Energy is Conserved in General Relativity

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Abstract
In this article, the author demonstrates that there is a huge contradiction between the statements made in the famous literature about general relativity regarding the vanishing covariant divergence of the energy-momentum tensor of matter representing a conservation law. It is reasoned which of these contradictory standpoints are correct and which are not. The author points out why pseudotensors cannot represent the energy density of the gravitational field. Contrary to the statements in the famous literature about general relativity, the energy density of the gravitational field is shown to be described by a tensor. Moreover, the author demonstrates that in general relativity there necessarily exists the conservation of total energy, momentum, and stress regarding the completed version of Einstein's field equations which is that one with the cosmological constant, whereby the latter one takes on a completely new meaning that solves the cosmological constant problem. This new interpretation of the cosmological constant also explains the dark energy and the dark matter phenomenon. The modified Poisson equation, that is obtained from Einstein's field equations with the cosmological constant in the limit of weak gravitational fields, approximately meets the requirement of conservation of total energy in Newton's theory of gravity, whereby flat rotation curves of spiral galaxies are obtained.

Keywords
General relativity, Einstein's field equations, energy-momentum tensor of matter, pseudotensors, energy-momentum tensor of the gravitational field, total energy-momentum tensor, conservation law, cosmological constant, dark energy, dark matter, Newton's theory of gravity, modified Poisson equation, rotation curves of spiral galaxies.

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Contents

1. Matter tensor

The energy-momentum tensor of matter $T^{\mu\nu}$, also simply termed matter tensor, consists by definition of all kinds of matter-energy, but does *not* contain the energy of the gravitational field [1, 2],

$$T^{\mu\nu} = T^{\mu\nu}_{(\mathrm{pf})} + T^{\mu\nu}_{(\mathrm{em})} + \ldots ,$$

where

$$T^{\mu\nu}_{(\mathrm{pf})} = \left(\varrho + \frac{P}{c^2}\right) u^\mu u^\nu + P g^{\mu\nu}$$

is the energy-momentum tensor of a perfect fluid and

$$T^{\mu\nu}_{(\mathrm{em})} = \frac{1}{\mu_0}\left(F^{\mu\alpha} F^{\nu}_{\ \alpha} - \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}\right)$$

is the energy-momentum tensor of the electromagnetic field [3].

In special relativity, i.e. in an inertial frame as well as in a local inertial frame, the vanishing partial divergence of the matter tensor $T^{\mu\nu}{}_{,\nu} = 0$ demonstrates that the sum of all kinds of matter-energy are conserved [1, 4].

However, regarding the vanishing covariant divergence of the matter tensor representing a conservation law in general relativity there is a huge contradiction between the statements made in the books of Landau and Lifshitz [2], Møller [5], and Straumann [3] which are shown in Sec. 1.1, and those made in the books of Weinberg [6], Fließbach [1], and Lavenda [7] which are demonstrated in Sec. 1.3. Additionally, in Sec. 1.2 are pointed out the corresponding statements in the comprehensive script of Blau [8] whose standpoint is ambiguous. In Sec. 1.4, it is cleared up which of these positions are correct and which are not.

1.1 Standpoint

1.1.1 Landau and Lifshitz

Landau and Lifshitz write in § 96 in their famous book [2]:

> In the absence of a gravitational field, the law of conservation of energy and momentum of the material (and electromagnetic field) is expressed by the equation $\partial T^{ik}/\partial x^k = 0$. The generalization of this equation to the case where a gravitational field is present is equation (94.7):
>
> $$T^{k}_{i;\,k} = \frac{1}{\sqrt{-g}} \frac{\partial (T^k_i \sqrt{-g})}{\partial x^k} - \frac{1}{2} \frac{\partial g_{kl}}{\partial x^i} T^{kl} = 0. \qquad (96.1)$$
>
> In this form, however, this equation does not generally express any conservation law whatever. Because the integral $\int T^k_i \sqrt{-g}\, dS_k$ is conserved only if the condition
>
> $$\frac{\partial(\sqrt{-g}\, T^k_i)}{\partial x^k} = 0$$
>
> is fulfilled, and not (96.1). This is related to the fact that in a gravitational field the four-momentum of the matter alone must not be conserved, but rather the four-momentum of matter plus gravitational field; the latter is not included in the expression for T^k_i.

According to that, Landau and Lifshitz claim in § 96 of Ref. [2], that the vanishing covariant divergence of the matter tensor (96.1) represents no conservation law at all by stating:

> In this form, however, this equation does not generally express any conservation law whatever.

