A Brief Mathematical Look at
the Dirac Equation

Introduction

The Dirac equation is a linear matrix partial differential equation (or 4-vector linear partial differential equation, or system of linear partial differential equations) in non-negative dependant variable(s) (matrix / 4-vector) which also satisfy the generalized Helmholtz / Klein-Gordon Equation.

1.

The Dirac Equation is generally written:

$$\left(i\sum_{\mu=0}^{3}\gamma^{\mu}\partial_{\mu} - \mathrm{I}m\right)\boldsymbol{\psi} = \mathbf{0}$$

$$where: \quad \sum_{\mu=0}^{3}(\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu}) = 2g^{\mu\nu}\mathrm{I} \quad , \quad g^{\mu\nu} \equiv \delta^{\mu\nu}(-1)^{\left(1-\delta_0^{\mu}\right)}$$

$$and: \quad \gamma^{\mu'} = \mathbf{U}\gamma^{\mu}\mathbf{U}^{-1} \quad , \quad \boldsymbol{\psi}' = \mathbf{U}\boldsymbol{\psi}$$

$$\forall\mathbf{U} \ni \mathbf{U}\mathbf{U}^{-1} = \mathrm{I}$$

$$\boldsymbol{\gamma}^5 \equiv i\boldsymbol{\gamma}^0\boldsymbol{\gamma}^1\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3$$

(alternatively: $\boldsymbol{\gamma}^5 \equiv \boldsymbol{\gamma}^0\boldsymbol{\gamma}^1\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3$)

(is in limited use   )

additional useful definitions:

$$\boldsymbol{\beta} \equiv \boldsymbol{\gamma}^0 \quad , \quad \boldsymbol{\alpha}^k \equiv \boldsymbol{\gamma}^0\boldsymbol{\gamma}^k$$

$$\boldsymbol{\Sigma}^k \equiv -\boldsymbol{\alpha}^k\boldsymbol{\gamma}^5 \quad , \quad k \in \{1,2,3\}$$

$$\therefore \ \Sigma^k = -\boldsymbol{\gamma}^0\boldsymbol{\gamma}^k\boldsymbol{\gamma}^5 = -i\boldsymbol{\gamma}^0\boldsymbol{\gamma}^k\boldsymbol{\gamma}^0\boldsymbol{\gamma}^1\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3$$

$$= i\boldsymbol{\gamma}^0\boldsymbol{\gamma}^0\boldsymbol{\gamma}^k\boldsymbol{\gamma}^1\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3 = i\boldsymbol{\gamma}^k\boldsymbol{\gamma}^1\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3$$

($\boldsymbol{\gamma}^k, \mathrm{I}$ are $2n \times 2n$ identity matrices)

So:

$$\mathbf{0} = \boldsymbol{\gamma}^0\left(i\sum_{\mu=0}^{3}\gamma^{\mu}\partial_{\mu} - \mathrm{I}m\right)\boldsymbol{\psi} = i\sum_{\mu=1}^{3}\boldsymbol{\gamma}^0\gamma^{\mu}\partial_{\mu}\boldsymbol{\psi} + i\gamma^0\boldsymbol{\gamma}^0\partial_0\boldsymbol{\psi} - \boldsymbol{\gamma}^0 m\boldsymbol{\psi}$$

$$\mathbf{0} = i\boldsymbol{\gamma}^0\gamma^{\mu}\partial_{\mu}\boldsymbol{\psi} + i\partial_0\boldsymbol{\psi} - \boldsymbol{\gamma}^0 m\boldsymbol{\psi} = i\boldsymbol{\alpha}^{\mu}\partial_{\mu}\boldsymbol{\psi} + i\mathrm{I}\partial_0\boldsymbol{\psi} - \boldsymbol{\beta}m\boldsymbol{\psi}$$

$$\mathbf{0} = i\boldsymbol{\alpha}\bullet\vec{\nabla}\boldsymbol{\psi} + i\partial_0\boldsymbol{\psi} - \boldsymbol{\beta}m\boldsymbol{\psi} = i\boldsymbol{\alpha}\bullet\vec{\nabla}\boldsymbol{\psi} + i\frac{\partial}{\partial x^0}\boldsymbol{\psi} - \boldsymbol{\beta}m\boldsymbol{\psi}$$

$$\mathbf{0} = i\boldsymbol{\alpha}\bullet\vec{\nabla}\boldsymbol{\psi} + i\frac{\partial}{\partial t}\boldsymbol{\psi} - \boldsymbol{\beta}m\boldsymbol{\psi}$$

$\therefore$

$$i\frac{\partial}{\partial t}\boldsymbol{\psi} = \left(-i\boldsymbol{\alpha}\bullet\vec{\nabla} + \boldsymbol{\beta}m\right)\boldsymbol{\psi}$$

or, defining: $\vec{\mathbf{p}} \equiv -i\vec{\nabla}$, then:

$\mathbf{H} \equiv -\boldsymbol{\alpha}\bullet\vec{\mathbf{p}} + \boldsymbol{\beta}m$ and, so:

$$i\frac{\partial}{\partial t}\boldsymbol{\psi} = \mathbf{H}\boldsymbol{\psi}$$

Just as generalized coordinate systems take form from their relationship to a specific coordinate system (as polar coordinates are defined by its transformation to rectangular cartesian coordinates; as are cylindrical, spherical, parabolic cylindircal, paraboloidal, elliptic cylindrical, prolate spheroidal, ellipsoidal, bipolar, ...) the Dirac equation takes form from a specific choice of representations: γ^{μ}

The Dirac and Weyl representations are inmost common use, but there are a number of others:

The Dirac representation:

$$\gamma_D^0 \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad , \quad \gamma_D^1 \equiv \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_D^2 \equiv \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \quad , \quad \gamma_D^3 \equiv \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\gamma_D^5 = i\boldsymbol{\gamma}_D^0\boldsymbol{\gamma}_D^1\boldsymbol{\gamma}_D^2\boldsymbol{\gamma}_D^3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

The Weyl / chiral representation:

under the transformation:

$$\gamma_W^{\mu} = \mathbf{U}\gamma_D^{\mu}\mathbf{U}^{-1} \Leftrightarrow \gamma_D^{\mu} = \mathbf{U}^{-1}\gamma_W^{\mu}\mathbf{U}$$

$$where:$$

$$\mathbf{U} \equiv \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \quad , \quad \mathbf{U}^{-1} \equiv \sqrt{2} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

or:

$$\mathbf{U} \equiv \begin{pmatrix} \mathrm{I}_2 & \mathrm{I}_2 \\ -\mathrm{I}_2 & \mathrm{I}_2 \end{pmatrix} \quad , \quad \mathbf{U}^{-1} \equiv \begin{pmatrix} \mathrm{I}_2 & -\mathrm{I}_2 \\ \mathrm{I}_2 & \mathrm{I}_2 \end{pmatrix}$$

$$\gamma_W^0 \equiv \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad , \quad \gamma_W^1 \equiv \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_W^2 \equiv \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \quad , \quad \gamma_W^3 \equiv \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\gamma_W^5 = i\gamma_W^0\gamma_W^1\gamma_W^2\gamma_W^3 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The Light-plane representation:

$$\gamma_L^0 \equiv i\gamma_D^0\gamma_D^5 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \quad , \quad \gamma_L^1 \equiv i\gamma_D^1\gamma_D^5 = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}$$

$$\gamma_L^2 \equiv i\gamma_D^2\gamma_D^5 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad , \quad \gamma_L^3 \equiv -i\gamma_D^5 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$\gamma_L^5 = i\gamma_L^0\gamma_L^1\gamma_L^2\gamma_L^3 = i\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

A Majorana representation is a representation where the γ^μ are all purely imaginary, so the Dirac equation becomes purely real:

The Majorana representation (**1**):

$$\gamma_{M1}^0 \equiv i\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \gamma_{M1}^1 \equiv i\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\gamma_{M1}^2 \equiv i\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \gamma_{M1}^3 \equiv i\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\gamma_{M1}^{5} = i\boldsymbol{\gamma}_{M1}^{0}\boldsymbol{\gamma}_{M1}^{1}\boldsymbol{\gamma}_{M1}^{2}\boldsymbol{\gamma}_{M1}^{3} = i\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

The Majorana representation (**2**):

$$\gamma_{M2}^{0} \equiv \gamma_{D}^{0}\gamma_{D}^{2} = i\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \gamma_{M2}^{1} \equiv i\gamma_{D}^{0}\gamma_{D}^{1} = i\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_{M2}^{2} \equiv i\gamma_{D}^{0} = \quad i\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad , \quad \gamma_{M2}^{3} \equiv i\gamma_{D}^{0}\gamma_{D}^{3} = i\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\gamma_{M2}^{5} = i\boldsymbol{\gamma}_{M2}^{0}\boldsymbol{\gamma}_{M2}^{1}\boldsymbol{\gamma}_{M2}^{2}\boldsymbol{\gamma}_{M2}^{3} = i\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

The Majorana representation (**3**):

$$\gamma_{M3}^{0} \equiv i\gamma_{D}^{1} = i\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \gamma_{M3}^{1} \equiv i\gamma_{D}^{0} = i\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\gamma_{M3}^{2} \equiv \gamma_{D}^{2} = i\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \gamma_{M3}^{3} \equiv -i\gamma_{D}^{5} = i\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\gamma_{M3}^{5} = i\boldsymbol{\gamma}_{M3}^{0}\boldsymbol{\gamma}_{M3}^{1}\boldsymbol{\gamma}_{M3}^{2}\boldsymbol{\gamma}_{M3}^{3} = i\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

2.

The **Pauli Matrices** may be defined from these $\gamma - matrices$
(or vice-versa), as:

$$\boldsymbol{\gamma}_D^0 \equiv \sigma^0 \begin{pmatrix} \mathrm{I}_2 & \mathbf{0} \\ \mathbf{0} & -\mathrm{I}_2 \end{pmatrix} \Leftrightarrow \sigma^0 \begin{pmatrix} \mathrm{I}_2 & \mathbf{0} \\ \mathbf{0} & -\mathrm{I}_2 \end{pmatrix} \equiv \boldsymbol{\gamma}_D^0$$

$$\boldsymbol{\gamma}_W^0 \equiv \sigma^0 \begin{pmatrix} \mathbf{0} & \mathrm{I}_2 \\ \mathrm{I}_2 & \mathbf{0} \end{pmatrix} \Leftrightarrow \sigma^0 \begin{pmatrix} \mathbf{0} & \mathrm{I}_2 \\ \mathrm{I}_2 & \mathbf{0} \end{pmatrix} \equiv \boldsymbol{\gamma}_W^0$$

$$\boldsymbol{\gamma}_{M1}^0 \equiv \sigma^2 \begin{pmatrix} \mathbf{0} & \mathrm{I}_2 \\ \mathrm{I}_2 & \mathbf{0} \end{pmatrix} \Leftrightarrow \sigma^2 \begin{pmatrix} \mathbf{0} & \mathrm{I}_2 \\ \mathrm{I}_2 & \mathbf{0} \end{pmatrix} \equiv \boldsymbol{\gamma}_{M1}^0$$

$$\boldsymbol{\gamma}_{M2}^0 \equiv \sigma^2 \begin{pmatrix} \mathbf{0} & \mathrm{I}_2 \\ \mathrm{I}_2 & \mathbf{0} \end{pmatrix} \Leftrightarrow \sigma^2 \begin{pmatrix} \mathbf{0} & \mathrm{I}_2 \\ \mathrm{I}_2 & \mathbf{0} \end{pmatrix} \equiv \boldsymbol{\gamma}_{M2}^0$$

$$\boldsymbol{\gamma}_{D/W}^\mu \equiv \boldsymbol{\sigma}^\mu \begin{pmatrix} \mathbf{0} & \mathrm{I}_2 \\ \mathrm{I}_2 & \mathbf{0} \end{pmatrix} \Leftrightarrow \sigma^\mu \begin{pmatrix} \mathbf{0} & \mathrm{I}_2 \\ \mathrm{I}_2 & \mathbf{0} \end{pmatrix} \equiv \boldsymbol{\gamma}_{D/W}^\mu \qquad (\mu = 1,2,3)$$

$$\boldsymbol{\gamma}_{M1}^1 \equiv i\boldsymbol{\sigma}^3 \begin{pmatrix} \mathrm{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathrm{I}_2 \end{pmatrix} \Leftrightarrow \mathbf{i}\boldsymbol{\sigma}^3 \begin{pmatrix} \mathrm{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathrm{I}_2 \end{pmatrix} \equiv \boldsymbol{\gamma}_{M1}^1$$

$$\boldsymbol{\gamma}_{M1}^2 \equiv \boldsymbol{\sigma}^2 \begin{pmatrix} \mathbf{0} & \mathrm{I}_2 \\ -\mathrm{I}_2 & \mathbf{0} \end{pmatrix} \Leftrightarrow \boldsymbol{\sigma}^2 \begin{pmatrix} \mathbf{0} & \mathrm{I}_2 \\ -\mathrm{I}_2 & \mathbf{0} \end{pmatrix} \equiv \boldsymbol{\gamma}_{M1}^2$$

$$\boldsymbol{\gamma}_{M1}^3 \equiv -i\boldsymbol{\sigma}^1 \begin{pmatrix} \mathrm{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathrm{I}_2 \end{pmatrix} \Leftrightarrow -i\boldsymbol{\sigma}^1 \begin{pmatrix} \mathrm{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathrm{I}_2 \end{pmatrix} \equiv \boldsymbol{\gamma}_{M1}^3$$

