A dynamic metric with variable flow of time (VFT): Proposal for a revised cosmological model and a solution to the dark energy enigma

A Preprint

Ralph Gramigna
Zurich
Switzerland
ralph.gramigna@kellerhals-carrard.ch

July 27, 2022

Abstract

A dynamic metric for a cosmological model is proposed, which includes a newly introduced time scale factor $T(t)$ and the space scale factor $S(t)$, the latter known as the cosmological scale factor. The assumption for this proposal is that the expansion of the time dimension in a homogeneous and isotropic universe depends on the energy density and pressure, as it is the case for the expansion of space. The Christoffel symbols, the Ricci tensor and the Ricci scalar are derived. Evaluating the metric using Einstein’s field equations and the energy momentum tensor, we obtain a set of equations that describe the expansion and contraction of space and time in a homogeneous and isotropic model of the universe. It is shown that the luminosity distance in the light of the proposed metric and cosmological model is calculated considering additionally the expansion of the time dimension, in contrast to the calculation based on the Friedmann–Lemaître–Robertson–Walker (FLRW) metric and the Friedmann equations. The reason that high-redshifted supernovae should be brighter than what is observed can be explained by this amended calculation, which provides for a redshift of e.g. $z = 0.5$ a deviation of the luminosity distance amounting to $\sim 22.5\%$. Of course, this conclusion is subject to further review.

Keywords  Cosmology · Dark energy · Spacetime

1 Introduction and assumptions

The Friedmann–Lemaître–Robertson–Walker (FLRW) metric and the Friedmann equations form the basis of modern cosmology and the $\Lambda$CDM model. The FLRW metric ensures in relation to its time-axis $t > 0$ a development of spacetime, whereas the (mathematical) singularity is at $t = 0$. The development of the spacetime is represented by the cosmological scale factor, usually denoted as $a(t)$ or $R(t)$. The FLRW-universe is spatially homogeneous and isotropic but evolving in time. Hence, the geometry of the FLRW space is 'stretched' in the time direction. However, time in the FLRW metric is of Newtonian nature as it flows equably without regard to anything external.

The metrics describing a spacetime geometry in the region surrounding a mass, such as the Schwarschild or Kerr spacetime, reveal a curvature of the dimensions of space and time. The Schwarschild metric describes the gravitational field, i.e. the curved spacetime, outside a spherical, non-rotating and non-charged mass. In Schwarschild coordinates, with the center of the mass located at $r = 0$, and with $r_s = 2GMc^{-2}$, the metric shows a spherical symmetry, the center of which is the space-like singularity at $r = 0$. The proper radial distance, the distance measured by an observer at rest at radius $r$ (with $r > r_s$), between two points in space separated by an interval $dr$ in the radial direction is $dr / \sqrt{1 - r_s/r}$ (with $d\theta = d\phi = dt = 0$). Accordingly, the proper radial time, the time measured by an observer at rest at radius $r$, during an interval $dt$ of universal time is $\sqrt{1 - r_s/r}dt$. Thus, the geometry of space and time is 'stretched' in the radial direction.
This raises the question whether the dimension of time in a metric describing an isotropic and homogenous universe is also affected by the gravitational effect of energy and mass. This question has apparently not yet been the subject of consideration. Let us therefore make the following assumptions and accept the listed principles and postulates:

1. The cosmological principle: The universe is homogeneous and isotropic when viewed on a large enough scale (hereinafter referred to as the "Cosmological Principle").

2. The constant light speed postulate: The light speed \( c \) is constant when measured in any inertial frame.

3. The assumption of the variability of time flow: The time flow in a homogeneous and isotropic universe depends on the energy density and pressure. Further considerations are set out in section [2.1].

4. The assumption of the numerical identity of the space scale factor and the time scale factor: The space scale factor measured at a point in time is numerically equal to the time scale factor at that time (hereinafter referred to as the "Identity Assumption"). This assumption is used in sections [8.1 - 11] and we will elaborate on this assumption in section 8.1.

Please consider that, by adopting the constant light speed postulate, the proposal contemplated herein is not based on variable speed of light (VSL) theories (see for VSL theories e.g. [Albrecht & Magueijo 1999, Barrow 1999, Magueijo 2003]). The assumption of a variable flow of time is a more fundamental ansatz.

2 Spacetime scale factors and spacetime parameters

Let us consider a variable time flow between two events at time \( t(e) \) and \( t(a) \) and a sufficiently short time interval, which amounts to \( \Delta t_e = t(e_2) - t(e_1) \) at \( t(e_2) \) and to \( \Delta t_a = t(a_2) - t(a_1) \) at \( t(a_2) \). The subscripts \( a \) and \( e \) refer to absorbed and emitted, respectively to the reference frames \( a \) and \( e \). Let us define a dimensionless time scale factor \( \mathcal{T} \) as relation between the proper time interval and the comoving time interval. Let us assume a universe with a variable time flow, but which is not subject to an expansion or contraction of space. In such case, the wavelength of the variable speed of light (VSL) theories (see for VSL theories e.g. [Albrecht & Magueijo 1999, Barrow 1999, Magueijo 2003]). The assumption of a variable flow of time is a more fundamental ansatz.

\[
\frac{T_e}{T_a} := \frac{\Delta t_a}{\Delta t_e}, \text{ with } T(t) \neq 0.
\]

(2.1)

The space scale factor is introduced as relation of proper distance and comoving distance, e.g. the wavelengths \( \lambda_e \) and \( \lambda_a \) as \( \Delta x_e = x_{e_2} - x_{e_1} \) and \( \Delta x_a = x_{a_2} - x_{a_1} \) of a photon at time \( t(e) \) and \( t(a) \), in an expanding or contracting space. It is assumed that the expansion or contraction of space causes the wavelength of a photon to expand or contract accordingly. If it shall be required that \( S_e < S_a \) leads to a redshift, it is defined as follows:

\[
\frac{S_e}{S_a} := \frac{\Delta x_e}{\Delta x_a}, \text{ with } S(t) \neq 0.
\]

(2.2)

\( T(t) \) or \( T \) and \( S(t) \) or \( S \) will be used without indices, if the scale factors \( T_a \) and \( S_a \) refer to today's scale factors \( (T(t_0) := 1 \) and \( S(t_0) := 1) \). \( T \) and \( S \) are time-dependent functions.

In a last step, the spacetime parameters represented by the time parameter \( \mathcal{G}(t) \) and the space parameter \( \mathcal{J}(t) \) are defined:

\[
\mathcal{G}(t) := \frac{\dot{T}}{T},
\]

(2.3)

\[
\mathcal{J}(t) := \frac{\dot{S}}{S}.
\]

(2.4)

The 'over-dot' is used as a notation for the first time derivative (e.g. \( \dot{T} \)), respectively for the second time derivative (e.g. \( \ddot{T} \)). The space scale factor \( S \) and space parameter \( \mathcal{G} \) will be used in the context of the herewith proposed metric and
cosmological model with variable time flow, whereas, for purposes of clarity, the cosmological scale factor denoted as $R$ or $a$ and the Hubble parameter $H$ will be used in the context of the FLRW metric and the Friedmann equations.

3 The dynamic metric

The FLRW metric $ds^2 = -c^2dt^2 + R^2(\frac{1}{1-k\rho}dr^2 + r^2d\theta^2 + r^2 \sin^2 \theta d\phi^2)$ (see e.g. Mueller & Grave 2009 p. 38) shall be extended by the time scale factor $T(t)$ and, for the purposes of clarity, the cosmological scale factor $a$ shall be replaced by the space scale factor $S(t)$. It is expected (see for the FLRW metric e.g. Carroll 2019 p. 382) that the spacetime is of the form $T \times \Sigma$, where $T$ represents the time direction and $\Sigma$ is a maximally symmetric three-manifold. It seems to be necessary that the partial derivatives of a position vector have to be determined. Therefrom, the covariant basis vectors and the co- and contravariant metric tensor can be derived. We will elaborate on the basis of pseudo-cartesian coordinates. If the expectation is true and the spacetime is of the form $T \times \Sigma$, the transformation of the metric to pseudo-spherical coordinates is straightforward.

Let us consider a four-dimensional manifold with the signature of the form $(-,+,+,+)$ and an event in spacetime $E(ct, x, y, z)$. Let us further consider the scale factors $T_e$ and $S_e$ in the reference frame $e$, $T_a$ and $S_a$ in the reference frame $a$ as well as a sufficiently short time interval $dt_e$ and a distance $dx_e$. With equations 2.1 and 2.2 the following transformation rules are obtained:

$$
\frac{dt_a}{T_a} dt_e, \quad dx_a = \frac{S_a}{S_e} dx_e, \quad dy_a = \frac{S_a}{S_e} dy_e, \quad dz_a = \frac{S_a}{S_e} dz_e,
$$

(3.1)

and vice versa

$$
\frac{dt_e}{T_e} dt_a, \quad dx_e = \frac{S_e}{S_a} dx_a, \quad dy_e = \frac{S_e}{S_a} dy_a, \quad dz_e = \frac{S_e}{S_a} dz_a.
$$

(3.2)

In order to obtain the basis vectors, the partial derivatives of a position vector $\vec{E}$ (the notation of which will be omitted), considering equations 3.1 and 3.2 will have to be calculated. In the following equations the indices $a$ and $e$ are omitted; for the partial time-derivative in direction of $e$ the notation $\partial_t$ , and in direction of $a$ the notation $\partial_t$ will be used instead. Further, we apply the chain rule and consider the Cosmological Principle and obtain:

$$
\frac{\partial}{\partial t} = \frac{T_a}{T_e} \frac{\partial}{\partial t_e}, \quad \frac{\partial}{\partial \vec{x}} = \frac{S_e}{S_a} \frac{\partial}{\partial \vec{x}_a}, \quad \frac{\partial}{\partial \vec{y}} = \frac{S_e}{S_a} \frac{\partial}{\partial \vec{y}_a}, \quad \frac{\partial}{\partial \vec{z}} = \frac{S_e}{S_a} \frac{\partial}{\partial \vec{z}_a}.
$$

(3.3)

With these results, the covariant basis vectors for the time and the time direction result in:

$$
\vec{e}_t = \frac{T_a}{T_e} \vec{e}_t, \quad \vec{e}_x = \frac{S_e}{S_a} \vec{e}_x, \quad \vec{e}_y = \frac{S_e}{S_a} \vec{e}_y, \quad \vec{e}_z = \frac{S_e}{S_a} \vec{e}_z.
$$

(3.4)

With the covariant basis vectors, the covariant metric tensor of signature $(-,+,+,+)$ calculated with the dot product $g_{\mu \nu} = \vec{e}_\mu \cdot \vec{e}_\nu$. The indices in $S_e$ and $T_e$ will be omitted; $S_a$ and $T_a$ are set to 1.

$$
g_{\mu \nu} = \begin{bmatrix}
-\frac{T^2}{S^2} & 0 & 0 & 0 \\
0 & S^2 & 0 & 0 \\
0 & 0 & S^2 & 0 \\
0 & 0 & 0 & S^2
\end{bmatrix}.
$$

(3.5)

This leads to the contravariant metric tensor as the inverse of the covariant form $g_{\mu \nu} g^{\mu \sigma} = \delta^\sigma_\nu$:

$$
g^{\mu \nu} = \begin{bmatrix}
-\frac{1}{T^2} & 0 & 0 & 0 \\
0 & \frac{1}{S^2} & 0 & 0 \\
0 & 0 & \frac{1}{S^2} & 0 \\
0 & 0 & 0 & \frac{1}{S^2}
\end{bmatrix}.
$$

(3.6)
Indeed, the expectation according to which the spacetime is of the form $T \times \Sigma$ turns out to be valid, so the metric can be transformed using the usual transformation rules (see e.g. [Mueller & Grave 2009] p. 7), which then reads:

$$ds^2 = -\frac{c^2}{T^2} dt^2 + S^2 \left( dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right),$$  \hspace{1cm} (3.7)

respectively, if a parameter for a constant curvature $k$ is introduced (see for details [Carroll 2019] p. 329-332), the following dynamic metric yields:

$$ds^2 = -\frac{c^2}{T^2} dt^2 + S^2 \left( \frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right).$$  \hspace{1cm} (3.8)

As a result of the calculations, an extended FLRW metric is obtained. The only difference in relation to the FLRW metric is that the time direction is scaled by the time scale factor. The metric describes a homogeneous, isotropic, expanding (or otherwise, contracting) universe. I am of the opinion that the variable time flow reflected by this metric is not in contradiction to the Cosmological Principle. In the FLRW metric, the spatial dimensions of the universe’s spacetime may be contracted or expanded in scale, in the metric derived above, the dimensions of space and/or time may be contracted or expanded.

4 The Christoffel symbols (of the second kind)

The relation between the metric tensor and the Christoffel symbols (of the second kind) is given (see [Mueller & Grave 2009] section 1.3 on p. 2) by:

$$\Gamma^\nu_{\mu\lambda} = \frac{1}{2} g^{\mu\rho} \left[ \partial_\lambda g_{\nu\rho} + \partial_\nu g_{\rho\lambda} - \partial_\rho g_{\nu\lambda} \right], \text{ whereas } \Gamma^\mu_{\nu\lambda} = \Gamma^\mu_{\lambda\nu}$$  \hspace{1cm} (4.1)

With the metric tensors (equations (3.5) and (3.6)), the non-zero components of the contra- and covariant metric tensor are:

$$g_{00} = -\frac{1}{T^2}, \hspace{0.5cm} g_{11} = \frac{S^2}{1-kr^2}, \hspace{0.5cm} g_{22} = S^2 r^2, \hspace{0.5cm} g_{33} = S^2 r^2 \sin^2 \theta,$$  \hspace{1cm} (4.2)

and

$$g^{00} = -T^2, \hspace{0.5cm} g^{11} = \frac{1-kr^2}{S^2}, \hspace{0.5cm} g^{22} = \frac{1}{S^2 r^2}, \hspace{0.5cm} g^{33} = \frac{1}{S^2 r^2 \sin^2 \theta}.$$  \hspace{1cm} (4.3)

The detailed step-by-step calculation of all Christoffel symbols is set out in Appendix A. Please note that the index 0 is used equivalently to the index $t$, and consequently 1 for $r$, 2 for $\theta$ and 3 for $\varphi$. When calculating the partial time derivatives $\partial_0$, the results contain additionally a factor $c^{-1}$. This ensures consistency of the units of the Christoffel symbols.

$$\Gamma^t_{tt} = \frac{\dot{T}}{c T},$$  \hspace{1cm} (4.4)

$$\Gamma^0_{11} = \Gamma^r_{rr} = \frac{T^2 \dot{S} S}{c (1-kr^2)},$$  \hspace{1cm} (4.5)

$$\Gamma^0_{22} = \Gamma^\theta_{\theta\theta} = \frac{T^2 \dot{S} S r^2}{c},$$  \hspace{1cm} (4.6)

$$\Gamma^0_{33} = \Gamma^\varphi_{\varphi\varphi} = \frac{T^2 \dot{S} S r^2 \sin^2 \theta}{c}.$$  \hspace{1cm} (4.7)

\footnote{Obviously, there are other methods - one can enclose the factor $c^{-2}$ when multiplying $g^{00}$ with the respective partial derivative $\partial_0 g_{aa}$, see e.g. results obtained for $\Gamma_0^a$ and $\Gamma^0_a$ in [Mueller & Grave 2009] equations 2.9.3 on p. 38). However, this leads to the situation that the units of a Christoffel symbol may differ from another.}
\[\Gamma^i_{01} = \Gamma^i_{10} = \Gamma^i_{rr} = \Gamma^r_{rt} = \frac{\dot{S}}{c S},\] (4.8)

\[\Gamma^1_{11} = \Gamma^r_{rr} = \frac{kr}{1 - kr^2},\] (4.9)

\[\Gamma^1_{22} = \Gamma^r_{\theta\theta} = r(kr^2 - 1),\] (4.10)

\[\Gamma^1_{33} = \Gamma^r_{\varphi\varphi} = r(kr^2 - 1)\sin^2 \theta,\] (4.11)

\[\Gamma^2_{02} = \Gamma^2_{20} = \Gamma^\theta_{t\theta} = \Gamma^\theta_{\theta t} = \frac{\dot{S}}{c S},\] (4.12)

\[\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r},\] (4.13)

\[\Gamma^3_{33} = \Gamma^\varphi_{\varphi\varphi} = - \sin \theta \cos \theta,\] (4.14)

\[\Gamma^3_{03} = \Gamma^3_{30} = \Gamma^\varphi_{t\varphi} = \Gamma^\varphi_{\varphi t} = \frac{\dot{S}}{c S},\] (4.15)

\[\Gamma^3_{13} = \Gamma^3_{31} = \Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r} = \frac{1}{r},\] (4.16)

\[\Gamma^3_{23} = \Gamma^3_{32} = \Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi \theta} = \cot \theta.\] (4.17)

The Christoffel symbols of the dynamic metric proposed herein have - in comparison to the FLRW metric - an additional term \(\Gamma^r_{tt}, \Gamma^r_{rr}, \Gamma^\theta_{\theta t}\) as well \(\Gamma^\varphi_{\varphi\varphi}\) contain additionally a factor of \(T^2\). All of the other Christoffel symbols remain unchanged in comparison to the FLRW metric.

