Model $P(\varphi)_4$ Quantum Field Theory

A Nonstandard Approach Based on Nonstandard Pointwise-Defined Quantum Fields

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Abstract. A new non-Archimedean approach to interacted quantum fields is presented. In proposed approach, a field operator $\varphi(x,t)$ no longer a standard tempered operator-valued distribution, but a non-classical operator-valued function. We prove using this novel approach that the quantum field theory with Hamiltonian $P(\varphi)_4$ exists and that the corresponding C^*- algebra of bounded observables satisfies all the Haag-Kastler axioms except Lorentz covariance. We prove that the $\lambda(\varphi^4)_4$ quantum field theory model is Lorentz covariant.

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CHAPTER I.

CHAPTER II.

§ 1. INTRODUCTION

Extending the real numbers $\mathbb{R}$ to include infinite and infinitesimal quantities originally enabled D. Laugwitz [1] to view the delta distribution $\delta(x)$ as a nonstandard point function. Independently A. Robinson [2] demonstrated that distributions could be viewed as generalized polynomials. Luxemburg [3] and Sloan [4] presented an alternate re-presentative of distributions as internal functions within the context of canonical Robinson's theory of nonstandard analysis. For further information on nonstandard real analysis, we refer to [5]-[6].

Abbreviation 1.1.1In this paper we adopt the following notations. For a standard set E we often write E_{st}. For a set E_{st} let ${}^{\sigma}E_{\mathrm{st}}$ be a set${}^{\sigma}E_{\mathrm{st}} = \{{}^{*}x | x \in E_{\mathrm{st}}\}$. We identify z with ${}^{\sigma}z$ i.e., $z \equiv {}^{\sigma}z$ for all $z \in \mathbb{C}$. Hence, ${}^{\sigma}E_{\mathbf{st}} = E_{\mathrm{st}}$ if $E \subseteq \mathbb{C}$, e.g., ${}^{\sigma}\mathbb{C} = \mathbb{C}$, ${}^{\sigma}\mathbb{R} = \mathbb{R}$, ${}^{\sigma}P = P$, ${}^{\sigma}L_{+}^{\uparrow} = L_{+}^{\uparrow}$, etc. Let ${}^{*}\mathbb{R}_{\approx}$, ${}^{*}\mathbb{R}_{\approx+}$, ${}^{*}\mathbb{R}_{\mathrm{fin}}$, ${}^{*}\mathbb{R}_{\infty}$, and ${}^{*}\mathbb{N}_{\infty}$ denote the sets of infinitesimal hyper-real numbers, positive infinitesimal hyper-real numbers, finite hyper-real numbers, infinite hyper-real numbers and infinite hyper natural numbers, respectively. Note that ${}^{*}\mathbb{R}_{\mathrm{fin}} = {}^{*}\mathbb{R}\backslash{}^{*}\mathbb{R}_{\infty}$, ${}^{*}\mathbb{C} = {}^{*}\mathbb{R} + \mathrm{i}{}^{*}\mathbb{R}$, ${}^{*}\mathbb{C}_{\mathrm{fin}} = {}^{*}\mathbb{R}_{\mathrm{fin}} + \mathrm{i}{}^{*}\mathbb{R}_{\mathrm{fin}}$.

Definition 1.1 Let $\{X, O\}$ be a standard topological space and let ${}^{*}X$ be the nonstandard extension of X. Let O_x denote the set of open neighbourhoods of point $x \in X$. The monad $mon_O(x)$ of x is the subset of ${}^{*}X$ defined by $mon_O(x) = \cap\{{}^{*}O | O \subset O_x\}$.The set of near standard points of ${}^{*}X$ is the subset of ${}^{*}X$ defined by $nst\,({}^{*}X) = \cup\{mon_O(x) | x \in X\}$. It is shown that $\{X, O\}$ is Hausdorff space if and only if $x \neq y$ implies $mon_O(x) \cap mon_O(y) = \emptyset$. Thus for any Hausdorff space$\{X, O\}$, we can define the equivalence relation $\approx_O$ on $nst\,({}^{*}X)$ so that $x \approx_O y$ if and only if $x \in mon_O(z)$ and $y \in mon_O(z)$ for some $z \in X$.