So:

$$\boldsymbol{\sigma}^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{I}_2$$

$$\boldsymbol{\sigma}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \boldsymbol{\sigma}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \boldsymbol{\sigma}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and:

$$\boldsymbol{\gamma}_D^1 \equiv \begin{pmatrix} \mathbf{0} & \sigma^1 \\ -\sigma^1 & \mathbf{0} \end{pmatrix} \quad , \quad \boldsymbol{\gamma}_D^2 \equiv \begin{pmatrix} \mathbf{0} & \sigma^2 \\ -\sigma^2 & \mathbf{0} \end{pmatrix} \quad , \quad \boldsymbol{\gamma}_D^3 \equiv \begin{pmatrix} \mathbf{0} & \sigma^3 \\ -\sigma^3 & \mathbf{0} \end{pmatrix}$$

$$\boldsymbol{\gamma}_W^1 \equiv \begin{pmatrix} \mathbf{0} & \sigma^1 \\ -\sigma^1 & \mathbf{0} \end{pmatrix} \quad , \quad \boldsymbol{\gamma}_W^2 \equiv \begin{pmatrix} \mathbf{0} & \sigma^2 \\ -\sigma^2 & \mathbf{0} \end{pmatrix} \quad , \quad \boldsymbol{\gamma}_W^3 \equiv \begin{pmatrix} \mathbf{0} & \sigma^3 \\ -\sigma^3 & \mathbf{0} \end{pmatrix}$$

$$\boldsymbol{\gamma}_{M1}^1 \equiv i\begin{pmatrix} \sigma^3 & \mathbf{0} \\ \mathbf{0} & \sigma^3 \end{pmatrix} \quad , \quad \boldsymbol{\gamma}_{M1}^2 \equiv \begin{pmatrix} \mathbf{0} & -\sigma^2 \\ \sigma^2 & \mathbf{0} \end{pmatrix} \quad , \quad \boldsymbol{\gamma}_{M1}^3 \equiv i\begin{pmatrix} -\mathrm{I}_2 & \mathbf{0} \\ \mathbf{0} & -\mathrm{I}_2 \end{pmatrix}$$

$$\boldsymbol{\gamma}_{M2}^1 \equiv i\begin{pmatrix} \mathbf{0} & \sigma^1 \\ \sigma^1 & \mathbf{0} \end{pmatrix} \quad , \quad \boldsymbol{\gamma}_{M2}^2 \equiv i\begin{pmatrix} \mathrm{I}_2 & \mathbf{0} \\ \mathbf{0} & -\mathrm{I}_2 \end{pmatrix} \quad , \quad \boldsymbol{\gamma}_{M2}^3 \equiv i\begin{pmatrix} \mathbf{0} & \sigma^3 \\ \sigma^3 & \mathbf{0} \end{pmatrix}$$

and, so:

$$\boldsymbol{\beta}_D \equiv \boldsymbol{\gamma}_D^0 = \begin{pmatrix} \mathrm{I}_2 & \mathbf{0} \\ \mathbf{0} & -\mathrm{I}_2 \end{pmatrix} \quad , \quad \boldsymbol{\beta}_W \equiv \boldsymbol{\gamma}_W^0 = \begin{pmatrix} \mathbf{0} & \mathrm{I}_2 \\ \mathrm{I}_2 & \mathbf{0} \end{pmatrix}$$

$$\boldsymbol{\alpha}_D^1 \equiv \boldsymbol{\gamma}_D^0\boldsymbol{\gamma}_D^1 = \begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma}^1 \\ \boldsymbol{\sigma}^1 & \mathbf{0} \end{pmatrix} \quad , \quad \boldsymbol{\alpha}_W^1 \equiv \boldsymbol{\gamma}_W^0\boldsymbol{\gamma}_W^1 = \begin{pmatrix} -\boldsymbol{\sigma}^1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma}^1 \end{pmatrix}$$

$$\boldsymbol{\alpha}_D^2 \equiv \boldsymbol{\gamma}_D^0\boldsymbol{\gamma}_D^2 = \begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma}^2 \\ \boldsymbol{\sigma}^2 & \mathbf{0} \end{pmatrix} \quad , \quad \boldsymbol{\alpha}_W^2 \equiv \boldsymbol{\gamma}_W^0\boldsymbol{\gamma}_W^2 = \begin{pmatrix} -\boldsymbol{\sigma}^2 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma}^2 \end{pmatrix}$$

$$\boldsymbol{\alpha}_D^3 \equiv \boldsymbol{\gamma}_D^0\boldsymbol{\gamma}_D^3 = \begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma}^3 \\ \boldsymbol{\sigma}^3 & \mathbf{0} \end{pmatrix} \quad , \quad \boldsymbol{\alpha}_W^3 \equiv \boldsymbol{\gamma}_W^0\boldsymbol{\gamma}_W^3 = \begin{pmatrix} -\boldsymbol{\sigma}^3 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma}^3 \end{pmatrix}$$

$$\boldsymbol{\beta}_{M1} \equiv \boldsymbol{\gamma}_{M1}^0 = \sigma^2\begin{pmatrix} \mathbf{0} & \mathrm{I}_2 \\ \mathrm{I}_2 & \mathbf{0} \end{pmatrix}$$

$$\boldsymbol{\alpha}_{M1}^1 \equiv \boldsymbol{\gamma}_{M1}^0\boldsymbol{\gamma}_{M1}^1 = \begin{pmatrix} \mathbf{0} & \sigma^2 \\ \sigma^2 & \mathbf{0} \end{pmatrix}\begin{pmatrix} i\boldsymbol{\sigma}^3 & \mathbf{0} \\ \mathbf{0} & i\boldsymbol{\sigma}^3 \end{pmatrix} = \begin{pmatrix} \mathbf{0} & i\sigma^2\boldsymbol{\sigma}^3 \\ i\sigma^2\boldsymbol{\sigma}^3 & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & -\sigma^1 \\ -\sigma^1 & \mathbf{0} \end{pmatrix}$$

$$\boldsymbol{\alpha}_{M1}^2 \equiv \boldsymbol{\gamma}_{M1}^0\boldsymbol{\gamma}_{M1}^2 = \begin{pmatrix} \mathbf{0} & \sigma^2 \\ \sigma^2 & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{0} & -\sigma^2 \\ \sigma^2 & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathrm{I}_2 & \mathbf{0} \\ \mathbf{0} & -\mathrm{I}_2 \end{pmatrix}$$

$$\boldsymbol{\alpha}_{M1}^3 \equiv \boldsymbol{\gamma}_{M1}^0\boldsymbol{\gamma}_{M1}^3 = \begin{pmatrix} \mathbf{0} & \sigma^2 \\ \sigma^2 & \mathbf{0} \end{pmatrix}\begin{pmatrix} -i\boldsymbol{\sigma}^1 & \mathbf{0} \\ \mathbf{0} & -i\boldsymbol{\sigma}^1 \end{pmatrix} = \begin{pmatrix} \mathbf{0} & -i\sigma^2\boldsymbol{\sigma}^1 \\ -i\sigma^2\boldsymbol{\sigma}^1 & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & -\sigma^3 \\ -\sigma^3 & \mathbf{0} \end{pmatrix}$$

$$\boldsymbol{\beta}_{M2} \equiv \boldsymbol{\gamma}_{M2}^0 = \sigma^2\begin{pmatrix} \mathbf{0} & \mathrm{I}_2 \\ \mathrm{I}_2 & \mathbf{0} \end{pmatrix}$$

$$\boldsymbol{\alpha}_{M2}^1 \equiv \boldsymbol{\gamma}_{M2}^0\boldsymbol{\gamma}_{M2}^1 = \begin{pmatrix} \mathbf{0} & \sigma^2 \\ \sigma^2 & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{0} & i\sigma^1 \\ i\sigma^1 & \mathbf{0} \end{pmatrix} = \begin{pmatrix} i\sigma^2\sigma^1 & \mathbf{0} \\ \mathbf{0} & i\sigma^2\sigma^1 \end{pmatrix} = \begin{pmatrix} \sigma^3 & \mathbf{0} \\ \mathbf{0} & \sigma^3 \end{pmatrix}$$

$$\boldsymbol{\alpha}_{M2}^2 \equiv \boldsymbol{\gamma}_{M2}^0\boldsymbol{\gamma}_{M2}^2 = \begin{pmatrix} \mathbf{0} & \sigma^2 \\ \sigma^2 & \mathbf{0} \end{pmatrix}\begin{pmatrix} i\mathrm{I}_2 & \mathbf{0} \\ \mathbf{0} & -i\mathrm{I}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{0} & -i\sigma^2 \\ i\sigma^2 & \mathbf{0} \end{pmatrix}$$

$$\boldsymbol{\alpha}_{M2}^3 \equiv \boldsymbol{\gamma}_{M2}^0\boldsymbol{\gamma}_{M2}^3 = \begin{pmatrix} \mathbf{0} & \sigma^2 \\ \sigma^2 & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{0} & i\sigma^3 \\ i\sigma^3 & \mathbf{0} \end{pmatrix} = \begin{pmatrix} i\sigma^2\sigma^3 & \mathbf{0} \\ \mathbf{0} & i\sigma^2\sigma^3 \end{pmatrix} = \begin{pmatrix} -\sigma^1 & \mathbf{0} \\ \mathbf{0} & -\sigma^1 \end{pmatrix}$$

other Unitary transformations:

$$\mathbf{U}_\theta \equiv \begin{pmatrix} \mathrm{I}_2\cos\theta & \mathrm{I}_2 i\sin\theta \\ \mathrm{I}_2 i\sin\theta & \mathrm{I}_2\cos\theta \end{pmatrix} \quad , \quad \mathbf{U}_\theta^{-1} \equiv \begin{pmatrix} \mathrm{I}_2\cos\theta & -\mathrm{I}_2 i\sin\theta \\ -\mathrm{I}_2 i\sin\theta & \mathrm{I}_2\cos\theta \end{pmatrix}$$

3.

Note:

(Dirac) $$i\begin{pmatrix} \mathrm{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathrm{I}_2 \end{pmatrix}\frac{\partial}{\partial t}\boldsymbol{\psi}_D = -i\begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma}^1 \\ \boldsymbol{\sigma}^1 & \mathbf{0} \end{pmatrix}\frac{\partial}{\partial x^1}\boldsymbol{\psi}_D - i\begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma}^2 \\ \boldsymbol{\sigma}^2 & \mathbf{0} \end{pmatrix}\frac{\partial}{\partial x^2}\boldsymbol{\psi}_D + \\ -i\begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma}^3 \\ \boldsymbol{\sigma}^3 & \mathbf{0} \end{pmatrix}\frac{\partial}{\partial x^3}\boldsymbol{\psi}_D + \begin{pmatrix} \mathrm{I}_2 & \mathbf{0} \\ \mathbf{0} & -\mathrm{I}_2 \end{pmatrix} m\boldsymbol{\psi}_D$$

then: $$\begin{pmatrix} i\mathrm{I}_2\frac{\partial}{\partial t} - \mathrm{I}_2 m & i\left(\boldsymbol{\sigma}^1\frac{\partial}{\partial x^1} + \boldsymbol{\sigma}^2\frac{\partial}{\partial x^2} + \boldsymbol{\sigma}^3\frac{\partial}{\partial x^3}\right) \\ i\left(\boldsymbol{\sigma}^1\frac{\partial}{\partial x^1} + \boldsymbol{\sigma}^2\frac{\partial}{\partial x^2} + \boldsymbol{\sigma}^3\frac{\partial}{\partial x^3}\right) & i\mathrm{I}_2\frac{\partial}{\partial t} + \mathrm{I}_2 m \end{pmatrix}\boldsymbol{\psi}_D = \mathbf{0}$$

but

$$\boldsymbol{\psi}_D = \begin{pmatrix} \psi_D^0 \\ \psi_D^1 \\ \psi_D^2 \\ \psi_D^3 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \psi_D^0 \\ \psi_D^1 \end{pmatrix} \\ \begin{pmatrix} \psi_D^2 \\ \psi_D^3 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\phi}_D^A \\ \boldsymbol{\phi}_D^B \end{pmatrix}$$

so: $$\begin{pmatrix} i\mathrm{I}_2\frac{\partial}{\partial t} - \mathrm{I}_2 m & i\left(\boldsymbol{\sigma}^1\frac{\partial}{\partial x^1} + \boldsymbol{\sigma}^2\frac{\partial}{\partial x^2} + \boldsymbol{\sigma}^3\frac{\partial}{\partial x^3}\right) \\ i\left(\boldsymbol{\sigma}^1\frac{\partial}{\partial x^1} + \boldsymbol{\sigma}^2\frac{\partial}{\partial x^2} + \boldsymbol{\sigma}^3\frac{\partial}{\partial x^3}\right) & i\mathrm{I}_2\frac{\partial}{\partial t} + \mathrm{I}_2 m \end{pmatrix}\begin{pmatrix} \boldsymbol{\phi}_D^A \\ \boldsymbol{\phi}_D^B \end{pmatrix} = \mathbf{0}$$

$\therefore$

$$\left(i\mathrm{I}_2\frac{\partial}{\partial t} - \mathrm{I}_2 m\right)\boldsymbol{\phi}_D^A + i\left(\boldsymbol{\sigma}^1\frac{\partial}{\partial x^1} + \boldsymbol{\sigma}^2\frac{\partial}{\partial x^2} + \boldsymbol{\sigma}^3\frac{\partial}{\partial x^3}\right)\boldsymbol{\phi}_D^B = \mathbf{0}$$
$$\left(i\mathrm{I}_2\frac{\partial}{\partial t} + \mathrm{I}_2 m\right)\boldsymbol{\phi}_D^B + i\left(\boldsymbol{\sigma}^1\frac{\partial}{\partial x^1} + \boldsymbol{\sigma}^2\frac{\partial}{\partial x^2} + \boldsymbol{\sigma}^3\frac{\partial}{\partial x^3}\right)\boldsymbol{\phi}_D^A = \mathbf{0}$$

$\therefore$

$$\begin{pmatrix} \mathrm{I}_2\left(-i\frac{\partial}{\partial t} + m\right) & -i\boldsymbol{\sigma}\bullet\vec{\boldsymbol{\nabla}} \\ i\boldsymbol{\sigma}\bullet\vec{\boldsymbol{\nabla}} & \mathrm{I}_2\left(i\frac{\partial}{\partial t} + m\right) \end{pmatrix}\begin{pmatrix} \boldsymbol{\phi}_D^A \\ \boldsymbol{\phi}_D^B \end{pmatrix} = \mathbf{0}$$

so:

$$\begin{pmatrix} \mathrm{I}_2\left(i\frac{\partial}{\partial t} + m\right) & i\boldsymbol{\sigma}\bullet\vec{\boldsymbol{\nabla}} \\ -i\boldsymbol{\sigma}\bullet\vec{\boldsymbol{\nabla}} & \mathrm{I}_2\left(-i\frac{\partial}{\partial t} + m\right) \end{pmatrix}\begin{pmatrix} \mathrm{I}_2\left(-i\frac{\partial}{\partial t} + m\right) & -i\boldsymbol{\sigma}\bullet\vec{\boldsymbol{\nabla}} \\ i\boldsymbol{\sigma}\bullet\vec{\boldsymbol{\nabla}} & \mathrm{I}_2\left(i\frac{\partial}{\partial t} + m\right) \end{pmatrix}\begin{pmatrix} \boldsymbol{\phi}_D^A \\ \boldsymbol{\phi}_D^B \end{pmatrix} = \mathbf{0}$$

