Harmonic Representation of Prime Numbers

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Abstract

We extend in two dimensions the problem of prime numbers by introducing a triangular algorithm which attributes to each prime the number 1 and to each nonprime the number 0:

n	μ_n
2	$1 = 1$
3	$1 = 1 \cdot 1$
4	$0 = 1 \cdot 0 \cdot 1$
5	$1 = 1 \cdot 1 \cdot 1 \cdot 1$
6	$0 = 1 \cdot 0 \cdot 0 \cdot 1 \cdot 1$
7	$1 = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1$
8	$0 = 1 \cdot 0 \cdot 1 \cdot 0 \cdot 1 \cdot 1 \cdot 1$
9	$0 = 1 \cdot 1 \cdot 0 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1$
10	$0 = 1 \cdot 0 \cdot 1 \cdot 1 \cdot 0 \cdot 1 \cdot 1 \cdot 1 \cdot 1$

It is shown that the positions of the primes within the set of integers are not arbitrary, but are the result of precise combinations of the vertical harmonic sequences forming this algorithm. The characteristic function μ_n defining the positions of the primes is obtained from the product of the elements of each row of the algorithm: $\mu_2 = 1$;

$$\mu_n = \prod_{\kappa=2}^{n-1}\left[1 - \frac{1}{\kappa}\sum_{\ell=0}^{\kappa-1} cos\left(2\pi\frac{n}{\kappa}\ell\right)\right] = \begin{cases} 1\,;\; n\ is\ a\ prime \\ 0\,;\; n\ is\ not\ a\ prime \end{cases} \quad ;\; n = 3,4,5,\ldots$$

The *prime counting function* $\varpi(N)$ [equal to the number of primes that are smaller or equal to a given integer N≥2] and the index s= $\varpi(p)$ of a prime p, are expressed as sums over μ_n:

$$\varpi(N) = \sum_{n=2}^{N}\mu_n \qquad\qquad s = \sum_{n=2}^{p}\mu_n$$

so that $\varpi(N)$=(1, 2, 2, 3, 3, 4, 4, 4, 4, 5, 5,…) ; N=2, 3, 4, 5,… is calculated exactly without resorting to the ζ-function and to Riemann's hypothesis. Inversely, it is shown that the s^{th} prime is given by

$$p_s = 2 + \frac{2}{\pi}\sum_{N=2}^{\infty}\int_0^{\pi/2}\frac{sin[(2s-1)x]}{sinx}cos[2x\,\varpi(N)]\,dx$$

and the full set of primes p_s = (2, 3, 5, 7, 11, 13, 17, 19,…) ; s=1, 2, 3,… is derived in terms of harmonic functions.

An alternative form of μ_n is also obtained by an additive harmonic triangular algorithm and the solution of equation

$$\sum_{\kappa=2}^{n-1}\sum_{\ell=0}^{\kappa-1} cos\left(2\pi\frac{n}{\kappa}\ell\right) = 0$$

is shown to define all the primes n=3, 5, 7, 11, 13, 17,…except n=2.

1. Introduction

In his interesting book “The music of the primes” [1] Marcus du Sautoy describes the epic journey of finding order in the distribution of the most important numbers of mathematics: *the primes*. Are they infinite? Is there harmony in their positions among integers or they are governed by statistical laws? These questions led us from the original work of Euclid and Eratosthenes to the fundamental theories of Euler, Gauss, Dirichlet and Riemann. In particular, Gauss and Legendre by inspection of the tables of primes, proposed at the end of the 18^{th} century an asymptotic formula for the number of primes $\pi(N)$ not exceeding a certain positive integer $N \geq 2$: $\pi(N) \approx N/\ln N$; $N \gg 1$. This conjecture was solved about hundred years later (1896) by Hadamard and de la Vallee Poussin and is now known as the *prime number theorem* [2]. On the other hand, Riemann [3] (1859) also proposed that the calculation of the nontrivial roots of the ζ-function, all existing on the line $x = \frac{1}{2}$ of the complex plane (Riemann hypothesis), may lead to the derivation of $\pi(N)$ and the determination of the distribution of primes. Since then, many mathematicians have developed various theories about prime numbers leading, as it was well put by John Derbyshire in his excellent book [4], to a veritable “Prime Obsession”.

Reading about the efforts made to study the problem of the primes, one may observe a connection to another equally difficult problem, this time in physics, that occupied the minds of many scientists over centuries: the question of the motion of the planets. Is their motion random? Or some periodic elements that are observed are related to deeper harmonic laws? Again, this problem started from the Greek philosophers and Ptolemy and it was completely solved by Copernicus, Kepler, Galileo and Newton. It is now recognized that the crucial step in the solution of the planetary motion was to transfer the origin of the frame of reference from the Earth to the Sun and subsequently to identify the orbits of the planets as two-dimensional (2-D) curves located on a plane where the harmonic nature of the motion is fully manifested.

The approach in the present article follows similar logic. Consider the following harmonic sequences:

$$h_1(n) = (1, 1, 1, 1, \dots) \quad ; \quad n = 1, 2, 3,\dots$$

$$h_2(n) = (2, 0, 2, 0,\dots) \quad ; \quad n = 2, 3, 4,\dots$$

$$h_3(n) = (3, 0, 0, 3, 0, 0, \dots) \quad ; \quad n = 3, 4, 5,\dots$$

$$h_4(n) = (4, 0, 0, 0, 4, 0, 0, 0,\dots) \quad ; \quad n = 4, 5, 6,\dots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \tag{1a}$$

written compactly as

$$h_\kappa(n) = \sum_{\ell=0}^{\kappa-1} \cos\left(2\pi \frac{n}{\kappa}\ell\right) \quad ; \quad n = \kappa, \kappa+1, \kappa+2, \dots \; ; \; \kappa = 1, 2, 3 \dots \tag{1b}$$

