ON THE GENERAL ERDŐS-MOSER EQUATION VIA THE NOTION OF OLLOIDS

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ABSTRACT. We introduce and develop the notion of the olloid. We apply this notion to study a variant and a generalized version of the Erdős-Moser equation under some special local condition.

1. Introduction

The Erdős-Moser equation is an equation of the form

$$1^k + 2^k + \cdots + m^k = (m + 1)^k$$

where $m$ and $k$ are positive integers. The only known solution to the equation is $1^1 + 2^1 = 3^1$ and Paul Erdős is known to have conjectured that the equation has no further solution. The exponent $k$ and the arguments in the Erdős-Moser equation has also been studied quite extensively. In other words, several constraints on the exponent $k$ and the argument $m$ of the Erdős-Moser equation have been studied under a presumption that other solutions - if any - exists. In particular, it has been shown that $k$ must be divisible by 2 and that there is no solution with $m < 10^{1000000}$ [1]. The methods introduced by Moser were later refined and adapted to show that $m > 1.485 \times 10^{9321155}$ [2]. This was improved to the lower bound $m > 2.7139 \times 10^{1.667, 658, 416}$ in [5] via large scale computation of $\ln(2)$. It is also shown (see [3]) that $6 \leq k + 2 < m < 2k$. It is also known that $\text{lcm}(1, 2, \cdots, 200)$ must divide $k$ and that any prime factor of $m+1$ must be irregular and $> 1000$ [4]. In 2002, it was shown that all primes $200 < p < 1000$ must divide the exponent $k$ in the Erdős-Moser equation

$$1^k + 2^k + \cdots + m^k = (m + 1)^k$$

where $m$ and $k$ are positive integers.

In this paper we introduce and study the notion of the olloid and develop a technique for extending the solution of the generalized Erdős-Moser equation upto exponents $k$ under some special local conditions of the underlying generator. In particular, we obtain the following result

**Theorem 1.1** (The generalized extension method). Let $h : \mathbb{N} \rightarrow \mathbb{R}^+$ have continuous derivative on $[1, s]$ and decreasing on $\mathbb{R}^+$. If the equation

$$\sum_{i=1}^{s} h(i)^k = h(s + 1)^k$$

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for $k > 1$ has a solution and there exist some $r \in \mathbb{N}$ such that

$$1 - \frac{1}{g(s)^r} > \int_1^s \frac{g'(t)}{g(t)^2} dt + \frac{1}{g(s)} \int_1^s \frac{g'(t)}{g(t)^2} dt + \cdots + \frac{1}{g(s)^r-1} \int_1^s \frac{g'(t)}{g(t)^2} dt$$

with

$$g(i) := \frac{h(i)}{h(i + 1)}$$

for $1 \leq i \leq s$. Then the equation

$$\sum_{i=1}^s h(i)^{k+r} = h(s + 1)^{k+r}$$

also has a solution.

This result is a consequence of the more fundamental result using the notion of the olloid.

**Lemma 1.2** (Expansion principle). Let $\mathbb{F}_s^k$ be an $s$-dimensional olloid of degree $k$ for a fixed $k \in \mathbb{N}$ with $k > 1$. If $g : \mathbb{N} \rightarrow \mathbb{R}^+$ is a generator with continuous derivative on $[1, s]$ and decreasing on $\mathbb{R}^+$ such that

$$1 - \frac{1}{g(s)^r} > \int_1^s \frac{g'(t)}{g(t)^2} dt + \frac{1}{g(s)} \int_1^s \frac{g'(t)}{g(t)^2} dt + \cdots + \frac{1}{g(s)^r-1} \int_1^s \frac{g'(t)}{g(t)^2} dt$$

for $r \in \mathbb{N}$ then $g : \mathbb{N} \rightarrow \mathbb{R}^+$ is also a generator of the olloid $\mathbb{F}_s^{k+r}$ of degree $k + r$.

2. The notion of the olloid

In this section we launch the notion of the olloid and prove a fundamental lemma, which will be relevant for our studies in the sequel.

**Definition 2.1.** Let $\mathbb{F}_s^k := \{(u_1, u_2, \ldots, u_s) \in \mathbb{R}^s \mid \sum_{i=1}^s u_i^k = 1, \ k > 1\}$. Then we call $\mathbb{F}_s^k$ an $s$-dimensional olloid of degree $k > 1$. We say $g : \mathbb{N} \rightarrow \mathbb{R}$ is a generator of the $s$-dimensional olloid of degree $k$ if there exists some vector $(v_1, v_2, \ldots, v_s) \in \mathbb{F}_s^k$ such that $v_i = g(i)$ for each $1 \leq i \leq s$.