$\therefore$

$$\begin{pmatrix} \mathrm{I}_2\left(\frac{\partial^2}{\partial t^2} + m^2 - \sum_{\mu=1}^{3}\sum_{\nu=1}^{3}\boldsymbol{\sigma}^\mu\boldsymbol{\sigma}^\nu\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\right) & \mathbf{0} \\ \mathbf{0} & \mathrm{I}_2\left(\frac{\partial^2}{\partial t^2} + m^2 - \sum_{\mu=1}^{3}\sum_{\nu=1}^{3}\boldsymbol{\sigma}^\mu\boldsymbol{\sigma}^\nu\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\right) \end{pmatrix}\begin{pmatrix} \boldsymbol{\phi}_D^A \\ \boldsymbol{\phi}_D^B \end{pmatrix} = \mathbf{0}$$

so:

$$\left(\nabla^2 - \frac{\partial^2}{\partial t^2} - m^2\right)\boldsymbol{\psi}_D = \mathbf{0}$$, $\boldsymbol{\psi}_D$ satisfies the Klein-Gordon equation And:

(Weyl / chiral)
$$i\begin{pmatrix} \mathrm{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathrm{I}_2 \end{pmatrix}\frac{\partial}{\partial t}\boldsymbol{\psi}_W = -i\begin{pmatrix} -\boldsymbol{\sigma}^1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma}^1 \end{pmatrix}\frac{\partial}{\partial x^1}\boldsymbol{\psi}_W - i\begin{pmatrix} -\boldsymbol{\sigma}^2 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma}^2 \end{pmatrix}\frac{\partial}{\partial x^2}\boldsymbol{\psi}_W + \\ -i\begin{pmatrix} -\boldsymbol{\sigma}^3 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma}^3 \end{pmatrix}\frac{\partial}{\partial x^3}\boldsymbol{\psi}_W + \begin{pmatrix} \mathbf{0} & \mathrm{I}_2 \\ \mathrm{I}_2 & \mathbf{0} \end{pmatrix} m\boldsymbol{\psi}_W$$

then:
$$\begin{pmatrix} i\mathrm{I}_2\frac{\partial}{\partial t} + i\left(\boldsymbol{\sigma}^1\frac{\partial}{\partial x^1} + \boldsymbol{\sigma}^2\frac{\partial}{\partial x^2} + \boldsymbol{\sigma}^3\frac{\partial}{\partial x^3}\right) & -\mathrm{I}_2 m \\ -\mathrm{I}_2 m & i\mathrm{I}_2\frac{\partial}{\partial t} - i\left(\boldsymbol{\sigma}^1\frac{\partial}{\partial x^1} + \boldsymbol{\sigma}^2\frac{\partial}{\partial x^2} + \boldsymbol{\sigma}^3\frac{\partial}{\partial x^3}\right) \end{pmatrix}\boldsymbol{\psi}_W = \mathbf{0}$$

but

$$\boldsymbol{\psi}_W = \begin{pmatrix} \psi_W^0 \\ \psi_W^1 \\ \psi_W^2 \\ \psi_W^3 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \psi_W^0 \\ \psi_W^1 \end{pmatrix} \\ \begin{pmatrix} \psi_W^2 \\ \psi_W^3 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\phi}_W^A \\ \boldsymbol{\phi}_W^B \end{pmatrix}$$

so:

$$\begin{pmatrix} i\mathrm{I}_2\frac{\partial}{\partial t} + i\boldsymbol{\sigma}\bullet\vec{\nabla} & -\mathrm{I}_2 m \\ -\mathrm{I}_2 m & i\mathrm{I}_2\frac{\partial}{\partial t} - i\boldsymbol{\sigma}\bullet\vec{\nabla} \end{pmatrix}\begin{pmatrix} \boldsymbol{\phi}_W^A \\ \boldsymbol{\phi}_W^B \end{pmatrix} = \mathbf{0}$$

∴

$$\begin{pmatrix} i\mathrm{I}_2\frac{\partial}{\partial t} - i\boldsymbol{\sigma}\bullet\vec{\nabla} & \mathrm{I}_2 m \\ \mathrm{I}_2 m & i\mathrm{I}_2\frac{\partial}{\partial t} + i\boldsymbol{\sigma}\bullet\vec{\nabla} \end{pmatrix}\begin{pmatrix} i\mathrm{I}_2\frac{\partial}{\partial t} + i\boldsymbol{\sigma}\bullet\vec{\nabla} & -\mathrm{I}_2 m \\ -\mathrm{I}_2 m & i\mathrm{I}_2\frac{\partial}{\partial t} - i\boldsymbol{\sigma}\bullet\vec{\nabla} \end{pmatrix}\begin{pmatrix} \boldsymbol{\phi}_W^A \\ \boldsymbol{\phi}_W^B \end{pmatrix} = \mathbf{0}$$

∴

$$\begin{pmatrix} \mathrm{I}_2\left(-\frac{\partial^2}{\partial t^2} + \sum_{\mu=1}^{3}\sum_{\nu=1}^{3}\boldsymbol{\sigma}^\mu\boldsymbol{\sigma}^\nu\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} - \mathrm{I}_2 m^2\right) & \mathbf{0} \\ \mathbf{0} & \mathrm{I}_2\left(-\mathrm{I}_2 m^2 - \frac{\partial^2}{\partial t^2} + \sum_{\mu=1}^{3}\sum_{\nu=1}^{3}\boldsymbol{\sigma}^\mu\boldsymbol{\sigma}^\nu\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\right) \end{pmatrix}\begin{pmatrix} \boldsymbol{\phi}_W^A \\ \boldsymbol{\phi}_W^B \end{pmatrix} = \mathbf{0}$$

so:

$$\left(\nabla^2 - \frac{\partial^2}{\partial t^2} - m^2\right)\boldsymbol{\psi}_W = \mathbf{0}$$, $\boldsymbol{\psi}_W$ satisfies the Klein-Gordon equation And:

(Majorana (1))
$$i\begin{pmatrix} \mathrm{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathrm{I}_2 \end{pmatrix}\frac{\partial}{\partial t}\boldsymbol{\psi}_{M1} = -i\begin{pmatrix} \mathbf{0} & -\sigma^1 \\ -\sigma^1 & \mathbf{0} \end{pmatrix}\frac{\partial}{\partial x^1}\boldsymbol{\psi}_{M1} - i\begin{pmatrix} \mathrm{I}_2 & \mathbf{0} \\ \mathbf{0} & -\mathrm{I}_2 \end{pmatrix}\frac{\partial}{\partial x^2}\boldsymbol{\psi}_{M1} + \\ -i\begin{pmatrix} \mathbf{0} & -\sigma^3 \\ -\sigma^3 & \mathbf{0} \end{pmatrix}\frac{\partial}{\partial x^3}\boldsymbol{\psi}_{M1} + \sigma^2\begin{pmatrix} \mathbf{0} & \mathrm{I}_2 \\ \mathrm{I}_2 & \mathbf{0} \end{pmatrix} m\boldsymbol{\psi}_{M1}$$

then: $$\begin{pmatrix} \left(iI_2\frac{\partial}{\partial t} + iI_2\frac{\partial}{\partial x^2}\right) & \left(-i\sigma^1\frac{\partial}{\partial x^1} - i\sigma^3\frac{\partial}{\partial x^3} - \sigma^2 m\right) \\ \left(-i\sigma^1\frac{\partial}{\partial x^1} - i\sigma^3\frac{\partial}{\partial x^3} - \sigma^2 m\right) & \left(iI_2\frac{\partial}{\partial t} - iI_2\frac{\partial}{\partial x^2}\right) \end{pmatrix}\boldsymbol{\psi}_{M1} = \mathbf{0}$$

but

$$\boldsymbol{\psi}_{M1} = \begin{pmatrix} \psi^0_{M1} \\ \psi^1_{M1} \\ \psi^2_{M1} \\ \psi^3_{M1} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \psi^0_{M1} \\ \psi^1_{M1} \end{pmatrix} \\ \begin{pmatrix} \psi^2_{M1} \\ \psi^3_{M1} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\phi}^A_{M1} \\ \boldsymbol{\phi}^B_{M1} \end{pmatrix}$$

so:

$$\left(iI_2\frac{\partial}{\partial t} + iI_2\frac{\partial}{\partial x^2}\right)\boldsymbol{\phi}^A_{M1} + \left(-i\sigma^1\frac{\partial}{\partial x^1} - i\sigma^3\frac{\partial}{\partial x^3} - \sigma^2 m\right)\boldsymbol{\phi}^B_{M1} = \mathbf{0}$$

and:

$$\left(-i\sigma^1\frac{\partial}{\partial x^1} - i\sigma^3\frac{\partial}{\partial x^3} - \sigma^2 m\right)\boldsymbol{\phi}^A_{M1} + \left(iI_2\frac{\partial}{\partial t} - iI_2\frac{\partial}{\partial x^2}\right)\boldsymbol{\phi}^B_{M1} = \mathbf{0}$$

$\therefore$

$$\left[\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x^2}\right)\right]\boldsymbol{\phi}^A_{M1} + \left[\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}\frac{\partial}{\partial x^1} + \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\frac{\partial}{\partial x^3} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}m\right]\boldsymbol{\phi}^B_{M1} = \mathbf{0}$$

and:

$$\left[\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}\frac{\partial}{\partial x^1} + \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\frac{\partial}{\partial x^3} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}m\right]\boldsymbol{\phi}^A_{M1} + \left[\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x^2}\right)\right]\boldsymbol{\phi}^B_{M1} = \mathbf{0}$$

so:

$$\begin{pmatrix} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x^2}\right) & 0 \\ 0 & \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x^2}\right) \end{pmatrix}\begin{pmatrix} \psi^0_{M1} \\ \psi^1_{M1} \end{pmatrix} + \begin{pmatrix} -\frac{\partial}{\partial x^3} & \left(-\frac{\partial}{\partial x^1} + m\right) \\ \left(-\frac{\partial}{\partial x^1} - m\right) & \frac{\partial}{\partial x^3} \end{pmatrix}\begin{pmatrix} \psi^2_{M1} \\ \psi^3_{M1} \end{pmatrix} = \mathbf{0}$$

and:

$$\begin{pmatrix} -\frac{\partial}{\partial x^3} & \left(-\frac{\partial}{\partial x^1} + m\right) \\ \left(-\frac{\partial}{\partial x^1} - m\right) & \frac{\partial}{\partial x^3} \end{pmatrix}\begin{pmatrix} \psi^0_{M1} \\ \psi^1_{M1} \end{pmatrix} + \begin{pmatrix} \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x^2}\right) & 0 \\ 0 & \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x^2}\right) \end{pmatrix}\begin{pmatrix} \psi^2_{M1} \\ \psi^3_{M1} \end{pmatrix} = \mathbf{0}$$

$\therefore$

$$\begin{pmatrix} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x^2}\right)\psi^0_{M1} - \frac{\partial}{\partial x^3}\psi^2_{M1} + \left(-\frac{\partial}{\partial x^1} + m\right)\psi^3_{M1} \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x^2}\right)\psi^1_{M1} + \left(-\frac{\partial}{\partial x^1} - m\right)\psi^2_{M1} + \frac{\partial}{\partial x^3}\psi^3_{M1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and:

$$\begin{pmatrix} -\frac{\partial}{\partial x^3}\psi^0_{M1} + \left(-\frac{\partial}{\partial x^1} + m\right)\psi^1_{M1} + \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x^2}\right)\psi^2_{M1} \\ \left(-\frac{\partial}{\partial x^1} - m\right)\psi^0_{M1} + \frac{\partial}{\partial x^3}\psi^1_{M1} + \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x^2}\right)\psi^3_{M1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

∴

$$(\partial_t - \partial_2)(\partial_t + \partial_2)\psi^0_{M1} - (\partial_t - \partial_2)\partial_3\psi^2_{M1} + (\partial_t - \partial_2)(-\partial_1 + m)\psi^3_{M1} = 0$$
$$-(-\partial_1 + m)(-\partial_1 - m)\psi^0_{M1} - (-\partial_1 + m)\partial_3\psi^1_{M1} - (-\partial_1 + m)(\partial_t - \partial_2)\psi^3_{M1} = 0$$

so:

$$(\partial_t^2 - \partial_2^2)\psi^0_{M1} - (\partial_t - \partial_2)\partial_3\psi^2_{M1} - (\partial_1^2 - m^2)\psi^0_{M1} - (-\partial_1 + m)\partial_3\psi^1_{M1} = 0$$

but:

$$-\partial_3^2\psi^0_{M1} - \partial_3(-\partial_1 + m)\psi^1_{M1} - \partial_3(\partial_t - \partial_2)\psi^2_{M1} = 0$$

so:

$$(\partial_t^2 - \partial_2^2)\psi^0_{M1} - \partial_3^2\psi^0_{M1} - (\partial_1^2 - m^2)\psi^0_{M1} = 0$$

∴

$$(\nabla^2 - \partial_t^2 - m^2)\psi^0_{M1} = 0$$

satisfies the Klein-Gordon equation

($\psi^1_{M2}, \psi^2_{M2}, \psi^3_{M2}$ left for the reader)

And:

