# A Proposal for More Economic Fuel Use at Lagrange Points

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This paper demonstrates why – regarding fuel consumption – it is more senseful to perform stationkeeping at Lagrange point as often as possible, i.e. when thrust needed is greater than 12 cm/s for the James Webb Space Telescope. Fuel can be saved by striving to correct the orbit each time as early as manageable. With such change, the conservative estimate for the mission lifetime increase is over 12%.

## INTRODUCTION

An interesting placement of telescopes in space are Lagrange points. One of them, namely Sun-Earth's L2 point has been chosen as most appropriate for the JWST, James Webb Space Telescope [1]. This stationary position is unstable [2], therefore small perturbations increase with time. For this reason station keeping is required and fuel is needed. It is not yet realistic to expect refueling of JWST's fuel reservoir, so its lifetime is limited by fuel consumption. In case of JWST the expected lifetime is around a decade due to Telescope's very successful deployment [3].

Current station keeping plan is for the maneuver to happen every three weeks and skipped if telescope's orbit is still close enough to L2 [4]. Such plan can be seen as rigid.

The goal of this paper is to propose a more economic way of fuel consumption. As an argument, the fuel needed per some time period is compared between two basic station keeping plans, namely an instant impulse by fuel kick at fixed distances from the stationary point.

## EVALUATION

Let us consider the potential around the stationary point L2. The effective potential is best described in the rotating frame, taking gravitation and centrifugal force into account, without Coriolis force for simplicity [5].

To estimate needed fuel, the test body of mass $m$ – i.e. the telescope – is first left for some time to diverge away from the starting position close to the stationary point L2, where it starts at rest. At a predetermined point some fuel is burnt to kick it back to the initial almost-stationary-position. To return there it needs velocity $v$, if $-v$ was the velocity away from L2 before the kick. The velocity $|v|$ is computed using $\Delta T + \Delta V = 0$, energy conservation in effective potential $V$,

$$\frac{mv^2}{2} = \Delta T = -\Delta V = -(V - V_0) = V_0 - V.$$

Next, the needed fuel impulse $F\delta t$ is computed through required momentum change $\Delta\Gamma$, taking $\Delta v = v-(-v) = 2v$,

$$F\delta t = \Delta\Gamma = m\,\Delta v = 2mv = \sqrt{8m(V_0 - V)}.$$

To compare efficiency of fuel powered impulses at various divergence thresholds, the impulse should be normalised with time in which the test object freely diverges from the initial almost stationary point. This time can be evaluated solving the equation of motion. It is linear in the vicinity of L2, since effective potential there can be approximated with a quadratic function,

$$V(x) = -\frac{1}{2}\alpha x^2,$$

where $x = 0$ is taken to be a stationary point and $\alpha$ is a constant, in principle determined from the second order derivative of the effective potential. Solving the equation of motion, i.e. Newton's law

$$\ddot{x} = -\frac{1}{m}\frac{\mathrm{d}V}{\mathrm{d}x} = \frac{\alpha}{m}x$$

for $x = x(t)$ gives exponential relation. The time at which the test object gets to some point $x$ can then be derived by inverting the solution,

$$t = \sqrt{\frac{m}{\alpha}}\,\log\frac{x}{x_\mathrm{m}},$$

with $x_\mathrm{m}$ a parameter chosen to be small enough for the quadratic approximation of the effective potential to still be appropriate in the considered region from $x_\mathrm{m}$ to the stationary point at 0. Furthermore at $t = 0$ the test object would get to $x_\mathrm{m}$. Time starts at some negative value, when $x$ is a small deviation from L2 which is at 0. As time $t$ increases, the test object moves toward a chosen threshold $x_i$. After the momentary fuel burn, the same amount of time is needed for the test object to return to the initial point, due to reversibility of the differential equation's solutions.

Fuel consumption can be compared between plans of different $x_i$ via time normalized impulses, which are computed as

$$\frac{F\delta t}{2\,\Delta t} = \frac{\sqrt{8m(V_0 - V)}}{2\sqrt{\frac{m}{\alpha}}\left(\log\frac{x_i}{x_\mathrm{m}} - \log\frac{x_0}{x_\mathrm{m}}\right)} = \frac{\alpha\,x_i}{\log\frac{x_i}{x_\mathrm{m}} - \log\frac{x_0}{x_\mathrm{m}}},$$

where $x_i$ are turning points and $x_0 \approx 0$ the starting point close to stationary, which is also why $V_0$ is taken as 0. Let us set $x_0 = 10^{-3} x_\mathrm{m}$ and furthermore compare $x_1 = 10^{-1} x_\mathrm{m}$ and $x_2 = 2 \cdot x_1 = 2 \cdot 10^{-1} x_\mathrm{m}$.

