ERRATUM TO “TABLES OF INTEGRAL TRANSFORMS” BY A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER & F. G. TRICOMI (1953), P. 61 (4)

RICHARD J. MATHAR

The table of integral transforms contains on page 61, equation (4), the Fourier cosine integral of an exponential multiplied by a sum of two Parabolic-Cylinder Functions [4]

\[ \int_{0}^{\infty} e^{-x^2/2} \cos(bx) [D_{2\nu-1/2}(x) + D_{2\nu-1/2}(-x)] dx \]

\[ \Rightarrow \sin[(\nu + 1/4)\pi] 2^{1/4 - 2\nu} \sqrt{\pi}^{2\nu - 1/2} e^{-b^2/4}. \]

It also appears in that form in the Gradshteyn-Ryzhik tables [5, (7.741.5.)]. Standard numerical comparison with Riemann Approximations with a few hundred sampling points for finite ranges up to a few hundred and \( b \) and \( \nu \) in ranges around 0.5 demonstrate that this right hand side is not correct. A correct expression is derived here: equations (8) and (12).

The Parabolic-Cylinder Function is rephrased as a difference of two Confluent Hypergeometric Functions [5, (9.240)] [1, 19.12.4] [6, §8.1.1]

\[ D_p(z) = 2^{p/2} e^{-z^2/4} \left[ \frac{\sqrt{\pi}}{\Gamma(1/2)} \right] {}_1 F_1 (-p/2; 1/2; z^2/2) - \frac{\sqrt{2\pi} z}{\Gamma(-p/2)} {}_1 F_1 (1/2; 3/2; z^2/2) \]

in the standard notation [8]. In the even component, the sum of the two Parabolic-Cylinder Functions, one Confluent Hypergeometric Function remains:

\[ D_{2\nu-1/2}(x) + D_{2\nu-1/2}(-x) = 2^{\nu+3/4} \sqrt{\pi} e^{-x^2/4} {}_1 F_1 (-\nu + 1/4; 1/2; x^2/2). \]

The series expansion of the Confluent Hypergeometric Function yields

\[ I_{\alpha,\nu,b} \equiv \int_{0}^{\infty} e^{-\alpha x^2} \cos(bx) [D_{2\nu-1/2}(x) + D_{2\nu-1/2}(-x)] dx = \]

\[ = \int_{0}^{\infty} e^{-\alpha x^2} \cos(bx) \frac{2^{\nu+3/4} \sqrt{\pi}}{\Gamma(3/4 - \nu)} e^{-x^2/4} {}_1 F_1 (-\nu + 1/4; 1/2; x^2/2) dx = \]

\[ = \frac{2^{\nu+3/4} \sqrt{\pi}}{\Gamma(3/4 - \nu)} \sum_{k=0}^{\infty} \int_{0}^{\infty} e^{-(\alpha+1/4)x^2} \cos(bx) \frac{(-\nu + 1/4)_k}{(1/2)_k k!} (x^2/2)^k \]

\[ = \frac{2^{\nu+3/4} \sqrt{\pi}}{\Gamma(3/4 - \nu)} \sum_{k=0}^{\infty} \frac{(-\nu + 1/4)_k}{(1/2)_k k!} \int_{0}^{\infty} e^{-(\alpha+1/4)x^2} \cos(bx) x^{2k}. \]

Date: July 25, 2022.
Interchange of summation and integration leads to integrals that are again Confluent Hypergeometric Functions [5, 3.952.8][3, p. 15 (14)]

(5)
\[ I_{\alpha,\nu,b} = \frac{2^{\nu+3/4} \sqrt{\pi}}{\Gamma(3/4 - \nu)} \sum_{k=0}^{\infty} \frac{(-\nu + 1/4)k}{(1/2)_k k! 2^{k+1}} \frac{\Gamma(\frac{2k+1}{2})}{2(\alpha + 1/4)^{k+1/2}} F_1(k+1/2; 1/2; -\frac{b^2}{4\alpha + 1}) \]
\[ = \frac{2^{\nu-1/4} \Gamma(3/4 - \nu) \sqrt{\alpha + 1/4}}{\Gamma(3/4 - \nu) \sqrt{\alpha + 1/4}} \sum_{k=0}^{\infty} \frac{(-\nu + 1/4)k}{k!} \frac{1}{(2\alpha + 1/2)^k} F_1(k+1/2; 1/2; -\frac{b^2}{4\alpha + 1}). \]