(Majorana (2))
$$i\begin{pmatrix} I_2 & \mathbf{0} \\ \mathbf{0} & I_2 \end{pmatrix}\frac{\partial}{\partial t}\boldsymbol{\psi}_{M2} = -i\begin{pmatrix} i\sigma^2\sigma^1 & \mathbf{0} \\ \mathbf{0} & i\sigma^2\sigma^1 \end{pmatrix}\frac{\partial}{\partial x^1}\boldsymbol{\psi}_{M2} - i\begin{pmatrix} \mathbf{0} & -i\sigma^2 \\ i\sigma^2 & \mathbf{0} \end{pmatrix}\frac{\partial}{\partial x^2}\boldsymbol{\psi}_{M2} + $$
$$-i\begin{pmatrix} i\sigma^2\sigma^3 & \mathbf{0} \\ \mathbf{0} & i\sigma^2\sigma^3 \end{pmatrix}\frac{\partial}{\partial x^3}\boldsymbol{\psi}_{M2} + \begin{pmatrix} \mathbf{0} & \sigma^2 \\ \sigma^2 & \mathbf{0} \end{pmatrix} m\boldsymbol{\psi}_{M2}$$

then:
$$\begin{pmatrix} \left(iI_2\frac{\partial}{\partial t} - \sigma^2\sigma^1\frac{\partial}{\partial x^1} - \sigma^2\sigma^3\frac{\partial}{\partial x^3}\right) & \left(-\sigma^2\frac{\partial}{\partial x^2} - \sigma^2 m\right) \\ \left(\sigma^2\frac{\partial}{\partial x^2} - \sigma^2 m\right) & \left(iI_2\frac{\partial}{\partial t} - \sigma^2\sigma^1\frac{\partial}{\partial x^1} - \sigma^2\sigma^3\frac{\partial}{\partial x^3}\right) \end{pmatrix}\boldsymbol{\psi}_{M2} = \mathbf{0}$$

but

$$\boldsymbol{\psi}_{M2} = \begin{pmatrix} \psi^0_{M2} \\ \psi^1_{M2} \\ \psi^2_{M2} \\ \psi^3_{M2} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \psi^0_{M2} \\ \psi^1_{M2} \end{pmatrix} \\ \begin{pmatrix} \psi^2_{M2} \\ \psi^3_{M2} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\phi}^A_{M2} \\ \boldsymbol{\phi}^B_{M2} \end{pmatrix}$$

so:

$$\left(iI_2\frac{\partial}{\partial t} - \sigma^2\sigma^1\frac{\partial}{\partial x^1} - \sigma^2\sigma^3\frac{\partial}{\partial x^3}\right)\boldsymbol{\phi}^A_{M2} + \left(-\sigma^2\frac{\partial}{\partial x^2} - \sigma^2 m\right)\boldsymbol{\phi}^B_{M2} = \mathbf{0}$$

and:

$$\left(\sigma^2\frac{\partial}{\partial x^2} - \sigma^2 m\right)\boldsymbol{\phi}^A_{M2} + \left(iI_2\frac{\partial}{\partial t} - \sigma^2\sigma^1\frac{\partial}{\partial x^1} - \sigma^2\sigma^3\frac{\partial}{\partial x^3}\right)\boldsymbol{\phi}^B_{M2} = \mathbf{0}$$

∴

$$\left[\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}\frac{\partial}{\partial t} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\frac{\partial}{\partial x^1} - \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\frac{\partial}{\partial x^3}\right]\boldsymbol{\phi}^A_{M2} + \left[\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\left(\frac{\partial}{\partial x^2} + m\right)\right]\boldsymbol{\phi}^B_{M2} = \mathbf{0}$$

and:

$$\left[\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\left(\frac{\partial}{\partial x^2} - m\right)\right]\boldsymbol{\phi}^A_{M2} + \left[\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}\frac{\partial}{\partial t} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\frac{\partial}{\partial x^1} - \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\frac{\partial}{\partial x^3}\right]\boldsymbol{\phi}^B_{M2} = \mathbf{0}$$

so:

$$\begin{pmatrix} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x^1}\right) & -\frac{\partial}{\partial x^3} \\ -\frac{\partial}{\partial x^3} & \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x^1}\right) \end{pmatrix}\begin{pmatrix} \psi^0_{M2} \\ \psi^1_{M2} \end{pmatrix} + \begin{pmatrix} 0 & \left(\frac{\partial}{\partial x^2} + m\right) \\ -\left(\frac{\partial}{\partial x^2} + m\right) & 0 \end{pmatrix}\begin{pmatrix} \psi^2_{M2} \\ \psi^3_{M2} \end{pmatrix} = \mathbf{0}$$

and:

$$\begin{pmatrix} 0 & -\left(\frac{\partial}{\partial x^2} - m\right) \\ \left(\frac{\partial}{\partial x^2} - m\right) & 0 \end{pmatrix}\begin{pmatrix} \psi^0_{M2} \\ \psi^1_{M2} \end{pmatrix} + \begin{pmatrix} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x^1}\right) & -\frac{\partial}{\partial x^3} \\ -\frac{\partial}{\partial x^3} & \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x^1}\right) \end{pmatrix}\begin{pmatrix} \psi^2_{M2} \\ \psi^3_{M2} \end{pmatrix} = \mathbf{0}$$

∴

$$\begin{pmatrix} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x^1}\right)\psi^0_{M2} - \frac{\partial}{\partial x^3}\psi^1_{M2} + \left(\frac{\partial}{\partial x^2} + m\right)\psi^3_{M2} \\ -\frac{\partial}{\partial x^3}\psi^0_{M2} + \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x^1}\right)\psi^1_{M2} - \left(\frac{\partial}{\partial x^2} + m\right)\psi^2_{M2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and:

$$\begin{pmatrix} -\left(\frac{\partial}{\partial x^2} - m\right)\psi^1_{M2} + \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x^1}\right)\psi^2_{M2} - \frac{\partial}{\partial x^3}\psi^3_{M2} \\ \left(\frac{\partial}{\partial x^2} - m\right)\psi^0_{M2} - \frac{\partial}{\partial x^3}\psi^2_{M2} + \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x^1}\right)\psi^3_{M2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

∴

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x^1}\right)\psi^0_{M2} - \frac{\partial}{\partial x^3}\psi^1_{M2} + \left(\frac{\partial}{\partial x^2} + m\right)\psi^3_{M2} = 0$$
$$-\left(\frac{\partial}{\partial x^2} - m\right)\psi^1_{M2} + \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x^1}\right)\psi^2_{M2} - \frac{\partial}{\partial x^3}\psi^3_{M2} = 0$$

so:

$$\partial_3(\partial_t + \partial_1)\psi^0_{M2} - \partial_3^2\psi^1_{M2} + \partial_3(\partial_2 + m)\psi^3_{M2} = 0$$
$$-(\partial_2 + m)(\partial_2 - m)\psi^1_{M2} + (\partial_2 + m)(\partial_t + \partial_1)\psi^2_{M2} - (\partial_2 + m)\partial_3\psi^3_{M2} = 0$$

∴

$$\partial_3(\partial_t + \partial_1)\psi^0_{M2} - \partial_3^2\psi^1_{M2} - (\partial_2 + m)(\partial_2 - m)\psi^1_{M2} + (\partial_2 + m)(\partial_t + \partial_1)\psi^2_{M2} = 0$$

but:

$$-\partial_3\psi^0_{M2} + (\partial_t - \partial_1)\psi^1_{M2} - (\partial_2 + m)\psi^2_{M2} = 0$$

∴

$$-(\partial_t + \partial_1)\partial_3\psi^0_{M2} + (\partial_t + \partial_1)(\partial_t - \partial_1)\psi^1_{M2} - (\partial_t + \partial_1)(\partial_2 + m)\psi^2_{M2} = 0$$

so:

$$-\partial_3^2\psi^1_{M2} - (\partial_2 + m)(\partial_2 - m)\psi^1_{M2} + (\partial_t + \partial_1)(\partial_t - \partial_1)\psi^1_{M2} = 0$$

∴

$$-\partial_3^2\psi_{M2}^1 - (\partial_2^2 - m^2)\psi_{M2}^1 + (\partial_t^2 - \partial_1^2)\psi_{M2}^1 = 0$$

and:

$$(-\partial_1^2 - \partial_2^2 - \partial_3^2 + \partial_t^2 + m^2)\psi_{M2}^1 = 0 \Rightarrow -(\nabla^2 - \partial_t^2 - m^2)\psi_{M2}^1 = 0$$

satisfies the Klein-Gordon equation

also:

$$(\partial_2 - m)(\partial_t + \partial_1)\psi_{M2}^0 - (\partial_2 - m)\partial_3\psi_{M2}^1 + (\partial_2 - m)(\partial_2 + m)\psi_{M2}^3 = 0$$

$$\partial_3(\partial_2 - m)\psi_{M2}^1 - \partial_3(\partial_t + \partial_1)\psi_{M2}^2 + \partial_3^2\psi_{M2}^3 = 0$$

so:

$$(\partial_2 - m)(\partial_t + \partial_1)\psi_{M2}^0 + (\partial_2 - m)(\partial_2 + m)\psi_{M2}^3 - \partial_3(\partial_t + \partial_1)\psi_{M2}^2 + \partial_3^2\psi_{M2}^3 = 0$$

but:

$$-(\partial_t + \partial_1)(\partial_2 - m)\psi_{M2}^0 + (\partial_t + \partial_1)\partial_3\psi_{M2}^2 - (\partial_t + \partial_1)(\partial_t - \partial_1)\psi_{M2}^3 = 0$$

$\therefore$

$$(\partial_2 - m)(\partial_2 + m)\psi_{M2}^3 + \partial_3^2\psi_{M2}^3 - (\partial_t + \partial_1)(\partial_t - \partial_1)\psi_{M2}^3 = 0$$

so:

$$(\partial_2^2 - m^2)\psi_{M2}^3 + \partial_3^2\psi_{M2}^3 - (\partial_t^2 - \partial_1^2)\psi_{M2}^3 = 0$$

$\therefore$

$$(\nabla^2 - \partial_t^2 - m^2)\psi_{M2}^3 = 0$$

satisfies the Klein-Gordon equation

also:

$$(\partial_t + \partial_1)\psi_{M2}^0 - \partial_3\psi_{M2}^1 + (\partial_2 + m)\psi_{M2}^3 = 0$$

and:

$$(\partial_2 - m)\psi_{M2}^0 - \partial_3\psi_{M2}^2 + (\partial_t - \partial_1)\psi_{M2}^3 = 0$$

so:

$$(\partial_t - \partial_1)(\partial_t + \partial_1)\psi_{M2}^0 - (\partial_t - \partial_1)\partial_3\psi_{M2}^1 + (\partial_t - \partial_1)(\partial_2 + m)\psi_{M2}^3 = 0$$

and:

$$-(\partial_2 + m)(\partial_2 - m)\psi_{M2}^0 + (\partial_2 + m)\partial_3\psi_{M2}^2 - (\partial_2 + m)(\partial_t - \partial_1)\psi_{M2}^3 = 0$$

$\therefore$

$$(\partial_t^2 - \partial_1^2)\psi_{M2}^0 - (\partial_t - \partial_1)\partial_3\psi_{M2}^1 - (\partial_2^2 - m^2)\psi_{M2}^0 + (\partial_2 + m)\partial_3\psi_{M2}^2 = 0$$

but:

$$-\partial_3^2\psi_{M2}^0 + \partial_3(\partial_t - \partial_1)\psi_{M2}^1 - \partial_3(\partial_2 + m)\psi_{M2}^2 = 0$$

so:

$$(\partial_t^2 - \partial_1^2)\psi_{M2}^0 - (\partial_2^2 - m^2)\psi_{M2}^0 - \partial_3^2\psi_{M2}^0 = 0$$

$\therefore$

$$(\nabla^2 - \partial_t^2 - m^2)\psi_{M2}^0 = 0$$

satisfies the Klein-Gordon equation

(ψ_{M2}^2 is left for the reader)

4.

Now, with:

$$\mathbf{C}^0 \equiv \begin{pmatrix} \mathrm{I}_2\cos\frac{\pi}{2} & \mathrm{I}_2\sin\frac{\pi}{2} \\ \mathrm{I}_2\sin\frac{\pi}{2} & \mathrm{I}_2\cos\frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathrm{I}_2 \\ \mathrm{I}_2 & \mathbf{0} \end{pmatrix}$$

two **chiral projection operators** may be defined:

$$\mathbf{P}^- \equiv \tfrac{1}{2}\gamma^0\mathbf{C}^0(\mathbf{C}^0\gamma^0 - \gamma^5) \quad , \quad \mathbf{P}^+ \equiv \tfrac{1}{2}\gamma^0\mathbf{C}^0(\mathbf{C}^0\gamma^0 + \gamma^5)$$

so:

$$\begin{aligned}\mathbf{P}_D^- &\equiv \tfrac{1}{2}\gamma_D^0\mathbf{C}^0\left(\mathbf{C}^0\gamma_D^0 - \gamma_D^5\right) \\ &= \tfrac{1}{2}\begin{pmatrix} \mathrm{I}_2 & \mathbf{0} \\ \mathbf{0} & -\mathrm{I}_2 \end{pmatrix}\begin{pmatrix} \mathbf{0} & \mathrm{I}_2 \\ \mathrm{I}_2 & \mathbf{0} \end{pmatrix}\left[\begin{pmatrix} \mathbf{0} & \mathrm{I}_2 \\ \mathrm{I}_2 & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathrm{I}_2 & \mathbf{0} \\ \mathbf{0} & -\mathrm{I}_2 \end{pmatrix} - \begin{pmatrix} \mathbf{0} & -\mathrm{I}_2 \\ -\mathrm{I}_2 & \mathbf{0} \end{pmatrix}\right] \\ &= \tfrac{1}{2}\begin{pmatrix} \mathbf{0} & \mathrm{I}_2 \\ -\mathrm{I}_2 & \mathbf{0} \end{pmatrix}\left[\begin{pmatrix} \mathbf{0} & -\mathrm{I}_2 \\ \mathrm{I}_2 & \mathbf{0} \end{pmatrix} - \begin{pmatrix} \mathbf{0} & -\mathrm{I}_2 \\ -\mathrm{I}_2 & \mathbf{0} \end{pmatrix}\right] \\ &= \tfrac{1}{2}\begin{pmatrix} \mathbf{0} & \mathrm{I}_2 \\ -\mathrm{I}_2 & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ 2\mathrm{I}_2 & \mathbf{0} \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} 2\mathrm{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathrm{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\mathbf{P}_D^+ &\equiv \tfrac{1}{2}\gamma_D^0\mathbf{C}^0\left(\mathbf{C}^0\gamma_D^0 + \gamma_D^5\right) \\ &= \tfrac{1}{2}\begin{pmatrix} \mathrm{I}_2 & \mathbf{0} \\ \mathbf{0} & -\mathrm{I}_2 \end{pmatrix}\begin{pmatrix} \mathbf{0} & \mathrm{I}_2 \\ \mathrm{I}_2 & \mathbf{0} \end{pmatrix}\left[\begin{pmatrix} \mathbf{0} & \mathrm{I}_2 \\ \mathrm{I}_2 & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathrm{I}_2 & \mathbf{0} \\ \mathbf{0} & -\mathrm{I}_2 \end{pmatrix} + \begin{pmatrix} \mathbf{0} & -\mathrm{I}_2 \\ -\mathrm{I}_2 & \mathbf{0} \end{pmatrix}\right] \\ &= \tfrac{1}{2}\begin{pmatrix} \mathbf{0} & \mathrm{I}_2 \\ -\mathrm{I}_2 & \mathbf{0} \end{pmatrix}\left[\begin{pmatrix} \mathbf{0} & -\mathrm{I}_2 \\ \mathrm{I}_2 & \mathbf{0} \end{pmatrix} + \begin{pmatrix} \mathbf{0} & -\mathrm{I}_2 \\ -\mathrm{I}_2 & \mathbf{0} \end{pmatrix}\right] \\ &= \tfrac{1}{2}\begin{pmatrix} \mathbf{0} & \mathrm{I}_2 \\ -\mathrm{I}_2 & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{0} & -2\mathrm{I}_2 \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2\mathrm{I}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathrm{I}_2 \end{pmatrix}\end{aligned}$$