If it is assumed that time scale factor is constant \((T = 1)\), and consequently the derivative of the time scale factor becomes zero \((\dot{T} = 0)\), all Christoffel symbols listed above will become the Christoffel symbols of the FLRW metric. Note that this is true for most part of the equations in this document.

### 5 Ricci tensor and Ricci scalar

The detailed step-by-step calculations of the Ricci tensor and the Ricci scalar are set out in Appendix [B]. The relation between the Riemann and the Ricci tensor is in accordance with [Mueller & Grave 2009] equations 1.2.2, 1.3.5 and 1.3.9 on p. 2:

\[R_{\mu\nu} = R^\rho_{\mu\rho\nu} = \partial_\rho \Gamma^\rho_{\mu\nu} - \partial_\nu \Gamma^\rho_{\mu\rho} + \Gamma^\rho_{\lambda\mu} \Gamma^\lambda_{\rho\nu} - \Gamma^\rho_{\lambda\nu} \Gamma^\lambda_{\rho\mu}.\] (5.1)

The non-zero components of the Ricci tensor are as follows:

\[R_{tt} = \frac{3}{c^2} \left( - \frac{\dot{S}}{S} + \frac{\ddot{T} S}{T S} \right),\] (5.2)

\[R_{rr} = \frac{T^2 S^2}{c^2} \left( \frac{\dot{S}}{S} + 3 \frac{\ddot{T} S}{T S} + 2 \frac{\dot{T} S^2}{S^2} + \frac{2c^2 k}{T^2 S^2} \right) \frac{1}{1 - kr^2},\] (5.3)

\[R_{\theta\theta} = \frac{T^2 S^2}{c^2} \left( \frac{\dot{S}}{S} + 3 \frac{\ddot{T} S}{T S} + 2 \frac{\dot{T} S^2}{S^2} + \frac{2c^2 k}{T^2 S^2} \right) r^2.\] (5.4)
\[ R_{\phi\phi} = \frac{\mathcal{T}^2 S^2}{c^2} \left( \frac{\dot{S}}{S} + \frac{\mathcal{T} \dot{S}}{T S} + \frac{2 \dot{S}^2}{T^2 S^2} + \frac{2 c^2 k}{T^2 S^2} \right) r^2 \sin^2 \theta. \]  

(5.5)

The Ricci scalar (or scalar curvature) is the contraction of the Ricci tensor (see Mueller & Grave 2009, equations 1.3.10 on p. 2):

\[ R = g^{\mu\nu} R_{\mu\nu}, \]  

(5.6)

which results in:

\[ R = \frac{6 \mathcal{T}^2}{c^2} \left( \frac{\dot{S}}{S} + \frac{\mathcal{T} \dot{S}}{T S} + \frac{\dot{S}^2}{T^2 S^2} \right) + \frac{6 k}{S^2}. \]  

(5.7)

### 6 Energy momentum tensor

In order to be able to solve the field equations, we will have to take a closer look at the energy-momentum tensor. The approach is the same as for the FLRW metric where a perfect and continuous fluid is contemplated. We follow step-by-step Carroll (2019, p. 35, 117 - 119).

\[ T^{\mu\nu} = (\rho + p) U^{\mu} U^{\nu} + p g^{\mu\nu}, \]  

(6.1)

where \( \rho(t) \) is the energy density, \( p(t) \) is the pressure and \( U^\mu \) the four-velocity of the perfect fluid. With the inverse of the Minkowski metric, the energy-momentum tensor becomes

\[ T^{\mu\nu} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix}. \]  

(6.2)

By applying the contravariant (equation 4.1) and the covariant (equation 4.2) metric tensors, the components of the tensor yield

\[
T^{\mu\nu} = \begin{bmatrix}
\mathcal{T}^2 \dot{\rho} & 0 & 0 & 0 \\
0 & p S^{-2} (1 - kr^2) & 0 & 0 \\
0 & 0 & p (Sr)^{-2} & 0 \\
0 & 0 & 0 & p (Sr \sin^2 \theta)^{-2}
\end{bmatrix}
\]  

(6.3)

and

\[
T_{\mu\nu} = \begin{bmatrix}
\rho \mathcal{T}^{-2} & 0 & 0 & 0 \\
0 & p S^2 (1 - kr^2)^{-1} & 0 & 0 \\
0 & 0 & p S^2 r^2 & 0 \\
0 & 0 & 0 & p S^2 r^2 \sin^2 \theta
\end{bmatrix}.
\]  

(6.4)

The trace is accordingly \( g_{\mu\nu} T^{\mu\nu} = -\rho + 3p \). Now, the divergence of the energy-momentum tensor can be examined. The starting point is the conservation equation (Carroll 2019, p.118):

\[ \nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^{\mu}_{\mu\lambda} T^{\lambda\nu} + \Gamma^{\nu}_{\mu\lambda} T^{\mu\lambda} \neq 0. \]  

(6.5)

In a first step, the components of the terms \( \nu = 0 \) are contemplated:

\[
\partial_\mu T^{\mu0} = \partial_\mu T^{00} = \mathcal{T}^2 \frac{\dot{\rho}}{c} + \frac{2 \mathcal{T} \dot{\mathcal{T}}}{c} \rho.
\]  

(6.6)
For the second term, the Christoffel symbols, which were obtained in section 4, can be applied:

\[
\Gamma^\mu_{\mu\lambda} T^{\lambda 0} = \Gamma^\mu_{\mu 0} T^{0 0} \\
= \left( \Gamma^0_{0 0} + \Gamma^1_{0 1} + \Gamma^2_{0 2} + \Gamma^3_{0 3} \right) T^{0 0} \\
= \frac{T^2}{c} \left( \frac{\dot{T}}{T} + 3 \frac{\dot{S}}{S} \right) \rho,
\]

and the third term results in

\[
\Gamma^0_{\mu\lambda} T^{\mu 0} = \Gamma^0_{0 0} T^{0 0} + \Gamma^0_{1 1} T^{1 1} + \Gamma^0_{2 2} T^{2 2} + \Gamma^0_{3 3} T^{3 3} \\
= \frac{1}{c} \frac{\dot{T}}{T} T^2 \rho + \frac{T^2 \ddot{S} S}{c (1 - kr^2)} \frac{1 - kr^2}{S^2} \\
+ \frac{1}{c} \ddot{T} \dddot{S} r^2 \frac{1}{S^2 r^2} + \frac{1}{c} \dot{T} \dddot{S} r^2 \sin^2 \theta \frac{1}{S^2 r^2 \sin^2 \theta} \\
= \frac{T^2}{c} \left( \frac{\dot{T}}{T} \rho + 3 \frac{\dot{S}}{S} \rho \right).
\]

The calculation of the spatial components of the energy-momentum tensor is provided in Appendix C. All of the partial derivatives amount to:

\[
\dot{\rho} = 0.
\]

We can now derive the overall result:

\[
\nabla_\mu T^{\mu\nu} = \frac{T^2}{c} \frac{\dot{T}}{T} \rho + \frac{2}{c} \frac{\dot{T}}{T} \dot{\rho} + \frac{T^2}{c} \left( \frac{\dot{T}}{T} + 3 \frac{\dot{S}}{S} \right) \rho + \frac{T^2}{c} \left( \frac{\dot{T}}{T} \rho + 3 \frac{\dot{S}}{S} \rho \right) = 0,
\]

which amounts to

\[
\dot{\rho} = -4 \frac{\dot{T}}{T} \rho - 3 \frac{\dot{S}}{S} (\rho + p).
\]

Next step is the introduction of the equation of state in order to express the relationship between the energy density and the pressure for radiation, mass and vacuum energy (Carroll 2019, p. 334):

\[
p = \omega \rho,
\]

with

Table 1: Energy density - pressure relationship

<table>
<thead>
<tr>
<th>energy density</th>
<th>( \omega )</th>
<th>( \rho_{\omega} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>radiation</td>
<td>( \frac{1}{3} )</td>
<td>( \rho_{rad} )</td>
</tr>
<tr>
<td>mass</td>
<td>0</td>
<td>( \rho_m )</td>
</tr>
<tr>
<td>vacuum</td>
<td>-1</td>
<td>( \rho_{vac} )</td>
</tr>
</tbody>
</table>

Finally, we obtain from equation 6.11 considering equation 6.12 and the definitions 2.3 and 2.4

\[
\frac{\dot{\rho}}{\rho} = -4 \dot{\frac{\rho}{\omega}} - 3 \frac{\ddot{\rho}}{\rho} (\omega + 1).
\]

This result will be discussed in section 8.3. Please note (again), that if we set \( \dot{T} \) constant (e.g. \( = 1 \)), the time-derivative would become \( \dot{T} = 0 \) and we would find exactly the solution for the FLRW metric as the basis for the Friedmann equations.
7 The dynamic metric and Einstein’s field equations

Now, we can solve Einstein’s field equations in their general forms:

\[ R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} = 8\pi G \frac{c^4}{c^4} T_{\mu \nu}, \] (7.1)

\[ \Leftrightarrow R_{\mu \nu} = 8\pi G \frac{c^4}{c^4} \left( T_{\mu \nu} - \frac{1}{2} g_{\mu \nu} T \right). \] (7.2)

Please note that the two forms can be used equally (see Carroll [2019], p. 158 and 335). In a first step, the equation of the form given in equation 7.2 will be solved for \( T_{00} = \frac{\rho}{T^2} \) and \( T = -\rho + 3p \):

\[ R_{00} = 8\pi G \frac{c^4}{c^4} \left( T_{00} - \frac{1}{2} g_{00} T \right), \] (7.3)

\[ \Rightarrow \frac{1}{c^2} \left( -3 \frac{\dot{S}}{S} + 3 \frac{T \dot{S}}{T S} \right) = 8\pi G \frac{c^4}{c^4} \left( \frac{\rho}{T^2} + \frac{-\rho + 3p}{2 T^2} \right), \] (7.4)

\[ \Rightarrow -\frac{\dot{S}}{S} + \frac{T \dot{S}}{T S} = 4\pi G \frac{(\rho + 3p)}{3c^2} \frac{1}{T^2}. \] (7.5)

Let us now assess the field equations for \( T_{11} = \frac{S^2}{1 - kr^2} p \) and \( T = -\rho + 3p \):

\[ R_{11} = 8\pi G \frac{c^4}{c^4} \left( T_{11} - \frac{1}{2} g_{11} T \right), \] (7.6)

\[ \Rightarrow \frac{1}{c^2} T^2 S^2 \left( \frac{\dot{S}}{S} + 3 \frac{T \dot{S}}{T S} + 2 \frac{S^2}{S^2} + \frac{2c^2k}{T^2 S^2} \right) \frac{1}{1 - kr^2} \]

\[ = \frac{8\pi G}{c^4} \left( \frac{S^2}{1 - kr^2} p - \frac{1}{2} \frac{S^2}{1 - kr^2} [-\rho + 3p] \right), \] (7.7)

\[ \Rightarrow \frac{\dot{S}}{S} + \frac{T \dot{S}}{T S} + 2 \left( \frac{S^2}{S^2} \right) = 4\pi G \frac{\rho - p}{c^2} - \frac{2c^2 k}{T^2 S^2} \] . (7.8)

The two results constitute the conditions of the development of the spacetime parameters. The results for \( R_{22} \) and \( R_{33} \) are equal to \( R_{11} \).

8 Cosmological model based on the dynamic metric

8.1 Some considerations on the relation between the time and the space scale factor

In section 1 the Identity Assumption was made. So far, all results were derived without the consideration of the relation between the time scale factor \( T \) and the space scale factor \( S \). Before we take a closer look at the cosmological model, this relation should be discussed. One could argue that there is no such relation, e.g. \( T = const. \) (i.e. 1 in the FLRW metric) and \( \dot{T} = 0 \), or, the relation could be chaotic (there are not many arguments for this point) or that it can be expressed by a constant or by any other function of some quantity. One indication is that the Schwarzschild metric includes a symmetrical curvature of the radial direction \( (dr/\sqrt{1 - r_s/r}) \) and the time direction \( (\sqrt{1 - r_s/r} dt) \). If it is assumed, that the time flow in a homogeneous and isotropic universe depends on the energy density and pressure, then the assumption, that the space scale factor is numerically equal to the time scale factor, seems not to be unreasonable. Of course, the Identity Assumption will have to be tested against observations and experiments. In the following sections the Identity Assumption is adopted.
8.2 Proposing a revised cosmological model

In order to solve equation 6.11, it is required that \( \omega \) is constant (see Carroll, 2019 p. 334). By integrating equation 6.11 we obtain \( \ln \rho + \ln(\text{const.}) \propto (-4 \ln T - 3(\omega + 1) \ln S) \) and therefore \( \rho(T) = \rho(S) \propto \text{const.} (T^{-4} S^{-3(\omega + 1)}) \). If we normalize the scale factors to the value for today \( T_0 = 1 \) and \( S_0 = 1 \), the constant becomes \( \rho_0^{-1} \) - which then finally results in

\[
\rho(T) = \rho(S) = \rho_0 \left( T^{-4} S^{-3(\omega + 1)} \right).
\] (8.1)

Finally, considering the values for \( \omega \), we obtain

\[
\rho_{\text{rad}} = \frac{\rho_{\text{rad},0}}{T^4 S^4}, \quad \rho_m = \frac{\rho_{m,0}}{T^4 S^3}, \quad \rho_{\text{vac}} = \frac{\rho_{\text{vac},0}}{T^4}.
\] (8.2)

You will have noticed that \( \rho_m \) and \( \rho_{\text{vac}} \) are both also divided by \( T^4 \). This will be discussed in section 8.3. If equation 7.5 solved for \( S/t \), is plugged into equation 7.8, we obtain

\[
2 \frac{\dot{T}S}{T^3} + \left( \frac{S}{T^2} \right)^2 = \frac{8 \pi G}{3 c^2} \frac{\rho}{T^2} - \frac{c^2 k}{T^2 S^2},
\] (8.3)

and repeat equation 7.5

\[
-\frac{\ddot{S}}{S} + \frac{\dot{T}S}{T S} = \frac{4 \pi G}{3 c^2} \left( \rho + 3 p \right).
\] (8.4)

If we set \( T = 1 \) and consequently \( \dot{T} = 0 \), the two equations become the first and the second Friedmann equations.

Please note that the equations contain a factor \( T^{-2} \) on the right side. We take a closer look at the time scale factor in section 8.3.

In a last step, we consider the equation of state and the values of \( \omega \) for radiation, matter and vacuum energy (see equation 6.12), and find for the equations obtained so far:

\[
2 \frac{\dot{T}S}{T^3} + \left( \frac{S}{T^2} \right)^2 = \frac{8 \pi G}{3 c^2} \left( \frac{\rho_{\text{rad},0}}{T^6 S^4} + \frac{\rho_{m,0}}{T^6 S^3} + \frac{\rho_{\text{vac},0}}{T^6} \right) - \frac{c^2 k}{T^2 S^2},
\] (8.5)

\[
-\frac{\ddot{S}}{S} + \frac{\dot{T}S}{T S} = \frac{4 \pi G}{3 c^2} \left( 2 \frac{\rho_{\text{rad},0}}{T^6 S^4} + \frac{\rho_{m,0}}{T^6 S^3} - 2 \frac{\rho_{\text{vac},0}}{T^6} \right).
\] (8.6)

8.3 Some considerations on the time scale factor

We have seen in equations 8.5 and 8.6 that the energy density dilutes by a factor of \( T^6 \).