1.1.2 Møller

Møller explicitly writes in § 126 in his well-known book [5]:

> The law of conservation of energy and momentum, which has the form (X. 41) or
>
> $$\frac{\partial}{\partial x^k}\{\sqrt{(-g)}T^k_i\} = \frac{1}{2}\frac{\partial g_{kl}}{\partial x^i}\sqrt{(-g)}T^{kl} \equiv k_i \qquad (143)$$
>
> is, however, not in general equivalent to the vanishing of an ordinary divergence and will therefore not immediately give rise to any conservation laws by integration over the space coordinates. Only in the case of a stationary system considered in § 114 is the right-hand side of (143) zero for $i = 4$, and by subsequent integration over the space coordinates we get a constant of the motion which may be interpreted as the total energy.
>
> The occurrence of the term on the right-hand side of (143) indicates that the system is not strictly closed, this term being analogous to the external four-force density on a non-closed system in the special theory of relativity (cf. Chapter VII).

1.1.3 Straumann

Straumann shows the vanishing covariant divergence of the matter tensor $T^{\mu\nu}{}_{;\nu} = 0$ in his famous book [3] in Eqs. (2.36):

$$\frac{1}{\sqrt{-g}}\partial_\nu\left(\sqrt{-g}T^{\mu\nu}\right) + \Gamma^{\mu}_{\ \nu\lambda} T^{\lambda\nu} = 0. \qquad (2.36)$$

He emphasizes on p. 27 in Ref. [3]:

> Because of the second term in (2.36), this is *no longer* a conservation law. We cannot form any constants of the motion from (2.36). This should also not be expected, since the system under consideration can exchange energy and momentum with the gravitational field.

1.2 Ambiguous standpoint

1.2.1 Blau

In Sec. 7.9 of his comprehensive script [8], Blau denotes the vanishing covariant divergence of the matter tensor a "conservation law":

> In particular, in general relativity, and assuming that $T_{\mu\nu}$ is the complete matter energy-momentum tensor (otherwise we certainly cannot expect to derive any conservation law), we will have a "conservation law" of the form
>
> $$\nabla_\mu T^{\mu\nu} = g^{-1/2}\partial_\mu(g^{1/2}T^{\mu\nu}) + \Gamma^\nu_{\mu\lambda}T^{\mu\lambda} = 0 \ . \tag{7.144}$$
>
> We see that, due to the second term, this does not define four conserved currents in the ordinary or covariant sense (and we will return to the interpretation of this equation, and the related issue of energy and energy density of the gravitational field, in section 22.6).

In Sec. 22.6 of Ref. [8], Blau then denotes that, what he terms in Sec. 7.9 a "conservation law" (cf. Eqs. (7.144)), now being a "non-conservation law" although this just represents the rearranged Eqs. (7.144):

> To get us started, let us return to the covariant conservation law $\nabla_\mu T^{\mu\nu} = 0$ for the matter energy-momentum tensor which played such a key role above, and which we now write more explicitly as the "non-conservation law" (cf. (7.144))
>
> $$\nabla_\mu T^{\mu\nu} = 0 \ \Leftrightarrow\ \partial_\mu(\sqrt{g}T^{\mu\nu}) = -\sqrt{g}\,\Gamma^\nu_{\mu\lambda}T^{\mu\lambda} \ . \tag{22.205}$$

1.3 Opposite standpoint

1.3.1 Weinberg

By performing an infinitesimal transformation of the dynamical variables in the change in the scalar matter action

$$\delta I_M = \tfrac{1}{2}\int d^4x\,\sqrt{g(x)}\,T^{\mu\nu}(x)\delta g_{\mu\nu}(x) \tag{12.2.2}$$

Weinberg demonstrates in Sec. 12.3 in his famous book [6], that the covariant divergence of the matter tensor

$$0 = (T^\nu{}_\lambda)_{;\nu} \tag{12.3.2}$$

represents a conservation law. He emphasizes in Sec. 12.3 of Ref. [6]:

> *Thus the energy-momentum tensor defined by Eq.* (12.2.2) *is conserved* (*in the covariant sense*) *if and only if the matter action is a scalar*. Also, with I_M a scalar, (12.2.2) shows immediately that $T^{\mu\nu}$ is a symmetric *tensor*, so this definition of the energy-momentum tensor has all the properties for which one could wish.
>
> This proof, that general covariance implies energy-momentum conservation, has an exact analog in the proof that gauge invariance implies charge conservation.

1.3.2 Fließbach

Fließbach's book [1] is a highly recommended German standard textbook and maybe is used most at German universities for lectures of general relativity. One can read therein on p. 113:

> In einem Inertialsystem (wie auch im Lokalen IS) gilt für den Energie-Impuls-Tensor $T^{\alpha\beta}$ der Erhaltungssatz $T^{\alpha\beta}{}_{|\beta} = 0$, (8.9). Nach dem Kovarianzprinzip wird dies zu
>
> $$T^{\mu\nu}{}_{||\nu} = 0 \tag{20.31}$$

English translation:

> In an inertial frame (as well as in the local inertial frame) the conservation law $T^{\alpha\beta}{}_{|\beta} = 0$ applies to the energy-momentum tensor $T^{\alpha\beta}$, (8.9). According to the principle of covariance, this becomes
>
> $$T^{\mu\nu}{}_{||\nu} = 0 \tag{20.31}$$

Also on p. 275 in Ref. [1], Fließbach denotes the vanishing covariant divergence of the matter tensor as conservation of energy-momentum.