and:

$$\begin{aligned}\mathbf{P}_W^- &\equiv \tfrac{1}{2}\gamma_W^0\mathbf{C}^0\left(\mathbf{C}^0\gamma_W^0 - \gamma_W^5\right) \\ &= \tfrac{1}{2}\begin{pmatrix} \mathbf{0} & \mathrm{I}_2 \\ \mathrm{I}_2 & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{0} & \mathrm{I}_2 \\ \mathrm{I}_2 & \mathbf{0} \end{pmatrix}\left[\begin{pmatrix} \mathbf{0} & \mathrm{I}_2 \\ \mathrm{I}_2 & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{0} & \mathrm{I}_2 \\ \mathrm{I}_2 & \mathbf{0} \end{pmatrix} - \begin{pmatrix} -\mathrm{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathrm{I}_2 \end{pmatrix}\right] \\ &= \tfrac{1}{2}\begin{pmatrix} \mathrm{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathrm{I}_2 \end{pmatrix}\left[\begin{pmatrix} \mathrm{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathrm{I}_2 \end{pmatrix} - \begin{pmatrix} -\mathrm{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathrm{I}_2 \end{pmatrix}\right] \\ &= \tfrac{1}{2}\begin{pmatrix} 2\mathrm{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathrm{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\mathbf{P}_W^+ &\equiv \tfrac{1}{2}\gamma_W^0\mathbf{C}^0\left(\mathbf{C}^0\gamma_W^0 + \gamma_W^5\right) \\ &= \tfrac{1}{2}\begin{pmatrix} \mathbf{0} & \mathrm{I}_2 \\ \mathrm{I}_2 & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{0} & \mathrm{I}_2 \\ \mathrm{I}_2 & \mathbf{0} \end{pmatrix}\left[\begin{pmatrix} \mathbf{0} & \mathrm{I}_2 \\ \mathrm{I}_2 & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{0} & \mathrm{I}_2 \\ \mathrm{I}_2 & \mathbf{0} \end{pmatrix} + \begin{pmatrix} -\mathrm{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathrm{I}_2 \end{pmatrix}\right]\end{aligned}$$

$$= \tfrac{1}{2}\begin{pmatrix} I_2 & \mathbf{0} \\ \mathbf{0} & I_2 \end{pmatrix}\left[\begin{pmatrix} I_2 & \mathbf{0} \\ \mathbf{0} & I_2 \end{pmatrix} + \begin{pmatrix} -I_2 & \mathbf{0} \\ \mathbf{0} & I_2 \end{pmatrix}\right]$$

$$= \tfrac{1}{2}\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2I_2 \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_2 \end{pmatrix}$$

$$\mathbf{P}^{-}_{M1} \equiv \tfrac{1}{2}\gamma^{0}_{M1}\mathbf{C}^{0}\left(\mathbf{C}^{0}\gamma^{0}_{M1} - \gamma^{5}_{M1}\right)$$

$$= \tfrac{1}{2}\begin{pmatrix} \mathbf{0} & \sigma^2 \\ \sigma^2 & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{0} & I_2 \\ I_2 & \mathbf{0} \end{pmatrix}\left[\begin{pmatrix} \mathbf{0} & I_2 \\ I_2 & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{0} & \sigma^2 \\ \sigma^2 & \mathbf{0} \end{pmatrix} - \begin{pmatrix} \sigma^2 & \mathbf{0} \\ \mathbf{0} & -\sigma^2 \end{pmatrix}\right]$$

$$= \tfrac{1}{2}\begin{pmatrix} \sigma^2 & \mathbf{0} \\ \mathbf{0} & \sigma^2 \end{pmatrix}\left[\begin{pmatrix} \sigma^2 & \mathbf{0} \\ \mathbf{0} & \sigma^2 \end{pmatrix} - \begin{pmatrix} \sigma^2 & \mathbf{0} \\ \mathbf{0} & -\sigma^2 \end{pmatrix}\right]$$

$$= \tfrac{1}{2}\begin{pmatrix} \sigma^2 & \mathbf{0} \\ \mathbf{0} & \sigma^2 \end{pmatrix}\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2\sigma^2 \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2I_2 \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_2 \end{pmatrix}$$

$$\mathbf{P}^{+}_{M1} \equiv \tfrac{1}{2}\gamma^{0}_{M1}\mathbf{C}^{0}\left(\mathbf{C}^{0}\gamma^{0}_{M1} + \gamma^{5}_{M1}\right)$$

$$= \tfrac{1}{2}\begin{pmatrix} \mathbf{0} & \sigma^2 \\ \sigma^2 & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{0} & I_2 \\ I_2 & \mathbf{0} \end{pmatrix}\left[\begin{pmatrix} \mathbf{0} & I_2 \\ I_2 & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{0} & \sigma^2 \\ \sigma^2 & \mathbf{0} \end{pmatrix} + \begin{pmatrix} \sigma^2 & \mathbf{0} \\ \mathbf{0} & -\sigma^2 \end{pmatrix}\right]$$

$$= \tfrac{1}{2}\begin{pmatrix} \sigma^2 & \mathbf{0} \\ \mathbf{0} & \sigma^2 \end{pmatrix}\left[\begin{pmatrix} \sigma^2 & \mathbf{0} \\ \mathbf{0} & \sigma^2 \end{pmatrix} + \begin{pmatrix} \sigma^2 & \mathbf{0} \\ \mathbf{0} & -\sigma^2 \end{pmatrix}\right]$$

$$= \tfrac{1}{2}\begin{pmatrix} \sigma^2 & \mathbf{0} \\ \mathbf{0} & \sigma^2 \end{pmatrix}\begin{pmatrix} 2\sigma^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} 2I_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} I_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

so:

$$\mathbf{P}^{-}_{D/W}\boldsymbol{\psi} = \begin{pmatrix} \boldsymbol{\phi}^{A} \\ \mathbf{0} \end{pmatrix} \quad , \quad \mathbf{P}^{+}_{D/W}\boldsymbol{\psi} = \begin{pmatrix} \mathbf{0} \\ \boldsymbol{\phi}^{B} \end{pmatrix}$$

and:

$$\mathbf{P}^{-}_{M1}\boldsymbol{\psi} = \begin{pmatrix} \mathbf{0} \\ \boldsymbol{\phi}^{B} \end{pmatrix} \quad , \quad \mathbf{P}^{+}_{M1}\boldsymbol{\psi} = \begin{pmatrix} \boldsymbol{\phi}^{A} \\ \mathbf{0} \end{pmatrix}$$

so, these projection operators split $\boldsymbol{\psi}$

5.

Note:
(Dirac)

$$\begin{pmatrix} I_2\left(-i\frac{\partial}{\partial t}+m\right) & -i\boldsymbol{\sigma}\bullet\vec{\nabla} \\ i\boldsymbol{\sigma}\bullet\vec{\nabla} & I_2\left(i\frac{\partial}{\partial t}+m\right) \end{pmatrix}\begin{pmatrix} \boldsymbol{\phi}_D^A \\ \boldsymbol{\phi}_D^B \end{pmatrix} = \mathbf{0}$$

(seperates space and mass-time)

so:

$$I_2\left(-i\frac{\partial}{\partial t}+m\right)\boldsymbol{\phi}_D^A - i\boldsymbol{\sigma}\bullet\vec{\nabla}\boldsymbol{\phi}_D^B = \mathbf{0}$$

and:

$$i\boldsymbol{\sigma}\bullet\vec{\nabla}\boldsymbol{\phi}_D^A + I_2\left(i\frac{\partial}{\partial t}+m\right)\boldsymbol{\phi}_D^B = \mathbf{0}$$

$\therefore$

$$\begin{pmatrix} (-i\partial_t+m) & 0 \\ 0 & (-i\partial_t+m) \end{pmatrix}\begin{pmatrix} \psi_D^0 \\ \psi_D^1 \end{pmatrix} - i\begin{pmatrix} 0 & \partial_1 \\ \partial_1 & 0 \end{pmatrix}\begin{pmatrix} \psi_D^2 \\ \psi_D^3 \end{pmatrix} +$$

$$- i\begin{pmatrix} 0 & -i\partial_2 \\ i\partial_2 & 0 \end{pmatrix}\begin{pmatrix} \psi_D^2 \\ \psi_D^3 \end{pmatrix} - i\begin{pmatrix} \partial_3 & 0 \\ 0 & -\partial_3 \end{pmatrix}\begin{pmatrix} \psi_D^2 \\ \psi_D^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and:

$$i\begin{pmatrix} 0 & \partial_1 \\ \partial_1 & 0 \end{pmatrix}\begin{pmatrix} \psi_D^0 \\ \psi_D^1 \end{pmatrix} + i\begin{pmatrix} 0 & -i\partial_2 \\ i\partial_2 & 0 \end{pmatrix}\begin{pmatrix} \psi_D^0 \\ \psi_D^1 \end{pmatrix} +$$

$$+ i\begin{pmatrix} \partial_3 & 0 \\ 0 & -\partial_3 \end{pmatrix}\begin{pmatrix} \psi_D^0 \\ \psi_D^1 \end{pmatrix} + \begin{pmatrix} (i\partial_t+m) & 0 \\ 0 & (i\partial_t+m) \end{pmatrix}\begin{pmatrix} \psi_D^2 \\ \psi_D^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

so:

$$\begin{pmatrix} (-i\partial_t+m) & 0 \\ 0 & (-i\partial_t+m) \end{pmatrix}\begin{pmatrix} \psi_D^0 \\ \psi_D^1 \end{pmatrix} +$$

$$+ \begin{pmatrix} -i\partial_3 & (-\partial_2 - i\partial_1) \\ (\partial_2 - i\partial_1) & i\partial_3 \end{pmatrix}\begin{pmatrix} \psi_D^2 \\ \psi_D^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and:

$$\begin{pmatrix} i\partial_3 & (i\partial_1 + \partial_2) \\ (i\partial_1 - \partial_2) & -i\partial_3 \end{pmatrix}\begin{pmatrix} \psi_D^0 \\ \psi_D^1 \end{pmatrix} +$$

$$+ \begin{pmatrix} (i\partial_t+m) & 0 \\ 0 & (i\partial_t+m) \end{pmatrix}\begin{pmatrix} \psi_D^2 \\ \psi_D^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore$

$$(-i\partial_t+m)\psi_D^0 - i\partial_3\psi_D^2 + (-\partial_2 - i\partial_1)\psi_D^3 = 0$$
$$(-i\partial_t+m)\psi_D^1 + (\partial_2 - i\partial_1)\psi_D^2 + i\partial_3\psi_D^3 = 0$$
$$i\partial_3\psi_D^0 + (i\partial_1 + \partial_2)\psi_D^1 + (i\partial_t+m)\psi_D^2 = 0$$

$$(i\partial_1 - \partial_2)\psi_D^0 - i\partial_3\psi_D^1 + (i\partial_t + m)\psi_D^3 = 0$$

And:

(Weyl / chiral)

$$\begin{pmatrix} i\mathrm{I}_2\frac{\partial}{\partial t} + i\boldsymbol{\sigma}\bullet\vec{\nabla} & -\mathrm{I}_2 m \\ -\mathrm{I}_2 m & i\mathrm{I}_2\frac{\partial}{\partial t} - i\boldsymbol{\sigma}\bullet\vec{\nabla} \end{pmatrix}\begin{pmatrix} \boldsymbol{\phi}_W^A \\ \boldsymbol{\phi}_W^B \end{pmatrix} = \mathbf{0}$$

(seperates mass and space-time)

so:

$$\left(i\mathrm{I}_2\frac{\partial}{\partial t} + i\boldsymbol{\sigma}\bullet\vec{\nabla}\right)\boldsymbol{\phi}_W^A - \mathrm{I}_2 m\boldsymbol{\phi}_W^B = \mathbf{0}$$

and:

$$(-\mathrm{I}_2 m)\boldsymbol{\phi}_W^A + \left(i\mathrm{I}_2\frac{\partial}{\partial t} - i\boldsymbol{\sigma}\bullet\vec{\nabla}\right)\boldsymbol{\phi}_W^B = \mathbf{0}$$