Thereof, the factor \( T^{-2} \) was obtained by solving Einstein field equations (see equations 7.5 and 7.8). The factor \( T^{-2} \) has the congruent effect like the space scale factor: With an increasing time scale factor (the flow of time is getting faster) the energy density is decreasing over time; with a decreasing time scale factor (the flow of time is getting slower) an increase of the energy density will be observed. The cause is the contraction or expansion of the time dimension.

The remaining factor \( T^{-4} \) results from the calculation of the divergence of the energy momentum tensor (see equations 6.13 and 8.1). Mathematically, two of the time parameters \( \mathcal{G} \) were obtained by the derivative of 0,0-component of the energy-momentum tensor, \( \partial_0 T^{00} \). The remaining two \( \mathcal{G} \) resulted from multiplying the 0,0-component of the tensor with the 0, 0, 0-Christoffel symbol. Consider for the purposes of interpretation, a universe where the time parameter is (always) the reciprocal of the space parameter (and assume that such a universe may exist). In such a case, the effect of the redshifting of a light wave caused by the expansion of space would be compensated by blueshifting of said light wave by the expansion of time. And the diluting of the light waves in a (non-comoving) volume caused by the expansion of space, which results in a decrease of the energy density in such volume by a factor of \( S^3 \), would be compensated by the blueshifting by a factor of \( T^3 \), which yields an increase of the energy density in the volume back to the original state. Thus, depending on the relation between \( T \) and \( S \), the factor \( T^{-4} \) reinforces, reduces or neutralizes the effects of the term \( S^{-3(\omega + 1)} \) (equations 8.5 and 8.6).
In line with the Identity Assumption, a reasonable explanation seems therefore to be that the time parameter does increase, in case of expansion of space, the effect of the space scale factor over time, respectively decrease such effect in case of contraction of space. Of course, this interpretation leads to the result that the energy density of mass should relate to \(-3\bar{G}(\omega + 1)\) in equations 6.13 and 8.1. However, this is not the result of the calculation of the divergence of the energy-momentum tensor, which leads to questions about the effects of the variable time flow on mass and vacuum energy. This topic will have to be raised further in section 12 but remains an open point in this paper.

9 Redshift

To derive the redshift, the geodesic equation for a light wave travelling radially \((d\theta = d\varphi = 0)\) is used, which then is according to equation [3.7]

\[
0 = ds^2 = -\frac{1}{T^2}c^2dt^2 + S^2\left(\frac{dr^2}{1-kr^2}\right).
\]

The crest of the light wave, observed at position \(r = r_a\) and time \(t = t_a\), was emitted at a time \(t = t_e\) in the past and at a distant position \(r = r_e\). Integrating over the path in both space and time that the light wave travels results in:

\[
c\int_{t_a}^{t_a} \frac{dt}{T} = \int_{r_e}^{r_a} \frac{dr}{\sqrt{1-kr^2}}.
\]

The next crest is emitted at \(t = t_e + \frac{\lambda_e}{c}\) and observed at \(t = t_a + \frac{\lambda_a}{c}\). Integrating over the path and time gives:

\[
c\int_{t_e}^{t_e + \frac{\lambda_e}{c}} \frac{dt}{T} = \int_{r_e}^{r_a} \frac{dr}{\sqrt{1-kr^2}}.
\]

Equations 9.2 and 9.3 can be combined to:

\[
0 = c\int_{t_e}^{t_a} \frac{dt}{T} - c\int_{t_e}^{t_a + \frac{\lambda_a}{c}} \frac{dt}{T} - c\int_{t_e}^{t_e + \frac{\lambda_e}{c}} \frac{dt}{T}.
\]

Let us assume that \(\frac{\lambda_a}{c}\) and \(\frac{\lambda_e}{c}\) are small enough, so that the result above is

\[
\frac{t_a + \lambda_a/c}{T_aS_a} - \frac{t_a}{T_aS_a} = \frac{t_e + \lambda_e/c}{T_eS_e} - \frac{t_e}{T_eS_e}.
\]

\[
\Rightarrow \frac{\lambda_a}{\lambda_e} = \frac{T_aS_a}{T_eS_e} \quad \Rightarrow z = \frac{1}{T_eS_e} - 1, \quad \text{with} \quad T_aS_a = 1.
\]

Let us define a component of the redshift related to the time scale factor

\[
z_T := \frac{1}{T} - 1, \quad T = \frac{1}{z_T + 1},
\]

and accordingly the component of the redshift related to the space scale factor:

\[
z_S := \frac{1}{S} - 1, \quad S = \frac{1}{z_S + 1}.
\]
Without any further restriction, let us consider a photon with a wavelength $\lambda_0$ at the beginning moving through the dynamic spacetime and meanwhile experiencing two causes for a redshift. The first cause results in a redshift $z_1$ and $\lambda_1$ and the second cause results in $z_2$ and $\lambda_2$. For the resulting $z_3$ it can be concluded:

$$z_1 = \frac{\lambda_1}{\lambda_0} - 1, \quad z_2 = \frac{\lambda_2}{\lambda_1} - 1, \quad z_3 = \frac{\lambda_2}{\lambda_0} - 1.$$  \hfill (9.13)

The second equation can be rearranged to obtain the result for $\lambda_1$ which then is plugged into the first equation:

$$z_1 = \frac{\lambda_2}{(z_2 + 1)\lambda_0} - 1 \Rightarrow \lambda_2 = \lambda_0(z_2 + 1)(z_1 + 1).$$  \hfill (9.14)

The result for $\lambda_2$ from equation (9.14) is now used to calculate $z_3$ given in equation (9.13):

$$z_3 = (z_1 + 1)(z_2 + 1) - 1,$$  \hfill (9.15)

which then can be combined with equations (9.12) and (9.13)

$$z = \frac{1}{ST} - 1.$$  \hfill (9.16)

This is the result of equation (9.10). An observed redshift composed of one component related to the space scale factor $z_S$ and another component related to the time scale factor component $z_T$ can be noted as:

$$z = (z_S + 1)(z_T + 1) - 1.$$  \hfill (9.17)

### 10 Space and time parameters, revised Hubble law

In the FLRW metric, the Hubble law can be derived through calculating the proper distance and assuming a flat geometry with $k = 0$ (see Spatschek, 2021, p. 571 - 576). In the dynamic metric (equation 3.7), the proper distance for a radial trajectory (one with $d\theta = d\varphi = 0$) is given by

$$d_p = S \int_0^r \frac{dr}{\sqrt{1 - kr^2}},$$  \hfill (10.1)

whereas the observer is at $r = 0$. Let us compare it with the light travel time. For photons $d\tau = 0$ applies.

$$S \int_{t_e}^{t_0} \frac{c}{ST} dt = -S \int_{r_e}^{r_0} \frac{dr}{\sqrt{1 - kr^2}}.$$  \hfill (10.2)

Note that $r_e$ and $t_e$ are the location and time of the emission of the photon for an observer at time $t_0$ and location $r = 0$. If we use the expression for proper time, the identity yields immediately

$$d_p(t) = S \int_0^r \frac{dr}{\sqrt{1 - kr^2}} \equiv S \int_{r_e}^{r_0} \frac{c}{ST} dt,$$  \hfill (10.3)

and

$$d_p(t_0) = \int_{r_e}^{r_0} \frac{c}{ST} dt, \quad S(t_0) = 1.$$  \hfill (10.4)

$d_p(t_0)$ becomes $r$, if a flat geometry is assumed.

Considering the definition of the space parameter $\mathcal{J}$ and that $dt = 1/\sqrt{SdT}$, we obtain

$$d_p(t_0) = c \int_{S_\epsilon}^{1} \frac{1}{S^2T\mathcal{J}} dS, \quad S(t_0) = 1.$$  \hfill (10.5)
Assuming that for low redshift \( z < 0.1 \) the derivative of the space scale factor \( \dot{S} \) is constant, we can substitute \( \mathcal{J} \) by \( \frac{\dot{S}}{S} \):

\[
d \approx \frac{c}{\mathcal{J}_0} \int_{S_0}^{1} \frac{1}{S T} dS, \quad z < 0.1. \tag{10.6}
\]

If we apply the Identity Assumption and substitute \( T \) by \( S \), the integral can be solved.

\[
d \approx \frac{c}{\mathcal{J}_0} \left[ -1 + \frac{1}{S e} \right], \quad z < 0.1. \tag{10.7}
\]

Considering the definition given inequation 9.12, we obtain finally the revised Hubble law:

\[
d \approx \frac{c}{\mathcal{J}_0} z_s, \quad z < 0.1. \tag{10.8}
\]

From equation 9.16 the relation between the space scalefactor and the redshift is given by \( ST = (z + 1)^{-1} \). In the FLRW metric, the relation is \( \alpha = (z + 1)^{-1} \). This relation is the basis for the comparison of the Hubble parameter with the space and time parameters. Given a redshift, the product of the spacetime scale factors corresponds to the cosmological scale factor of the FLRW universe. Accordingly, the sum of the space parameter and the time parameter corresponds to the Hubble parameter of the Friedmann equations, considering the definition of the Hubble parameter:

\[
\frac{(ST)}{ST} = \frac{\dot{S}}{S} + \frac{\dot{T}}{T} = \mathcal{J} + \mathcal{G} \tag{10.9}
\]

If we assume that the space parameter \( \mathcal{J} \) amounts to \( 35 \text{ km} \text{ mpc}^{-1} \), we obtain a value for the spacetime parameters of \( \mathcal{G} = \mathcal{J} = 1.134 \times 10^{-18} \text{ s}^{-1} \). Consequently, this results in

\[
\mathcal{G}^{-1} \approx 27.936 \text{ billion years, if } \mathcal{G} = 1.134 \times 10^{-18} \text{ s}^{-1} \tag{10.10}
\]

The \( \mathcal{G} \)-age amounts to the double of the Hubble-age.

11 Distances, redshift and the dark energy enigma

Considering equation 10.5 and the Identity Assumption, we can further substitute \( dS \) by \( dz \), with \( S = (z + 1)^{-1/2} \) and \( T = (z + 1)^{-1/2} \), and therefore \( dS/dz = -1/2(z + 1)^{-3/2} \).

\[
d_p(t_0) = -c \int_{z_e}^{0} \frac{1}{2\sqrt{z+1}} \frac{1}{S^2 T J} dz \tag{11.1}
\]

With the results found in equation 9.17 and the definition of equation 9.12 we obtain:

\[
d_p(t_0) = -c \int_{z_e}^{0} \frac{1}{2\mathcal{J}} dz. \tag{11.2}
\]

For the FLRW metric and the Friedmann equations, the result is (see Carroll [2019] p. 347):

\[
d_p(t_0) = -c \int_{z_e}^{0} \frac{1}{H} dz. \tag{11.3}
\]

If the results are compared and the Identity Assumption is adopted, we can conclude that the proper distance in the FLRW metric and the dynamic metric are the same.

The luminosity distance \( d_L \) is defined by the relationship between the flux \( F \) measured by an observer (the energy per unit time per unit area of some detector) and the absolute luminosity \( L \). Please refer to Hogg [1999], Carroll [2019] p. 346 and Spatschek [2021] p. 579.
\[ d_L^2 = \frac{L}{4\pi F}. \] (11.4)

In a space-like static universe, where the time flows equably without regard to anything external, the flux would show a \( \frac{1}{r^2} \)-dependency. In the proposed model, the flux will be diluted, however in a slightly different way than in the Friedmann universe, as we have seen at the end of section 8.3 and in equation 8.5. Conservation of photons means that all of the photons emitted by the source will eventually pass through a sphere at comoving distance \( x \) from the emitter (Carroll, 2019, p. 346). But the flux is diluted by the following additional effects: The individual photons redshift by a factor \((ST)^{-1}\), and the photons hit the sphere less frequently, since two photons emitted a time \( \delta t \) apart will be measured at a time \((ST)^{-1} \delta t \) apart. Lastly, we have to consider the effect of the expansion of the time dimension as discussed in section 8.3, the resulting effect of which is \( T^{-\frac{1}{2}} \). If we assume a flat geometry, the proper distance \( d_P \) becomes \( r \) (equation 10.2). This brought together yields to:

\[ F_S = \frac{L}{4\pi r_S^2(z_S + 1)^2(z_T + 1)^4}. \] (11.5)

Considering the Identity Assumption and that \((z_S + 1)(z_T + 1) = z + 1\), we obtain finally

\[ d_{L,S} = r_S(\sqrt{(z + 1)^3}). \] (11.6)

In the cosmological model which is based on the FLRW-metric, the luminosity distance is:

\[ d_{L,a} = r_a(z + 1). \] (11.7)

Whilst the proper distances in the FLRW universe and the dynamic universe are the same (see equations 11.2 and 11.3), the luminosity distance in the dynamic metric exceeds the luminosity distance in the FLRW universe. We can concretely calculate the relative deviation between the results using

\[ \text{dev}_{rel} = \frac{d_{L,a} - d_{L,S}}{d_{L,a}} = 1 - \frac{\sqrt{(z + 1)^3}}{z + 1} = 1 - \sqrt{(z + 1)}. \] (11.8)

In the table below, the relative deviations of the luminosity distances are shown for some examples of redshift.

<table>
<thead>
<tr>
<th>( z_{\text{obs}} )</th>
<th>( \text{dev}_{rel} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.000</td>
</tr>
<tr>
<td>0.005704</td>
<td>-0.003</td>
</tr>
<tr>
<td>0.1</td>
<td>-0.049</td>
</tr>
<tr>
<td>0.2</td>
<td>-0.095</td>
</tr>
<tr>
<td>0.3</td>
<td>-0.140</td>
</tr>
<tr>
<td>0.4</td>
<td>-0.183</td>
</tr>
<tr>
<td>0.5</td>
<td>-0.225</td>
</tr>
<tr>
<td>0.6</td>
<td>-0.265</td>
</tr>
<tr>
<td>0.7</td>
<td>-0.304</td>
</tr>
<tr>
<td>0.8</td>
<td>-0.342</td>
</tr>
<tr>
<td>0.9</td>
<td>-0.378</td>
</tr>
<tr>
<td>1</td>
<td>-0.414</td>
</tr>
<tr>
<td>1.4</td>
<td>-0.549</td>
</tr>
<tr>
<td>5</td>
<td>-1.449</td>
</tr>
<tr>
<td>10</td>
<td>-2.317</td>
</tr>
<tr>
<td>100</td>
<td>-9.050</td>
</tr>
<tr>
<td>1000</td>
<td>-30.639</td>
</tr>
</tbody>
</table>

Saul Perlmutter et. al., Brian Schmidt and Adam Riess et. al. discovered that distant supernovae were fainter than expected (Perlmutter et al., 1999; Riess et al., 1998, 2004). At redshifts \( z > 0.1 \), the cosmological predictions started to diverge. First, the data on high-redshift Type Ia supernovae were published: high-redshift \((z \approx 0.5)\) supernovae Ia...
are ∼25\% fainter than expected in a universe that has Ω_M = 0.3, and ∼15\% fainter than expected in a freely coasting universe (Ω_M = 0), suggesting that the expansion of the universe is accelerating with time (Filippenko 2001). The driving force behind the acceleration is unknown, but the current opinion is that the cause of the accelerated expansion is vacuum energy, called dark energy.

For a given redshift, the supernovae were too faint. But this could also mean that the distance as calculated by the means of luminosity was too small. If we base the dynamic metric and the revised cosmological model, it can be explained that the luminosity distance is bigger than in the FLRW universe. As it can be seen in the table above, a redshift of z = 0.5 leads to a deviation of ∼22.5\% or a redshift of z = 1 leads to a deviation of ∼44.1\%. Consequently, these deviations result in differing expectations of how faint or bright an observed supernova should be. The consequence and conclusion is that the dynamic metric and the proposed cosmological model render a more obvious explanation for the observations according to which high-z supernovae should have been ∼25\% brighter. It is therefore not necessary to explain the observations by a negative deceleration parameter q_0 and by an unexpectedly accelerated the expansion at the present epoch caused by dark energy. Of course, this conclusion is subject to further review and modelling with concrete data.