**Question 2.2.** Does there exists a fixed generator $g : \mathbb{N} \rightarrow \mathbb{R}$ with infinitely many olloids?

**Remark 2.3.** While it may be difficult to provide a general answer to question 2.2, we can in fact provide an answer by imposition certain conditions for which the generator of the olloid must satisfy. In particular, we launch a basic and a fundamental principle relevant for our studies in the sequel.

**Lemma 2.4** (Expansion principle). Let $\mathbb{F}_s^k$ be an $s$-dimensional olloid of degree $k > 1$ for a fixed $k \in \mathbb{N}$. If $g : \mathbb{N} \rightarrow \mathbb{R}^+$ is a generator with continuous derivative on $[1, s]$ and decreasing on $\mathbb{R}^+$ such that

$$1 - \frac{1}{g(s)^r} > \int_1^s \frac{g'(t)}{g(t)^2} dt + \frac{1}{g(s)} \int_1^s \frac{g'(t)}{g(t)^2} dt + \cdots + \frac{1}{g(s)^r-1} \int_1^s \frac{g'(t)}{g(t)^2} dt$$

for $r \in \mathbb{N}$ then $g : \mathbb{N} \rightarrow \mathbb{R}^+$ is also a generator of the olloid $\mathbb{F}_s^{k+r}$ of degree $k + r$. 
Proof. Suppose \( g : \mathbb{N} \rightarrow \mathbb{R}^+ \) is a generator of the olloid \( F_k^s \) with continuous derivative on \([1, s]\). Then there exists a vector \((v_1, v_2, \ldots, v_s) \in \mathbb{F}_s^k\) such that \( v_i = g(i) \) for each \( 1 \leq i \leq s \), so that we can write

\[
\sum_{i=1}^{s} g(i)^k = 1.
\]

Let us assume to the contrary that there exists no \( r \in \mathbb{N} \) such that \( g : \mathbb{N} \rightarrow \mathbb{R}^+ \) is a generator of the olloid \( F_k^{s+r} \). By applying the summation by parts, we obtain the inequality

\[
\frac{1}{g(s)} \sum_{i=1}^{s} g(i)^{k+1} \geq 1 - \int_{1}^{s} \frac{g'(t)}{g(t)^2} dt
\]

by using the inequality

\[
\sum_{i=1}^{s} g(i)^{k+1} < \sum_{i=1}^{s} g(i)^k = 1.
\]

By applying summation by parts on the left side of (2.1) and using the contrary assumption, we obtain further the inequality

\[
\frac{1}{g(s)^2} \sum_{i=1}^{s} g(i)^{k+2} \geq 1 - \int_{1}^{s} \frac{g'(t)}{g(t)^2} dt - \frac{1}{g(s)} \int_{1}^{s} \frac{g'(t)}{g(t)^2} dt.
\]

By induction we can write the inequality as

\[
\frac{1}{g(s)^r} \sum_{i=1}^{s} g(i)^{k+r} \geq 1 - \int_{1}^{s} \frac{g'(t)}{g(t)^2} dt - \frac{1}{g(s)} \int_{1}^{s} \frac{g'(t)}{g(t)^2} dt - \cdots - \frac{1}{g(s)^{r-1}} \int_{1}^{s} \frac{g'(t)}{g(t)^2} dt
\]

for any \( r \geq 2 \) with \( r \in \mathbb{N} \). Since \( g : \mathbb{N} \rightarrow \mathbb{R}^+ \) is decreasing, it follows that

\[
1 - \int_{1}^{s} \frac{g'(t)}{g(t)^2} dt - \frac{1}{g(s)} \int_{1}^{s} \frac{g'(t)}{g(t)^2} dt - \cdots - \frac{1}{g(s)^{r-1}} \int_{1}^{s} \frac{g'(t)}{g(t)^2} dt > 1
\]

and using the requirement

\[
1 - \frac{1}{g(s)^r} > \int_{1}^{s} \frac{g'(t)}{g(t)^2} dt + \frac{1}{g(s)} \int_{1}^{s} \frac{g'(t)}{g(t)^2} dt + \cdots + \frac{1}{g(s)^{r-1}} \int_{1}^{s} \frac{g'(t)}{g(t)^2} dt
\]

for \( r \in \mathbb{N} \), we have the inequality

\[
1 = \sum_{i=1}^{s} g(i)^k
\]

\[
\geq \sum_{i=1}^{s} g(i)^{k+r} > 1
\]

which is absurd. This completes the proof of the Lemma. \( \square \)
Lemma 2.4 - albeit fundamental - is ultimately useful for our study of variants and possibly extensions of the Erdős-Moser equation. It can be seen as a tool for extending the solution of equations of the form
\[
\sum_{i=1}^{s} g(i)^k = 1
\]
for \( k > 1 \) - under the presumption that it exists - to the solution of equations of the form
\[
\sum_{i=1}^{s} g(i)^{k+r} = 1
\]
for a fixed \( r \in \mathbb{N} \) under some special requirements of the generator \( g : \mathbb{N} \to \mathbb{R} \).