$\therefore$

$$\begin{pmatrix} i\partial_t & 0 \\ 0 & i\partial_t \end{pmatrix}\begin{pmatrix} \psi_W^0 \\ \psi_W^1 \end{pmatrix} + i\begin{pmatrix} 0 & \partial_1 \\ \partial_1 & 0 \end{pmatrix}\begin{pmatrix} \psi_W^0 \\ \psi_W^1 \end{pmatrix} + i\begin{pmatrix} 0 & -i\partial_2 \\ i\partial_2 & 0 \end{pmatrix}\begin{pmatrix} \psi_W^0 \\ \psi_W^1 \end{pmatrix} +$$
$$+ i\begin{pmatrix} \partial_3 & 0 \\ 0 & -\partial_3 \end{pmatrix}\begin{pmatrix} \psi_W^0 \\ \psi_W^1 \end{pmatrix} - \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} \psi_W^2 \\ \psi_W^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and:

$$-\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} \psi_W^0 \\ \psi_W^1 \end{pmatrix} + \begin{pmatrix} i\partial_t & 0 \\ 0 & i\partial_t \end{pmatrix}\begin{pmatrix} \psi_W^2 \\ \psi_W^3 \end{pmatrix} +$$
$$- i\begin{pmatrix} 0 & \partial_1 \\ \partial_1 & 0 \end{pmatrix}\begin{pmatrix} \psi_W^2 \\ \psi_W^3 \end{pmatrix} - i\begin{pmatrix} 0 & -i\partial_2 \\ i\partial_2 & 0 \end{pmatrix}\begin{pmatrix} \psi_W^2 \\ \psi_W^3 \end{pmatrix} - i\begin{pmatrix} \partial_3 & 0 \\ 0 & -\partial_3 \end{pmatrix}\begin{pmatrix} \psi_W^2 \\ \psi_W^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

so:

$$\begin{pmatrix} (i\partial_t + i\partial_3) & (i\partial_1 + \partial_2) \\ (i\partial_1 - \partial_2) & (i\partial_t - i\partial_3) \end{pmatrix}\begin{pmatrix} \psi_W^0 \\ \psi_W^1 \end{pmatrix} - \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} \psi_W^2 \\ \psi_W^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and:

$$\begin{pmatrix} -m & 0 \\ 0 & -m \end{pmatrix}\begin{pmatrix} \psi_W^0 \\ \psi_W^1 \end{pmatrix} + \begin{pmatrix} (i\partial_t - i\partial_3) & (-i\partial_1 - \partial_2) \\ (-i\partial_1 + \partial_2) & (i\partial_t + i\partial_3) \end{pmatrix}\begin{pmatrix} \psi_W^2 \\ \psi_W^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore$

$$(i\partial_t + i\partial_3)\psi_W^0 + (i\partial_1 + \partial_2)\psi_W^1 - m\psi_W^2 = 0$$
$$(i\partial_1 - \partial_2)\psi_W^0 + (i\partial_t - i\partial_3)\psi_W^1 - m\psi_W^3 = 0$$
$$-m\psi_W^0 + (i\partial_t - i\partial_3)\psi_W^2 + (-i\partial_1 - \partial_2)\psi_W^3 = 0$$
$$-m\psi_W^1 + (-i\partial_1 + \partial_2)\psi_W^2 + (i\partial_t + i\partial_3)\psi_W^3 = 0$$

And:

(Majorana(1))
$$\begin{pmatrix} \left(i\mathrm{I}_2\frac{\partial}{\partial t} + i\mathrm{I}_2\frac{\partial}{\partial x^2}\right) & \left(-i\sigma^1\frac{\partial}{\partial x^1} - i\sigma^3\frac{\partial}{\partial x^3} - \sigma^2 m\right) \\ \left(-i\sigma^1\frac{\partial}{\partial x^1} - i\sigma^3\frac{\partial}{\partial x^3} - \sigma^2 m\right) & \left(i\mathrm{I}_2\frac{\partial}{\partial t} - i\mathrm{I}_2\frac{\partial}{\partial x^2}\right) \end{pmatrix}\boldsymbol{\psi}_{M1} = \mathbf{0}$$

(seperates amalgams x^2-time and x^1-x^3-mass)

so:

$$\left(i\mathrm{I}_2\frac{\partial}{\partial t}+i\mathrm{I}_2\frac{\partial}{\partial x^2}\right)\boldsymbol{\phi}^A_{M1}+\left(-i\sigma^1\frac{\partial}{\partial x^1}-i\sigma^3\frac{\partial}{\partial x^3}-\sigma^2 m\right)\boldsymbol{\phi}^B_{M1}=\mathbf{0}$$

and:

$$\left(-i\sigma^1\frac{\partial}{\partial x^1}-i\sigma^3\frac{\partial}{\partial x^3}-\sigma^2 m\right)\boldsymbol{\phi}^A_{M1}+\left(i\mathrm{I}_2\frac{\partial}{\partial t}-i\mathrm{I}_2\frac{\partial}{\partial x^2}\right)\boldsymbol{\phi}^B_{M1}=\mathbf{0}$$

∴

$$\begin{pmatrix}(i\partial_t+i\partial_2) & 0\\ 0 & (i\partial_t+i\partial_2)\end{pmatrix}\begin{pmatrix}\psi^0_{M1}\\ \psi^1_{M1}\end{pmatrix}+\begin{pmatrix}0 & -i\partial_1\\ -i\partial_1 & 0\end{pmatrix}\begin{pmatrix}\psi^2_{M1}\\ \psi^3_{M1}\end{pmatrix}+$$

$$+\begin{pmatrix}-i\partial_3 & 0\\ 0 & i\partial_3\end{pmatrix}\begin{pmatrix}\psi^2_{M1}\\ \psi^3_{M1}\end{pmatrix}+\begin{pmatrix}0 & -im\\ im & 0\end{pmatrix}\begin{pmatrix}\psi^2_{M1}\\ \psi^3_{M1}\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

and:

$$\begin{pmatrix}0 & -i\partial_1\\ -i\partial_1 & 0\end{pmatrix}\begin{pmatrix}\psi^0_{M1}\\ \psi^1_{M1}\end{pmatrix}+\begin{pmatrix}-i\partial_3 & 0\\ 0 & i\partial_3\end{pmatrix}\begin{pmatrix}\psi^0_{M1}\\ \psi^1_{M1}\end{pmatrix}+$$

$$+\begin{pmatrix}0 & -im\\ im & 0\end{pmatrix}\begin{pmatrix}\psi^0_{M1}\\ \psi^1_{M1}\end{pmatrix}+\begin{pmatrix}(i\partial_t-i\partial_2) & 0\\ 0 & (i\partial_t-i\partial_2)\end{pmatrix}\begin{pmatrix}\psi^2_{M1}\\ \psi^3_{M1}\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

so:

$$\begin{pmatrix}(i\partial_t+i\partial_2) & 0\\ 0 & (i\partial_t+i\partial_2)\end{pmatrix}\begin{pmatrix}\psi^0_{M1}\\ \psi^1_{M1}\end{pmatrix}+\begin{pmatrix}-i\partial_3 & (-i\partial_1-im)\\ (-i\partial_1+im) & i\partial_3\end{pmatrix}\begin{pmatrix}\psi^2_{M1}\\ \psi^3_{M1}\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

and:

$$\begin{pmatrix}-i\partial_3 & (-i\partial_1-im)\\ (-i\partial_1+im) & i\partial_3\end{pmatrix}\begin{pmatrix}\psi^0_{M1}\\ \psi^1_{M1}\end{pmatrix}+\begin{pmatrix}(i\partial_t-i\partial_2) & 0\\ 0 & (i\partial_t-i\partial_2)\end{pmatrix}\begin{pmatrix}\psi^2_{M1}\\ \psi^3_{M1}\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

∴

$$(\partial_t+\partial_2)\psi^0_{M1}-\partial_3\psi^2_{M1}+(-\partial_1-m)\psi^3_{M1}=0$$
$$(\partial_t+\partial_2)\psi^1_{M1}+(-\partial_1+m)\psi^2_{M1}+\partial_3\psi^3_{M1}=0$$
$$-\partial_3\psi^0_{M1}+(-\partial_1-m)\psi^1_{M1}+(\partial_t-\partial_2)\psi^2_{M1}=0$$
$$(-\partial_1+m)\psi^0_{M1}+\partial_3\psi^1_{M1}+(\partial_t-\partial_2)\psi^3_{M1}=0$$

And:

(Majorana(2))

$$\begin{pmatrix}\left(i\mathrm{I}_2\frac{\partial}{\partial t}-\sigma^2\sigma^1\frac{\partial}{\partial x^1}-\sigma^2\sigma^3\frac{\partial}{\partial x^3}\right) & \left(-\sigma^2\frac{\partial}{\partial x^2}-\sigma^2 m\right)\\ \left(\sigma^2\frac{\partial}{\partial x^2}-\sigma^2 m\right) & \left(i\mathrm{I}_2\frac{\partial}{\partial t}-\sigma^2\sigma^1\frac{\partial}{\partial x^1}-\sigma^2\sigma^3\frac{\partial}{\partial x^3}\right)\end{pmatrix}\boldsymbol{\psi}_{M2}=\mathbf{0}$$

(seperates amalgams x^2-mass and x^1-x^3-time)

so:

$$\left(i\mathrm{I}_2\frac{\partial}{\partial t}-\sigma^2\sigma^1\frac{\partial}{\partial x^1}-\sigma^2\sigma^3\frac{\partial}{\partial x^3}\right)\boldsymbol{\phi}^A_{M2}+\left(-\sigma^2\frac{\partial}{\partial x^2}-\sigma^2 m\right)\boldsymbol{\phi}^B_{M2}=\mathbf{0}$$

and:

$$\left(\sigma^2\frac{\partial}{\partial x^2}-\sigma^2 m\right)\boldsymbol{\phi}^A_{M2}+\left(i\mathrm{I}_2\frac{\partial}{\partial t}-\sigma^2\sigma^1\frac{\partial}{\partial x^1}-\sigma^2\sigma^3\frac{\partial}{\partial x^3}\right)\boldsymbol{\phi}^B_{M2}=\mathbf{0}$$

$\therefore$

$$\begin{pmatrix} i\partial_t & 0 \\ 0 & i\partial_t \end{pmatrix}\begin{pmatrix} \psi^0_{M2} \\ \psi^1_{M2} \end{pmatrix}+\begin{pmatrix} i\partial_1 & 0 \\ 0 & -i\partial_1 \end{pmatrix}\begin{pmatrix} \psi^0_{M2} \\ \psi^1_{M2} \end{pmatrix}+$$

$$+\begin{pmatrix} 0 & -i\partial_3 \\ -i\partial_3 & 0 \end{pmatrix}\begin{pmatrix} \psi^0_{M2} \\ \psi^1_{M2} \end{pmatrix}+\begin{pmatrix} 0 & i(\partial_2+m) \\ -i(\partial_2+m) & 0 \end{pmatrix}\begin{pmatrix} \psi^2_{M2} \\ \psi^3_{M2} \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and:

$$\begin{pmatrix} 0 & i(-\partial_2+m) \\ -i(-\partial_2+m) & 0 \end{pmatrix}\begin{pmatrix} \psi^0_{M2} \\ \psi^1_{M2} \end{pmatrix}+\begin{pmatrix} i\partial_t & 0 \\ 0 & i\partial_t \end{pmatrix}\begin{pmatrix} \psi^2_{M2} \\ \psi^3_{M2} \end{pmatrix}+$$

$$+\begin{pmatrix} i\partial_1 & 0 \\ 0 & -i\partial_1 \end{pmatrix}\begin{pmatrix} \psi^2_{M2} \\ \psi^3_{M2} \end{pmatrix}+\begin{pmatrix} 0 & -i\partial_3 \\ -i\partial_3 & 0 \end{pmatrix}\begin{pmatrix} \psi^2_{M2} \\ \psi^3_{M2} \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

so:

$$\begin{pmatrix} (i\partial_t+i\partial_1) & -i\partial_3 \\ -i\partial_3 & (i\partial_t-i\partial_1) \end{pmatrix}\begin{pmatrix} \psi^0_{M2} \\ \psi^1_{M2} \end{pmatrix}+\begin{pmatrix} 0 & i(\partial_2+m) \\ -i(\partial_2+m) & 0 \end{pmatrix}\begin{pmatrix} \psi^2_{M2} \\ \psi^3_{M2} \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and:

$$\begin{pmatrix} 0 & i(-\partial_2+m) \\ -i(-\partial_2+m) & 0 \end{pmatrix}\begin{pmatrix} \psi^0_{M2} \\ \psi^1_{M2} \end{pmatrix}+\begin{pmatrix} (i\partial_t+i\partial_1) & -i\partial_3 \\ -i\partial_3 & (i\partial_t-i\partial_1) \end{pmatrix}\begin{pmatrix} \psi^2_{M2} \\ \psi^3_{M2} \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore$

$$(\partial_t+\partial_1)\psi^0_{M2}-\partial_3\psi^1_{M2}+(\partial_2+m)\psi^3_{M2}=0$$
$$-\partial_3\psi^0_{M2}+(\partial_t-\partial_1)\psi^1_{M2}-(\partial_2+m)\psi^2_{M2}=0$$
$$(-\partial_2+m)\psi^1_{M2}+(\partial_t+\partial_1)\psi^2_{M2}-\partial_3\psi^3_{M2}=0$$
$$-(-\partial_2+m)\psi^0_{M2}-\partial_3\psi^2_{M2}+(\partial_t-\partial_1)\psi^3_{M2}=0$$

6.