12 Conclusion and open questions

A dimensionless time scale factor \( T(t) \) was introduced and a dynamic metric and revised cosmological model were proposed which describe the expansion and contraction of the space and time dimensions in a homogeneous and isotropic model of the universe. It was possible to provide indications that the accelerated expansion of the universe as it results from the FLRW metric and the Friedmann equations, may be explained by a different calculation of the luminosity distance in the proposed cosmological model. Of course, this conclusion is subject to further review. The new cosmological model includes a revised Hubble law and implies, that the universe is much older than calculated by the means of the Friedmann equations. The reciprocal of the time parameter \( G^{-1} \) amounts tp approximately 27.9 billion years.

Several questions and topics were not yet discussed in this paper. Possible effects of the variable time flow on mass and vacuum energy were not considered. In section 8.3 it remains an open question whether all of the time scale factors (\( T^6 \)) in equations 8.5 and 8.6 apply also in a matter-dominated universe. It is somehow understandable that the energy momentum tensor "desires" to establish the mass-energy equivalence; however from a (purely) classical view, it is difficult to comprehend why mass itself should be redshifted by a variable flow of time, unless one were to argue that the major portion of mass consists of kinetic or field energy of the quarks and gluons, and that such kinetic or field energy is subjected to the expansion of the dimension of time - as it is valid for the electromagnetic field. In any case, this topic will also have to be further assessed in the light of particle and quantum physics. Further, possible effects of the variable time flow on fundamental physical constants were not taken into consideration. In the case of the light speed, the decision was taken by adopting the constant light speed postulate; however, even this postulate does reveal other consequences in spacetime with variable time flow, as clocks in different inertial frames tick differently (if \( T_e \neq T_a \)). Further studies and research seem to be necessary to reveal such possible effects, e.g. on the Newtonian constant of gravitation and on the propagation of curvature at light speed. Further, in the VLS-theories (see e.g. Albrecht & Magueijo 1999, Barrow 1999, Magueijo 2003) impressive approaches and answers were provided to solve the flatness, horizon, homogeneity and isotropy problems which also will have to be discussed in the light of the dynamic metric with a variable flow of time.

References

Barrow J. D., 1999, Phys. Rev. D 59, 043515
Appendix A  Calculation of the Christoffel symbols of the second kind

As stated in the Main Document, the relation between the metric tensor and the Christoffel symbols is given by:

\[ \Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\rho} \left[ \partial_\lambda g_{\nu\rho} + \partial_\nu g_{\rho\lambda} - \partial_\rho g_{\nu\lambda} \right], \quad \text{whereas} \quad \Gamma^\mu_{\nu,\lambda} = \Gamma^\mu_{\lambda,\nu} \quad (A.1) \]

In all formulas in this Appendix a dot denotes differentiation with respect to \( t \). Further, it is considered that \( g_{\nu\rho} = 0 \), if \( \nu \neq \rho \), and that \( g^{\mu\rho} = 0 \), if \( \mu \neq \rho \). The respective terms, which become zero, will be omitted.

A.1  The 16 Christoffel symbols of the form \( \Gamma^0_{\nu,\lambda} \)

Let us start with \( \Gamma^0_{00} \). As we assume a variable time flow, the partial time derivative of the time scale factor is not zero:

\[
\Gamma^0_{00} = \frac{1}{2} g^{0\rho} \left[ \partial_0 g_{0\rho} + \partial_\rho g_{00} - \partial_\rho g_{00} \right] \quad (A.2)
\]

\[
= \frac{1}{2} g^{00} \left[ \partial_0 g_{00} + \partial_0 g_{00} - \partial_0 g_{00} \right] \quad (A.3)
\]

\[
= \frac{1}{2} g^{00} \left[ \partial_0 g_{00} \right] \quad (A.4)
\]

\[
= \frac{1}{2} \mathcal{T}^2 \left[ \partial_0 \left( \frac{1}{\mathcal{T}^2} \right) \right] \quad (A.5)
\]

\[
= -\frac{1}{2} \mathcal{T}^2 \left[ \frac{-2 \mathcal{T}^2}{\mathcal{T}^4} \right] \quad (A.6)
\]

\[
= \frac{\mathcal{T}^2}{c} \quad (A.7)
\]

The Christoffel symbols of the form \( \Gamma^0_{0k} = \Gamma^0_{k0} \) with \( k = 1, 2, 3 \) are calculated considering, that the time scale factor is constant at given point of (cosmological) time, thus \( \partial_k g_{00} = 0 \):

\[
\Gamma^0_{0k} = \Gamma^0_{k0} = \frac{1}{2} g^{0\rho} \left[ \partial_k g_{0\rho} + \partial_\rho g_{0k} - \partial_\rho g_{0k} \right] \quad (A.8)
\]

\[
= \frac{1}{2} g^{00} \left[ \partial_k g_{00} + \partial_0 g_{0k} - \partial_0 g_{0k} \right] \quad (A.9)
\]

\[
= \frac{1}{2} g^{00} \left[ \partial_k g_{00} \right] \quad (A.10)
\]

\[
= 0 \quad (A.11)
\]

\( \Gamma^0_{11} \) is therefore:

\[
\Gamma^0_{11} = \frac{1}{2} g^{0\rho} \left[ \partial_1 g_{1\rho} + \partial_\rho g_{11} - \partial_\rho g_{11} \right] \quad (A.12)
\]

\[
= \frac{1}{2} g^{00} \left[ \partial_1 g_{10} + \partial_0 g_{11} - \partial_0 g_{11} \right] \quad (A.13)
\]

\[
= \frac{1}{2} g^{00} \left[ -\partial_0 g_{11} \right] \quad (A.14)
\]

\[
= \frac{1}{2} \mathcal{T}^2 \left[ \partial_0 \left( \frac{S^2}{1-kr^2} \right) \right] \quad (A.15)
\]

\[
= \frac{\mathcal{T}^2 \dot{S} S}{c(1-kr^2)} \quad (A.16)
\]
The Christoffel symbols of the form $\Gamma^0_{il} = \Gamma^0_{li}$ with $l = 2, 3$ result in:

$$\Gamma^0_{il} = \Gamma^0_{li} = \frac{1}{2} g^{0\rho} \left[ \partial_t g_{\rho l} + \partial_l g_{\rho t} - \partial_{\rho} g_{tl} \right]$$  
(A.17)

$$= \frac{1}{2} g^{00} \left[ \partial_l g_{0t} + \partial_t g_{0l} - \partial_0 g_{lt} \right]$$  
(A.18)

$$= 0$$  
(A.19)

For $\Gamma^0_{22}$ we obtain:

$$\Gamma^0_{22} = \frac{1}{2} g^{0\rho} \left[ \partial_2 g_{2\rho} + \partial_\rho g_{22} - \partial_\rho g_{22} \right]$$  
(A.20)

$$= \frac{1}{2} g^{00} \left[ \partial_2 g_{02} + \partial_2 g_{02} - \partial_0 g_{22} \right]$$  
(A.21)

$$= \frac{1}{2} g^{00} \left[ -\partial_0 g_{22} \right]$$  
(A.22)

$$= \frac{1}{2} T^{-2} \left[ \partial_0 \left( S^2 r^2 \right) \right]$$  
(A.23)

$$= \frac{T^{-2} S S r^2}{c}$$  
(A.24)

$\Gamma^0_{33}$ results in:

$$\Gamma^0_{33} = \frac{1}{2} g^{0\rho} \left[ \partial_3 g_{3\rho} + \partial_\rho g_{33} - \partial_\rho g_{33} \right]$$  
(A.25)

$$= \frac{1}{2} g^{00} \left[ \partial_3 g_{03} + \partial_3 g_{03} - \partial_0 g_{33} \right]$$  
(A.26)

$$= \frac{1}{2} g^{00} \left[ -\partial_0 g_{33} \right]$$  
(A.27)

$$= \frac{1}{2} T^{-2} \left[ \partial_0 \left( S^2 r^2 \sin^2 \theta \right) \right]$$  
(A.28)

$$= \frac{T^{-2} S S r^2 \sin^2 \theta}{c}$$  
(A.29)

The remaining Christoffel symbols $\Gamma^0_{23} = \Gamma^0_{32}$ become:

$$\Gamma^0_{23} = \Gamma^0_{32} = \frac{1}{2} g^{0\rho} \left[ \partial_2 g_{\rho 3} + \partial_3 g_{\rho 2} - \partial_\rho g_{23} \right]$$  
(A.30)

$$= \frac{1}{2} g^{00} \left[ \partial_2 g_{03} + \partial_3 g_{02} - \partial_0 g_{23} \right]$$  
(A.31)

$$= 0$$  
(A.32)
A.2 The 16 Christoffel symbols of the form $\Gamma^1_{\nu \lambda}$

In general, the Christoffel symbols of the form $\Gamma^k_{00}$ with $k = 1, 2, 3$ are calculated considering, that the time scale factor is constant at given point of (cosmological) time, thus ($\partial_k g_{00} = 0$):

$$\Gamma^k_{00} = \frac{1}{2} g^{k \rho} \left[ \partial_0 g_{0 \rho} + \partial_0 g_{\rho 0} - \partial_\rho g_{00} \right]$$ (A.33)

$$= \frac{1}{2} g^{k k} \left[ \partial_0 g_{0 k} + \partial_0 g_{k 0} - \partial_k g_{00} \right]$$ (A.34)

$$= \frac{1}{2} g^{k k} [-\partial_k g_{00}]$$ (A.35)

$$= 0$$ (A.36)

The Christoffel symbols of the form $\Gamma^k_{0l} = \Gamma^k_{l0}$ with $k = 1, 2, 3$ do not include the term $\mathcal{T}$. For $\Gamma^1_{01} = \Gamma^1_{10}$ we obtain:

$$\Gamma^1_{01} = \Gamma^1_{10} = \frac{1}{2} g^{1 \rho} \left[ \partial_1 g_{0 \rho} + \partial_0 g_{\rho 1} - \partial_\rho g_{01} \right]$$ (A.37)

$$= \frac{1}{2} g^{1 1} \left[ \partial_1 g_{01} + \partial_0 g_{11} - \partial_1 g_{01} \right]$$ (A.38)

$$= \frac{1}{2} g^{1 1} \left[ \partial_0 g_{11} \right]$$ (A.39)

$$= \frac{1}{2} \left[ \frac{1 - kr^2}{S^2} \right] \left[ \partial_0 \left( \frac{S^2}{1 - kr^2} \right) \right]$$ (A.40)

$$= \frac{1}{2} \left[ \frac{1 - kr^2}{S^2} \right] \left[ \frac{2S \dot{S}}{c(1 - kr^2)} \right]$$ (A.41)

$$= \frac{\dot{S}}{c \ddot{S}}$$ (A.42)

The Christoffel symbols of the form $\Gamma^l_{0l} = \Gamma^l_{l0}$ with $l = 2, 3$ become:

$$\Gamma^l_{0l} = \Gamma^l_{l0} = \frac{1}{2} g^{l \rho} \left[ \partial_0 g_{l \rho} + \partial_0 g_{\rho l} - \partial_\rho g_{l0} \right]$$ (A.43)

$$= \frac{1}{2} g^{1 1} \left[ \partial_0 g_{l 1} + \partial_0 g_{1 l} - \partial_1 g_{0 l} \right]$$ (A.44)

$$= 0$$ (A.45)

$\Gamma^1_{11}$ results in:

$$\Gamma^1_{11} = \frac{1}{2} g^{1 \rho} \left[ \partial_1 g_{1 \rho} + \partial_1 g_{\rho 1} - \partial_\rho g_{11} \right]$$ (A.46)

$$= \frac{1}{2} g^{1 1} \left[ \partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11} \right]$$ (A.47)

$$= \frac{1}{2} g^{1 1} \left[ \partial_1 g_{11} \right]$$ (A.48)

$$= \frac{1}{2} \left[ \frac{1 - kr^2}{S^2} \right] \left[ \partial_1 \left( \frac{S^2}{1 - kr^2} \right) \right]$$ (A.49)

$$= \frac{1}{2} \left[ \frac{1 - kr^2}{S^2} \right] \left[ 0 - S^2 \frac{-2kr}{(1 - kr^2)^2} \right]$$ (A.50)

$$= \frac{kr}{1 - kr^2}$$ (A.51)
Further, we obtain for the Christoffel symbols of the form $\Gamma^l_{i1} = \Gamma^l_{1i}$ with $l = 2, 3$:  

\[
\Gamma^l_{i1} = \Gamma^l_{1i} = \frac{1}{2} g^{lp} \left[ \partial_l g_{ip} + \partial_i g_{pl} - \partial_p g_{il} \right]  
= \frac{1}{2} g^{1p} \left[ \partial_p g_{11} + \partial_1 g_{p1} - \partial_1 g_{1p} \right]  
= 0
\]  

(A.52)  

(A.53)  

(A.54)  

$\Gamma^1_{22}$ results in:  

\[
\Gamma^1_{22} = \frac{1}{2} g^{lp} \left[ \partial_2 g_{2p} + \partial_2 g_{p2} - \partial_p g_{22} \right]  
= \frac{1}{2} g^{11} \left[ \partial_2 g_{21} + \partial_2 g_{12} - \partial_1 g_{22} \right]  
= \frac{1}{2} g^{11} \left[ -\partial_1 g_{22} \right]  
= \frac{1}{2} \left[ \frac{1 - kr^2}{S^2} \right] \left[ -\partial_1 \left( S^2 r^3 \right) \right]  
= \frac{1}{2} \left[ \frac{1 - kr^2}{S^2} \right] \left[ -2 S^2 r \right]  
= r \left( kr^2 - 1 \right)
\]  

(A.55)  

(A.56)  

(A.57)  

(A.58)  

(A.59)  

For $\Gamma^l_{23} = \Gamma^l_{32}$ we calculate:  

\[
\Gamma^l_{23} = \Gamma^l_{32} = \frac{1}{2} g^{lp} \left[ \partial_3 g_{2p} + \partial_2 g_{p3} - \partial_p g_{23} \right]  
= \frac{1}{2} g^{11} \left[ \partial_3 g_{21} + \partial_2 g_{13} - \partial_1 g_{23} \right]  
= 0
\]  

(A.61)  

(A.62)  

(A.63)  

Finally $\Gamma^l_{33}$:  

\[
\Gamma^l_{33} = \frac{1}{2} g^{lp} \left[ \partial_3 g_{3p} + \partial_3 g_{p3} - \partial_p g_{33} \right]  
= \frac{1}{2} g^{11} \left[ \partial_3 g_{31} + \partial_3 g_{13} - \partial_1 g_{33} \right]  
= \frac{1}{2} g^{11} \left[ -\partial_1 g_{33} \right]  
= \frac{1}{2} \left[ \frac{1 - kr^2}{S^2} \right] \left[ -\partial_1 \left( S^2 r^2 \sin^2 \theta \right) \right]  
= \frac{1}{2} \left[ \frac{1 - kr^2}{S^2} \right] \left[ -2 S^2 r \sin^2 \theta \right]  
= r \left( kr^2 - 1 \right) \sin^2 \theta
\]  

(A.64)  

(A.65)  

(A.66)  

(A.67)  

(A.68)  

(A.69)
A.3 The 16 Christoffel symbols of the form $\Gamma_{\nu t}^2$

For $\Gamma_{00}^2 = 0$ please refer to the calculation of $\Gamma_{00}^4$ in the first equation of subsection A.1.