1.3.3 Lavenda

In Sec. 3.1 in the highly recommended book of Lavenda [7], he thematizes the vanishing covariant divergence of the matter tensor:

$$\nabla_k T^k_i = \frac{1}{\sqrt{-g}}\frac{\partial(T^k_i\sqrt{-g})}{\partial x^k} - \frac{1}{2}\frac{\partial g_{kl}}{\partial x^i}T^{kl} = 0. \tag{3.2}$$

> According to Landau and Lifshitz [LL75], and just about everyone else, "this equation does not express any conservation law whatever." This is "because the integral $\int T^k_i\sqrt{-g}\,dS$ is conserved only if the condition
>
> $$\frac{\partial T^k_i\sqrt{-g}}{\partial x^k} = 0 \tag{3.3}$$
>
> is fulfilled, and not (3.2)." Møller [lle58] comes out and says that if the second term in (3.2) is different from zero, "then it expresses the fact that matter energy is not conserved."

Lavenda in Sec. 3.2 of Ref. [7] states:

> Conservation equations of the form (3.3) are invariant with respect to linear coordinate transformations. Einstein was considering such types of invariant equations during the period 1913-1914. According to Folomeshkin, "Einstein non-critically carried over equations (3.3) into the newly general covariant theory. As a result, the 'problem' of energy-momentum arose."

1.4 Resolution of the contradiction

Regarding Weinberg's proof in Sec. 12.3 of Ref. [6] and Fließbach's argument of applying the principle of covariance on the vanishing partial divergence of the matter tensor in a local inertial frame on p. 113 in his book [1], one must conclude, that conservation laws in general relativity are only such ones, which underlie the principle of covariance, i.e., that a conservation law in general relativity must be a tensor, the covariant divergence of which vanishes. According to that, Lavenda concludes in Sec. 3.2 of Ref. [7]:

> The vanishing of the covariant derivative does lead to a conservation law, but not one we Euclideans are use to dealing with.

One should not be bothered about the non-vanishing second term in the expression of the vanishing covariant divergence of the matter tensor. This term must be present as it is because otherwise the latter would not be a (covariant) conservation law.

From all these citations, statements, proofs, and arguments one must conclude, that the vanishing covariant divergence of the matter tensor represents the law of energy-momentum conservation of matter in general relativity, contrary to the wording of Landau and Lifshitz in § 96 in their famous book [2] regarding their Eqs. (96.1):

> In this form, however, this equation does not generally express any conservation law whatever.

However, they actually relate this statement to conservation of *total* energy by writing in § 96 of Ref. [2]:

> This is related to the fact that in a gravitational field the four-momentum of the matter alone must not be conserved, but rather the four-momentum of matter plus gravitational field; the latter is not included in the expression for T_i^k.

This is also why total energy equals energy of matter plus energy of the gravitational field.

2. Completed field equations

2.1 Denial of the energy tensor of the gravitational field

Misner, Thorne, and Wheeler claim regarding "local gravitational energy-momentum" in §20.4. in their famous book [4]:

> There is no unique formula for it, but a multitude of quite distinct formulas. The two cited are only two among an infinity. Moreover, "local gravitational energy-momentum" has no weight. It does not curve space. It does not serve as a source term on the righthand side of Einstein's field equations. It does not produce any relative geodesic deviation of two nearby world lines that pass through the region of space in question. It is not observable.
>
> Anybody who looks for a magic formula for "local gravitational energy-momentum" is looking for the right answer to the wrong question. Unhappily, enormous time and effort were devoted in the past to trying to "answer this question" before investigators realized the futility of the enterprise. Toward the end, above all mathematical arguments, one came to appreciate the quiet but rock-like strength of Einstein's equivalence principle. One can always find in any given locality a frame of reference in which all local "gravitational fields" (all Christoffel symbols; all ${\Gamma^\alpha}_{\mu\nu}$) disappear. No Γ's means no "gravitational field" and no local gravitational field means no "local gravitational energy-momentum".
>
> Nobody can deny or wants to deny that gravitational forces make a contribution to the mass-energy of a gravitationally interacting system. The mass-energy of the Earth-moon system is less than the mass-energy that the system would have if the two objects were at infinite separation. The mass-energy of a neutron star is less than the mass-energy of the same number of baryons at infinite separartion. Surrounding a region of empty space where there is a concentration of gravitational waves, there is a net attraction, betokening a positive net mass-energy in that region (see Chapter 35). At issue is not the existence of gravitational energy, but the localizability of gravitational energy. It is not localizable. The equivalence principle forbids.