3. Application to solutions of the generalized Erdős-Moser equation

In this section we apply the notion of the \textit{ooloid} to study solutions of the Erdős-Moser equation. We launch the following method as an outgrowth of Lemma 2.4.

**Theorem 3.1** (The generalized extension method). Let \( h : \mathbb{N} \to \mathbb{R}^+ \) have continuous derivative on \([1, s]\) and decreasing on \(\mathbb{R}^+\). If the equation
\[
\sum_{i=1}^{s} h(i)^k = h(s + 1)^k
\]
for \( k > 1 \) has a solution and there exist some \( r \in \mathbb{N} \) such that
\[
1 - \frac{1}{g(s)^r} > \int_{1}^{s} \frac{g'(t)}{g(t)^2} \, dt + \frac{1}{g(s)} \int_{1}^{s} \frac{g'(t)}{g(t)^2} \, dt + \cdots + \frac{1}{g(s)^{r-1}} \int_{1}^{s} \frac{g'(t)}{g(t)^2} \, dt
\]
with
\[
g(i) := \frac{h(i)}{h(s + 1)}
\]
for \( 1 \leq i \leq s \). Then the equation
\[
\sum_{i=1}^{s} h(i)^{k+r} = h(s + 1)^{k+r}
\]
also has a solution.

**Proof.** Suppose the equation
\[
(3.1) \quad \sum_{i=1}^{s} h(i)^k = h(s + 1)^k
\]
has a solution. Then equation (3.1) can be recast as
\[
(3.2) \quad \sum_{i=1}^{s} \left( \frac{h(i)}{h(s + 1)} \right)^k = 1
\]
which can also be transformed into the sum
\[
\sum_{i=1}^{s} g(i)^k = 1
\]
with
\[ g(i) := \frac{h(i)}{h(s + 1)}. \]
The function
\[ g(i) := \frac{h(i)}{h(s + 1)} \]
for \(1 \leq i \leq s\) is decreasing and has continuous derivative on \([1, s]\) since \(h : \mathbb{N} \rightarrow \mathbb{R}^+\) have continuous derivative on \([1, s]\) and decreasing on \(\mathbb{R}^+\), so that if there exists some \(r \in \mathbb{N}\) such that
\[
1 - \frac{1}{g(s)^r} > \int_1^s \frac{g'(t)}{g(t)^2} \, dt + \frac{1}{g(s)} \int_1^s \frac{g'(t)}{g(t)^2} \, dt + \cdots + \frac{1}{g(s)^{r-1}} \int_1^s \frac{g'(t)}{g(t)^2} \, dt
\]
with
\[ g(i) := \frac{h(i)}{h(s + 1)} \]
for \(1 \leq i \leq s\), then by appealing to Lemma 2.4 the equation
\[
\sum_{i=1}^s g(i)^{k+r} = 1 \tag{3.3}
\]
also has a solution. We note that equation (3.3) can also be transformed to the equation
\[
\sum_{i=1}^s \left( \frac{h(i)}{h(s + 1)} \right)^{k+r} = 1 \tag{3.4}
\]
so that it has a solution. Since equation (3.4) can be recast as
\[
\sum_{i=1}^s h(i)^{k+r} = h(s + 1)^{k+r}
\]
and the claim follows immediately. \(\square\)

It is important to note that if the values of \(h\) on the positive integers is still a positive integer and have continuous derivative on \([1, s]\) and decreasing on \(\mathbb{R}^+\), then the integer solution of the more general Erdős-Moser equation
\[
\sum_{i=1}^s h(i)^k = h(s + 1)^k
\]
can be extended to the integer solutions of the equation
\[
\sum_{i=1}^s h(i)^{k+r} = h(s + 1)^{k+r}
\]
under the local condition of the normalized values of \(h\) on \([1, s+1]\). One could also examine the problem with the sequence \(h : \mathbb{N} \rightarrow \mathbb{R}^+\) and ask if it is possible to take \(h\) to be an arithmetic progression. A similar question could be ask for sequences \(h : \mathbb{N} \rightarrow \mathbb{R}^+\) of general types. It is important to recognize that the tool we have developed only allows us to extend solutions of the general Erdős-Moser equation under a certain local condition of normalized generators of the olloid.
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