The ratio of time-normalized-fuel-consumption between station-keeping plans of fuel-boost at $x_1$ and $x_2 = 2x_1$ is

$$\frac{x_1}{x_2} \frac{\log \frac{x_2}{x_\mathrm{m}} - \log \frac{x_0}{x_\mathrm{m}}}{\log \frac{x_1}{x_\mathrm{m}} - \log \frac{x_0}{x_\mathrm{m}}} = \frac{1}{2} \frac{\log 2 \cdot 10^{-1} - \log 10^{-3}}{\log 10^{-1} - \log 10^{-3}} \approx 0.58$$

and is lower than 1. This means the fuel consumption is lower if the boost is made earlier, i.e. more often.

### Addition of thruster start up

For a fuel boost the initial cost of starting the thruster has to be considered in practice. This can be added to the evaluation as $F\delta t \mapsto F\delta t + c$ for some constant $c$. The full result for the ratio changes into

$$\frac{\alpha x_1 + c\sqrt{\frac{\alpha}{m}}}{\alpha x_2 + c\sqrt{\frac{\alpha}{m}}} \cdot \frac{\log \frac{x_2}{x_\mathrm{m}} - \log \frac{x_0}{x_\mathrm{m}}}{\log \frac{x_1}{x_\mathrm{m}} - \log \frac{x_0}{x_\mathrm{m}}}.$$

For large $c$ the first factor is close to 1 and the ratio up to 1.15. Since this is larger than 1 it would mean the boost has too much of initial cost and running it rarely would be more economic.

If $x_\mathrm{m}$ is large enough however, the first factor can be lower. In the limit $x_i \gg c/\sqrt{\alpha m}$ the factor evaluates to $\frac{1}{2}$ as in the first evaluation. For $x_1$ and $x_2$ as above, the factor is

$$\frac{x_1 + \frac{c}{\sqrt{\alpha m}}}{x_2 + \frac{c}{\sqrt{\alpha m}}} = \frac{1 + \frac{c}{\sqrt{\alpha m} x_1}}{2 + \frac{c}{\sqrt{\alpha m} x_1}}$$

and is lower than $1.15^{-1} \approx 0.87$ for $c < 5.6 \cdot \sqrt{\alpha m}\, x_1$, meaning the consumption ratio is less than 1. Furthermore it means the fuel consumption is lower with earlier boosts, provided the return thresholds $x_i$ are large enough for the thruster start-up cost to relatively diminish.

### Lifetime expectancy

The improvement of the telescope's lifetime expectancy can be evaluated by comparing the average expected fuel burn. JWST's fuel budget is $2.43 \frac{\mathrm{m}}{\mathrm{s}}$ annually which means on average $14 \frac{\mathrm{cm}}{\mathrm{s}}$ per stationkeeping manoeuvre, which can be compared to the threshold for using JWST's thruster, $12 \frac{\mathrm{cm}}{\mathrm{s}}$ [4]. Taking $x_2 = \frac{14}{12} \cdot x_1$ in the first example gives the consumption ratio 0.886, meaning over 11% of fuel can be saved if stationkeeping is performed regularly at the threshold $12 \frac{\mathrm{cm}}{\mathrm{s}}$.

The expected lifetime expectancy increase in this case is 12.9%, which is a conservative estimate. Due to nonlinearity, the distribution of thruster burns around the average $14 \frac{\mathrm{cm}}{\mathrm{s}}$ according to the current plan contributes to even larger ratio, so the improvement can be expected even larger.

## CONCLUSION

It can be concluded that it is more economic to plan stationkeeping as early as possible, considering of course that running the engine also has some initial cost to it. The threshold for using JWST's thruster is $\Delta v = 12 \frac{\mathrm{cm}}{\mathrm{s}}$. Fuel consumption increases if the free movement is undergone for a longer period. An improved plan *regarding fuel use* would be to use the thruster as soon as the boost needed to return it closer to a desired orbit is over the above $\Delta v$ threshold.

It remains open to check how fuel cost increases in the full potential around the Lagrange point, together with Coriolis force. The improvement of fuel consumption enlengthens lifetime expectancy, conservatively estimated by over 12%. Better estimate can be derived as data about ongoing stationkeeping fuel budget spending is known.

Costs should be seen as investments. It pays off to use fuel as early as manageable, compared to using it per fixed schedule, since waiting can be costly. When the sun is up, the early bird gets the worm.

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