Definition 1.2 The standard Schwartz space of rapidly decreasing test functions on $\mathbb{R}^n, n \in \mathbb{N}$ is the standard function space is defined by $S(\mathbb{R}^n, \mathbb{C}) = \{f \in C^{\infty}(\mathbb{R}^n, \mathbb{C}) | \forall \alpha, \beta \in \mathbb{N}^n[\|f\|_{\alpha,\beta} < \infty]\}$, where

$$\|f\|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^n}\left|x^{\alpha}\left(D^{\beta}f(x)\right)\right|.$$

Remark 1.1 If f is a rapidly decreasing function, then for all $\alpha \in \mathbb{N}^n$ the integral of $\left|x^\alpha D^\beta f(x)\right|$ exists

$$\int_{\mathbb{R}^n}\left|x^\alpha D^\beta f(x)\right| d^n x < \infty.$$

Definition 1.3 The internal Schwartz space of rapidly decreasing test functions on ${}^*\mathbb{R}^n, n \in {}^*\mathbb{N}$ is the function space defined by ${}^*S({}^*\mathbb{R}^n, {}^*\mathbb{C}) = \left\{{}^*f \in {}^*C^{{}^*\infty}({}^*\mathbb{R}^n, {}^*\mathbb{C}) | \forall \alpha, \beta \in {}^*\mathbb{N}^n\left[{}^*\|{}^*f\|_{\alpha,\beta} < {}^*\infty\right]\right\}$, where

$${}^*\|{}^*f\|_{\alpha,\beta} = \sup{}^*\left\{x^\alpha\left(D^\beta f(x)\right) | x \in \mathbb{R}^n\right\}.$$

Remark 1.2 If f is a rapidly decreasing function, $f \in S(\mathbb{R}^n, \mathbb{C})$, then for all $\alpha, \beta \in {}^*\mathbb{N}^n$ the internal integral of $\left|{}^*x^\alpha D^\beta\, {}^*f(x)\right|$ exists

$${}^*\!\!\int_{{}^*\mathbb{R}^n}\left|{}^*x^\alpha D^\beta\, {}^*f(x)\right| d^n x < {}^*\infty.$$

Here $D^\beta\, {}^*f(x) = {}^*\!\left(D^\beta f(x)\right)$.

Definition 1.4 The Schwartz space of essentially rapidly decreasing test functions on ${}^*\mathbb{R}^n, n \in {}^*\mathbb{N}$ is the function space defined by

$${}^*S_{\text{fin}}({}^*\mathbb{R}^n, {}^*\mathbb{C}) =$$

$$\left\{{}^*f \in {}^*C^{{}^*\infty}({}^*\mathbb{R}^n, {}^*\mathbb{C}) | \forall(\alpha,\beta)(\alpha,\beta \in {}^*\mathbb{N}^n)\exists c_{\alpha\beta}\left(c_{\alpha\beta} \in {}^*\mathbb{R}_{\text{fin}}\right)\forall x(x \in {}^*\mathbb{R}^n)\left[\left|x^\alpha\left({}^*D^\beta\, {}^*f(x)\right)\right| < c_{\alpha\beta}\right]\right\}.$$

Remark 1.3 If ${}^*f \in {}^*S_{\text{fin}}({}^*\mathbb{R}^n, {}^*\mathbb{C})$, then for all $\alpha \in {}^*\mathbb{N}^n$ the internal integral of $\left|{}^*x^\alpha D^\beta\, {}^*f(x)\right|$ exists and finitely bounded above

$${}^*\!\!\int_{{}^*\mathbb{R}^n}\left|{}^*x^\alpha D^\beta\, {}^*f(x)\right| d^n x < d_{\alpha\beta}, d_{\alpha\beta} \in {}^*\mathbb{R}_{\text{fin}}.$$