Now $\Gamma_{02}^2 = \Gamma_{20}^2$:

$$
\Gamma_{02}^2 = \Gamma_{20}^2 = \frac{1}{2} g^{2\rho} \left[ \partial_2 g_{0\rho} + \partial_0 g_{2\rho} - \partial_\rho g_{02} \right]
$$

(A.70)

$$
= \frac{1}{2} g^{22} \left[ \partial_2 g_{02} + \partial_0 g_{22} - \partial_2 g_{02} \right]
$$

(A.71)

$$
= \frac{1}{2} g^{22} [\partial_0 g_{22}]
$$

(A.72)

$$
= \frac{1}{2} \left[ \frac{1}{S^2 r^2} \right] \left[ \partial_0 \left( S^2 r^2 \right) \right]
$$

(A.73)

$$
= \frac{1}{2} \frac{S}{c} S
$$

(A.74)

We obtain for the Christoffel symbols of the form $\Gamma_{0l}^2 = \Gamma_{l0}^2$ with $l = 1, 3$:

$$
\Gamma_{01}^2 = \Gamma_{10}^2 = \frac{1}{2} g^{2\rho} \left[ \partial_1 g_{0\rho} + \partial_0 g_{1\rho} - \partial_\rho g_{01} \right]
$$

(A.76)

$$
= \frac{1}{2} g^{22} \left[ \partial_1 g_{02} + \partial_0 g_{21} - \partial_2 g_{01} \right]
$$

(A.77)

$$
= 0
$$

(A.78)

$\Gamma_{11}^2$ is:

$$
\Gamma_{11}^2 = \frac{1}{2} g^{2\rho} \left[ \partial_1 g_{1\rho} + \partial_1 g_{\rho 1} - \partial_\rho g_{11} \right]
$$

(A.79)

$$
= \frac{1}{2} g^{22} \left[ \partial_1 g_{12} + \partial_1 g_{21} - \partial_2 g_{11} \right]
$$

(A.80)

$$
= \frac{1}{2} g^{22} [\partial_2 g_{11}]
$$

(A.81)

$$
= \frac{1}{2} \left[ \frac{1}{S^2 r^2} \right] \left[ -\partial_2 \left( \frac{S^2}{1 - kr^2} \right) \right]
$$

(A.82)

$$
= \frac{1}{2} \left[ \frac{1}{S^2 r^2} \right] [0]
$$

(A.83)

$$
= 0
$$

(A.84)

Then, the result for $\Gamma_{12}^2 = \Gamma_{21}^2$ is:

$$
\Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{2} g^{2\rho} \left[ \partial_2 g_{1\rho} + \partial_1 g_{\rho 2} - \partial_\rho g_{12} \right]
$$

(A.85)

$$
= \frac{1}{2} g^{22} \left[ \partial_2 g_{12} + \partial_1 g_{22} - \partial_2 g_{12} \right]
$$

(A.86)

$$
= \frac{1}{2} g^{22} [\partial_1 g_{22}]
$$

(A.87)

$$
= \frac{1}{2} \frac{1}{S^2 r^2} \left[ \partial_1 \left( S^2 r^2 \right) \right]
$$

(A.88)

$$
= \frac{1}{2} \left[ \frac{1}{S^2 r^2} \right] [2 S^2 r]
$$

(A.89)

$$
= \frac{1}{r}
$$

(A.90)
And $\Gamma^2_{13} = \Gamma^2_{31}$:

$$\Gamma^2_{13} = \Gamma^2_{31} = \frac{1}{2} g^{2p} \left[ \partial_3 g_{1p} + \partial_1 g_{p3} - \partial_p g_{13} \right]$$  \hspace{1cm} (A.91)

$$= \frac{1}{2} g^{22} \left[ \partial_3 g_{12} + \partial_1 g_{23} - \partial_2 g_{13} \right]$$  \hspace{1cm} (A.92)

$$= 0$$  \hspace{1cm} (A.93)

Also for $\Gamma^2_{22}$:

$$\Gamma^2_{22} = \frac{1}{2} g^{2p} \left[ \partial_2 g_{2p} + \partial_1 g_{p2} - \partial_p g_{22} \right]$$  \hspace{1cm} (A.94)

$$= \frac{1}{2} g^{22} \left[ \partial_2 g_{22} + \partial_2 g_{22} - \partial_2 g_{22} \right]$$  \hspace{1cm} (A.95)

$$= \frac{1}{2} g^{22} \left[ \partial_2 g_{22} \right]$$  \hspace{1cm} (A.96)

$$= \frac{1}{2} \left[ \frac{1}{S^2 r^2} \right] \left[ \partial_2 \left( S^2 r^2 \right) \right]$$  \hspace{1cm} (A.97)

$$= \frac{1}{2} \left[ \frac{1 - kr^2}{S^2} \right] \left[ 0 \right]$$  \hspace{1cm} (A.98)

$$= 0$$  \hspace{1cm} (A.99)

Again for $\Gamma^2_{23} = \Gamma^2_{32}$:

$$\Gamma^2_{23} = \Gamma^2_{32} = \frac{1}{2} g^{2p} \left[ \partial_3 g_{2p} + \partial_2 g_{p3} - \partial_p g_{23} \right]$$  \hspace{1cm} (A.100)

$$= \frac{1}{2} g^{22} \left[ \partial_3 g_{22} + \partial_2 g_{23} - \partial_2 g_{23} \right]$$  \hspace{1cm} (A.101)

$$= \frac{1}{2} g^{22} \left[ \partial_3 g_{22} \right]$$  \hspace{1cm} (A.102)

$$= \frac{1}{2} \left[ \frac{1}{S^2 r^2} \right] \left[ \partial_3 \left( S^2 r^2 \right) \right]$$  \hspace{1cm} (A.103)

$$= \frac{1}{2} \left[ \frac{1}{S^2 r^2} \right] \left[ 0 \right]$$  \hspace{1cm} (A.104)

$$= 0$$  \hspace{1cm} (A.105)

Finally, we calculate $\Gamma^2_{33}$:

$$\Gamma^2_{33} = \frac{1}{2} g^{2p} \left[ \partial_3 g_{3p} + \partial_3 g_{p3} - \partial_p g_{33} \right]$$  \hspace{1cm} (A.106)

$$= \frac{1}{2} g^{22} \left[ \partial_3 g_{32} + \partial_3 g_{23} - \partial_3 g_{23} \right]$$  \hspace{1cm} (A.107)

$$= \frac{1}{2} g^{22} \left[ \partial_3 g_{32} \right]$$  \hspace{1cm} (A.108)

$$= \frac{1}{2} \left[ \frac{1}{S^2 r^2} \right] \left[ -\partial_3 \left( S^2 r^2 sin^2 \theta \right) \right]$$  \hspace{1cm} (A.109)

$$= \frac{1}{2} \left[ \frac{1}{S^2 r^2} \right] \left[ -2S^2 r^2 sin \theta cos \theta \right]$$  \hspace{1cm} (A.110)

$$= -sin\theta cos \theta$$  \hspace{1cm} (A.111)
### A.4 The 16 Christoffel symbols of the form $\Gamma^3_{\nu \lambda}$

$\Gamma^3_{00} = 0$ was already calculated above. For $\Gamma^3_{00} = 0$ please refer to the calculation of $\Gamma^3_{00}$ in the first equation of subsection A.1. The Christoffel symbols of the form $\Gamma^3_{0l} = \Gamma^3_{l0}$ with $l = 1, 2$ result in:

$$\Gamma^3_{0l} = \Gamma^3_{l0} = \frac{1}{2} g^{3p} \left[ \partial_t g_{0p} + \partial_0 g_{p0} - \partial_p g_{00} \right]$$  \hspace{1cm} (A.112)

$$= \frac{1}{2} g^{33} \left[ \partial_t g_{03} + \partial_0 g_{30} - \partial_3 g_{00} \right]$$  \hspace{1cm} (A.113)

$$= 0$$  \hspace{1cm} (A.114)

For $\Gamma^3_{03} = \Gamma^3_{30}$ we calculate:

$$\Gamma^3_{03} = \Gamma^3_{30} = \frac{1}{2} g^{3p} \left[ \partial_3 g_{0p} + \partial_0 g_{p3} - \partial_p g_{03} \right]$$  \hspace{1cm} (A.115)

$$= \frac{1}{2} g^{33} \left[ \partial_3 g_{03} + \partial_0 g_{33} - \partial_3 g_{03} \right]$$  \hspace{1cm} (A.116)

$$= \frac{1}{2} g^{33} \left[ \partial_0 g_{33} \right]$$  \hspace{1cm} (A.117)

$$= \frac{1}{2} \left[ \frac{1}{S^2 r^2 \sin^2 \theta} \right] \left[ \partial_0 \left( S^2 r^2 \sin^2 \theta \right) \right]$$  \hspace{1cm} (A.118)

$$= \frac{1}{2} \left[ \frac{1}{S^2 r^2 \sin^2 \theta} \right] \left[ S S^r r^2 \sin^2 \theta \right]$$  \hspace{1cm} (A.119)

$$= \frac{S}{c S}$$  \hspace{1cm} (A.120)

We obtain for $\Gamma^3_{11}$:

$$\Gamma^3_{11} = \frac{1}{2} g^{3p} \left[ \partial_1 g_{1p} + \partial_1 g_{p1} - \partial_p g_{11} \right]$$  \hspace{1cm} (A.121)

$$= \frac{1}{2} g^{33} \left[ \partial_1 g_{13} + \partial_1 g_{31} - \partial_3 g_{11} \right]$$  \hspace{1cm} (A.122)

$$= \frac{1}{2} g^{33} \left[ -\partial_3 g_{11} \right]$$  \hspace{1cm} (A.123)

$$= \frac{1}{2} \left[ \frac{1}{S^2 r^2 \sin^2 \theta} \right] \left[ -\partial_3 \left( \frac{S^2}{1 - kr^2} \right) \right]$$  \hspace{1cm} (A.124)

$$= \frac{1}{2} \left[ \frac{1}{S^2 r^2 \sin^2 \theta} \right] \left[ 0 \right]$$  \hspace{1cm} (A.125)

$$= 0$$  \hspace{1cm} (A.126)

Same for $\Gamma^3_{12} = \Gamma^3_{21}$:

$$\Gamma^3_{12} = \Gamma^3_{21} = \frac{1}{2} g^{3p} \left[ \partial_2 g_{1p} + \partial_1 g_{p2} - \partial_p g_{12} \right]$$  \hspace{1cm} (A.127)

$$= \frac{1}{2} g^{33} \left[ \partial_2 g_{13} + \partial_1 g_{32} - \partial_3 g_{12} \right]$$  \hspace{1cm} (A.128)

$$= 0$$  \hspace{1cm} (A.129)
Then $\Gamma^3_{13} = \Gamma^3_{31}$ is calculated as follows:

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{2} \delta^{3p} \left[ \partial_3 g_{1p} + \partial_1 g_{p3} - \partial_p g_{13} \right]$$  \hspace{1cm} (A.130)

$$= \frac{1}{2} g^{33} \left[ \partial_3 g_{13} + \partial_1 g_{33} - \partial_3 g_{13} \right]$$  \hspace{1cm} (A.131)

$$= \frac{1}{2} g^{33} \left[ \partial_1 g_{33} \right]$$  \hspace{1cm} (A.132)

$$= \frac{1}{2} \left[ \frac{1}{S^2 r^2 \sin^2 \theta} \right] \left[ \partial_1 \left( S^2 r^2 \sin^2 \theta \right) \right]$$  \hspace{1cm} (A.133)

$$= \frac{1}{2} \left[ \frac{1}{S^2 r^2 \sin^2 \theta} \right] \left[ 2S^2 r \sin^2 \theta \right]$$  \hspace{1cm} (A.134)

$$= \frac{1}{2}$$  \hspace{1cm} (A.135)

$\Gamma^3_{22}$ results in:

$$\Gamma^3_{22} = \frac{1}{2} \delta^{3p} \left[ \partial_2 g_{2p} + \partial_2 g_{p2} - \partial_p g_{22} \right]$$  \hspace{1cm} (A.136)

$$= \frac{1}{2} g^{33} \left[ \partial_2 g_{23} + \partial_2 g_{32} - \partial_3 g_{22} \right]$$  \hspace{1cm} (A.137)

$$= \frac{1}{2} g^{33} \left[ - \partial_3 g_{22} \right]$$  \hspace{1cm} (A.138)

$$= \frac{1}{2} \left[ \frac{1}{S^2 r^2 \sin^2 \theta} \right] \left[ \partial_3 \left( S^2 r^2 \right) \right]$$  \hspace{1cm} (A.139)

$$= \frac{1}{2} \left[ \frac{1}{S^2 r^2 \sin^2 \theta} \right] \left[ 0 \right]$$  \hspace{1cm} (A.140)

$$= 0$$  \hspace{1cm} (A.141)

The Christoffel symbols $\Gamma^3_{23} = \Gamma^3_{32}$ are:

$$\Gamma^3_{23} = \Gamma^3_{32} = \frac{1}{2} \delta^{3p} \left[ \partial_3 g_{2p} + \partial_2 g_{p3} - \partial_p g_{23} \right]$$  \hspace{1cm} (A.142)

$$= \frac{1}{2} g^{33} \left[ \partial_3 g_{23} + \partial_2 g_{33} - \partial_3 g_{23} \right]$$  \hspace{1cm} (A.143)

$$= \frac{1}{2} g^{33} \left[ \partial_2 g_{33} \right]$$  \hspace{1cm} (A.144)

$$= \frac{1}{2} \left[ \frac{1}{S^2 r^2 \sin^2 \theta} \right] \left[ \partial_2 \left( S^2 r^2 \sin^2 \theta \right) \right]$$  \hspace{1cm} (A.145)

$$= \frac{1}{2} \left[ \frac{1}{S^2 r^2 \sin^2 \theta} \right] \left[ 2S^2 r \sin \theta \cos \theta \right]$$  \hspace{1cm} (A.146)

$$= \frac{\cos \theta}{\sin \theta}$$  \hspace{1cm} (A.147)

$$= \cot \theta$$  \hspace{1cm} (A.148)
Finally, $\Gamma_{33}^3$ becomes:

$$\Gamma_{33}^3 = \frac{1}{2} g^{3\rho} \left[ \partial_3 g_{3\rho} + \partial_{3\rho} g_{33} - \partial_\rho g_{33} \right]$$  \hspace{1cm} (A.149)

$$= \frac{1}{2} g^{33} \left[ \partial_3 g_{33} + \partial_3 g_{33} - \partial_3 g_{33} \right]$$  \hspace{1cm} (A.150)

$$= \frac{1}{2} g^{33} [\partial_3 g_{33}]$$  \hspace{1cm} (A.151)

$$= \frac{1}{2} \left[ \frac{1}{S^2 r^2 \sin^2 \theta} \right] [\partial_3 \left( S^2 r^2 \sin^2 \theta \right)]$$  \hspace{1cm} (A.152)

$$= \frac{1}{2} \left[ \frac{1}{S^2 r^2 \sin^2 \theta} \right] [0]$$  \hspace{1cm} (A.153)

$$= 0$$  \hspace{1cm} (A.154)

***
Appendix B  Calculation of the Ricci tensor and the Ricci scalar

B.1  Ricci tensor - general remarks

In the following sections, the components of the Ricci tensor will be calculated. The relation between the Riemann- and the Ricci tensor are based on the following.

\[ R_{\mu\nu} = R'_{\mu\nu} = \partial_{\rho} \Gamma^\rho_{\mu\nu} - \partial_{\nu} \Gamma^\rho_{\mu\rho} + \Gamma^\rho_{\lambda\mu} \Gamma^\lambda_{\nu\rho} - \Gamma^\rho_{\lambda\nu} \Gamma^\lambda_{\mu\rho} \quad (B.1) \]

The approach is for all calculations the same. In a first step, the components of the relevant Ricci-tensor are plugged into the equation given above. In a second step, the terms are summed over \( \rho = 0, 1, 2, 3 \). Then, after omitting the terms containing a Christoffel symbol which is 0, the relevant terms are summed over \( \lambda = 0, 1, 2, 3 \).

Please note that the indices \( t \) may be denoted as 0, \( r \) as 1, \( \theta \) as 2 and \( \varphi \) as 3 and vice versa.