Similarly, Straumann in Sec. 3.4 of Ref. [3] states:

> In SR the conservation laws for energy and momentum of a closed system are a consequence of the invariance with respect to translations in time and space. Except for special solutions, translations do not act as isometries on a Lorentz manifold and for this reason a general conservation law for energy and momentum does *not* exist in GR. This has been disturbing to many people, but one simply has to get used to this fact. There is no "energy-momentum tensor for the gravitational field". Independently of any formal arguments, Einstein's equivalence principle tells us directly

that there is no way to localize the energy of the gravitational field: The "gravitational field" (the connection $\Gamma^{\mu}_{\alpha\beta}$) can be locally transformed away. But if there is no field, there is locally no energy and no momentum. This is closely analogous to the situation with regard to charge conservation in non-Abelian gauge theories.

2.2 Counterarguments

In Newton's theory of gravity, the energy density of the gravitational field amounts to

$$\varepsilon_{\mathrm{gf}}(\mathbf{r}) = -\frac{[\boldsymbol{\nabla}\Phi(\mathbf{r})]^2}{8\pi G}\,, \tag{1}$$

see Eq. (17) in Ref. [9] and the Solution to Problem 1 in § 106 of Ref. [2]. However, in the Poisson equation of Newton's theory of gravity,

$$\Delta\Phi(\mathbf{r}) = 4\pi G\varrho(\mathbf{r})\,, \tag{2}$$

there does not appear the energy density of the gravitational field, but only the mass distribution on the right-hand side as a source of gravity. Eq. (2) is only an approximation of general relativity in the limit of weak gravitational fields. However, all kinds of energy have to be taken into account in order to satisfy the requirements of a precise theory of gravity. Not only for this reason the energy-momentum *tensor* of the gravitational field must appear in Einstein's field equations, but also to satisfy the correspondence principle because of the existence and the localizability of the energy density of the gravitational field in Newton's theory of gravity, see Eq. (1).

Regarding the statement of Straumann in Sec. 3.4 of Ref. [3] that "... a general conservation law for energy and momentum does *not* exist in GR" it is necessary to point out the following: In physics, there exist four interactions, namely gravity, electromagnetism, the weak interaction, and the strong interaction. Conservation of *total* energy, which is matter-energy plus energy of the gravitational field, does *not* occur in any of the last three interactions. Consequently, it must exist in the former and "highest-ranked" interaction, i.e. gravity, otherwise our universe would be a chaos if there were no conservation of total energy.

Regarding these shortcomings and contradictions, there consequently must have been made a mistake or a logical fallacy in general relativity.

2.3 Why pseudotensors do not solve the problem

Einstein claims in § 6. of Ref. [10] that

$$\sum_{\nu} \frac{\partial\,(\mathfrak{T}^{\nu}_{\sigma} + \mathfrak{t}^{\nu}_{\sigma})}{\partial x_{\nu}} = 0 \quad (\sigma = 1,\,2,\,3,\,4) \tag{35}$$

are conservation laws:

> Mr. Levi-Civita (and before him, with less emphasis, already H. A. Lorentz) has suggested a formulation of the conservation theorems that deviates from (35). He (and with him other colleagues) is opposed to an emphasis of equations (35), and are also opposed to the above interpretation because the $\mathfrak{t}^{\nu}_{\sigma}$ do not form a *tensor*. The latter is readily conceded; but I do not understand why only quantities with the transformation characteristics of tensor components should be granted physical meaning. Necessary is only that equation systems are valid for any choice of a system of reference which for the equation system (35) is true. Levi-Civita suggests the following formulation for the energy-momentum theorem. He writes the field equations of gravitation in the form
>
> $$T_{im} + A_{im} = 0\,, \tag{37}$$
>
> where T_{im} is the energy tensor of matter and A_{im} is a covariant tensor that depends only upon the $g_{\mu\nu}$ and their first two derivatives with respect to the coordinates. The A_{im} are called the energy components of the gravitational field.
>
> A *logical* objection can, of course, not be raised against such wording. But I find that (37) does not allow us to draw these conclusions which we are used to drawing from the conservation theorems. This is connected to the fact that in (37) the components of the *total energy* vanish everywhere. The equations (37), for example, do not exclude the possibility (and this in contrast to the equations [35]) that a material system dissolves into just nothing without leaving any trace. Because the total energy in (37)—but not in (35)—is zero from the beginning: the conservation of this energy value does not demand the continued existence of the system in any form.

In Eqs. (35) non-tensorial quantities, i.e. pseudotensors, are utilized in order to represent the energy density of the gravitational field. Lavenda rightly writes regarding Eqs. (35) in Sec. 3.2 in his book [7] that this "is not a covariant conservation equation, it would essentially allow energy to be created out of nothing!" The reason for this is that pseudotensors vanish in a local inertial frame while they are non-zero in other reference frames. Moreover in §20.4. of Ref. [4], Misner, Thorne, and Wheeler justifiably state regarding pseudotensors which are intended to represent the "local gravitational energy-momentum":

> There is no unique formula for it, but a multitude of quite distinct formulas. The two cited are only two among an infinity.