Abbreviation 1.2 The standard Schwartz space of rapidly decreasing test functions on $\mathbb{R}^n$ we will be denote by $S(\mathbb{R}^n)$. Let ${}^*S({}^*\mathbb{R}^n), n \in {}^*\mathbb{N}$ denote the space of ${}^*\mathbb{C}$-valued rapidly decreasing internal test functions on ${}^*\mathbb{R}^n, n \in {}^*\mathbb{N}$ and let ${}^*S_{\text{fin}}({}^*\mathbb{R}^n), n \in {}^*\mathbb{N}$ denote the set of ${}^*\mathbb{C}_{\text{fin}}$-valued essentially rapidly decreasing test functions on ${}^*\mathbb{R}^n, n \in {}^*\mathbb{N}$. If $h(\omega, x): \mathbb{R} \times \mathbb{R}^n$ and $f: \mathbb{R}^n \to \mathbb{C}$ are Lebesgue measurable on $\mathbb{R}^{4n}$ we shall write $\langle {}^*h, {}^*f\rangle$ for internal Lebesgue integral ${}^*\!\int_{{}^*\mathbb{R}^n} {}^*h\, {}^*f\, d^n x$ with ${}^*f \in {}^*S_{\text{fin}}({}^*\mathbb{R}^n)$. Certain internal functions ${}^*h(\omega, x): {}^*\mathbb{R} \times {}^*\mathbb{R}^n \to {}^*\mathbb{C}$ define classical distribution $\tau(f)$ by the rule [3][4]:

$$\tau(f) = \text{st}(\langle {}^*h, {}^*f\rangle). \tag{1.1}$$

Here $\text{st}(a)$ is the standard part of a and $\text{st}(\langle {}^*h, {}^*f\rangle)$ exists [5].
Definition 1.5 We shall say that ${}^*h(\omega, x)$ with $\omega = \varpi \in {}^*\mathbb{R}_\infty$ is an internal representative to distribution $\tau(f)$ and we will write symbolically $\tau(x_1, \dots, x_n) \approx {}^*h(\omega, x_1, \dots, x_n)$ if the equation (1) holds.
Definition 1.6 [6] We shall say that certain internal functions ${}^*h(\omega, x): {}^*\mathbb{R} \times {}^*\mathbb{R}^n \to {}^*\mathbb{C}$ is a finite tempered distribution if ${}^*f \in {}^*S_{\text{fin}}({}^*\mathbb{R}^n)$ implies $|{}^*h, {}^*f| \in {}^\sigma\mathbb{R} = \mathbb{R}$. A functions ${}^*h(\omega, x): {}^*\mathbb{R} \times {}^*\mathbb{R}^n \to {}^*\mathbb{C}$ is called infinitesimal tempered distribution if ${}^*f \in {}^*S_{\text{fin}}({}^*\mathbb{R}^n)$ implies $|{}^*h, {}^*f| \in {}^*\mathbb{R}_\approx$ .The space of infinitesimal tempered distribution is denoted by ${}^*S_\approx({}^*\mathbb{R}^n)$.
Definition 1.7 We shall say that certain internal functions ${}^*h(\omega, x): {}^*\mathbb{R} \times {}^*\mathbb{R}^{4n} \to {}^*\mathbb{C}$ is a Lorentz $\approx$ -invariant tempered distribution if ${}^*f \in {}^*S_{\text{fin}}({}^*\mathbb{R}^n)$ and $\Lambda \in {}^\sigma L_+^\uparrow$ implies $\langle {}^*h, {}^*f(\Lambda x_1, \dots, \Lambda x_n)\rangle \approx \langle {}^*h, {}^*f(x_1, \dots, x_n)\rangle$.
Example 1.1 Let us consider Lorentz invariant distribution

$$D(x) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{i\boldsymbol{kr}}\frac{\sin\omega t}{\omega}d^3k = \frac{1}{2\pi}\delta(r^2-t^2)\mathrm{sign}(t). \tag{1.2}$$

Here $\omega = |\boldsymbol{k}| = \sqrt{k_1^2+k_2^2+k_3^2}$ and $\boldsymbol{r} = (x_1,x_2,x_3)$, $r = \sqrt{x_1^2+x_2^2+x_3^2}$. It easily verify that distribution $D(x)$ has the following internal representative

$$D(x,\varpi) = \frac{1}{(2\pi)^3}{}^*\!\int_{|\boldsymbol{k}|\leq\varpi} e^{i\boldsymbol{kr}}\frac{\sin\omega t}{\omega}d^3k. \tag{1.3}$$

Here $\varpi \in {}^*\mathbb{R}_\infty$. By integrating in (1.3) over angle variables we get

$$D(x,\varpi) = \frac{1}{8\pi^2 r}{}^*\!\int_0^{\varpi}\left\{e^{i\omega(r-t)} + e^{-i\omega(r-t)} - e^{i\omega(r+t)} - e^{-i\omega(r+t)}\right\}d\omega. \tag{1.4}$$