Matrix Transpose:

if $\mathbf{A} \equiv (A_{ij})$ then: $\mathbf{A}^T \equiv (A_{ji})$

Complex Conjugate:

if $x, y \in R, u \in C \ni u \equiv x + iy$ then: $u^* \equiv x - iy \quad (i \bullet i = -1)$

Matrix Complex Conjugate:

if $\mathbf{A} \equiv (A_{ij})$ then: $\mathbf{A}^* \equiv (A_{ij}^*)$

Matrix Hermitian Adjoint:

if $\mathbf{A} \equiv (A_{ij})$ then: $\mathbf{A}^\dagger \equiv (A_{ji}^*)$

So:

$$(A+B)^* = A^* + B^* \quad , \quad (\mathbf{A}+\mathbf{B})^* = \mathbf{A}^* + \mathbf{B}^*$$
$$(AB)^* = A^*B^* \quad , \quad (\mathbf{AB})^* = \mathbf{A}^*\mathbf{B}^*$$
$$(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$$
$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$$
$$(\mathbf{A}+\mathbf{B})^\dagger = \mathbf{A}^\dagger + \mathbf{B}^\dagger$$
$$(\mathbf{AB})^\dagger = \mathbf{B}^\dagger\mathbf{A}^\dagger$$

$\therefore$

$$\left[\left(i \sum_{\mu=0}^{3} \gamma^\mu \partial_\mu - \mathrm{I}m \right) \boldsymbol{\psi} \right]^\dagger = \mathbf{0}^\dagger = \mathbf{0}$$
$$= \left(i\gamma^0 \partial_0 \boldsymbol{\psi} + i \sum_{\mu=1}^{3} \gamma^\mu \partial_\mu \boldsymbol{\psi} - \mathrm{I}m\boldsymbol{\psi} \right)^\dagger$$
$$= (i\gamma^0 \partial_0 \boldsymbol{\psi})^\dagger + i^\dagger \sum_{\mu=1}^{3} (\gamma^\mu \partial_\mu \boldsymbol{\psi})^\dagger - \mathrm{I}m\boldsymbol{\psi}^\dagger$$
$$= (-i)(\partial_0 \boldsymbol{\psi}^\dagger)(\gamma^0)^\dagger + (-i) \sum_{\mu=1}^{3} (\partial_\mu \boldsymbol{\psi}^\dagger)(\gamma^\mu)^\dagger - \mathrm{I}m\boldsymbol{\psi}^\dagger$$

in any representation, where:

$$(\gamma^0)^\dagger = \gamma^0 \quad , \quad (\gamma^1)^\dagger = -\gamma^1 \quad , \quad (\gamma^2)^\dagger = -\gamma^2 \quad , \quad (\gamma^3)^\dagger = -\gamma^3$$

which is true for every one of the above representations

then:

$$-i(\partial_0 \boldsymbol{\psi}^\dagger)\gamma^0 + i \sum_{\mu=1}^{3} (\partial_\mu \boldsymbol{\psi}^\dagger)\gamma^\mu - \mathrm{I}m\boldsymbol{\psi}^\dagger = \mathbf{0}$$

and:

$$\left(i \sum_{\mu=0}^{3} \gamma^\mu \partial_\mu - \mathrm{I}m \right) \boldsymbol{\psi} = \mathbf{0}$$
$$= i\gamma^0 \partial_0 \boldsymbol{\psi} + i \sum_{\mu=0}^{3} \gamma^\mu \partial_\mu \boldsymbol{\psi} - \mathrm{I}m\boldsymbol{\psi}$$

$\therefore$

$$\left[-i\partial_0(\boldsymbol{\psi}^\dagger \gamma^0) + i \sum_{\mu=1}^{3} \partial_\mu(\boldsymbol{\psi}^\dagger \gamma^\mu) - \mathrm{I}m\boldsymbol{\psi}^\dagger \right] (\gamma^0 \boldsymbol{\psi}) = \mathbf{0}$$

and:

$$(\boldsymbol{\psi}^{\dagger}\gamma^{0})\left[\; i\partial_{0}(\gamma^{0}\boldsymbol{\psi})+i\sum_{\mu=0}^{3}\partial_{\mu}(\gamma^{\mu}\boldsymbol{\psi})-\mathrm{I}m\boldsymbol{\psi}\;\right]=\mathbf{0}$$

so, subtracting:

$$i(\boldsymbol{\psi}^{\dagger}\gamma^{0})\partial_{0}(\gamma^{0}\boldsymbol{\psi})+i\partial_{0}(\boldsymbol{\psi}^{\dagger}\gamma^{0})(\gamma^{0}\boldsymbol{\psi})+$$

$$+\,i(\boldsymbol{\psi}^{\dagger}\gamma^{0})\sum_{\mu=0}^{3}\partial_{\mu}(\gamma^{\mu}\boldsymbol{\psi})-\;i\sum_{\mu=1}^{3}\partial_{\mu}(\boldsymbol{\psi}^{\dagger}\gamma^{\mu})(\gamma^{0}\boldsymbol{\psi})=\mathbf{0}$$

$\therefore$

$$\partial_{0}(\boldsymbol{\psi}^{\dagger}\gamma^{0}\gamma^{0}\boldsymbol{\psi})+$$

$$+\sum_{\mu=1}^{3}[(\boldsymbol{\psi}^{\dagger}\gamma^{0})\partial_{\mu}(\gamma^{\mu}\boldsymbol{\psi})-\partial_{\mu}(\boldsymbol{\psi}^{\dagger}\gamma^{\mu})(\gamma^{0}\boldsymbol{\psi})]=\mathbf{0}$$

so:

$$\partial_{0}(\boldsymbol{\psi}^{\dagger}\gamma^{0}\gamma^{0}\boldsymbol{\psi})+$$

$$+\sum_{\mu=1}^{3}[(\boldsymbol{\psi}^{\dagger}\gamma^{0})\partial_{\mu}(\gamma^{\mu}\boldsymbol{\psi})-\partial_{\mu}(\boldsymbol{\psi}^{\dagger})(\gamma^{\mu}\gamma^{0}\boldsymbol{\psi})]=\mathbf{0}$$

and, noting: $\forall\mu\in\{1,2,3\},\gamma^{\mu}\gamma^{0}=-\gamma^{0}\gamma^{\mu}$, so:

$$\partial_{0}(\boldsymbol{\psi}^{\dagger}\gamma^{0}\gamma^{0}\boldsymbol{\psi})+$$

$$+\sum_{\mu=1}^{3}[(\boldsymbol{\psi}^{\dagger}\gamma^{0})\partial_{\mu}(\gamma^{\mu}\boldsymbol{\psi})+\partial_{\mu}(\boldsymbol{\psi}^{\dagger}\gamma^{0})(\gamma^{\mu}\boldsymbol{\psi})]=\mathbf{0}$$

$\therefore$

$$\partial_{0}(\boldsymbol{\psi}^{\dagger}\gamma^{0}\gamma^{0}\boldsymbol{\psi})+\sum_{\mu=1}^{3}\partial_{\mu}[(\boldsymbol{\psi}^{\dagger}\gamma^{0})(\gamma^{\mu}\boldsymbol{\psi})]=\mathbf{0}$$

so, defining:

$$\rho\equiv\boldsymbol{\psi}^{\dagger}(\gamma^{0}\gamma^{0})\boldsymbol{\psi}=\boldsymbol{\psi}^{\dagger}\boldsymbol{\psi}$$

and:

$$j^{\mu}\equiv\boldsymbol{\psi}^{\dagger}(\gamma^{0}\gamma^{\mu})\boldsymbol{\psi}$$

$\partial_{0}\rho+\sum_{\mu=1}^{3}\partial_{\mu}j^{\mu}=\mathbf{0}$ has the form of an equation of continuity

ρ is called a probability density

$\vec{\mathbf{j}}$ is called a probability current density

7.

The **expectation value** of a physically meaningful quantity is given by:

$$\langle \mathbf{F} \rangle \equiv \iiint_V \boldsymbol{\psi}^\dagger \mathbf{F} \boldsymbol{\psi} dV$$

so:

$$\frac{d}{dt}\langle \mathbf{F} \rangle \equiv \frac{d}{dt}\iiint_V \boldsymbol{\psi}^\dagger \mathbf{F} \boldsymbol{\psi} dV = \iiint_V \frac{\partial}{\partial t}(\boldsymbol{\psi}^\dagger \mathbf{F} \boldsymbol{\psi}) dV$$

$$= \iiint_V \frac{\partial}{\partial t}(\boldsymbol{\psi}^\dagger)\mathbf{F}\boldsymbol{\psi} dV + \iiint_V \boldsymbol{\psi}^\dagger \frac{\partial}{\partial t}(\mathbf{F})\boldsymbol{\psi} dV + \iiint_V \boldsymbol{\psi}^\dagger \mathbf{F} \frac{\partial}{\partial t}(\boldsymbol{\psi}) dV$$

$$= \iiint_V \left(\frac{\partial}{\partial t}\boldsymbol{\psi}\right)^\dagger \mathbf{F}\boldsymbol{\psi} dV + \iiint_V \boldsymbol{\psi}^\dagger \mathbf{F}\left(\frac{\partial}{\partial t}\boldsymbol{\psi}\right) dV + \left\langle \frac{\partial \mathbf{F}}{\partial t} \right\rangle$$

but: $i\frac{\partial}{\partial t}\boldsymbol{\psi} = \mathbf{H}\boldsymbol{\psi}$, so:

$$\frac{d}{dt}\langle \mathbf{F} \rangle = \iiint_V \left(\frac{1}{i}\mathbf{H}\boldsymbol{\psi}\right)^\dagger \mathbf{F}\boldsymbol{\psi} dV + \iiint_V \boldsymbol{\psi}^\dagger \mathbf{F}\left(\frac{1}{i}\mathbf{H}\boldsymbol{\psi}\right) dV + \left\langle \frac{\partial \mathbf{F}}{\partial t} \right\rangle$$

$$= \iiint_V (-i\mathbf{H}\boldsymbol{\psi})^\dagger \mathbf{F}\boldsymbol{\psi} dV + \iiint_V \boldsymbol{\psi}^\dagger \mathbf{F}(-i\mathbf{H}\boldsymbol{\psi}) dV + \left\langle \frac{\partial \mathbf{F}}{\partial t} \right\rangle$$

$$= \iiint_V i\boldsymbol{\psi}^\dagger \mathbf{H}^\dagger \mathbf{F}\boldsymbol{\psi} dV + \iiint_V -i\boldsymbol{\psi}^\dagger \mathbf{F}\mathbf{H}\boldsymbol{\psi} dV + \left\langle \frac{\partial \mathbf{F}}{\partial t} \right\rangle$$

and: $(\boldsymbol{\alpha}_D^\mu)^\dagger = \boldsymbol{\alpha}_D^\mu$, $\boldsymbol{\beta}_D^\dagger = \boldsymbol{\beta}_D$, are both Hermitian, so:

$\mathbf{H}_D^\dagger \equiv \left(\boldsymbol{\alpha}^\mu \mathbf{p}_\mu + \boldsymbol{\beta} m\right)^\dagger = \boldsymbol{\alpha}^\mu \mathbf{p}_\mu + \boldsymbol{\beta} m = \mathbf{H}_D$ is Hermitian

$\therefore$

$$\frac{d}{dt}\langle \mathbf{F} \rangle = i\iiint_V \boldsymbol{\psi}^\dagger (\mathbf{H}\mathbf{F} - \mathbf{F}\mathbf{H})\boldsymbol{\psi} dV + \left\langle \frac{\partial \mathbf{F}}{\partial t} \right\rangle$$

Defining:

$$[\mathbf{A}, \mathbf{B}] \equiv \mathbf{A}\mathbf{B} - \mathbf{B}\mathbf{A}$$

then:

$$\frac{d}{dt}\langle \mathbf{F} \rangle = i\iiint_V \boldsymbol{\psi}^\dagger [\mathbf{H}, \mathbf{F}]\boldsymbol{\psi} dV + \left\langle \frac{\partial \mathbf{F}}{\partial t} \right\rangle$$

so, if $\mathbf{F}$ is not explicitly a function of t , then:

$$\frac{d}{dt}\langle \mathbf{F} \rangle = i\iiint_V \boldsymbol{\psi}^\dagger [\mathbf{H}, \mathbf{F}]\boldsymbol{\psi} dV$$

and, therefore, if $[\mathbf{H}, \mathbf{F}] = \mathbf{H}\mathbf{F} - \mathbf{F}\mathbf{H} = \mathbf{0}$

then:

$$\frac{d}{dt}\langle \mathbf{F} \rangle = \mathbf{0}$$

so, $\mathbf{F}$ is called a constant of the motion, whenever: $[\mathbf{H}, \mathbf{F}] = \mathbf{0}$

Note that:

$$[\mathbf{H}, \mathbf{p}_\nu] = \mathbf{H}\mathbf{p}_\nu - \mathbf{p}_\nu \mathbf{H} = \left(\boldsymbol{\alpha}^\mu \mathbf{p}_\mu + \boldsymbol{\beta} m\right)\mathbf{p}_\nu - \mathbf{p}_\nu\left(\boldsymbol{\alpha}^\mu \mathbf{p}_\mu + \boldsymbol{\beta} m\right)$$

$$= \boldsymbol{\alpha}^\mu \mathbf{p}_\mu \mathbf{p}_\nu + \boldsymbol{\beta} m \mathbf{p}_\nu - \mathbf{p}_\nu \boldsymbol{\alpha}^\mu \mathbf{p}_\mu + \mathbf{p}_\nu \boldsymbol{\beta} m = \mathbf{0}$$

so $\mathbf{p}_\nu$, and, thus $\mathbf{p}$, is a constant of the motion.