2.4 Completion of the field equations

Einstein's field equations

$$G_{im} = \kappa T_{im} \tag{3}$$

can be rearranged to get Levi-Civita's field equations (37), where A_{im} is proportional to the Einstein tensor G_{im}. Einstein rightly objects that "... in (37) the components of the *total energy* vanish everywhere." However, this physical shortcoming can be remedied by a simple modification: One just needs to introduce the *non-zero* total energy-momentum tensor L_{im} on the right-hand side of Eqs. (37), so that the completed Levi-Civita field equations read

$$T_{im} + A_{im} = L_{im}\,. \tag{4}$$

With this modification it is clear that Einstein's field equations (3) must be incomplete and moreover violate the conservation law of total energy.

According to Lovelock's theorem – see e.g. Sec. 3.2.2 in Ref. [3], in particular Theorem 3.1 and Eqs. (3.51) – Einstein's field equations in their maximum possible modified form read

$$G_{im} = \kappa T_{im} - \Lambda g_{im}\,, \tag{5}$$

which can be rearranged in order to obtain

$$T_{im} - \kappa^{-1} G_{im} = \kappa^{-1}\Lambda g_{im}\,, \tag{6}$$

so that on comparison with the completed Levi-Civita field equations (4) one can read off

$$A_{im} = -\kappa^{-1} G_{im}\,, \qquad L_{im} = \kappa^{-1}\Lambda g_{im}\,.$$

Eqs. (5) are Einstein's field equations with the cosmological constant. Einstein's gravitational constant $\kappa = 8\pi G/c^4$ is a universal constant which regulates the strength of the gravitational interaction, whereas the cosmological constant Λ is proven to be a constant of integration and thus a parameter and no universal constant [11], reflecting the fact, that different gravitational systems hold different total energy densities. Thus, there exists a different metric ds^2 with a different *non-zero* Λ for each gravitational system [12]. This finding is of course not in conflict with Lovelock's theorem. However, on the one hand it is questionable whether the term "cosmological constant" is still appropriate, but on the other hand one has got used to this designation.

Additionally, Einstein's condition for emptiness, $G_{im} = 0$, must be novated in order to satisfy the requirement of the conservation of total energy, $G_{im} = -\Lambda g_{im}$. In fact, "empty" space-time is not really empty because it consists of the energy of the gravitational field, wherefore it is more appropriate to designate it *matter-free* instead of "empty" space-time, where $T_{im} = 0$ and consequently the total energy density equals the energy density of the gravitational field.

3. Energy-momentum tensor of the gravitational field

3.1 Importance of the mixed-tensor representation

By considering the Schwarzschild metric, the metric tensor contains the Newtonian gravitational potential,

$$g_{00} = -\left(1 + \frac{2\Phi}{c^2}\right), \qquad \Phi(r) = -\frac{GM}{r}\,,$$

wherefore the metric tensor is a quantity which belongs to the gravitational field and hence to its energy density. It is of great importance to recognize that in the mixed-tensor representation of Einstein's field equations (6),

$$\begin{aligned} T_i^{\,k} + A_i^{\,k} &= L_i^{\,k}\,, \\ T_i^{\,k} - \kappa^{-1} G_i^{\,k} &= \kappa^{-1}\Lambda\,\delta_i^{\,k}\,, \end{aligned} \tag{7}$$

all metric tensors and their first two derivatives therein appear only in the Einstein tensor, whereas in $T_i^{\,k}$ and $L_i^{\,k}$ there are no quantities left that represent the energy density of the gravitational field, wherefore this suitable separation of the metric tensors from other quantities enables to assign the physical meaning to the respective tensors in Einstein's field equations (7), see Ref. [13].

Moreover, the conservation law of total energy, momentum, and stress can only be obtained in the mixed-tensor representation of Einstein's field equations (7) because of the special property of the Kronecker tensor, $\delta^{\,k}_{i;j} = \delta^{\,k}_{i,j} = 0$. The derivation of this fundamental conservation law is shown in detail in Sec. 4.1.

3.2 Properties

The tensor $A_i^{\,k} = -\kappa^{-1} G_i^{\,k}$ has the following properties, which are inherent to the energy-momentum tensor of the gravitational field, confirming that $A_i^{\,k}$ really represents it [13]:

- First of all, $A_i^{\,k}$ is a *tensor*. In Einstein's field equations (7), the matter-energy is represented by the matter tensor $T_i^{\,k}$. Consequently, also the energy of the gravitational field as well as the total energy must be represented by a *tensor*, otherwise one would toss apples and pears together.

- The metric tensor is a quantity that belongs to the gravitational field and hence to its energy density. *All* metric tensors and their first two derivatives in Einstein's field equations (7) appear only in the tensor $A_i^{\,k} = -\kappa^{-1} G_i^{\,k}$, whereas in $T_i^{\,k}$ and $L_i^{\,k}$ there are no quantities left that represent the energy density of the gravitational field.

- The tensor $A_i^{\,k}$ contains terms with Christoffel symbols squared. This is in conformance with Newton's theory of gravity, because there appears the analoguous expression $[\nabla\Phi(\mathbf{r})]^2$ in the energy density of the gravitational field (1).