From (1.4) by canonical calculation finally we get

$$D(x,\varpi) \approx \frac{1}{4\pi^2 r}\left[\frac{\sin\varpi(r-t)}{r-t} - \frac{\sin\varpi(r+t)}{r+t}\right] \approx \frac{\delta(r-t)-\delta(r+t)}{4\pi^2 r} = \frac{1}{2\pi}\delta(r^2-t^2)\mathrm{sign}(t). \tag{1.5}$$

Example 1.2 We consider now the following Lorentz invariant distribution:

$$D_1(x) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{i\boldsymbol{kr}}\frac{\cos\omega t}{\omega}d^3k = \frac{1}{2\pi^2}\frac{1}{x^2}. \tag{1.6}$$

It easily verify that distribution $D(x)$ has the following internal representative

$$D_1(x,\varpi) = \frac{1}{(2\pi)^3}{}^*\!\int_{|\boldsymbol{k}|\leq\varpi} e^{i\boldsymbol{kr}}\frac{\cos\omega t}{\omega}d^3k. \tag{1.7}$$

Here $\varpi \in {}^*\mathbb{R}_\infty$. By integrating in (1.7) over angle variables we get

$$D_1(x,\varpi) \approx -\frac{i}{8\pi^2 r}{}^*\!\int_0^{\varpi}\left\{e^{i\omega(r-t)} - e^{-i\omega(r-t)} + e^{i\omega(r+t)} - e^{-i\omega(r+t)}\right\}d\omega. \tag{1.8}$$

From (1.8) finally we get

$$D_1(x,\varpi) \approx -\frac{i}{8\pi^2 r}\left[\frac{-2}{i(r-t)} + \frac{-2}{i(r+t)} + \frac{2\cos\varpi(r-t)}{i(r-t)} + \frac{2\cos\varpi(r+t)}{i(r+t)}\right] \approx \frac{1}{2\pi^2}\frac{1}{x^2}. \tag{1.9}$$

Example 3.We consider now the following Lorentz invariant distribution

$$\Delta_c(x) = \frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3} e^{i(\boldsymbol{kr}-\varepsilon(\boldsymbol{k})|t|)}\frac{d^3k}{\varepsilon(\boldsymbol{k})} = -\frac{m}{8\pi}\frac{H_1^{(2)}\left(-im\sqrt{|x^2|}\right)}{m\sqrt{|x^2|}}. \tag{1.10}$$

Here $-x^2 < 0$, $\varepsilon(\boldsymbol{k}) = \sqrt{|\boldsymbol{k}^2| + m^2}$ and $H_1^{(2)}$ is a Hankel function of the second kind. It easily verify that distribution $\Delta_c(x)$ has the following internal representative

$$\Delta_c(x,\varpi) = \frac{1}{2(2\pi)^3}{}^*\!\int_{|\boldsymbol{k}|\leq\varpi} e^{i(\boldsymbol{kr}-\varepsilon(\boldsymbol{k})|t|)}\frac{d^3k}{\varepsilon(\boldsymbol{k})} \tag{1.11}$$

From (1.10)-(1.11) it follows ${}^*\Delta_c(x) = \Delta_c(x,\varpi) + \breve{\Delta}_c(x)$ where

$$\breve{\Delta}_c(x) = \frac{1}{2(2\pi)^3}{}^*\!\int_{|\boldsymbol{k}|>\varpi} e^{i(\boldsymbol{kr}-\varepsilon(\boldsymbol{k})|t|)}\frac{d^3k}{\varepsilon(\boldsymbol{k})}. \tag{1.12}$$

Note that for all $\Lambda \in {}^{\sigma}L_{+}^{\uparrow}$, $\breve{\Delta}_c(\Lambda x) \in {}^{*}S_{\approx}({}^{*}\mathbb{R}^n)$ and therefore for all $\Lambda \in {}^{\sigma}L_{+}^{\uparrow}$, $\Delta_c(\Lambda x, \varpi) \approx \Delta_c(x, \varpi)$, i.e., $\Delta_c(x, \varpi)$ is a Lorentz $\approx$ -invariant tempered distribution, see definition 4. Thus we can set $t = 0$ in (1.11). By integrating in (1.11) over angle variables and using substitution of variables $|\boldsymbol{k}| = m \sinh(u)$ we get

$$\Delta_c(x, \varpi) \approx \frac{m}{8\pi^2 ir} {}^{*}\!\int_{-\ln\varpi}^{\ln\varpi} \exp\bigl(imr\sinh(u)\bigr) du. \tag{1.13}$$