Kummer’s transformation [1, 13.1.27][5, 9.212][7, 13.2.39][6, 6.1.2] demonstrates that these are terminating Hypergeometric Functions: [3, §6.3]

(6)
\[ I_{\alpha,\nu,b} = \frac{2^{\nu-1/4} \Gamma(3/4 - \nu) \sqrt{\alpha + 1/4}}{\Gamma(3/4 - \nu) \sqrt{\alpha + 1/4}} e^{-b^2/(4\alpha + 1)} \sum_{k=0}^{\infty} \frac{(-\nu + 1/4)k}{k!} \frac{1}{(2\alpha + 1/2)^k} F_1(-k; 1/2; -\frac{b^2}{4\alpha + 1}). \]

This sum over Confluent Hypergeometric Functions is (again) a Confluent Hypergeometric Function [2][6, §6.4.3]

(7)
\[ I_{\alpha,\nu,b} = \frac{2^{\nu-1/4} \Gamma(3/4 - \nu) \sqrt{\alpha + 1/4}}{\Gamma(3/4 - \nu) \sqrt{\alpha + 1/4}} e^{-b^2/(4\alpha + 1)} (\frac{\alpha - 1/4}{\alpha + 1/4})^{\nu-1/4} F_1(-\nu+1/4; 1/2; -\frac{b^2}{4\alpha + 1} - \frac{2}{4\alpha - 1}), \]

and our main result is

(8)
\[ I_{\alpha,\nu,b} = \frac{2^{\nu-1/4} \Gamma(3/4 - \nu) \sqrt{\alpha + 1/4}}{\Gamma(3/4 - \nu) \sqrt{\alpha + 1/4}} e^{-b^2/(4\alpha + 1)} \frac{1}{(\alpha - 1/4)^{\nu-1/4}} F_1(-\nu+1/4; 1/2; -\frac{2b^2}{(4\alpha + 1)(4\alpha - 1)}), \]
\[ \alpha > 1/4, \Re \nu > 1/4, b > 0. \]

In particular at \( \alpha = 1/2 \) a correct expression of the right hand side of (1) is

(9)
\[ I_{1/2,\nu,b} = \frac{2^{\nu-1/4} \Gamma(1/4 - \nu) \sqrt{1/4}}{\Gamma(3/4 - \nu) \sqrt{3/4}} e^{-b^2/3} F_1(-\nu + 1/4; 1/2; -\frac{2b^2}{3}) \]
\[ = \frac{\sqrt{2\pi}}{\Gamma(3/4 - \nu)} \frac{2}{3}^{\nu+1/4} e^{-b^2/3} F_1(-\nu + 1/4; 1/2; -\frac{2b^2}{3}). \]

In the limit \( \alpha \to 1/4^+ \) the Confluent Hypergeometric Series in (8) has an asymptotic value of [1, 13.1.5][6, §6.8.2]

(10)
\[ I_{1/4,\nu,b} \to \frac{2^{\nu-1/2} \Gamma(1/4 - \nu) \sqrt{1/2}}{\Gamma(3/4 - \nu) \sqrt{1/2}} e^{-b^2/2} F_1(-\nu + 1/4; 1/2; -\frac{2b^2}{1/4}) \]
\[ = \frac{\sqrt{2\pi}}{\Gamma(3/4 - \nu)} \frac{2}{\Gamma(1/4 + \nu)} e^{-b^2/2} \frac{1}{b^{2\nu - 1/2}}. \]

The reflection formula of the \( \Gamma \)-function yields [1, 6.1.17][7, 5.5.3]

(11)
\[ I_{1/4,\nu,b} = \frac{\sqrt{2\pi}}{\pi} \sin[\pi(\nu + 1/4)] e^{-b^2/2} b^{2\nu-1/2}, \]
so a suitable substitution for both sides (!) of (1) is

(12)
\[ I_{1/4,\nu,b} = \sqrt{2\pi} \sin[\pi(\nu + 1/4)] e^{-b^2/2} b^{2\nu-1/2}, \Re \nu > 1/4, b > 0. \]
References


Max-Planck Inst. Astronomy, Königstuhl 17, 69117 Heidelberg, Germany
Email address: mathar@mpia-hd.mpg.de