B.2  Ricci tensor - the \( tt \)-component

\[ R_{tt} = \partial_{\rho} \Gamma^\rho_{00} - \partial_{0} \Gamma^\rho_{0\rho} + \Gamma^\rho_{\lambda0} \Gamma^\lambda_{00} - \Gamma^\rho_{\lambda0} \Gamma^\lambda_{0\rho} \quad (B.2) \]

\[ = -\partial_{0} \Gamma^0_{00} + \partial_{1} \Gamma^1_{00} + \partial_{2} \Gamma^2_{00} + \partial_{3} \Gamma^3_{00} \]
\[ - \partial_{0} \Gamma^0_{01} - \partial_{1} \Gamma^1_{01} - \partial_{2} \Gamma^2_{01} - \partial_{3} \Gamma^3_{01} \]
\[ + \Gamma^0_{01} \Gamma^1_{00} + \Gamma^1_{01} \Gamma^2_{00} + \Gamma^2_{01} \Gamma^3_{00} + \Gamma^3_{01} \Gamma^4_{00} \]
\[ - \Gamma^0_{00} \Gamma^1_{01} - \Gamma^1_{00} \Gamma^2_{01} - \Gamma^2_{00} \Gamma^3_{01} - \Gamma^3_{00} \Gamma^4_{01} \quad (B.3) \]

\[ = -\partial_{0} \Gamma^0_{00} + \partial_{1} \Gamma^1_{00} + \partial_{2} \Gamma^2_{00} + \partial_{3} \Gamma^3_{00} \]
\[ + \Gamma^0_{01} \Gamma^1_{00} + \Gamma^1_{01} \Gamma^2_{00} + \Gamma^2_{01} \Gamma^3_{00} + \Gamma^3_{01} \Gamma^4_{00} \]
\[ - \Gamma^0_{00} \Gamma^1_{01} - \Gamma^1_{00} \Gamma^2_{01} - \Gamma^2_{00} \Gamma^3_{01} - \Gamma^3_{00} \Gamma^4_{01} \quad (B.4) \]

\[ = -\partial_{0} \Gamma^0_{00} + \partial_{1} \Gamma^1_{00} + \partial_{2} \Gamma^2_{00} + \partial_{3} \Gamma^3_{00} \]
\[ + \Gamma^0_{01} \Gamma^1_{00} + \Gamma^1_{01} \Gamma^2_{00} + \Gamma^2_{01} \Gamma^3_{00} + \Gamma^3_{01} \Gamma^4_{00} \]
\[ - \Gamma^0_{00} \Gamma^1_{01} - \Gamma^1_{00} \Gamma^2_{01} - \Gamma^2_{00} \Gamma^3_{01} - \Gamma^3_{00} \Gamma^4_{01} \quad (B.5) \]

\[ = -\partial_{0} \Gamma^0_{00} + \partial_{1} \Gamma^1_{00} + \partial_{2} \Gamma^2_{00} + \partial_{3} \Gamma^3_{00} \]
\[ + \Gamma^0_{01} \Gamma^1_{00} + \Gamma^1_{01} \Gamma^2_{00} + \Gamma^2_{01} \Gamma^3_{00} + \Gamma^3_{01} \Gamma^4_{00} \]
\[ - \Gamma^0_{00} \Gamma^1_{01} - \Gamma^1_{00} \Gamma^2_{01} - \Gamma^2_{00} \Gamma^3_{01} - \Gamma^3_{00} \Gamma^4_{01} \quad (B.6) \]

\[ = -3\partial_{0} \left( \frac{\dot{S}}{c} \right) + 3 \frac{1}{c} \frac{\dot{S}}{S} \frac{\dot{\varphi}}{\varphi} - 3 \frac{1}{c} \frac{\dot{S}}{S} \frac{\dot{\varphi}}{\varphi} \quad (B.7) \]
B.3 Ricci tensor - the $rr$-component

$$R_{rr} = \partial_\mu \Gamma^\mu - \partial_1 \Gamma^1_1 + \Gamma^\nu_\mu \Gamma^\mu_\nu - \Gamma^\rho_\nu \Gamma^\nu_\rho$$  \hspace{1cm} (B.10)

$$= \partial_0 \Gamma^0_{11} + \partial_1 \Gamma^1_{11} + \partial_2 \Gamma^2_{11} + \partial_3 \Gamma^3_{11}$$
$$- \partial_0 \Gamma^0_{10} - \partial_1 \Gamma^1_{11} - \partial_1 \Gamma^1_{12} - \partial_1 \Gamma^1_{13}$$
$$+ \Gamma^0_{01} \Gamma^1_{11} + \Gamma^1_{11} \Gamma^1_{11} + \Gamma^1_{21} \Gamma^1_{11} + \Gamma^1_{31} \Gamma^1_{11}$$
$$- \Gamma^0_{11} \Gamma^1_{01} - \Gamma^1_{11} \Gamma^1_{11} - \Gamma^1_{21} \Gamma^1_{21} - \Gamma^1_{31} \Gamma^1_{31}$$  \hspace{1cm} (B.11)

$$= \partial_0 \Gamma^0_{11} + \partial_1 \Gamma^1_{11}$$
$$- \partial_1 \Gamma^1_{11} - \partial_1 \Gamma^2_1 - \partial_1 \Gamma^3_1$$
$$+ \Gamma^0_{01} \Gamma^1_{11} + \Gamma^1_{11} \Gamma^1_{11} + \Gamma^1_{21} \Gamma^1_{11} + \Gamma^1_{31} \Gamma^1_{11}$$
$$- \Gamma^0_{11} \Gamma^1_{01} - \Gamma^1_{11} \Gamma^1_{11} - \Gamma^1_{21} \Gamma^1_{21} - \Gamma^1_{31} \Gamma^1_{31}$$  \hspace{1cm} (B.12)

$$= \partial_0 \Gamma^0_{11} + \partial_1 \Gamma^1_{11}$$
$$- \partial_1 \Gamma^1_{11} - \partial_1 \Gamma^2_1 - \partial_1 \Gamma^3_1$$
$$+ \Gamma^0_{01} \Gamma^1_{11} + \Gamma^1_{11} \Gamma^1_{11} + \Gamma^1_{21} \Gamma^1_{11} + \Gamma^1_{31} \Gamma^1_{11}$$
$$+ \Gamma^0_{01} \Gamma^0_{11} + \Gamma^1_{11} \Gamma^1_{11} + \Gamma^1_{21} \Gamma^1_{11} + \Gamma^1_{31} \Gamma^1_{11}$$
$$- \Gamma^0_{11} \Gamma^1_{01} - \Gamma^1_{11} \Gamma^1_{11} - \Gamma^1_{21} \Gamma^1_{21} - \Gamma^1_{31} \Gamma^1_{31}$$  \hspace{1cm} (B.13)

$$= \partial_0 \Gamma^0_{11}$$
$$- \partial_1 \Gamma^1_{11}$$
$$+ \Gamma^0_{01} \Gamma^1_{11} + \Gamma^1_{11} \Gamma^1_{11} + \Gamma^1_{21} \Gamma^1_{11} + \Gamma^1_{31} \Gamma^1_{11}$$
$$+ \Gamma^2_{11} \Gamma^1_{11} + \Gamma^3_{11} \Gamma^1_{11}$$  \hspace{1cm} (B.14)
\[ R_{\theta\theta} = \frac{1}{c^2} \left( \frac{T^2 S^2}{S^2} + \frac{3}{S} \frac{\dot{T}}{T} \frac{\dot{S}}{S} + \frac{2}{T^2 S^2} + \frac{2c^2 k}{1-kr^2} \right) \frac{1}{1-kr^2} \]

(B.16)

\[ = \frac{T^2 S^2}{c^2} \left( \frac{\dot{S}}{S} + \frac{3}{T} \frac{\dot{T}}{S} + \frac{2}{T^2 S^2} + \frac{2c^2 k}{1-kr^2} \right) \frac{1}{1-kr^2} \]

(B.17)

\[ B.4 \quad \text{Ricci tensor - the } \theta\theta\text{-component} \]

\[ R_{\theta\theta} = \partial_{\rho} \Gamma_{\theta}^{\rho} - \partial_{\theta} \Gamma_{\rho}^{\theta} + \Gamma_{\theta\rho}^{\nu} \Gamma_{\nu}^{\theta} - \Gamma_{\theta\nu}^{\rho} \Gamma_{\nu}^{\theta} \]

(B.19)

\[ = \partial_{\theta} \Gamma_{\theta}^{\theta} + \partial_{\theta} \Gamma_{\theta}^{\rho} + \partial_{\theta} \Gamma_{\rho}^{\theta} + \partial_{\theta} \Gamma_{\theta}^{\theta} \\
- \partial_{\theta} \partial_{\rho} \Gamma_{\theta}^{\theta} - \partial_{\theta} \partial_{\rho} \Gamma_{\theta}^{\theta} - \partial_{\theta} \partial_{\rho} \Gamma_{\theta}^{\theta} \]

(B.20)

\[ = \partial_{\theta} \partial_{\theta} \Gamma_{\theta}^{\theta} + \partial_{\theta} \partial_{\theta} \Gamma_{\theta}^{\theta} + \partial_{\theta} \partial_{\theta} \Gamma_{\theta}^{\theta} \]

(B.21)
Cosmological model with variable time flow

B.5 Ricci tensor - the $\varphi\varphi$-component

\[ R_{\varphi\varphi} = \partial_{\rho}\Gamma_{\varphi 33}^{\rho} - \partial_{3}\Gamma_{\varphi 3\rho}^{\rho} + \Gamma_{\varphi \rho 3}^{\rho} - \Gamma_{A3}^{\rho} \Gamma_{\rho 3}^{A} \]  

(B.28)
\[ = \dot{\alpha} \Gamma_{33} + \dot{\alpha} \Gamma_{13} + \dot{\alpha} \Gamma_{23} + \dot{\alpha} \Gamma_{33} \]
\[- \dot{\alpha} \Gamma_{30} - \dot{\alpha} \Gamma_{10} - \dot{\alpha} \Gamma_{20} - \dot{\alpha} \Gamma_{30} \]
\[+ \Gamma_{00} \Gamma_{33} + \Gamma_{01} \Gamma_{13} + \Gamma_{02} \Gamma_{23} \]
\[+ \Gamma_{10} \Gamma_{33} + \Gamma_{11} \Gamma_{13} + \Gamma_{12} \Gamma_{23} \]
\[+ \Gamma_{02} \Gamma_{23} + \Gamma_{21} \Gamma_{23} + \Gamma_{22} \Gamma_{23} \]
\[+ \Gamma_{03} \Gamma_{23} + \Gamma_{13} \Gamma_{13} + \Gamma_{23} \Gamma_{23} \]
\[+ \Gamma_{03} \Gamma_{33} + \Gamma_{04} \Gamma_{33} - \Gamma_{23} \Gamma_{23} \]
\[- \Gamma_{03} \Gamma_{33} - \Gamma_{04} \Gamma_{33} + \Gamma_{14} \Gamma_{23} \]
\[= \dot{\alpha} \Gamma_{33} + \dot{\alpha} \Gamma_{13} + \dot{\alpha} \Gamma_{23} + \dot{\alpha} \Gamma_{33} \]
\[+ \Gamma_{00} \Gamma_{33} + \Gamma_{01} \Gamma_{13} + \Gamma_{02} \Gamma_{23} \]
\[+ \Gamma_{10} \Gamma_{33} + \Gamma_{11} \Gamma_{13} + \Gamma_{12} \Gamma_{23} \]
\[+ \Gamma_{02} \Gamma_{23} + \Gamma_{21} \Gamma_{23} + \Gamma_{22} \Gamma_{23} \]
\[+ \Gamma_{03} \Gamma_{23} + \Gamma_{13} \Gamma_{13} + \Gamma_{23} \Gamma_{23} \]
\[= \dot{\alpha} \Gamma_{33} + \dot{\alpha} \Gamma_{13} + \dot{\alpha} \Gamma_{23} + \dot{\alpha} \Gamma_{33} \]
\[+ \Gamma_{00} \Gamma_{33} + \Gamma_{01} \Gamma_{13} + \Gamma_{02} \Gamma_{23} \]
\[+ \Gamma_{10} \Gamma_{33} + \Gamma_{11} \Gamma_{13} + \Gamma_{12} \Gamma_{23} \]
\[+ \Gamma_{02} \Gamma_{23} + \Gamma_{21} \Gamma_{23} + \Gamma_{22} \Gamma_{23} \]
\[+ \Gamma_{03} \Gamma_{23} + \Gamma_{13} \Gamma_{13} + \Gamma_{23} \Gamma_{23} \]
\[+ \Gamma_{03} \Gamma_{33} + \Gamma_{04} \Gamma_{33} - \Gamma_{23} \Gamma_{23} \]
\[= \dot{\alpha} \left( \frac{1}{c} \dot{T}^2 \dot{S} \dot{S} r^2 \sin^2 \theta \right) + \dot{\alpha} \left( \frac{c}{r} \left( k r^2 - 1 \right) \sin^2 \theta \right) \]
\[+ \frac{1}{c} \frac{\dot{T}}{T} \frac{1}{c} \dot{T}^2 \dot{S} \dot{S} r^2 \sin^2 \theta + \frac{1}{c} \frac{\dot{S}}{S} \frac{1}{c} \dot{T}^2 \dot{S} \dot{S} r^2 \sin^2 \theta \]
\[+ \frac{1}{c} \frac{\dot{r}}{r} \frac{1}{c} \dot{T}^2 \dot{S} \dot{S} r^2 \sin^2 \theta + \frac{1}{c} \frac{\dot{T}}{T} \frac{1}{c} \dot{T}^2 \dot{S} \dot{S} r^2 \sin^2 \theta \]
\[= \frac{1}{c^2} \left( 2 \dot{T} \dot{T} \dot{S} \dot{S} + \dot{T}^2 \dot{S} \dot{S} + \dot{T}^2 \dot{S} \dot{S} \right) r^2 \sin^2 \theta \]
\[+ \left( 3 k r^2 - 1 \right) \sin^2 \theta \]
\[- \cos^2 \theta + \sin^2 \theta \]
\[+ \frac{1}{c} \left( \dot{T} \dot{T} \dot{S} \dot{S} + 2 \dot{T}^2 \dot{S} \dot{S} \right) r^2 \sin^2 \theta \]
\[- k r^2 \sin^2 \theta + \left( k r^2 - 1 \right) \sin^2 \theta \]
\[- \frac{1}{c^2} \dot{T}^2 \dot{S} \dot{S} r^2 \sin^2 \theta - \left( k r^2 - 1 \right) \sin^2 \theta + \cos^2 \theta \]
\[= \frac{1}{c^2} \left( \dot{T}^2 \dot{S} \dot{S} + 3 \dot{T} \dot{T} \dot{S} \dot{S} + 2 \dot{T}^2 \dot{S} \dot{S} \right) r^2 \sin^2 \theta \]
\[+ \left( 2 k r^2 - 1 \right) \sin^2 \theta \]
\[+ \sin^2 \theta \]
\[\frac{1}{c^2} \left( \mathcal{T}^2 \dot{\mathcal{S}} \ddot{\mathcal{S}} + 3 \mathcal{T} \dot{\mathcal{T}} \ddot{\mathcal{S}} + 2 \mathcal{T}^2 \dot{\mathcal{S}}^2 + 2c^2 k \right) r^2 \sin^2 \theta \]  
(B.35)

\[\frac{\dot{\mathcal{T}}^2 \mathcal{S}^2}{c^2} \left( \frac{\ddot{\mathcal{S}}}{\mathcal{S}} + 3 \frac{\dot{\mathcal{T}} \dot{\mathcal{S}}}{\mathcal{S}} + \frac{2 \dot{\mathcal{S}}^2}{\mathcal{T}^2 \mathcal{S}^2} \right) r^2 \sin^2 \theta \]  
(B.36)