Note that

$${}^{*}H_1^{(2)}(x) = \frac{\pi}{i} \int_{-{}^{*}\infty}^{{}^{*}\infty} \exp\bigl(imr\sinh(u)\bigr) du = \Delta_c(x, \varpi) + \Xi(x, \varpi), \tag{1.14}$$

$$\Xi(x, \varpi) = \frac{\pi}{i} \int_{-{}^{*}\mathbb{R}}^{-\ln\varpi} \exp\bigl(imr\sinh(u)\bigr) du + \int_{\ln\varpi}^{{}^{*}\mathbb{R}} \exp\bigl(imr\sinh(u)\bigr) du. \tag{1.15}$$

From (1.13)-(1.15) finally we obtain $\Delta_c(x, \varpi) \approx H_1^{(2)}(x)$, since $\Xi(x, \varpi) \in {}^{*}S_{\approx}({}^{*}\mathbb{R}^n)$.

Example 1.4 Let us consider Lorentz invariant distribution

$$\Delta(x - y) = \int \{\exp[-ip(x - y)] - \exp[ip(x - y)]\} \, \delta(p^2 - m^2)\vartheta(p^0) d^4p. \tag{1.16}$$

From (1.16) one obtains $\Delta(x - y) = \Xi_1(x - y) - \Xi_2(x - y)$, where

$$\Xi_1(x - y) = \int \bigl\{\exp\{[i\boldsymbol{p}(\boldsymbol{x} - \boldsymbol{y})] - i\omega(\boldsymbol{p})(x^0 - y^0)\}\bigr\} \frac{d^3p}{\sqrt{\boldsymbol{p}^2 + m^2}}, \tag{1.17}$$

$$\Xi_2(x - y) = \int \bigl\{\exp\{[-i\boldsymbol{p}(\boldsymbol{x} - \boldsymbol{y})] + i\omega(\boldsymbol{p})(x^0 - y^0)\}\bigr\} \frac{d^3p}{\sqrt{\boldsymbol{p}^2 + m^2}}, \tag{1.18}$$

$\omega(\boldsymbol{p}) = \sqrt{\boldsymbol{p}^2 + m^2}$. It easily verify that distribution (1.17) and (1.18) has the following internal representatives

$$\Xi_1(x - y, \varpi) = {}^{*}\!\int_{|\boldsymbol{k}| \leq \varpi} \{\exp\{[i\boldsymbol{p}(\boldsymbol{x} - \boldsymbol{y})] - i\omega(\boldsymbol{p})(x^0 - y^0)\}\} \frac{d^3p}{\sqrt{\boldsymbol{p}^2 + m^2}}. \tag{1.19}$$

$$\Xi_2(x - y, \varpi) = {}^{*}\!\int_{|\boldsymbol{k}| \leq \varpi} \{-\exp\bigl[[i\boldsymbol{p}(\boldsymbol{x} - \boldsymbol{y})] + i\omega(\boldsymbol{p})(x^0 - y^0)\bigr]\} \frac{d^3p}{\sqrt{\boldsymbol{p}^2 + m^2}}. \tag{1.20}$$

Note that ${}^{*}\Delta(x - y) = [\Xi_1(x - y, \varpi) + \Xi_2(x - y, \varpi)] + \bigl[\breve{\Xi}_1(x - y, \varpi) + \breve{\Xi}_2(x - y, \varpi)\bigr]$, where

$$\breve{\Xi}_1(x - y, \varpi) = {}^{*}\!\int_{|\boldsymbol{k}| > \varpi} \{\exp\{[i\boldsymbol{p}(\boldsymbol{x} - \boldsymbol{y})] - i\omega(\boldsymbol{p})(x^0 - y^0)\}\} \frac{d^3p}{\sqrt{\boldsymbol{p}^2 + m^2}}, \tag{1.21}$$

$$\breve{\Xi}_2(x - y, \varpi) = {}^{*}\!\int_{|\boldsymbol{k}| > \varpi} \{-\exp\bigl[[i\boldsymbol{p}(\boldsymbol{x} - \boldsymbol{y})] + i\omega(\boldsymbol{p})(x^0 - y^0)\bigr]\} \frac{d^3p}{\sqrt{\boldsymbol{p}^2 + m^2}}. \tag{1.22}$$