- The tensor $A_i^{\,k}$ does not vanish in a local inertial frame because it contains non-vanishing terms with second derivatives of the metric tensor, otherwise $L_i^{\,k} = \kappa^{-1}\Lambda\,\delta_i^{\,k}$ would vanish. Regarding "the conservation of this energy value", Einstein rightly objects, that it "does not demand the continued existence of the system in any form". Consequently, there would be no free fall because then "a material system dissolves into just nothing without leaving any trace". In fact, this finding rules out the statement of Misner, Thorne, and Wheeler in §20.4. of Ref. [4]:

> ... no local gravitational field means no "local gravitational energy-momentum".

and that of Straumann in Sec. 3.4 of Ref. [3]:

> But if there is no field, there is locally no energy and no momentum.

- The tensor A_i^k has the unit of measurement of an energy density, which is required in order to represent a tensor of any kind of energy.
- The vanishing covariant divergence of the matter tensor $T^k_{i;k} = 0$ means, that the matter tensor is conserved. Because of the Bianchi identity, $A^k_{i;k} = -\kappa^{-1} G^k_{i;k} = 0$, which demonstrates, that also the energy-momentum tensor of the gravitational field A_i^k is conserved. Consequently, matter-energy is *not* converted into energy of the gravitational field and vice versa.
- Contrary to the statements of Misner, Thorne, and Wheeler in §20.4. of Ref. [4], the tensor $A_i^k = -\kappa^{-1} G_i^k$ demonstrates the necessarily existing unique formula for local gravitational energy-momentum. Thereby, it is localizable and neither in conflict with nor forbidden by the equivalence principle.

4. Total energy-momentum tensor

4.1 Conservation law of total energy

By definition, the energy of the gravitational field is *not* contained in the matter tensor T_i^k. Consequently, the matter tensor T_i^k *plus* the energy-momentum tensor of the gravitational field A_i^k must show the *total* energy-momentum tensor $L_i^k = \kappa^{-1} \Lambda \delta_i^k$. The latter one is *conserved*, wherefore its covariant divergence vanishes [13],

$$L^k_{i;k} = \nabla_k (T_i^k + A_i^k) = 0\,,$$

and which is nothing else than the covariant divergence of Einstein's field equations in the mixed-tensor representation (7). By using $\delta^k_{i;k} = \delta^k_{i,k} = 0$, this conservation law of total energy can be simplified,

$$L^k_{i;k} = L^k_{i,k} = \frac{\partial (T_i^k + A_i^k)}{\partial x^k} = 0\,.$$

One can even go further and take the derivative instead of the divergence because $\delta^k_{i;j} = \delta^k_{i,j} = 0$. Thus, not only the divergences but also the derivatives vanish, so that

$$L^k_{i;j} = L^k_{i,j} = \frac{\partial (T_i^k + A_i^k)}{\partial x^j} = 0\,, \tag{8}$$

which shows the conservation law of total energy, momentum, and stress in general relativity in its differential form [13].

One can consider a closed region with volume V. The volume integration over $L^k_{i,0}$ in Eqs. (8) shows

$$\frac{\partial}{\partial t} \int_V dV\, L_i^k = \frac{\partial}{\partial t} \int_V dV\, (T_i^k + A_i^k) = 0\,,$$

whereby the conserved total energy, momentum, and stress within the closed region are obtained,

$$E_i^k = \kappa^{-1} \Lambda V \delta_i^k = \int_V dV\, L_i^k = \int_V dV\, (T_i^k + A_i^k) = \text{const.}$$

4.1.1 Example: Non-rotating star

As a simple example, a non-rotating star with mass M and radius R featuring no electromagnetic fields is considered which occupies a closed region with volume $V = \frac{4}{3}\pi R^3$. The total energy contained in V amounts to

$$E_{\text{tot}} = E_0^0 = E_{\text{M}} + E_{\text{gf}} = \frac{4\pi \Lambda R^3}{3\kappa} = \text{const.}\,,$$

where

$$E_{\text{M}} = \int_V T_0^0\, dV = -4\pi c^2 \int_0^R dr\, r^2 \varrho(r) = -Mc^2$$

is the mass-energy of the star and

$$E_{\text{gf}} = \int_V A_0^0\, dV = -4\pi\kappa^{-1} \int_0^R dr\, r^2 G_0^0(r)$$

is the energy of the gravitational field [9]. The mass

$$M = M_{\text{c}} + E_{\text{pot}}/c^2$$

is the gravitational and hence the physical mass of the star, while in the metric of the star

$$M_{\text{c}} = 4\pi \int_0^R dr\, r^2 \varrho(r) \sqrt{g_{11}(r)}$$

is its constituent mass which is the unbound mass of the star. In contrast to Newton's theory of gravity, in general relativity, the energy of the gravitational field is stored as curvature of space-time, wherefore it is *not* equivalent to the gravitational potential energy E_{pot} appearing as gravitational binding energy.