### B.6 Ricci tensor - the tr- and rt-components

\[ R_{tr} = R_{rt} = \partial_t \Gamma_{\rho 0}^\rho - \partial_\rho \Gamma_{0 \rho}^\rho + \Gamma_{\nu \rho}^\nu \Gamma_{\rho 0}^\nu - \Gamma_{\nu \nu}^\nu \Gamma_{\rho 0}^\rho \]  
(B.37)

\[ = \partial_t \Gamma_{0 0}^0 + \partial_t \Gamma_{0 1}^1 + \partial_t \Gamma_{0 2}^2 + \partial_t \Gamma_{0 3}^3 \]  
- \( \partial_t \Gamma_{0 0}^0 - \partial_t \Gamma_{0 1}^1 - \partial_t \Gamma_{0 2}^2 - \partial_t \Gamma_{0 3}^3 \)  
+ \( \Gamma_{1 t}^1 \Gamma_{0 0}^1 + \Gamma_{1 t}^1 \Gamma_{0 1}^1 + \Gamma_{1 t}^1 \Gamma_{2 0}^2 + \Gamma_{1 t}^1 \Gamma_{3 0}^3 \)  
- \( \Gamma_{1 t}^1 \Gamma_{1 0}^0 - \Gamma_{1 t}^1 \Gamma_{1 0}^1 - \Gamma_{1 t}^1 \Gamma_{1 0}^2 - \Gamma_{1 t}^1 \Gamma_{1 0}^3 \)  
+ \( \Gamma_{2 e}^2 \Gamma_{0 0}^2 + \Gamma_{2 e}^2 \Gamma_{0 1}^2 + \Gamma_{2 e}^2 \Gamma_{2 0}^3 + \Gamma_{2 e}^2 \Gamma_{3 0}^3 \)  
- \( \Gamma_{2 e}^2 \Gamma_{1 0}^0 - \Gamma_{2 e}^2 \Gamma_{1 0}^1 - \Gamma_{2 e}^2 \Gamma_{1 0}^2 - \Gamma_{2 e}^2 \Gamma_{1 0}^3 \)  
+ \( \Gamma_{3 e}^3 \Gamma_{0 0}^3 + \Gamma_{3 e}^3 \Gamma_{0 1}^3 + \Gamma_{3 e}^3 \Gamma_{2 0}^4 + \Gamma_{3 e}^3 \Gamma_{3 0}^4 \)  
- \( \Gamma_{3 e}^3 \Gamma_{1 0}^0 - \Gamma_{3 e}^3 \Gamma_{1 0}^1 - \Gamma_{3 e}^3 \Gamma_{1 0}^2 - \Gamma_{3 e}^3 \Gamma_{1 0}^3 \)  
(B.38)

\[ = \partial_t \Gamma_{0 0}^0 - \partial_t \Gamma_{0 1}^1 - \partial_t \Gamma_{0 2}^2 - \partial_t \Gamma_{0 3}^3 \]  
+ \( \Gamma_{1 t}^1 \Gamma_{0 0}^1 + \Gamma_{1 t}^1 \Gamma_{0 1}^1 + \Gamma_{1 t}^1 \Gamma_{2 0}^2 + \Gamma_{1 t}^1 \Gamma_{3 0}^3 \)  
- \( \Gamma_{1 t}^1 \Gamma_{1 0}^0 - \Gamma_{1 t}^1 \Gamma_{1 0}^1 - \Gamma_{1 t}^1 \Gamma_{1 0}^2 - \Gamma_{1 t}^1 \Gamma_{1 0}^3 \)  
+ \( \Gamma_{2 e}^2 \Gamma_{0 0}^2 + \Gamma_{2 e}^2 \Gamma_{0 1}^2 + \Gamma_{2 e}^2 \Gamma_{2 0}^3 + \Gamma_{2 e}^2 \Gamma_{3 0}^3 \)  
- \( \Gamma_{2 e}^2 \Gamma_{1 0}^0 - \Gamma_{2 e}^2 \Gamma_{1 0}^1 - \Gamma_{2 e}^2 \Gamma_{1 0}^2 - \Gamma_{2 e}^2 \Gamma_{1 0}^3 \)  
+ \( \Gamma_{3 e}^3 \Gamma_{0 0}^3 + \Gamma_{3 e}^3 \Gamma_{0 1}^3 + \Gamma_{3 e}^3 \Gamma_{2 0}^4 + \Gamma_{3 e}^3 \Gamma_{3 0}^4 \)  
- \( \Gamma_{3 e}^3 \Gamma_{1 0}^0 - \Gamma_{3 e}^3 \Gamma_{1 0}^1 - \Gamma_{3 e}^3 \Gamma_{1 0}^2 - \Gamma_{3 e}^3 \Gamma_{1 0}^3 \)  
(B.39)

\[ = \partial_t \Gamma_{0 0}^0 + \partial_t \Gamma_{0 1}^1 + \partial_t \Gamma_{0 2}^2 + \partial_t \Gamma_{0 3}^3 \]  
- \( \partial_t \Gamma_{0 0}^0 - \partial_t \Gamma_{0 1}^1 - \partial_t \Gamma_{0 2}^2 - \partial_t \Gamma_{0 3}^3 \)  
+ \( \Gamma_{1 t}^1 \Gamma_{0 0}^1 + \Gamma_{1 t}^1 \Gamma_{0 1}^1 + \Gamma_{1 t}^1 \Gamma_{2 0}^2 + \Gamma_{1 t}^1 \Gamma_{3 0}^3 \)  
- \( \Gamma_{1 t}^1 \Gamma_{1 0}^0 - \Gamma_{1 t}^1 \Gamma_{1 0}^1 - \Gamma_{1 t}^1 \Gamma_{1 0}^2 - \Gamma_{1 t}^1 \Gamma_{1 0}^3 \)  
+ \( \Gamma_{2 e}^2 \Gamma_{0 0}^2 + \Gamma_{2 e}^2 \Gamma_{0 1}^2 + \Gamma_{2 e}^2 \Gamma_{2 0}^3 + \Gamma_{2 e}^2 \Gamma_{3 0}^3 \)  
- \( \Gamma_{2 e}^2 \Gamma_{1 0}^0 - \Gamma_{2 e}^2 \Gamma_{1 0}^1 - \Gamma_{2 e}^2 \Gamma_{1 0}^2 - \Gamma_{2 e}^2 \Gamma_{1 0}^3 \)  
+ \( \Gamma_{3 e}^3 \Gamma_{0 0}^3 + \Gamma_{3 e}^3 \Gamma_{0 1}^3 + \Gamma_{3 e}^3 \Gamma_{2 0}^4 + \Gamma_{3 e}^3 \Gamma_{3 0}^4 \)  
- \( \Gamma_{3 e}^3 \Gamma_{1 0}^0 - \Gamma_{3 e}^3 \Gamma_{1 0}^1 - \Gamma_{3 e}^3 \Gamma_{1 0}^2 - \Gamma_{3 e}^3 \Gamma_{1 0}^3 \)  
(B.40)

\[ = \partial_t \left( \frac{\dot{\mathcal{S}}}{c} \right) \left( \frac{1}{\mathcal{S}} \right) \]  
- \( \partial_t \left( \frac{\dot{\mathcal{T}}}{c} \right) \)  
- \( \partial_t \left( \frac{\dot{\mathcal{T}}}{c} \right) \)  
- \( \partial_t \left( \frac{\dot{\mathcal{T}}}{c} \right) \)  
+ \( \partial_t \left( \frac{\dot{\mathcal{T}}}{c} \right) \)  
- \( \partial_t \left( \frac{\dot{\mathcal{T}}}{c} \right) \)  
- \( \partial_t \left( \frac{\dot{\mathcal{T}}}{c} \right) \)  
+ \( \partial_t \left( \frac{\dot{\mathcal{T}}}{c} \right) \)  
- \( \partial_t \left( \frac{\dot{\mathcal{T}}}{c} \right) \)  
(B.41)

\[ = \partial_t \left( \frac{\dot{\mathcal{S}}}{c} \right) \]  
- \( \partial_t \left( \frac{\dot{\mathcal{T}}}{c} \right) \)  
- \( \partial_t \left( \frac{\dot{\mathcal{T}}}{c} \right) \)  
+ \( \partial_t \left( \frac{\dot{\mathcal{T}}}{c} \right) \)  
+ \( \partial_t \left( \frac{\dot{\mathcal{T}}}{c} \right) \)  
- \( \partial_t \left( \frac{\dot{\mathcal{T}}}{c} \right) \)  
+ \( \partial_t \left( \frac{\dot{\mathcal{T}}}{c} \right) \)  
- \( \partial_t \left( \frac{\dot{\mathcal{T}}}{c} \right) \)  
- \( \partial_t \left( \frac{\dot{\mathcal{T}}}{c} \right) \)  
(B.42)
B.7 Ricci tensor - the $t\theta$-and $\theta t$-components

\[ R_{t\theta} = R_{\theta t} = \partial_\rho \Gamma^\rho_0^0 - \partial_2 \Gamma^0_0^0 + \Gamma^\rho_0^0 \Gamma^\rho_0^2 - \Gamma^\rho_0^2 \Gamma^\rho_0^0 \]  

(B.44)

\[ \begin{aligned}
&= \partial_1 \Gamma^1_0^2 + \partial_1 \Gamma^0_0^1 + \partial_2 \Gamma^2_0^2 + \partial_3 \Gamma^3_0^3 \\
&- \partial_2 \Gamma^0_0^0 - \partial_2 \Gamma^1_0^1 - \partial_2 \Gamma^2_0^2 - \partial_2 \Gamma^3_0^3 \\
&+ \Gamma^0_0^1 \Gamma^1_0^1 + \Gamma^2_0^3 \Gamma^3_0^3 + \Gamma^3_0^1 \Gamma^1_0^3 + \Gamma^3_0^2 \Gamma^2_0^2 \\
&- \Gamma^2_0^1 \Gamma^1_0^0 - \Gamma^2_0^2 \Gamma^2_0^0 - \Gamma^2_0^3 \Gamma^3_0^0 \\
&= \partial_2 \Gamma^2_0^2 \\
&- \partial_2 \Gamma^0_0^0 - \partial_2 \Gamma^1_0^1 - \partial_2 \Gamma^2_0^2 - \partial_2 \Gamma^3_0^3 \\
&+ \Gamma^3_0^1 \Gamma^1_0^1 + \Gamma^3_0^2 \Gamma^2_0^2 + \Gamma^3_0^3 \Gamma^3_0^3 \\
&- \Gamma^3_0^0 \Gamma^0_0^0 - \Gamma^3_0^1 \Gamma^1_0^1 - \Gamma^3_0^2 \Gamma^2_0^2 - \Gamma^3_0^3 \Gamma^3_0^3 \\
&= \partial_2 \Gamma^2_0^2 \\
&- \partial_2 \Gamma^0_0^0 - \partial_2 \Gamma^1_0^1 - \partial_2 \Gamma^2_0^2 - \partial_2 \Gamma^3_0^3 \\
&+ \Gamma^3_0^2 \Gamma^2_0^2 - \Gamma^3_0^3 \Gamma^3_0^3 \\
&= \partial_2 \left( \frac{1}{c} \frac{\dot{S}}{S} \right) \\
&- \partial_2 \left( \frac{1}{c} \frac{\dot{T}}{T} \right) - \partial_2 \left( \frac{1}{c} \frac{\dot{S}}{S} \right) - \partial_2 \left( \frac{1}{c} \frac{\dot{S}}{S} \right) \\
&+ \cot \theta \left( \frac{1}{c} \frac{\dot{S}}{S} \right) - \cot \theta \left( \frac{1}{c} \frac{\dot{S}}{S} \right) \\
&= 0 \\
\end{aligned} \]

(B.45)

(B.46)

(B.47)

(B.48)

(B.49)

(B.50)
B.8 Ricci tensor - the $t\phi$-and $\varphi t$-components

\[
R_{t\phi} = R_{\varphi t} = \partial_\rho \Gamma^\rho_{03} - \partial_3 \Gamma^\rho_{0\rho} + \Gamma^\rho_{\lambda \rho} \Gamma^\lambda_{\rho 03} - \Gamma^\rho_{\lambda 0} \Gamma^\lambda_{\rho 30} \quad (B.51)
\]

\[
= \partial_0 \Gamma^0_{03} + \partial_1 \Gamma^1_{03} + \partial_2 \Gamma^2_{03} + \partial_3 \Gamma^3_{03} - \partial_0 \Gamma^0_{30} - \partial_1 \Gamma^1_{30} - \partial_2 \Gamma^2_{30} - \partial_3 \Gamma^3_{30} + \Gamma^0_{40} \Gamma^4_{03} + \Gamma^1_{41} \Gamma^4_{03} + \Gamma^2_{42} \Gamma^4_{03} + \Gamma^3_{43} \Gamma^4_{03} - \Gamma^0_{40} \Gamma^4_{30} - \Gamma^1_{41} \Gamma^4_{30} - \Gamma^2_{42} \Gamma^4_{30} - \Gamma^3_{43} \Gamma^4_{30} \quad (B.52)
\]

\[
= \partial_2 \Gamma^2_{02} - \partial_2 \Gamma^0_{02} - \partial_2 \Gamma^1_{01} - \partial_2 \Gamma^2_{02} - \partial_2 \Gamma^3_{03} + \Gamma^3_{23} \Gamma^2_{02} - \Gamma^3_{32} \Gamma^3_{30} \quad (B.53)
\]

\[
= \partial_2 \Gamma^2_{02} - \partial_2 \Gamma^0_{02} - \partial_2 \Gamma^1_{01} - \partial_2 \Gamma^2_{02} - \partial_2 \Gamma^3_{03} + \Gamma^2_{11} \Gamma^0_{22} + \Gamma^2_{33} \Gamma^1_{22} - \Gamma^3_{12} \Gamma^0_{23} - \Gamma^3_{12} \Gamma^1_{23} - \Gamma^3_{12} \Gamma^2_{23} \quad (B.54)
\]

\[
= \partial_2 \Gamma^2_{02} - \partial_2 \Gamma^0_{02} - \partial_2 \Gamma^1_{01} - \partial_2 \Gamma^2_{02} - \partial_2 \Gamma^3_{03} + \Gamma^3_{23} \Gamma^2_{02} - \Gamma^3_{32} \Gamma^3_{30} \quad (B.55)
\]

\[
= \partial_2 \left( \frac{1}{c} \dot{S} \right) - \partial_2 \left( \frac{1}{c} \dot{\mathcal{T}} \right) - \partial_2 \left( \frac{1}{c} \dot{\mathcal{S}} \right) - \partial_2 \left( \frac{1}{c} \dot{\mathcal{T}} \right) + \cot \theta \frac{1}{c} \dot{\mathcal{S}} - \cot \theta \frac{1}{c} \dot{\mathcal{T}} \quad (B.56)
\]

\[
= 0 \quad (B.57)
\]
B.9 Ricci tensor - the $r \varphi$-and $\varphi r$-components

\[ R_{r \varphi} = R_{\varphi r} = \partial_\rho \Gamma^\rho_{12} - \partial_1 \Gamma^\rho_{\rho \rho} + \Gamma^\rho_{\rho \rho} \Gamma^1_{12} - \Gamma^\rho_{\rho \rho} \Gamma^2_{12} \]  
\[ (B.58) \]

\[ = \partial_3 \Gamma^0_{13} + \partial_1 \Gamma^1_{13} + \partial_2 \Gamma^2_{13} + \partial_3 \Gamma^3_{13} \]
\[ - \partial_3 \Gamma^0_{10} - \partial_1 \Gamma^1_{11} - \partial_2 \Gamma^2_{12} - \partial_3 \Gamma^3_{13} \]
\[ + \Gamma^0_{01} \Gamma^1_{13} + \Gamma^0_{13} + \Gamma^0_{02} \Gamma^2_{13} + \Gamma^0_{03} \Gamma^3_{13} \]
\[ + \Gamma^0_{10} \Gamma^1_{13} + \Gamma^1_{11} + \Gamma^2_{12} \Gamma^3_{13} + \Gamma^3_{13} \Gamma^4_{13} \]
\[ + \Gamma^0_{20} \Gamma^1_{13} + \Gamma^1_{21} + \Gamma^2_{22} \Gamma^3_{13} + \Gamma^3_{23} \Gamma^2_{13} \]
\[ + \Gamma^0_{30} \Gamma^1_{13} + \Gamma^1_{31} + \Gamma^2_{32} \Gamma^3_{13} + \Gamma^3_{33} \Gamma^3_{13} \]
\[ = \partial_3 \Gamma^0_{13} \]
\[ - \partial_3 \Gamma^0_{10} - \partial_1 \Gamma^1_{11} - \partial_2 \Gamma^2_{12} - \partial_3 \Gamma^3_{13} \]
\[ + \Gamma^0_{01} \Gamma^1_{13} + \Gamma^0_{13} + \Gamma^0_{02} \Gamma^2_{13} + \Gamma^0_{03} \Gamma^3_{13} \]
\[ + \Gamma^0_{10} \Gamma^1_{13} + \Gamma^1_{11} + \Gamma^2_{12} \Gamma^3_{13} + \Gamma^3_{13} \Gamma^4_{13} \]
\[ + \Gamma^0_{20} \Gamma^1_{13} + \Gamma^1_{21} + \Gamma^2_{22} \Gamma^3_{13} + \Gamma^3_{23} \Gamma^2_{13} \]
\[ + \Gamma^0_{30} \Gamma^1_{13} + \Gamma^1_{31} + \Gamma^2_{32} \Gamma^3_{13} + \Gamma^3_{33} \Gamma^3_{13} \]
\[ = -\partial_3 \Gamma^1_{11} - \partial_3 \Gamma^2_{12} \]
\[ (B.59) \]

\[ = -\partial_3 \left( \frac{k r}{1 - k r^2} \right) - \partial_3 \frac{1}{r} \]
\[ (B.60) \]

\[ = 0 \]
\[ (B.61) \]