Now, from Classical Mechanics, **Angular Momentum** is a conserved quantity, defined as:

$$\vec{\mathbf{L}} \equiv \vec{\mathbf{r}} \times \vec{\mathbf{p}}$$

so:

$$\vec{\mathbf{L}}_x = y\mathbf{p}_3 - z\mathbf{p}_2 \quad , \quad \vec{\mathbf{L}}_y = z\mathbf{p}_1 - x\mathbf{p}_3 \quad , \quad \vec{\mathbf{L}}_z = x\mathbf{p}_2 - y\mathbf{p}_1$$

$$\begin{aligned}
\therefore \left[\mathbf{H}, \vec{\mathbf{L}}_x\right] &= \mathbf{H}\vec{\mathbf{L}}_x - \vec{\mathbf{L}}_x\mathbf{H} \\
&= \left(\boldsymbol{\alpha}^\mu \mathbf{p}_\mu + \boldsymbol{\beta} m\right)(y\mathbf{p}_3 - z\mathbf{p}_2) - (y\mathbf{p}_3 - z\mathbf{p}_2)\left(\boldsymbol{\alpha}^\mu \mathbf{p}_\mu + \boldsymbol{\beta} m\right) \\
&= \left(-i\boldsymbol{\alpha}^\mu \frac{\partial}{\partial x^\mu} + \boldsymbol{\beta} m\right)(y\mathbf{p}_3 - z\mathbf{p}_2) - \left(-iy\frac{\partial}{\partial z} + iz\frac{\partial}{\partial y}\right)\left(\boldsymbol{\alpha}^\mu \mathbf{p}_\mu + \boldsymbol{\beta} m\right) \\
&= -i\boldsymbol{\alpha}^\mu \frac{\partial}{\partial x^\mu}(y\mathbf{p}_3 - z\mathbf{p}_2) + \boldsymbol{\beta} m(y\mathbf{p}_3 - z\mathbf{p}_2) + \\
&\quad -\left(-iy\frac{\partial}{\partial z} + iz\frac{\partial}{\partial y}\right)\left(\boldsymbol{\alpha}^\mu \mathbf{p}_\mu\right) - (y\mathbf{p}_3 - z\mathbf{p}_2)(\boldsymbol{\beta} m) \\
&= -i\boldsymbol{\alpha}^\mu \frac{\partial}{\partial x^\mu}(x^2\mathbf{p}_3) + i\boldsymbol{\alpha}^\mu \frac{\partial}{\partial x^\mu}(x^3\mathbf{p}_2) + \boldsymbol{\beta} m(y\mathbf{p}_3 - z\mathbf{p}_2) - 0 - (y\mathbf{p}_3 - z\mathbf{p}_2)(\boldsymbol{\beta} m) \\
&= -i\boldsymbol{\alpha}^\mu \delta^2_\mu \mathbf{p}_3 + i\boldsymbol{\alpha}^\mu \delta^3_\mu \mathbf{p}_2 + \boldsymbol{\beta} m(y\mathbf{p}_3 - z\mathbf{p}_2) - 0 - (y\mathbf{p}_3 - z\mathbf{p}_2)(\boldsymbol{\beta} m) \\
&= -i(\boldsymbol{\alpha}^2\mathbf{p}_3 - \boldsymbol{\alpha}^3\mathbf{p}_2)
\end{aligned}$$

similarly:

$$\left[\mathbf{H}, \vec{\mathbf{L}}_y\right] = -i(\boldsymbol{\alpha}^3\mathbf{p}_1 - \boldsymbol{\alpha}^1\mathbf{p}_3)$$

$$\left[\mathbf{H}, \vec{\mathbf{L}}_z\right] = -i(\boldsymbol{\alpha}^1\mathbf{p}_2 - \boldsymbol{\alpha}^2\mathbf{p}_1)$$

Now, let:

$$\vec{\boldsymbol{\Sigma}} \equiv \hat{\mathbf{i}}\Sigma^1 + \hat{\mathbf{j}}\Sigma^2 + \hat{\mathbf{k}}\Sigma^3$$

and:

$$\vec{\mathbf{S}} \equiv i\left(\vec{\Sigma} \times \vec{\boldsymbol{\Sigma}}\right) = \hat{\mathbf{i}}(\boldsymbol{\Sigma}^2\boldsymbol{\Sigma}^3 - \boldsymbol{\Sigma}^3\boldsymbol{\Sigma}^2)i + \hat{\mathbf{j}}(\boldsymbol{\Sigma}^3\boldsymbol{\Sigma}^1 - \boldsymbol{\Sigma}^1\boldsymbol{\Sigma}^3)i + \hat{\mathbf{k}}(\boldsymbol{\Sigma}^1\boldsymbol{\Sigma}^2 - \boldsymbol{\Sigma}^2\boldsymbol{\Sigma}^1)i$$

[because $\boldsymbol{\Sigma}^1, \boldsymbol{\Sigma}^2, \boldsymbol{\Sigma}^3$ are matrices (not reals): $\vec{\boldsymbol{\Sigma}} \times \vec{\boldsymbol{\Sigma}} \neq \mathbf{0}$]

$$\boldsymbol{\Sigma}^1 = i\boldsymbol{\gamma}^1\boldsymbol{\gamma}^1\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3 = i(\boldsymbol{\gamma}^1\boldsymbol{\gamma}^1)\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3 = -i\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3 = i\boldsymbol{\gamma}^3\boldsymbol{\gamma}^2$$

$$\boldsymbol{\Sigma}^2 = i\boldsymbol{\gamma}^2\boldsymbol{\gamma}^1\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3 = -i\boldsymbol{\gamma}^1(\boldsymbol{\gamma}^2\boldsymbol{\gamma}^2)\boldsymbol{\gamma}^3 = i\boldsymbol{\gamma}^1\boldsymbol{\gamma}^3$$

$$\boldsymbol{\Sigma}^3 = i\boldsymbol{\gamma}^3\boldsymbol{\gamma}^1\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3 = i\boldsymbol{\gamma}^1(\boldsymbol{\gamma}^3\boldsymbol{\gamma}^3)\boldsymbol{\gamma}^2 = i\boldsymbol{\gamma}^1\boldsymbol{\gamma}^2$$

$\therefore$

$$\begin{aligned}
\boldsymbol{\Sigma}^2\boldsymbol{\Sigma}^3 &= i\boldsymbol{\gamma}^2\boldsymbol{\gamma}^1\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3 i\boldsymbol{\gamma}^3\boldsymbol{\gamma}^1\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3 = -\boldsymbol{\gamma}^2\boldsymbol{\gamma}^1\boldsymbol{\gamma}^2(\boldsymbol{\gamma}^3\boldsymbol{\gamma}^3)\boldsymbol{\gamma}^1\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3 \\
&= \boldsymbol{\gamma}^2\boldsymbol{\gamma}^1\boldsymbol{\gamma}^2\boldsymbol{\gamma}^1\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3 = -(\boldsymbol{\gamma}^2\boldsymbol{\gamma}^2)(\boldsymbol{\gamma}^1\boldsymbol{\gamma}^1)\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3 = -\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3 = -i\boldsymbol{\Sigma}^1 \\
\boldsymbol{\Sigma}^3\boldsymbol{\Sigma}^2 &= i\boldsymbol{\gamma}^3\boldsymbol{\gamma}^1\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3 i\boldsymbol{\gamma}^2\boldsymbol{\gamma}^1\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3 = -\boldsymbol{\gamma}^3\boldsymbol{\gamma}^1\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3\boldsymbol{\gamma}^2\boldsymbol{\gamma}^1\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3 \\
&= \boldsymbol{\gamma}^3\boldsymbol{\gamma}^1\boldsymbol{\gamma}^3(\boldsymbol{\gamma}^2\boldsymbol{\gamma}^2)\boldsymbol{\gamma}^1\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3 = -\boldsymbol{\gamma}^3\boldsymbol{\gamma}^1\boldsymbol{\gamma}^3\boldsymbol{\gamma}^1\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3 \\
&= (\boldsymbol{\gamma}^3\boldsymbol{\gamma}^3)(\boldsymbol{\gamma}^1\boldsymbol{\gamma}^1)\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3 = \boldsymbol{\gamma}^2\boldsymbol{\gamma}^3 = i\boldsymbol{\Sigma}^1
\end{aligned}$$

so:

$$\vec{\mathbf{S}}_x = i(\boldsymbol{\Sigma}^2\boldsymbol{\Sigma}^3 - \boldsymbol{\Sigma}^3\boldsymbol{\Sigma}^2) = 2\boldsymbol{\Sigma}^1$$

and:

$$\begin{aligned}
\vec{\mathbf{S}}_y = i(\boldsymbol{\Sigma}^3\boldsymbol{\Sigma}^1 - \boldsymbol{\Sigma}^1\boldsymbol{\Sigma}^3) &= i(i\boldsymbol{\gamma}^1\boldsymbol{\gamma}^2 i\boldsymbol{\gamma}^3\boldsymbol{\gamma}^2 - i\boldsymbol{\gamma}^3\boldsymbol{\gamma}^2 i\boldsymbol{\gamma}^1\boldsymbol{\gamma}^2) \\
&= i(-\boldsymbol{\gamma}^1\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3\boldsymbol{\gamma}^2 + \boldsymbol{\gamma}^3\boldsymbol{\gamma}^2\boldsymbol{\gamma}^1\boldsymbol{\gamma}^2) = i[\boldsymbol{\gamma}^1(\boldsymbol{\gamma}^2\boldsymbol{\gamma}^2)\boldsymbol{\gamma}^3 - \boldsymbol{\gamma}^3\boldsymbol{\gamma}^1(\boldsymbol{\gamma}^2\boldsymbol{\gamma}^2)] \\
&= i[-2i(i\boldsymbol{\gamma}^1\boldsymbol{\gamma}^3)] = 2\boldsymbol{\Sigma}^2
\end{aligned}$$

and:

$$\begin{aligned}
\vec{\mathbf{S}}_z = i(\boldsymbol{\Sigma}^1\boldsymbol{\Sigma}^2 - \boldsymbol{\Sigma}^2\boldsymbol{\Sigma}^1) &= i(i\boldsymbol{\gamma}^3\boldsymbol{\gamma}^2 i\boldsymbol{\gamma}^1\boldsymbol{\gamma}^3 - i\boldsymbol{\gamma}^1\boldsymbol{\gamma}^3 i\boldsymbol{\gamma}^3\boldsymbol{\gamma}^2) \\
&= i(-\boldsymbol{\gamma}^3\boldsymbol{\gamma}^2\boldsymbol{\gamma}^1\boldsymbol{\gamma}^3 + \boldsymbol{\gamma}^1\boldsymbol{\gamma}^3\boldsymbol{\gamma}^3\boldsymbol{\gamma}^2 = -\boldsymbol{\gamma}^2(\boldsymbol{\gamma}^3\boldsymbol{\gamma}^3)\boldsymbol{\gamma}^1 + \boldsymbol{\gamma}^1(\boldsymbol{\gamma}^3\boldsymbol{\gamma}^3)\boldsymbol{\gamma}^2) \\
&= i[-2i(i\boldsymbol{\gamma}^1\boldsymbol{\gamma}^2)] = 2\boldsymbol{\Sigma}^3
\end{aligned}$$

So:

$$\vec{\mathbf{S}} \equiv i\left(\vec{\Sigma} \times \vec{\mathbf{\Sigma}}\right) = 2\left(\hat{\mathbf{i}}\mathbf{\Sigma}^1 + \hat{\mathbf{j}}\mathbf{\Sigma}^2 + \hat{\mathbf{k}}\mathbf{\Sigma}^3\right) = 2\vec{\Sigma}$$

and, then:

$$\begin{aligned}
[\mathbf{H},\mathbf{\Sigma}^1] &= \left(\boldsymbol{\alpha}^\mu \mathbf{p}_\mu + \boldsymbol{\beta} m\right)(-i\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3) - (-i\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3)\left(\boldsymbol{\alpha}^\mu \mathbf{p}_\mu + \boldsymbol{\beta} m\right) \\
&= \left(\boldsymbol{\gamma}^0\boldsymbol{\gamma}^\mu \mathbf{p}_\mu + \boldsymbol{\beta} m\right)(-i\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3) - (-i\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3)\left(\boldsymbol{\gamma}^0\boldsymbol{\gamma}^\mu \mathbf{p}_\mu + \boldsymbol{\beta} m\right) \\
&= (-i\boldsymbol{\gamma}^0\boldsymbol{\gamma}^\mu\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3)\mathbf{p}_\mu + m\boldsymbol{\gamma}^0(-i\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3) - (-i\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3\boldsymbol{\gamma}^0\boldsymbol{\gamma}^\mu)\mathbf{p}_\mu - (-i\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3)\boldsymbol{\gamma}^0 m \\
&= (-i\boldsymbol{\gamma}^0\boldsymbol{\gamma}^\mu\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3)\mathbf{p}_\mu + (-i\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3)\boldsymbol{\gamma}^0 m - (-i\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3\boldsymbol{\gamma}^0\boldsymbol{\gamma}^\mu)\mathbf{p}_\mu - (-i\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3)\boldsymbol{\gamma}^0 m \\
&= -i\boldsymbol{\gamma}^0[(\boldsymbol{\gamma}^\mu\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3) - (\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3\boldsymbol{\gamma}^\mu)]\mathbf{p}_\mu \\
&= -i\boldsymbol{\gamma}^0[(\boldsymbol{\gamma}^1\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3) - (\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3\boldsymbol{\gamma}^1)]\mathbf{p}_1 - i\boldsymbol{\gamma}^0[(\boldsymbol{\gamma}^2\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3) - (\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3\boldsymbol{\gamma}^2)]\mathbf{p}_2 - i\boldsymbol{\gamma}^0[(\boldsymbol{\gamma}^3\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3) - (\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3\boldsymbol{\gamma}^3)]\mathbf{p}_3 \\
&= -i\boldsymbol{\gamma}^0[(\boldsymbol{\gamma}^1\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3) - (\boldsymbol{\gamma}^1\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3)]\mathbf{p}_1 - i\boldsymbol{\gamma}^0[(\boldsymbol{\gamma}^2\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3) - (-\boldsymbol{\gamma}^2\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3)]\mathbf{p}_2 - i\boldsymbol{\gamma}^0[(-\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3\boldsymbol{\gamma}^3) - (\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3\boldsymbol{\gamma}^3)]\mathbf{p}_3 \\
&= \boldsymbol{\gamma}^0[-2i\boldsymbol{\gamma}^3]\mathbf{p}_2 + \boldsymbol{\gamma}^0[2i\boldsymbol{\gamma}^2]\mathbf{p}_3 \\
&= 2i\boldsymbol{\alpha}^2\mathbf{p}_3 - 2i\boldsymbol{\alpha}^3\mathbf{p}_2 = 2i(\boldsymbol{\alpha}^2\mathbf{p}_3 - \boldsymbol{\alpha}^3\mathbf{p}_2) = -2\left[\mathbf{H},\vec{\mathbf{L}}_x\right] \\
&= [\mathbf{H},\tfrac{1}{2}\mathbf{S}^1]
\end{aligned}$$

similarly:

$$[\mathbf{H},\mathbf{\Sigma}^2] = [\mathbf{H},\tfrac{1}{2}\mathbf{S}^2] = -2\left[\mathbf{H},\vec{\mathbf{L}}_y\right]$$

$$[\mathbf{H},\mathbf{\Sigma}^3] = [\mathbf{H},\tfrac{1}{2}\mathbf{S}^3] = -2\left[\mathbf{H},\vec{\mathbf{L}}_z\right]$$

so, let:

$$\vec{\mathbf{J}} \equiv \vec{\mathbf{L}} + \vec{\mathbf{S}}$$

then:

$$[\mathbf{H},\mathbf{J}^1] = [\mathbf{H},\mathbf{J}^2] = [\mathbf{H},\mathbf{J}^3] = \mathbf{0}$$

$\vec{\mathbf{S}}$ is called the **Spin Angular Momentum**

$\mathbf{J}$ is called the **Total Angular Momentum**, and is a constant of the motion

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