4.2 Cosmological constant problem and dark energy

By relating the cosmological constant to the vacuum energy density a huge mismatch between its theoretical and observed value is obtained, whereby the cosmological constant problem arises [1]. It is important to acknowledge general relativity as a classical and not as a quantum theory. By doing so, the cosmological constant cannot be related to the energy density of the vacuum because the latter one can only occur in a quantum theory.

As is already mentioned in Sec. 2.4, the cosmological constant Λ is no universal constant, but a constant of integration and therefore a parameter, representing the respective different total energy densities $L_0^0 = \kappa^{-1}\Lambda$ regarding the different metrics ds^2 of the respective gravitational systems.

The cosmological constant Λ regarding the Friedmann-Lemaître-Robertson-Walker (FLRW) metric of the universe is proportional to its total energy density and thereby solves the cosmological constant problem as well as that of dark energy [13]. The cosmological constant regarding the FLRW metric of the cosmos is positive, $\Lambda > 0$, because of its accelerated expansion, wherefore also $L^0_0 > 0$. By using Einstein's field equations in mixed-tensor representation (7) one simply concludes that also the energy density of the gravitational field regarding the FLRW metric of the universe is positive, $A^0_0 > 0$, because $T^0_0 = -\varrho c^2 < 0$.

4.3 Dark matter

In Newton's theory of gravity, the energy density of the gravitational field in matter-free space outside of a celestial object is always negative, $\varepsilon_{\mathrm{gf}} < 0$, see Eq. (1). Consequently in general relativity, the energy density of the gravitational field in matter-free space-time must be negative as well regarding the metric of a celestial object[1], $A^0_0 < 0$, so that by using Eqs. (7) together with the matter-free space-time condition, $T^\mu_\nu = 0$, one simply recognizes that $\Lambda < 0$, whereby the dark matter phenomenon can be explained [13]. With this finding, flat rotation curves of spiral galaxies are obtained in Ref. [14] which are thematized in Sec. 5 and also shown in Fig. 1.

This finding demonstrates that dark matter in fact is nothing else than a negative scalar curvature of space-time, while dark energy is nothing else than a positive scalar curvature of space-time, which both arise naturally due to the conservation of total energy regarding the respective metrics. But also very faint ordinary matter and, if present, exotic particles contribute to dark matter, however are taken into account in the matter tensor.

The cosmological constant regarding our solar system must be tiny, otherwise the computed angle of the perihelion shift of Mercury would not match the observed one. Moreover, the cosmological constant has no effect on the angle of light deflection at all [14].

5. Flat rotation curves

The *modified* Poisson equation,

$$\Delta\Phi(\mathbf{r}) = 4\pi G\varrho(\mathbf{r}) - \Lambda c^2\,, \qquad (9)$$

that is obtained from Einstein's field equations *with* the cosmological constant in the limit of weak gravitational fields, approximately meets the requirement of conservation of total energy in Newton's theory of gravity in contrast to Eq. (2).

The rotation curves of spiral galaxies are calculated by using Eq. (9) and Freeman's method [15, 16] considering spiral galaxies to be exponential discs with weak gravitational fields and the surface mass distribution of ordinary matter

$$\Sigma(r) = \Sigma_0 \exp\left(-\frac{r}{r_c}\right),$$

which is in good agreement with observations. The author denotes the quantity r_c as "characteristic radius" of the spiral galaxy, while in Ref. [15] it is termed "disc length scale", and in Ref. [16] "disc scale length". More details regarding the computations are given in Ref. [14].

The flat rotation curves of spiral galaxies are shown in Fig. 1, where different central surface mass densities Σ_0, characteristic radii r_c, and boosting factors λ regarding the negative value of the cosmological constant of the FLRW metric as a benchmark are utilized [14],

$$\Lambda = -\lambda\Lambda_{\mathrm{FLRW}}\,, \qquad \Lambda_{\mathrm{FLRW}} = 1.1056\cdot 10^{-52}\ \mathrm{m}^{-2}\,.$$

The effect of the cosmological constant on the rotation curves is negligible on "short" distances but becomes dominant on larger ones. The respective values of the cosmological constant regarding any celestial object have to be fitted to observations because they are initially unknown. This is also the reason why a boosting factor λ is used in order to demonstrate the effect of the cosmological constant on the rotation curves of spiral galaxies.

6. Conclusions and outlook

The vanishing covariant divergence of the matter tensor demonstrates that the matter tensor is conserved. Einstein's field equations with the cosmological constant Λ as a parameter satisfy the requirement of conservation of total energy, momentum, and stress, where the tensor

- T^k_i is the matter tensor,
- $A^k_i = -\kappa^{-1} G^k_i$ is the energy-momentum tensor of the gravitational field, and
- $L^k_i = \kappa^{-1}\Lambda\delta^k_i$ is the total energy-momentum tensor.

The *modified* Poisson equation (9) that is obtained from Einstein's field equations *with* the cosmological constant in the limit of weak gravitational fields, approximately meets the requirement of conservation of total energy in Newton's theory of gravity in contrast to Eq. (2), wherefore the former can be utilized for computing the flat rotation curves of spiral galaxies.