Using the above sequences as columns we construct a 2-D triangular matrix where n=1,2,3,... is the index of the rows and κ=1,2,3,... is the index of the columns. From the way the matrix is constructed it becomes clear that *the elements of each row of the matrix represent the divisors of the index n of that row:*

n												
1	1											
2	1	2										
3	1	0	3									
4	1	2	0	4								
5	1	0	0	0	5							
6	1	2	3	0	0	6						
7	1	0	0	0	0	0	7					
8	1	2	0	4	0	0	0	8				
9	1	0	3	0	0	0	0	0	9			
10	1	2	0	0	5	0	0	0	0	10		
11	1	0	0	0	0	0	0	0	0	0	11	
12	1	2	3	4	0	6	0	0	0	0	0	12

. . . . (2)

Therefore, the above *harmonic triangular matrix* leads to the important conclusion that *the divisors of successive positive integers are not arbitrary and independent of each other but they are inter-related by harmonic laws.* In the present article, extending this idea to prime numbers and using again only harmonic sequences as columns, we construct another harmonic 2-D triangular matrix having elements (0,1) where again the elements of each row correspond to the divisors of the index n of that row. Then, we *project* this matrix horizontally by taking the *product* of all elements of each row and form an *algorithm* defining a new column [μ_n] that identifies if the index n of each row is a prime number. It becomes therefore clear that the positions of primes within the set of integers are governed by precise harmonic laws in 2-D and it is the projection of 2-D to 1-D that makes the distribution of primes look complicated and even random.

Taking a free parallel [5], we could say that the idea of a 2-D harmonic algorithm resembles the structure of *polyphonic music* where the vertical variation is based on *harmony* and the horizontal variation produces the *melody* of each instrument.

In section 2 of the present work, we define in detail the triangular algorithm of prime numbers in terms of vertical harmonic sequences and some properties of its elements are discussed. Also we derive from this algorithm explicitly the characteristic function μ_n that defines the positions of primes within the set of integers. In section 3, we obtain exactly *the prime counting function* $\varpi(N)$ [equal to the number of primes that are smaller or equal to a given integer $N \geq 2$] as a sum over μ_n. In section 4, the index s of the sth prime p is expressed in terms of p also as a sum over μ_n and inversely p is expressed in terms of s by a series of harmonic integrals. Also, a formula for the next prime is derived.
In section 5, an alternative form of μ_n is obtained by changing the elements of the triangular algorithm of the primes according to: $1 \rightarrow 0$; $0 \rightarrow \kappa$ where κ is the index of the column.
Also, the product of the terms of each row is replaced by the sum of these terms.
In section 6, we derive a harmonic equation the solution of which *defines all prime numbers except 2.*

2. The triangular algorithm of prime numbers

Consider the harmonic sequences $\alpha_\kappa(n)$; $n \geq \kappa+1$; $\kappa=1, 2, 3, \ldots$

$$\alpha_1(n) = (1, 1, 1, 1, 1, \ldots) \ ; \ n=2, 3, 4, \ldots$$
$$\alpha_2(n) = (1, 0, 1, 0, 1, \ldots) \ ; \ n=3, 4, 5, \ldots$$
$$\alpha_3(n) = (1, 1, 0, 1, 1, 0\ldots) \ ; \ n=4, 5, 6, \ldots$$
$$\alpha_4(n) = (1, 1, 1, 0, 1, 1, 1, 0\ldots) \ ; \ n=5, 6, 7, \ldots$$
$$\cdots \qquad (3)$$

For $\kappa \geq 2$ these sequences have the following important property:

$$\alpha_\kappa(n) = \begin{cases} 1 \ ; \ \kappa \text{ is not a divisor of } n \\ 0 \ ; \ \kappa \text{ is a divisor of } n \end{cases} \tag{4}$$

We construct next an algorithm in the form of a 2-D triangular matrix by using as columns the sequences $\alpha_\kappa(n)$ of Eqs (3) where κ is the index of each column:

$$\begin{array}{ll}
n & \mu_n \\
2 & 1 = 1 \\
3 & 1 = 1 \cdot 1 \\
4 & 0 = 1 \cdot 0 \cdot 1 \\
5 & 1 = 1 \cdot 1 \cdot 1 \cdot 1 \\
6 & 0 = 1 \cdot 0 \cdot 0 \cdot 1 \cdot 1 \\
7 & 1 = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \\
8 & 0 = 1 \cdot 0 \cdot 1 \cdot 0 \cdot 1 \cdot 1 \cdot 1 \\
9 & 0 = 1 \cdot 1 \cdot 0 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \\
10 & 0 = 1 \cdot 0 \cdot 1 \cdot 1 \cdot 0 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \\
11 & 1 = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \\
12 & 0 = 1 \cdot 0 \cdot 0 \cdot 0 \cdot 1 \cdot 0 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \\
\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{array} \tag{5}$$

The separate column [n] represents the index of each row of the matrix starting from n=2 and the next column [μ_n] represents the product of all n-1 elements of the n^{th} row of the matrix:

$$\mu_n = \alpha_1(n) \cdot \alpha_2(n) \cdot \alpha_3(n) \cdot \ldots\ldots \cdot \alpha_{n-1}(n) \quad ; \qquad n=2, 3, 4, \ldots \tag{6}$$

By the products (6), the matrix of (5) becomes an *algorithm* for the calculation of the sequence μ_n. We observe that, apart from its first element, the elements of the n^{th} row of algorithm (5) characterize all divisors of n. This is due to the correct vertical alinement of the sequences $\alpha_\kappa(n)$ forming the columns of the algorithm in accordance with property (4).