Note that for all $\Lambda \in {}^{\sigma}L_{+}^{\uparrow}$, $\breve{\Xi}_1(\Lambda(x - y), \varpi) + \breve{\Xi}_1(\Lambda(x - y), \varpi) \in {}^{*}S_{\approx}({}^{*}\mathbb{R}^n)$ and therefore for all $\Lambda \in {}^{\sigma}L_{+}^{\uparrow}$, ${}^{*}\Delta\bigl(\Lambda(x - y)\bigr) \approx \Delta(\Lambda(x - y), \varpi) = \Xi_1(\Lambda(x - y), \varpi) + \Xi_2(\Lambda(x - y), \varpi)$, i.e., $\Delta(x - y, \varpi)$ is a Lorentz $\approx$-invariant tempered distribution, see definition 1.4. From (1.20) by replacement $\boldsymbol{p} \to -\boldsymbol{p}$ we obtain

$$\Xi_1(x - y, \varpi) = -{}^{*}\!\int_{|\boldsymbol{k}| \leq \varpi} \{\exp\{[i\boldsymbol{p}(\boldsymbol{x} - \boldsymbol{y})] + i\omega(\boldsymbol{p})(x^0 - y^0)\}\} \frac{d^3p}{\sqrt{\boldsymbol{p}^2 + m^2}}. \tag{1.23}$$

From (1.19) and (1.23) we get

$$\Delta(x - y, \varpi) = \Xi_1(x - y, \varpi) + \Xi_2(x - y, \varpi) = {}^{*}\!\int_{|\boldsymbol{k}| \leq \varpi} \sin[\omega(\boldsymbol{p})(x^0 - y^0)] \exp[i\boldsymbol{p}(\boldsymbol{x} - \boldsymbol{y})] \frac{d^3p}{\sqrt{\boldsymbol{p}^2 + m^2}}. \tag{1.24}$$

Thus for any points x and y separated by space-like interval from (1.24) we obtain that

$$\Delta(x-y,\varpi)\approx 0, \tag{1.25}$$

since $\Delta(x-y,\varpi)$ is a Lorentz $\approx$-invariant tempered distribution. From (1.25) for any points x and y separated by spacelike interval we obtain that: $\mathrm{st}\big(\Delta(x-y,\varpi)\big)\equiv 0$.

Definition 1.8 [8] Let for each $m>0$: $H_m=\{p\in\mathbb{R}^4|p\cdot\tilde{p}=m^2,m>,p_0>0\}$, where $\tilde{p}=(p^0,-p^1,-p^2,-p^3)$. Here the sets H_m which are standard mass hyperboloids, are invariant under ${}^{\sigma}L_+^{\uparrow}$. Let j_m be the homeomorphism of H_m onto $\mathbb{R}^3$ given by $j_m:(p_0,p_1,p_2,p_3)\to(p_1,p_2,p_3)=\boldsymbol{p}$. Define a measure $\Omega_m(E)$ on H_m by

$$\Omega_m(E)=\int_{j_m(E)}\frac{d^3\boldsymbol{p}}{\sqrt{|\boldsymbol{p}|^2+m^2}}.$$

The measure $\Omega_m(E)$ is ${}^{\sigma}L_+^{\uparrow}$-invariant [8].

Theorem 1.1[8] Let μ is a polynomially bounded measure with support in $\bar{V}_+$. If μ is ${}^{\sigma}L_+^{\uparrow}=L_+^{\uparrow}$- invariant, there exists a polynomially bounded measure ρ on $[0,\infty)$ and a constant c so that for any $f\in S(\mathbb{R}^4)$

$$\int_{\mathbb{R}^4}f\,d\,\mu=cf(0)+\int_0^{\infty}d\,\rho\,(m)\left(\int_{\mathbb{R}^3}\frac{f\big(\sqrt{|\boldsymbol{p}|^2+m^2},p_1,p_2,p_3\big)d^3\boldsymbol{p}}{\sqrt{|\boldsymbol{p}|^2+m^2}}\right). \tag{1.26}$$