An extension of general relativity is the Einstein-Cartan theory which takes torsion into consideration that causes a *repulsive* gravitational interaction within matter and thereby prevents the formation of singularities and explains the inflation of the early universe [17]. However, torsion only plays a significant role at huge mass densities. In matter-free space-time, there is no difference between Einstein-Cartan theory and general relativity. In order to satisfy the requirement of the conservation of total energy, momentum, and stress, the cosmological term has to be taken into account in the Einstein-Cartan theory.

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[1] An exceptional case are the metrics of large scale structures (superclusters), where $\Lambda > 0$ because of the accelerated cosmological expansion.

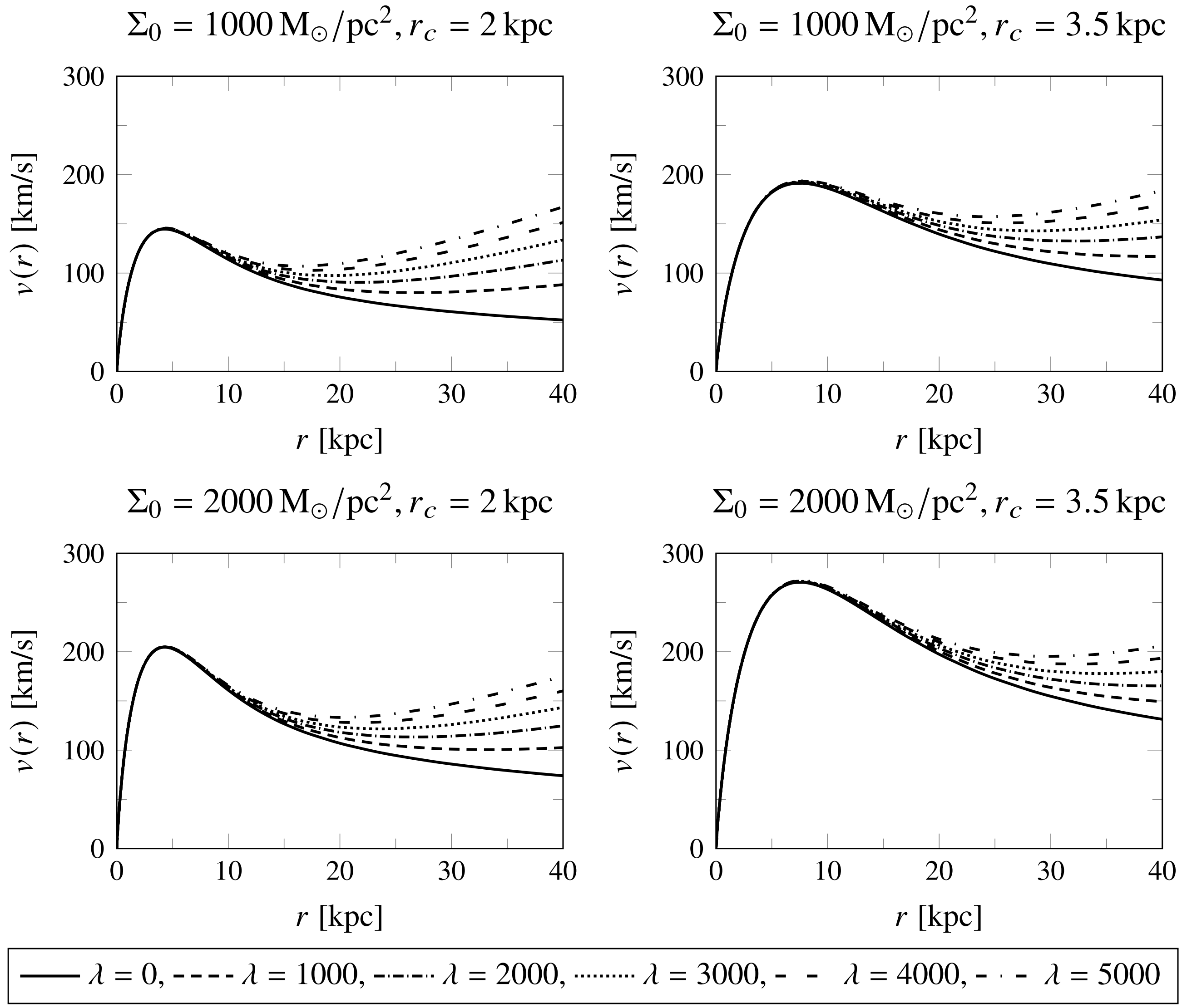

Figure 1. Rotation curves of spiral galaxies with different central surface mass densities Σ_0, characteristic radii r_c, and boosting factors λ regarding the negative value of the cosmological constant of the FLRW metric as a benchmark, $\Lambda = -\lambda\Lambda_{\mathrm{FLRW}}$, where $\Lambda_{\mathrm{FLRW}} = 1.1056 \cdot 10^{-52}\ \mathrm{m}^{-2}$, see Ref. [14].

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