Example: n=10 $\rightarrow$ row (1, 0, 1, 1, 0, 1, 1, 1, 1) with 9 elements:

the 2^{nd} element 0 means that 2 is a divisor of 10.

the 3^{rd} and 4^{th} elements 1,1 mean that 3,4 are not divisors of 10.

the 5^{th} element 0 means that 5 is a divisor of 10.

the rest elements 1,1,1,1 mean that 6,7,8,9 are not divisors of 10.

It is clear that if the n^{th} row of algorithm (5) contains at least one zero element, then n is not a prime and Eq.(6) implies that $\mu_n=0$, whereas if the n^{th} row has no zero elements, then n is a prime and Eq.(6) implies that $\mu_n=1$. Thus, μ_n is the *characteristic function* that defines exactly the positions of the primes within the set of positive integers:

$$\mu_n = \begin{cases} 1\ ;\ \text{n is a prime} \\ 0\ ;\ \text{n is not a prime} \end{cases} \quad ; \qquad n=2,\ 3,\ 4,\ 5\ldots.. \tag{7}$$

Definition (7) of μ_n will be discussed in detail later in this section after the derivation of an explicit formula for μ_n. It is interesting to notice that successive multiplication of the columns in algorithm (5) is equivalent to the mechanism of *Eratosthenes sieve* [4]. For instance, considering only the first two columns $\alpha_1(n)$, $\alpha_2(n)$ and keeping only $\alpha_1(n)\cdot\alpha_2(n)$ in the product of Eq.(6), we obtain $\mu_n=1$ for the set {n=2, and all odd numbers n=3,5,7, …} and we exclude the even numbers n=4,6,8… where $\mu_n=0$. Keeping next only $\alpha_1(n)\cdot\alpha_2(n)\cdot\alpha_3(n)$ in the product of Eq.(6), we further exclude all odd multiples of 3 so that here $\mu_n=1$ for n=2,3,5,7,11,13,17,19,23,25,29,… Increasing the terms of the product of Eq.(6) by adding new columns, we exclude gradually the multiples of 5,7,11, etc. until the full product of Eq.(6) removes *all* non primes from the set of positive integers so that $\mu_n=1$ only for the set of primes (present theory).

Apart from $\alpha_1(n)$, the harmonic sequences of Eqs (3) can be expressed compactly for $\kappa=2,3,4,\ldots.$ as follows:

$$\alpha_\kappa(n) = 1 - \frac{1}{\kappa}\sum_{\ell=0}^{\kappa-1}\cos\left(2\pi\frac{n}{\kappa}\ell\right) \quad ; \quad n = \kappa+1, \kappa+2, \ldots. \tag{8}$$

We prove that $\alpha_\kappa(n)$ defined by Eq.(8) has property (4):

I. If κ is not a divisor of n so that $\sin(\pi\frac{n}{\kappa}) \neq 0$ we use the formula [6]:

$$\sum_{\ell=0}^{\kappa-1} \cos\left(2\pi\frac{n}{\kappa}\ell\right) = \frac{\sin(\pi n)}{\sin(\pi\frac{n}{\kappa})}\cos[(\kappa-1)\pi\frac{n}{\kappa}] = 0 \tag{9}$$

which gives $\alpha_\kappa(n) = 1$ (10a)

II. If κ is a divisor of n viz. $\frac{n}{\kappa} = m$; m=2,3,4,... we have

$$\alpha_\kappa(n) = 1 - \frac{1}{\kappa}\{1 + \cos(2\pi m) + \cos(4\pi m) + \cdots + \cos[2(\kappa-1)\pi m]\} = 0 \tag{10b}$$

Hence, for κ=2,3,4,… Eq. (8) provides the harmonic sequences of Eqs(3) forming the columns of algorithm (5):

$$\alpha_\kappa(n) = (\underbrace{1, 1, 1, \ldots, 1}_{\kappa-1 \text{ terms}},\ 0,\ \underbrace{1, 1, 1, \ldots, 1}_{\kappa-1 \text{ terms}},\ 0, \ldots); \quad n=\kappa+1, \kappa+2, \ldots \tag{11}$$

Considering next the product of Eq.(6), with the elements $\alpha_\kappa(n)$ of the n^{th} row of algorithm (5) given by Eq.(8), the characteristic function μ_n [Eq.(7)] can be written explicitly as follows:

$$\mu_2 = 1$$

$$\mu_n = \prod_{\kappa=2}^{n-1}\left\{1 - \frac{1}{\kappa}\sum_{\ell=0}^{\kappa-1}\cos(2\pi\frac{n}{\kappa}\ell)\right\} \ ; \ n = 3,4,5,\ldots \tag{12}$$

The above result of Eq. (12) reproduces step by step the column $[\mu_n]$ of algorithm (5):

$$\mu_n = (1, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, \ldots) \ ; \ n=2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \ldots \tag{13}$$

which defines the positions of the primes within the set of integers.

In general we observe that for each index n=2, 3, 4, 5, … the κ-product of Eq. (12) has n-2 factors $\alpha_\kappa(n)$ defined by Eq. (8) and characterized by the numbers κ=2,3, …,n-1.

I. If n cannot be divided by any of the numbers κ=2, 3, …, n-1 which means that *n is a prime,* then all n-2 factors have value $\alpha_\kappa(n)$=1 [see Eq. (10a)] so that μ_n=1.

II. If n can be divided by one (or more than one)of the numbers κ=2, 3, ..., n-1 (i.e. $\frac{n}{\kappa} = m$; m=2, 3, ...) which means that *n is not a prime,* then the corresponding factor (or factors) where this division is possible has the value $\alpha_\kappa(n)=0$ [see Eq. (10b)] so that $\mu_n=0$.

Thus, μ_n expressed by Eq. (12) has the property (7) of the characteristic function of the prime numbers. An alternative form of μ_n is presented in section 5 of the present paper.