B.9 Ricci tensor - the $r \varphi$-and $\varphi r$-components

\[ R_{r \varphi} = R_{\varphi r} = \partial_\rho \Gamma^\rho_{12} - \partial_1 \Gamma^\rho_{\rho \rho} + \Gamma^\rho_{\rho \rho} \Gamma^1_{12} - \Gamma^\rho_{\rho \rho} \Gamma^2_{12} \]  
\[ (B.58) \]

\[ = \partial_3 \Gamma^0_{12} + \partial_1 \Gamma^1_{12} + \partial_2 \Gamma^2_{12} + \partial_3 \Gamma^3_{12} \]
\[ - \partial_3 \Gamma^0_{10} - \partial_1 \Gamma^1_{11} - \partial_2 \Gamma^2_{12} - \partial_3 \Gamma^3_{12} \]
\[ + \Gamma^0_{01} \Gamma^1_{12} + \Gamma^0_{12} + \Gamma^0_{02} \Gamma^2_{12} + \Gamma^0_{03} \Gamma^3_{12} \]
\[ + \Gamma^0_{10} \Gamma^1_{12} + \Gamma^1_{11} + \Gamma^2_{12} \Gamma^3_{12} + \Gamma^3_{12} \Gamma^4_{12} \]
\[ + \Gamma^0_{20} \Gamma^1_{12} + \Gamma^1_{21} + \Gamma^2_{22} \Gamma^3_{12} + \Gamma^3_{23} \Gamma^2_{12} \]
\[ + \Gamma^0_{30} \Gamma^1_{12} + \Gamma^1_{31} + \Gamma^2_{32} \Gamma^3_{12} + \Gamma^3_{33} \Gamma^3_{12} \]
\[ = \partial_3 \Gamma^0_{12} \]
\[ - \partial_3 \Gamma^0_{10} - \partial_1 \Gamma^1_{11} - \partial_2 \Gamma^2_{12} - \partial_3 \Gamma^3_{12} \]
\[ + \Gamma^0_{01} \Gamma^1_{12} + \Gamma^0_{12} + \Gamma^0_{02} \Gamma^2_{12} + \Gamma^0_{03} \Gamma^3_{12} \]
\[ + \Gamma^0_{10} \Gamma^1_{12} + \Gamma^1_{11} + \Gamma^2_{12} \Gamma^3_{12} + \Gamma^3_{12} \Gamma^4_{12} \]
\[ + \Gamma^0_{20} \Gamma^1_{12} + \Gamma^1_{21} + \Gamma^2_{22} \Gamma^3_{12} + \Gamma^3_{23} \Gamma^2_{12} \]
\[ + \Gamma^0_{30} \Gamma^1_{12} + \Gamma^1_{31} + \Gamma^2_{32} \Gamma^3_{12} + \Gamma^3_{33} \Gamma^3_{12} \]
\[ = -\partial_3 \Gamma^1_{12} \]
\[ = -\partial_3 \left( \frac{k r}{1 - k r^2} \right) - \partial_3 \frac{1}{r} \]
\[ = 0 \]
\begin{equation}
\partial_2 \Gamma_{12}^3
\end{equation}
\begin{equation}
- \partial_2 \Gamma_{11}^3 - \partial_2 \Gamma_{12}^3 - \partial_2 \Gamma_{13}^3
\end{equation}
\begin{equation}
+ \Gamma_{00}^0 \Gamma_{12}^0 + \Gamma_{10}^1 \Gamma_{01}^0 + \Gamma_{02}^2 \Gamma_{20}^0 + \Gamma_{03}^3 \Gamma_{30}^0
\end{equation}
\begin{equation}
+ \Gamma_{10}^1 \Gamma_{11}^1 + \Gamma_{12}^2 \Gamma_{21}^1 + \Gamma_{13}^3 \Gamma_{31}^1
\end{equation}
\begin{equation}
+ \Gamma_{20}^2 \Gamma_{21}^2 + \Gamma_{12}^2 \Gamma_{22}^1 + \Gamma_{23}^2 \Gamma_{23}^1
\end{equation}
\begin{equation}
+ \Gamma_{30}^3 \Gamma_{31}^1 + \Gamma_{32}^3 \Gamma_{32}^1 + \Gamma_{33}^3 \Gamma_{33}^1
\end{equation}
\begin{equation}
(B.66)
\end{equation}
\begin{equation}
= - \partial_2 \Gamma_{11}^3 - \partial_2 \Gamma_{13}^3
\end{equation}
\begin{equation}
+ \Gamma_{23}^2 \Gamma_{12}^3
\end{equation}
\begin{equation}
- \Gamma_{32}^3 \Gamma_{31}^3
\end{equation}
\begin{equation}
(B.67)
\end{equation}
\begin{equation}
= - \partial_2 \frac{kr}{1 + kr^2} - \partial_2 \frac{1}{r}
\end{equation}
\begin{equation}
+ \cot \theta \frac{1}{r}
\end{equation}
\begin{equation}
- \cot \theta \frac{1}{r}
\end{equation}
\begin{equation}
(B.68)
\end{equation}
\begin{equation}
= 0
\end{equation}
\begin{equation}
(B.69)
\end{equation}

**B.11 Ricci tensor - the \( \theta \varphi \) and \( \varphi \theta \)-components**

\begin{equation}
R_{\theta \varphi} = R_{\varphi \theta} = \partial_\theta \Gamma_{\varphi}^\rho - \partial_\varphi \Gamma_{\theta}^\rho + \Gamma_{\rho \lambda}^\varphi \Gamma_{\theta \lambda}^\lambda - \Gamma_{\rho \lambda}^\varphi \Gamma_{\theta \lambda}^\lambda
\end{equation}
\begin{equation}
(B.70)
\end{equation}
\begin{equation}
= \partial_\theta \Gamma_{0}^0 \Gamma_{23}^0 + \partial_\theta \Gamma_{1}^0 \Gamma_{23}^1 + \partial_\theta \Gamma_{2}^0 \Gamma_{23}^2 + \partial_\theta \Gamma_{3}^0 \Gamma_{23}^3
\end{equation}
\begin{equation}
- \partial_\varphi \Gamma_{0}^0 \Gamma_{23}^0 - \partial_\varphi \Gamma_{1}^0 \Gamma_{23}^1 - \partial_\varphi \Gamma_{2}^0 \Gamma_{23}^2 - \partial_\varphi \Gamma_{3}^0 \Gamma_{23}^3
\end{equation}
\begin{equation}
+ \Gamma_{00}^0 \Gamma_{23}^0 + \Gamma_{01}^1 \Gamma_{12}^1 + \Gamma_{02}^2 \Gamma_{22}^1 + \Gamma_{03}^3 \Gamma_{32}^1
\end{equation}
\begin{equation}
+ \Gamma_{20}^2 \Gamma_{23}^2 + \Gamma_{12}^2 \Gamma_{22}^1 + \Gamma_{23}^2 \Gamma_{23}^2
\end{equation}
\begin{equation}
+ \Gamma_{30}^3 \Gamma_{32}^2 + \Gamma_{32}^3 \Gamma_{32}^2 + \Gamma_{33}^3 \Gamma_{33}^2
\end{equation}
\begin{equation}
(B.71)
\end{equation}
\begin{equation}
= \Gamma_{00}^0 \Gamma_{01}^0 \Gamma_{23}^0 + \Gamma_{01}^1 \Gamma_{01}^0 \Gamma_{23}^1 + \Gamma_{02}^2 \Gamma_{02}^0 \Gamma_{23}^2 + \Gamma_{03}^3 \Gamma_{03}^0 \Gamma_{23}^3
\end{equation}
\begin{equation}
+ \Gamma_{10}^1 \Gamma_{11}^1 \Gamma_{23}^1 + \Gamma_{20}^2 \Gamma_{12}^2 \Gamma_{23}^2 + \Gamma_{30}^3 \Gamma_{13}^3 \Gamma_{23}^3
\end{equation}
\begin{equation}
(B.72)
\end{equation}
\begin{equation}
= 0
\end{equation}
\begin{equation}
(B.73)
\end{equation}
B.12 The non-zero components of the Ricci tensor

\[ R_{tt} = \frac{3}{c^2} \left( -\frac{\ddot{S}}{S} + \frac{T\ddot{S}}{TS} \right) \]  \hspace{1cm} (B.74)

\[ R_{rr} = \frac{T^2 S^2}{c^2} \left( \frac{\ddot{S}}{S} + 3 \frac{T\ddot{S}}{TS} + 2 \frac{\dot{S}^2}{S^2} + \frac{2c^2k}{T^2 S^2} \right) \frac{1}{1 - kr^2} \]  \hspace{1cm} (B.75)

\[ R_{\theta\theta} = \frac{T^2 S^2}{c^2} \left( \frac{\ddot{S}}{S} + 3 \frac{T\ddot{S}}{TS} + 2 \frac{\dot{S}^2}{S^2} + \frac{2c^2k}{T^2 S^2} \right) r^2 \]  \hspace{1cm} (B.76)

\[ R_{\phi\phi} = \frac{T^2 S^2}{c^2} \left( \frac{\ddot{S}}{S} + 3 \frac{T\ddot{S}}{TS} + 2 \frac{\dot{S}^2}{S^2} + \frac{2c^2k}{T^2 S^2} \right) r^2 \sin^2 \theta \]  \hspace{1cm} (B.77)

B.13 The Ricci scalar

The Ricci scalar (or scalar curvature) is the contraction of the Ricci tensor:

\[ R = g^{\mu\nu} R_{\mu\nu} \]  \hspace{1cm} (B.78)

\[ = -T^2 \frac{1}{c^2} \left( -3 \frac{\ddot{S}}{S} + 3 \frac{T\ddot{S}}{TS} \right) \]

\[ + \frac{1}{S^2} \frac{1}{r^2} \left[ \frac{1}{c^2} T^2 S^2 \left( \frac{\ddot{S}}{S} + 3 \frac{T\ddot{S}}{TS} + 2 \frac{\dot{S}^2}{S^2} + \frac{2c^2k}{T^2 S^2} \right) \frac{1}{1 - kr^2} \right] \]

\[ + \frac{1}{S^2} \left( \frac{1}{c^2} T^2 S^2 \left( \frac{\ddot{S}}{S} + 3 \frac{T\ddot{S}}{TS} + 2 \frac{\dot{S}^2}{S^2} + \frac{2c^2k}{T^2 S^2} \right) r^2 \right) \]

\[ + \frac{1}{S^2} \left[ \frac{1}{c^2} T^2 S^2 \left( \frac{\ddot{S}}{S} + 3 \frac{T\ddot{S}}{TS} + 2 \frac{\dot{S}^2}{S^2} + \frac{2c^2k}{T^2 S^2} \right) r^2 \sin^2 \theta \right] \]

\[ = \frac{1}{c^2} T^2 \left( 3 \frac{\ddot{S}}{S} - 3 \frac{T\ddot{S}}{TS} \right) \]  \hspace{1cm} (B.79)

\[ + \frac{1}{c^2} T^2 \left( 3 \frac{\ddot{S}}{S} + 9 \frac{T\ddot{S}}{TS} + 6 \frac{\dot{S}^2}{S^2} \right) + 6 \frac{k}{S^2} \]  \hspace{1cm} (B.80)

\[ = \frac{6T^2}{c^2} \left( \frac{\ddot{S}}{S} + \frac{T\ddot{S}}{TS} + \frac{\dot{S}^2}{S^2} \right) + 6 \frac{k}{S^2} \]  \hspace{1cm} (B.81)
Appendix C  Calculation of the spatial components of the energy-momentum tensor

As outlined in the Main Document, the spatial components of the energy momentum conservation equation for $\nu = 1$ become:

$$\partial_\mu T_{\mu 1} + \Gamma^\mu_{\mu 1} T^{11} + \Gamma^1_{\mu 1} T^{\mu 1} = 0.$$  \hspace{1cm} (C.1)

The first term is then the derivative of the 1, 1-component:

$$\partial_\mu T_{\mu 1} = \partial_1 T^{11} = \frac{1 - kr^2}{S^2}(\partial_1 p) - p \frac{2kr}{S^2}.$$  \hspace{1cm} (C.2)

For the second term, we obtain:

$$\Gamma^\mu_{\mu 1} T^{11} = \Gamma^\mu_{\mu 1} T^{11} = \left( \Gamma^{\nu}_{11} + \Gamma^{12}_{11} + \Gamma^{13}_{11} \right) \rho \frac{1 - kr^2}{S^2}$$

$$= \left( \frac{kr}{1 - kr^2} + \frac{1}{r} + \frac{1}{r} \right) \rho \frac{1 - kr^2}{S^2}$$

$$= \frac{2 - kr^2}{rS^2} \rho.$$  \hspace{1cm} (C.3)

Finally, the third term becomes:

$$\Gamma^1_{\mu 1} T^{\mu 1} = \Gamma^1_{\mu 1} T^{\mu 1} = \frac{k r}{S^2} p + \frac{2(kr^2 - 1)}{S^2 r} p.$$  \hspace{1cm} (C.4)

This now leads to the result:

$$\frac{1 - kr^2}{S^2}(\partial_1 p) = p \left( \frac{2kr}{S^2} - \frac{2 - kr^2}{rS^2} - \frac{kr}{S^2} - \frac{2(kr^2 - 1)}{S^2 r} \right)$$

$$\Rightarrow \partial_2 p = 0.$$  \hspace{1cm} (C.5)

For the spatial components of the energy-momentum conservation equation with $\nu = 2$, we obtain the same results, as it can be shown. We calculate all of the terms, step-by-step:

$$\partial_\mu T_{\mu 2} + \Gamma^\mu_{\mu 2} T^{12} + \Gamma^2_{\mu 2} T^{\mu 2} = 0.$$  \hspace{1cm} (C.7)

The partial derivative is therefore:

$$\partial_\mu T_{\mu 2} = \partial_2 T^{22} = \frac{1}{S^2 r^2}(\partial_2 p).$$  \hspace{1cm} (C.8)

The second term becomes:

$$\Gamma^\mu_{\mu 2} T^{12} = \Gamma^\mu_{\mu 2} T^{22} = \left( \Gamma^3_{23} \right) \rho \frac{1}{S^2 r^2}$$

$$= \cot \theta \frac{1}{S^2 r^2} p.$$  \hspace{1cm} (C.9)
and for the third term we obtain:

\[
\Gamma_{\mu A} T^{\mu A} = \Gamma_{00}^2 T^{00} + \Gamma_{11}^2 T^{11} + \Gamma_{22}^2 T^{22} + \Gamma_{33}^2 T^{33} \\
= \Gamma_{33}^2 T^{33} \\
= \frac{-\sin \theta \cos \theta}{S^2 r^2 \sin^2 \theta} p \\
= \frac{-\cot \theta}{S^2 r^2} p. 
\]  
(C.10)

Finally:

\[
\frac{1}{S^2 r^2} (\partial_3 p) = p \left( -\cot \theta + \cot \theta \right) \\
\Rightarrow \partial_1 p = 0 
\]  
(C.11)

For the sake of completeness, the spatial components with \( \nu = 3 \) shall be set out:

\[
\partial_\mu T^{\mu 3} + \Gamma_{\mu A} T^{33} + \Gamma_{\mu A} T^{\mu 4} = 0
\]  
(C.13)

The partial derivative in direction of \( \nu = 3 \) is:

\[
\partial_\mu T^{\mu 3} = \partial_3 T^{33} \\
= S^2 r^2 \sin^2 \theta (\partial_3 p).
\]  
(C.14)

The second term results in:

\[
\Gamma_{\mu A} T^{33} = \Gamma_{\mu 3} T^{33} \\
= 0
\]  
(C.15)

The third term becomes:

\[
\Gamma_{\mu A} T^{\mu 4} = \Gamma_{00}^3 T^{00} + \Gamma_{11}^3 T^{11} + \Gamma_{22}^3 T^{22} + \Gamma_{33}^3 T^{33} \\
= 0
\]  
(C.16)

Therefore, we obtain:

\[
S^2 r^2 \sin^2 \theta (\partial_3 p) = 0 \\
\Rightarrow \partial_3 p = 0 
\]  
(C.17)

\[***\]