldots + \frac{1}{g(s)^{r-1}} \int_{1}^{s} \frac{g'(t)}{g(t)^2} dt \]
with
\[ g(i) := \frac{h(i)}{f(s+1)} \]
for \( 1 \leq i \leq s \). Then the equation
\[ \sum_{i=1}^{s} h(i)^{k+r} = f(s+1)^{k+r} \]
also has a solution.

**Proof.** Suppose the equation
(3.1) \[ \sum_{i=1}^{s} h(i)^k = f(s+1)^k \]
has a solution. Then equation (3.1) can be recast as
(3.2) \[ \sum_{i=1}^{s} \left( \frac{h(i)}{f(s+1)} \right)^k = 1 \]
which can also be transformed into the sum
\[ \sum_{i=1}^{s} g(i)^k = 1 \]
with
\[ g(i) := \frac{h(i)}{f(s+1)}. \]
The function
\[ g(i) := \frac{h(i)}{f(s+1)} \]
for \(1 \leq i \leq s\) is decreasing and has continuous derivative on \([1, s]\) since \(h : \mathbb{N} \rightarrow \mathbb{R}^+\) have continuous derivative on \([1, s]\) and decreasing on \(\mathbb{R}^+\), so that if there exists some \(r \in \mathbb{N}\) such that
\[
1 - \frac{1}{g(s)^r} \geq \int_1^s \frac{g'(t)}{g(t)^2} dt + \frac{1}{g(s)} \int_1^s \frac{g'(t)}{g(t)^2} dt + \cdots + \frac{1}{g(s)^{r-1}} \int_1^s \frac{g'(t)}{g(t)^2} dt
\]
with
\[ g(i) := \frac{h(i)}{f(s+1)} \]
for \(1 \leq i \leq s\), then by appealing to Lemma 2.4 the equation
\[
\sum_{i=1}^s g(i)^{k+r} = 1 \tag{3.3}
\]
also has a solution. We note that equation (3.3) can also be transformed to the equation
\[
\sum_{i=1}^s \left( \frac{h(i)}{f(s+1)} \right)^{k+r} = 1 \tag{3.4}
\]
so that it has a solution. Since equation (3.4) can be recast as
\[
\sum_{i=1}^s h(i)^{k+r} = f(s+1)^{k+r}
\]
and the claim follows immediately. \(\square\)

It is important to note that if the values of \(h\) on the positive integers is still a positive integer and have continuous derivative on \([1, s]\) and decreasing on \(\mathbb{R}^+\), then the integer solution of the more general Erdős-Moser equation
\[
\sum_{i=1}^s h(i)^k = f(s+1)^k
\]
can be extended to the integer solutions of the equation
\[
\sum_{i=1}^s h(i)^{k+r} = f(s+1)^{k+r}
\]
under the local condition of the normalized values of \(h\) on \([1, s+1]\). One could also examine the problem with the sequence \(h : \mathbb{N} \rightarrow \mathbb{R}^+\) and ask if it is possible to take \(h\) to be an arithmetic progression. A similar question could be asked for sequences \(h : \mathbb{N} \rightarrow \mathbb{R}^+\) of general types. It is important to recognize that the tool we have developed only allows us to extend solutions of the general Erdős-Moser equation under a certain local condition of normalized generators of the olloid.
4. Data availability statement

The manuscript has no associated data.

5. Conflict of interest statement

The author declares no conflict of interest regarding this publication.

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