3. The prime counting function $\varpi(N)$

Let us consider next the number $\varpi(N)$ of primes that are smaller or equal to a given integer $N\geq 2$. As it was mentioned in the introduction, the asymptotic behaviour $\varpi(N)\approx N/\ln N$; $N\gg 1$ constitutes the prime number theorem [2], and it is expected [3] that the derivation of $\varpi(N)$ may be achieved by the calculation of the nontrivial roots of the ζ-function, all existing on the line $x=\frac{1}{2}$ of the complex plane (Riemann hypothesis).

In the present work on the other hand, the latter theory will not be used and instead $\varpi(N)$ will be expressed exactly as a sum over μ_n where μ_n is given by Eq. (12):

$$\varpi(N) = \sum_{n=2}^{N} \mu_n = (\mu_2,\ \mu_2+\mu_3,\ \mu_2+\mu_3+\mu_4,\ \mu_2+\mu_3+\mu_4+\mu_5,\ \mu_2+\mu_3+\mu_4+\mu_5+\mu_6, \dots)\ ;\ \ N = 2, 3, 4, \dots \tag{14}$$

In particular, replacing into Eq.(14) μ_n given by Eq.(12), we obtain explicitly $\varpi(N)$ in terms of harmonic sequences:

$$\varpi(2) = 1$$

$$\varpi(N) = 1 + \sum_{n=3}^{N} \prod_{\kappa=2}^{n-1} \left\{ 1 - \frac{1}{\kappa} \sum_{\ell=0}^{\kappa-1} \cos(2\pi \frac{n}{\kappa} \ell) \right\}\ ;\quad N = 3, 4, 5, \dots \tag{15}$$

Let us reproduce according to Eq. (15) the prime counting function $\pi(N)$ up to N=37:

N	2	3	4	5	6	7	8	9	10	11	12	13
$\pi(N)$	1	2	2	3	3	4	4	4	4	5	5	6

N	14	15	16	17	18	19	20	21	22	23	24	25
$\pi(N)$	6	6	6	7	7	8	8	8	8	9	9	9

N	26	27	28	29	30	31	32	33	34	35	36	37
$\pi(N)$	9	9	9	10	10	11	11	11	11	11	11	12

Table 1

Table 1 is represented by the diagram below:

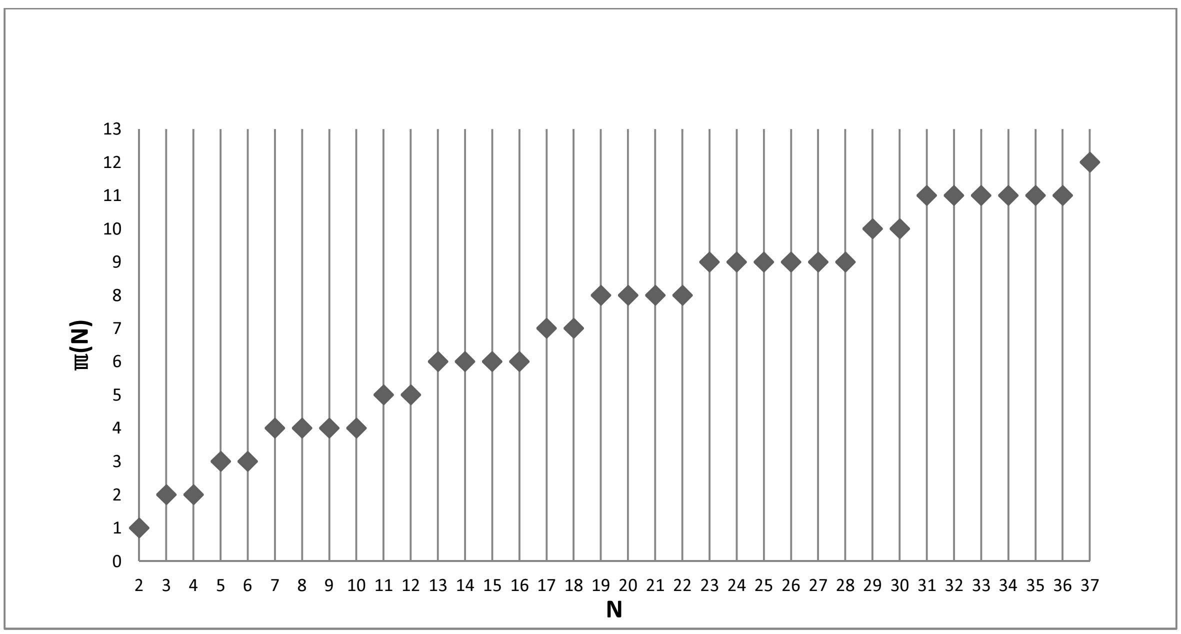

Fig. 1 The prime counting function $\pi(N)$; N=2, 3, 4,.... derived from Eq. (15).

We observe that $\pi(N)$ can be also obtained directly from the characteristic function μ_N [Eq.(13)] via a triangular algorithm where the sequence μ_N forms the columns displaced downwards by one step. Summing up the terms of each row we get:

N	$\pi(N)$		μ_N
2	1	=	1
3	2	=	1 + 1
4	2	=	0 + 1 + 1
5	3	=	1 + 0 + 1 + 1
6	3	=	0 + 1 + 0 + 1 + 1
7	4	=	1 + 0 + 1 + 0 + 1 + 1
8	4	=	0 + 1 + 0 + 1 + 0 + 1 + 1
9	4	=	0 + 0 + 1 + 0 + 1 + 0 + 1 + 1
10	4	=	0 + 0 + 0 + 1 + 0 + 1 + 0 + 1 + 1
11	5	=	1 + 0 + 0 + 0 + 1 + 0 + 1 + 0 + 1 + 1
12	5	=	0 + 1 + 0 + 0 + 0 + 1 + 0 + 1 + 0 + 1 + 1
13	6	=	1 + 0 + 1 + 0 + 0 + 0 + 1 + 0 + 1 + 0 + 1 + 1
14	6	=	0 + 1 + 0 + 1 + 0 + 0 + 0 + 1 + 0 + 1 + 0 + 1 + 1

. . . . (16)

The row corresponding to the index N contains N-1 terms: $\mu_2, \mu_3, \ldots, \mu_N$ so that the above algorithm is fully consistent with Eq.(14). Also we see that by applying successively algorithms (5,16) we can derive the prime counting function $\pi(N)$ without resorting to the ζ-function and to Riemann's hypothesis.

4. Harmonic representation of primes

From Eq.(14), the relation between the s^{th} prime p and its index s reads:

$$s = \pi(p) = \sum_{n=2}^{p} \mu_n \tag{17}$$

where $\pi(p)$ is given explicitly by Eq. (15) at N=p.

Note that $s = \pi(p)$ tell us the obvious fact that if there are s primes less or equal to a certain prime p, then p is the s^{th} prime. In particular, introducing the primes p=2,3,5,7,11, … in Eq.(17) we obtain the corresponding index s=1,2,3,4,5,… of each prime using Eq.(13) as follows:

$\mu_2 = 1$

$\mu_2 + \mu_3 = 2$

$\mu_2 + \mu_3 + \mu_4 + \mu_5 = 3$

$$\mu_2 + \mu_3 + \mu_4 + \mu_5 + \mu_6 + \mu_7 = 4$$

$$\mu_2 + \mu_3 + \mu_4 + \mu_5 + \mu_6 + \mu_7 + \mu_8 + \mu_9 + \mu_{10} + \mu_{11} = 5$$

$$\ldots \qquad (18)$$

Replacing next $\varpi(p)$ of Eq.(17) by $\varpi(N)$ of Eq.(15) at N=p, the index of the prime p is given explicitly by s=1; p=2 and

$$s = 1 + \sum_{n=3}^{p} \prod_{\kappa=2}^{n-1} \left\{ 1 - \frac{1}{\kappa} \sum_{\ell=0}^{\kappa-1} \cos(2\pi \frac{n}{\kappa} \ell) \right\} ; \quad p = 3, 5,7,11, \ldots \qquad (19)$$

From Eq.(19) and using Eq. (15), we can get for example the index of the primes p=3,5,7 as s= $\varpi(3)$=2 ; s= $\varpi(5)$=3 ; s= $\varpi(7)$=4.

The inverse of Eq.(17) relating each prime p_s to its index s, can be derived by defining *the differences c_m between successive primes*:

$$c_1 = p_2 - p_1 = 3 - 2 = 1$$

$$c_2 = p_3 - p_2 = 5 - 3 = 2$$

$$c_3 = p_4 - p_3 = 7 - 5 = 2$$

$$c_4 = p_5 - p_4 = 11 - 7 = 4$$

$$c_5 = p_6 - p_5 = 13 - 11 = 2$$

$$\ldots$$

$$c_{s-1} = p_s - p_{s-1} \qquad (20)$$

Adding up Eqs (20) we get:

$$p_1 = 2$$

$$p_s = 2 + \sum_{m=1}^{s-1} c_m ; \quad s = 2,3,4, \ldots \qquad (21)$$

We observe that the numbers c_1, c_2, c_3, ... defined by Eqs (20) coincide respectively with the number of repetitions of the numbers 1, 2, 3, ... occurring in Table 1 as values of $\varpi(N)$ for N=2, 3, 4, ... For instance, 1 occurs once and [c_1=1] ; 2 occurs twice [c_2=2] ; 3

occurs twice [c_3=2] ; 4 occurs four times [c_4=4] ; 5 occurs twice [c_5=2] etc. In the same context, the numbers c_1, c_2, c_3, ... defined by Eqs (20) coincide respectively with the number of points of the successive steps formed by $\pi(N)$ in Fig.1. For instance, the first step has one point at N=2 so that c_1=1 , the second step has two points at N=3, 4 so that c_2=2, the third step has also two points at N=5, 6 so that c_3=2, the fourth step has four points at N=7, 8, 9, 10 so that c_4=4, the fifth step has two points at N=11, 12 so that c_5=2 etc.

According to the above discussion, the calculation of the sequence c_m=[c_1, c_2, c_3, ...] defined by Eqs (20) can be obtained directly from the prime counting function $\pi(N)$; N=2, 3, 4, ... given in Table 1, by adding the terms of the successive columns forming the following triangular algorithm:

		c_1	c_2	c_3	c_4	c_5	c_6
$\pi(N)$	N \ m	1	2	3	4	5	6
1	2	1					
2	3	0	1				
2	4	0	1				
3	5	0	0	1			
3	6	0	0	1			
4	7	0	0	0	1		
4	8	0	0	0	1		
4	9	0	0	0	1		
4	10	0	0	0	1		
5	11	0	0	0	0	1	
5	12	0	0	0	0	1	
6	13	0	0	0	0	0	1
6	14	0	0	0	0	0	1
6	15	0	0	0	0	0	1
6	16	0	0	0	0	0	1

Table 2

The structure of the above algorithm where N=2,3,4,... is the index of each row and m=1,2,3,... is the index of each column, is fully defined by the rules:

I. The number of terms of the Nth row of the algorithm is equal to $\pi(N)$.

II. All terms of each row are equal to zero except the last term which is equal to one.

The sum of each column is shown on the top of Table 2 viz.

$$c_1 = 1 + 0 + 0 + 0 + 0 + 0 + \ldots = 1$$

$$c_2 = 1 + 1 + 0 + 0 + 0 + 0 + \ldots = 2$$

$$c_3 = 1 + 1 + 0 + 0 + 0 + 0 + \ldots = 2$$

$$c_4 = 1 + 1 + 1 + 1 + 0 + 0 + \ldots = 4$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \tag{22}$$

The functional link existing between c_m and $\mathbb{m}(N)$ may become precise by introducing the following sequence of harmonic functions based on $\mathbb{m}(N)$ [Table 1]:

$$\cos[2x\,\mathbb{m}(N)] = (\cos 2x,\ \cos 4x,\ \cos 4x,\ \cos 6x,\ \cos 6x,\ \cos 8x,\ \cos 8x,\ \cos 8x,\ \cos 8x,\ \cos 10x,\ \cos 10x,\ \ldots)\ ;\quad N=2,3,4,\ldots \tag{23}$$

Subsequently, we construct the orthogonal sequences:

$$\mathcal{L}m(N) = \frac{4}{\pi}\int_0^{\pi/2} \cos(2mx)\cos[2x\mathbb{m}(N)]\,dx = \begin{cases} 1\ ;\ \mathbb{m}(N) = m \\ 0\ ;\ \mathbb{m}(N) \neq m \end{cases} \tag{24α}$$

where $N=2,3,4,\ldots$; $m=1,2,3,4,\ldots$. In particular we have:

$$\mathcal{L}_1(N) = (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,\ldots)$$

$$\mathcal{L}_2(N) = (0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0,\ldots)$$

$$\mathcal{L}_3(N) = (0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0,\ldots)$$

$$\mathcal{L}_4(N) = (0, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0,\ldots)$$

$$\mathcal{L}_5(N) = (0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0,\ldots)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \tag{24b}$$

We observe that $\mathcal{L}_m(N)$ given by Eq.(24b) is a representation of the columns of the algorithm of Table 2. Thus we can define c_m in terms of $\mathbb{m}(N)$ as

$$c_m = \sum_{N=2}^{\infty}\mathcal{L}m(N) = \frac{4}{\pi}\sum_{N=2}^{\infty}\int_0^{\pi/2} \cos(2mx)\cos[2x\,\mathbb{m}(N)]dx \tag{25}$$

Note that since each sequence $\mathcal{L}_m(N)$ has a finite number of nonzero terms, each N-series of Eq.(25) defines a finite number c_m. Replacing c_m into Eq.(21) and simplifying by using the formula

$$\frac{\sin[(2s-1)x]}{\sin x} = \begin{cases} 1\,; \; s = 1 \\ 1 + 2\sum_{m=1}^{s-1} \cos(2mx)\,; \; s = 2,3,4,\ldots \end{cases} \tag{26}$$

we obtain exactly the prime numbers:

$$p_s = 2 + \frac{2}{\pi}\sum_{N=2}^{\infty}\int_0^{\pi/2} \frac{\sin[(2s-1)x]}{\sin x}\cos[2x\,\varpi(N)]dx \tag{27}$$

where s=1,2,3,4,,... and $\varpi(N)$ is given by Eq. (15) and Table 1.

Eq. (27) is the inverse of Eq.(17). Introducing the sequence (23) into Eq.(27), it is easy to derive the full set of primes:

$$p_s = (2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31,\ldots) \quad ; \quad s=1,2,3,4,\ldots \tag{28}$$

We observe that formula (27) can be also derived directly by writing p_s as follows:

$$p_s = 2 + \sum_{N=2}^{\infty} M_s(N) \tag{29}$$

where

$$M_s(N) = \begin{cases} 1\,; \; 2 \le N < p_s \\ 0\,; \; 2 \le p_s \le N \end{cases} \tag{30}$$

From the definition of $\varpi(N)$ [Eq.(14)] we have

$$2 \le N < p_s \quad \Rightarrow \quad \varpi(N) = \sum_{n=2}^{N} \mu_n < \sum_{n=2}^{p_s} \mu_n = s \quad \Rightarrow \quad 1 \le \varpi(N) \le s-1 \tag{31}$$

so that we can also write

$$M_s(N) = \begin{cases} 1\,; \; 1 \le \varpi(N) \le s-1 \\ 0\,; \; 1 \le s \le \varpi(N) \end{cases} \tag{32}$$

As given by Eq. (32), $M_s(N)$ can be expressed by the harmonic integral

$$M_s(N) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{\sin[(2s-1)x]}{\sin x}\cos[2x\varpi(N)]\,dx \qquad (33)$$

Indeed, expanding sin[(2s-1)x]/sinx according to Eq.(26) we can easily prove that Eq.(32) is valid. Therefore, replacing Eq.(33) into Eq.(29) formula (27) is obtained. Introducing next $\varpi(N)$ given by Eq.(15) into Eq.(27), we get an exact explicit formula for p_s in terms of harmonic functions:

$$p_1 = 2$$

$$p_s = 3 + \frac{2}{\pi}\sum_{N=3}^{\infty}\int_0^{\pi/2} \frac{\sin[(2s-1)x]}{\sin x}\cos\left\{2x\left[1+\sum_{n=3}^{N}\prod_{\kappa=2}^{n-1}\left(1-\frac{1}{\kappa}\sum_{\ell=0}^{\kappa-1}\cos\left(2\pi\frac{n}{\kappa}\ell\right)\right)\right]\right\}dx$$

where s=2, 3, 4,.. (34)

From Eq.(27) we can also derive a formula for the *next prime* in terms of $\varpi(N)$ given by Eq. (15):

$$p_{s+1} = p_s + \frac{2}{\pi}\sum_{N=2}^{\infty}\int_0^{\infty} \frac{\sin[(2s+1)x]-\sin[(2s-1)x]}{\sin x}\cos[2x\varpi(N)]\,dx$$

which leads to

$$p_{s+1} = p_s + \frac{4}{\pi}\sum_{N=2}^{\infty}\int_0^{\pi/2} \cos(2sx)\ \cos[2x\,\varpi(N)]\,dx \qquad (35)$$

where p_1=2; s=1,2,3,... In this case only the values of N for which $\varpi(N)=s$ give nonzero terms equal to 1 in the N-series. For example if s=6 we have $\varpi(N)=6$ at N=13,14,15,16 (Table 1):

$$p_7 = p_6 + \frac{4}{\pi}\sum_{N=2}^{\infty}\int_0^{\pi/2} \cos(12x)\cos[2x\,\varpi(N)]dx = 13+1+1+1+1 = 17 \qquad (36)$$

Considering the definition of c_s [Eq.(25)] we obtain from Eq.(35) $p_{s+1}=p_s+c_s$ which is fully consistent with Eqs (20).

We observe that the primes p_s can be also derived directly from a triangular algorithm based on the sequence c_m = (2, 1, 2, 2, 4, 2, 4, 2, 4, 6,...) of the differences between successive primes [Eqs (25)] enlarged by $c_0 = 2$.
In this case [c_m] forms *all* the columns displaced downwards. Summing up the terms of each row we get

$$
\begin{array}{llcl}
s & p_s & & \\
1 & 2 & = & 2 \\
2 & 3 & = & 1+2 \\
3 & 5 & = & 2+1+2 \\
4 & 7 & = & 2+2+1+2 \\
5 & 11 & = & 4+2+2+1+2 \\
6 & 13 & = & 2+4+2+2+1+2 \\
7 & 17 & = & 4+2+4+2+2+1+2 \\
8 & 19 & = & 2+4+2+4+2+2+1+2 \\
9 & 23 & = & 4+2+4+2+4+2+2+1+2 \\
10 & 29 & = & 6+4+2+4+2+4+2+2+1+2 \\
11 & 31 & = & 2+6+4+2+4+2+4+2+2+1+2 \\
12 & 37 & = & 6+2+6+4+2+4+2+4+2+2+1+2 \\
13 & 41 & = & 4+6+2+6+4+2+4+2+4+2+2+1+2 \\
14 & 43 & = & 2+4+6+2+6+4+2+4+2+4+2+2+1+2 \\
\ldots & & & \ldots
\end{array}
\tag{37}
$$

The row corresponding to the index s contains s terms: $c_0, c_1, c_2, c_3, \ldots, c_{s-1}$ so that the above algorithm is fully consistent with Eq.(21).
With the result of Eqs (37), we complete the four steps needed for the derivation of all prime numbers from harmonic sequences by using triangular algorithms.

Step 1 : Derivation of the characteristic function [μ_n] of the primes based on harmonic sequences [Eqs (5)].

Step 2 : Derivation of the prime counting function $\pi(N)$ from the characteristic function [μ_n] of the primes [Eqs (16)].

Step 3 : Derivation of the differences [c_m] between successive primes from the prime counting function $\pi(N)$ [Table 2].

Step 4 : Derivation of the sequence $[p_s]$ of all prime numbers from the differences $[c_m]$ between successive primes [Eqs (37)].

The result of the above four steps is expressed compactly by Eq.(34) where p_s depends explicitly on harmonic functions.

5. Alternative form of μ_n

The characteristic function μ_n [Eq.(7)] defining the primes and given by Eq. (12), can be also derived in a different form as follows:

I. We transform the elements of the triangular algorithm (5) using $1 \to 0$; $0 \to \kappa$ where κ is the index of the column.

II. We replace the products of the elements of each row of algorithm (5) by sums. Thus, we obtain the column $[\delta_n]$ as below:

$$
\begin{array}{ccll}
n & \delta_n & & \\
2 & 0 & = & 0 \\
3 & 0 & = & 0+0 \\
4 & 2 & = & 0+2+0 \\
5 & 0 & = & 0+0+0+0 \\
6 & 5 & = & 0+2+3+0+0 \\
7 & 0 & = & 0+0+0+0+0+0 \\
8 & 6 & = & 0+2+0+4+0+0+0 \\
9 & 3 & = & 0+0+3+0+0+0+0+0 \\
10 & 7 & = & 0+2+0+0+5+0+0+0+0 \\
11 & 0 & = & 0+0+0+0+0+0+0+0+0+0 \\
12 & 15 & = & 0+2+3+4+0+6+0+0+0+0+0 \\
\ldots & & &
\end{array}
\tag{38}
$$

We observe here that, contrary to algorithm (5), the primes correspond to $\delta_n=0$ and the nonprimes to $\delta_n\neq 0$ and in addition δ_n represents the sum of proper divisors of n i.e. divisors that divide n apart from 1,n.

The harmonic sequences $\beta_\kappa(n)$; $n \geq \kappa+1$; $\kappa=1,2,3,4,\ldots.$ forming the columns of algorithm (38) are given by

$\beta_1(n) = (0, 0, 0, 0, 0, 0,\ldots)$; $n=2, 3, 4, \ldots$

$\beta_2(n) = (0, 2, 0, 2, 0, 2, \ldots)$; $n=3, 4, 5, \ldots$

$\beta_3(n) = (0, 0, 3, 0, 0, 3,\ldots)$; $n=4, 5, 6, \ldots$

$\beta_4(n) = (0, 0, 0, 4, 0, 0, 0, 4,\ldots)$; $n=5, 6, 7, \ldots$

. . . . (39)

Apart from $\beta_1(n)$, the sequences of Eqs (39) can be expressed compactly for $\kappa=2,3,4,\ldots$ as follows:

$$\beta_\kappa(n) = \sum_{\ell=0}^{\kappa-1} \cos\left(2\pi\frac{n}{\kappa}\ell\right) \quad ; \quad n = \kappa+1, \kappa+2, \ldots. \qquad (40)$$

Note that $h_\kappa(n)$ defined by Eqs (1a,1b) in the introduction, is an enlarged version of $\beta_\kappa(n)$ so that each row of index n of the matrix (2) includes apart from the proper divisors of n also the elements (1,n).

Summing up all elements $\beta_\kappa(n)$; $\kappa=2,3,\ldots, n-1$ of the n^{th} row of algorithm (38) we obtain:

$$\delta_2 = 0$$

$$\delta_n = \sum_{\kappa=2}^{n-1} \sum_{\ell=0}^{\kappa-1} \cos\left(2\pi\frac{n}{\kappa}\ell\right) = \begin{cases} 0\,; & \text{n is a prime} \\ d(n)\,; & \text{n is not a prime} \end{cases} \quad ; \quad n = 3,4,5,\ldots \qquad (41)$$

where d(n) is the sum of proper divisors of n.

Eq. (41) is an exact analytic form of column $[\delta_n]$ in algorithm (38).
From the above results we can express in turn μ_n as

$$\mu_n = \frac{2}{\pi}\int_0^{\pi/2} \cos(2\delta_n x)\, dx \qquad (42)$$

If n is a prime, then according to Eq. (41) we have $\delta_n=0$ and Eq.(42) gives $\mu_n=1$, whereas if n is not a prime, then according to Eq. (41) we have $\delta_n\neq 0$ and since δ_n is an integer, Eq.(42) gives $\mu_n=0$. Hence, Eq.(42) obeys Eq.(7) and therefore provides with an alternative form of the characteristic function of prime numbers.

6. Harmonic equation of the primes

It is well known that the odd integers n=1,3,5,7,… can be obtained as solutions of the equation

$$1+\cos(\pi n) = 0 \tag{43a}$$

and the even integers n=2,4,6,8,… can be obtained as solutions of the equation

$$1-\cos(\pi n) = 0 \tag{43b}$$

We consider the question if within the context of the present work, a similar harmonic equation of the type $F(n) = 0$ exists having as solutions only the set of prime numbers. In this respect we observe that according to Eq.(41), equation

$$\sum_{\kappa=2}^{n-1}\sum_{\ell=0}^{\kappa-1}\cos\left(2\pi\frac{n}{\kappa}\ell\right) = 0 \tag{44}$$

indeed defines all the primes n=3,5,7,11,13,17,… except n=2 and is not satisfied if n is not a prime.

Proof: If n is a prime, the numbers κ=2,3,…, n-1 forming the κ-sum of Eq.(44) do not divide n, so that according to Eq.(9), for each κ=2,3,…, n-1 the ℓ-sum of Eq.(44) is equal to 0 and therefore n is a solution of Eq.(44). If n is not a prime, some of the numbers κ=2,3,…, n-1 divide n (i.e $\frac{n}{\kappa} = m$; m=2,3,…) and the ℓ-sum of Eq.(44) in this case is equal to the divisor κ of the number n [see Eq.(10b)]. Thus, if n is not a prime the LHS of Eq.(44) is equal to the sum d(n) of the proper divisors of n [Eq.(41)] so that n is not a solution of Eq.(44).

As in the previous theory, the structure of Eq.(44) shows the internal relation between prime numbers and harmonic functions.

7. Conclusions

We introduce an algorithm in the form of a triangular matrix [Eqs (5)] where the product of the elements of each row characterizes the index of that row. It is found that the positions of the primes within the set of integers are not arbitrary but are the result of precise combinations of the harmonic sequences [Eqs (3)] forming the columns of this algorithm. The characteristic function μ_n defining the positions of the primes [Eq. (7)] was obtained in terms of harmonic products [Eq. (12)]. The prime counting function $\varpi(N)$ representing the number of primes that are smaller or equal to a given integer $N\geq 2$, has been derived as a sum over μ_n [Eq. (14)] and was expressed exactly in terms of harmonic functions by Eq. (15) without resorting to the ζ-function and to Riemann's hypothesis. Also, it was shown that $\varpi(N)$ can be obtained directly from a triangular algorithm [Eq. (16)] where all the columns are formed by $[\mu_N]$ and the terms of each row are summed up. From the definition of the prime counting function $\varpi(N)$ it becomes clear that the relation between the s^{th} prime p and its index s is given by $s = \varpi(p)$ [Eq.(17)]. Inversely, the s^{th} prime p_s [Eqs(27,34)] as well as the next prime [Eq. (35)] were expressed in terms of harmonic integrals depending explicitly on s, and functionally on $\varpi(N)$. It was also shown that p_s can be derived directly from a triangular algorithm [Eqs (37)] where the columns are formed by the sequence of differences c_m between successive primes that are in turn obtained from $\varpi(N)$ in Table 2.

Finally, an alternative form of the characteristic function μ_n [Eq. (7)] has been derived in Eq. (42) by transforming the elements of the triangular algorithm [Eqs (5)] using $(1 \to 0\ ;\ 0 \to \kappa)$ where κ is the index of each column and by replacing the products of each row by sums [Eqs (38)]. This may lead to new expressions for $\varpi(N)$ and p_s. Also it was shown that all the prime numbers (except 2) and only them, are the solutions of a simple harmonic equation expressed by Eq.(44).

8. Discussion

In the present article a relation between prime numbers and harmonic functions is established, so that the notion of *the music of the primes* [1] is not only metaphoric but acquires a precise meaning. As mentioned in the introduction, the theory is based on the harmonic inter-relation between the divisors of positive integers that in turn constitute the building elements of the primes. The basic tools of the present work are the *harmonic triangular algorithms* where harmonic columns of a triangular matrix, correctly alined, are *projected* horizontally by a certain operation. Clearly, some results derived here are preliminary and deserve further analysis and justification. However, it is hoped that the ideas of the present theory will form the first step in exploring the harmonic world existing behind the prime numbers.

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