Theorem 1.2 Let μ is a polynomially bounded $L_+^{\uparrow}$- invariant measure with support in $\bar{V}_+$. Let $\mathcal{F}(f)$ be a linear $*$-continuous functional $\mathcal{F}:{}^*S_{\mathrm{fin}}\,({}^*\mathbb{R}^4)\to{}^*\mathbb{R}_{\mathrm{fin}}$ defined by ${}^*\!\int_{{}^*\mathbb{R}^4}{}^*f\,d\,\mu$ and there exists a polynomially bounded measure ρ on $[0,\infty)$ such that $\int_0^{{}^*\infty}d\,{}^*\rho\,(m)\in{}^*\mathbb{R}_{\mathrm{fin}}$ and a constant $c\in{}^*\mathbb{R}_{\mathrm{fin}}$. Then for any $f\in{}^*S_{\mathrm{fin}}({}^*\mathbb{R}^4)$ and for any $\varkappa\in{}^*\mathbb{R}_{\infty}$ the following property holds

$$\mathcal{F}({}^*f)\approx c\,{}^*f(0)+{}^*\!\!\int_0^{{}^*\infty}d\,{}^*\rho\,(m)\left({}^*\!\!\int_{|p|\le\varkappa}\frac{{}^*f\big(\sqrt{|\boldsymbol{p}|^2+m^2},p_1,p_2,p_3\big)d^{\#3}\boldsymbol{p}}{\sqrt{|\boldsymbol{p}|^2+m^2}}\right) \tag{1.27}$$

Definition 1.9 Let $\chi(\varkappa,\boldsymbol{p})$ be a function such that: $\chi(\varkappa,\boldsymbol{p})\equiv 1$ if $|\boldsymbol{p}|\le\varkappa$, $\chi(\varkappa,\boldsymbol{p})\equiv 0$ if $|\boldsymbol{p}|>\varkappa$, $\varkappa\in{}^*\mathbb{R}_{\infty}$.Define internal measure $\Omega_{m,\varkappa}$ on ${}^*H_m$ by

$$\Omega_{m,\varkappa}(E)={}^*\!\!\int_{{}^*H_m}\frac{\chi(\varkappa,\boldsymbol{p})d^3\boldsymbol{p}}{\sqrt{|\boldsymbol{p}|^2+m^2}}. \tag{1.28}$$

Theorem 1.3 [8] Let $W_2(x_1,x_2)$ be the two-point function of a field theory satisfying the Wightman axioms and the additional condition that $(\psi_0,\varphi(f)\psi_0)=0$ for all $f\in S(\mathbb{R}^4)$. Then there exists a polynomially bounded positive measure $\rho(m)$ on $[\![0,\infty)$ so that for all for all $f\in S(\mathbb{R}^4)$

$$W_2(f)=\big(\psi_0,\varphi(\bar{f})\varphi(f)\psi_0\big)=\int\bar{f}(x_1)f(x_2)\,W_2(x_1-x_2)d^4xd^4y=\int_0^{\infty}\left(\int_{H_m}\hat{f}d\Omega_m\right)d\rho(m). \tag{1.29}$$

Theorem 1.4 Let $W_2(x_1,x_2)$ be the two-point function of a field theory mentioned in Theorem 1.3. Then for all $f\in S_{\mathrm{fin}}({}^*\mathbb{R}^4)$ and for any $\varkappa\in{}^*\mathbb{R}_{\infty}$ the following property holds

$${}^*W_2(f)\approx{}^*\!\!\int_0^{{}^*\infty}\left({}^*\!\!\int_{{}^*H_m}\hat{f}d\Omega_{m,\varkappa}\right)d\,{}^*\rho(m). \tag{1.30}$$

Definition 1.10 (1) Let $L(H)$ be algebra of the all densely defined linear operators in standard Hilbert space H. Operator-valued distribution on $\mathbb{R}^n$, that is a map $\varphi:S(\mathbb{R}^n)\to L(H)$ such that there exists a dense subspace $D\subset H$ satisfying:

1. for each $f\in S(\mathbb{R}^n)$ the domain of φ contains D,

2. the induced map: $S \rightarrow End(D),\ f \rightarrow \varphi(f)$, is linear,
3. for each $h_1 \in D$ and $h_2 \in H$ the assignment $f \rightarrow \langle h_2, \varphi(f)h_1\rangle$ is a tempered distribution.
(2) Certain operator-valued internal function $\varphi\,({}^*f, \varpi)$: ${}^*S({}^*\mathbb{R}^n) \rightarrow {}^*L({}^*H)$ is an internal representative for standard operator valued distribution $\varphi\,(f)$ if for each near standard vectors $\tilde{h}_1 \in {}^*D$ and $\tilde{h}_2 \in {}^*H$ the equality holds

$$\langle h_2, \varphi(f)h_1\rangle = \mathrm{st}\left({}^*\langle \tilde{h}_2, \varphi({}^*f, \varpi)\tilde{h}_1\rangle\right), \tag{1.31}$$

where $h_1 \approx \tilde{h}_1$ and $h_2 \approx \tilde{h}_2$.
Definition 1.11[9] Let H be a Hilbert space and denote by H^n the n-fold tensor product $H^n = H\otimes H\otimes \cdots \otimes H$. Set $H^0 = \mathbb{C}$ and define $\mathcal{F}(H) = H^n$. $\mathcal{F}(H)$ is called the Fock space over Hilbert space H. Notice $\mathcal{F}(H)$ will be separable if H is. We set now $H = L_2(\mathbb{R}^3)$ then an element $\psi \in \mathcal{F}(H)$ is a sequence of $\mathbb{C}$-valued functions $\psi = \{\psi_0, \psi_1(x_1), \psi_2(x_1,x_2), \psi_2(x_1,x_2,x_3), \dots, \psi_n(x_1,\dots,x_n)\}, n \in \mathbb{N}$ and such that the following condition holds

$$|\psi_0|^2 + \textstyle\sum_{n\in\mathbb{N}}(\int|\psi_n(x_1,\dots,x_n)|^2 d^{3n}x) < \infty.$$

Definition 1.12 [8] Let us define now external operator $a(p)$ on $\mathcal{F}_s$ with domain D_S by

$$(a(p)\psi)^{(n)} = \sqrt{n+1}\,\psi^{(n+1)}(p, k_1, \dots k_n). \tag{1.32}$$

The formal adjoint of the operator $a(p)$ reads

$$(a^\dagger(p)\psi)^{(n)} = \tfrac{1}{\sqrt{n}}\textstyle\sum_{l=1}^{n}\delta^{(3)}(p-k_l)\psi^{(n-1)}(k_1,\dots,k_{l-1},k_{l+1},\dots,k_n) \tag{1.33}$$

Definition 1.13 [8] Let ψ^{fin} be a vector $\psi^{\mathrm{fin}} = \{\psi^{(n)}\}_{n=1}^{\infty}$ for which $\psi^{(n)} = 0$ for all except finitely many n is called a finite particle vector. We will denote the set of finite particle vectors by F_0. The vector $\Omega_0 = \langle 1,0,0,\dots\rangle$ is called the vacuum.
Definition 1.14 We let now ${}^*D_{^*S} = \{{}^*\psi | {}^*\psi \in {}^*F_0, {}^*\psi^{(n)} \in {}^*S\,({}^*\mathbb{R}^{3n}), n \in {}^*\mathbb{N}\}$ and for each $p \in {}^*\mathbb{R}^{3n}$ we define an internal operator ${}^*a(p)$ on ${}^*\mathcal{F}_s$ with domain ${}^*D_{^*S}$ by

$$({}^*a(p)\psi)^{(n)} = \sqrt{n+1}\,{}^*\psi^{(n+1)}(p, k_1, \dots k_n). \tag{1.34}$$

The formal $*$-adjoint of the operator ${}^*a$ reads

$$({}^*a^\dagger(p)\psi)^{(n)} = \tfrac{1}{\sqrt{n}}\textstyle\sum_{l=1}^{n}{}^*\delta^{(3)}(p-k_l){}^*\psi^{(n-1)}(k_1,\dots,k_{l-1},k_{l+1},\dots,k_n). \tag{1.35}$$

We express the free internal scalar field and the time zero fields with hyperfinite momentum cut-off $\varkappa \in {}^*\mathbb{R}_\infty$ in terms of ${}^*a^\dagger(p)$ and ${}^*a(p)$ as quadratic forms on ${}^*D_{^*S}$ by

$${}^*\Phi_{m,\varkappa}(x,t) =$$

$$(2\pi)^{-3/2}\,{}^*\!\!\int_{|p|\leq\varkappa}\left\{\left(\exp(\mu(p)t - ipx)\right){}^*a^\dagger(p) + \left(\exp(\mu(p)t + ipx)\right){}^*a\,(p)\right\}\tfrac{d^3p}{\sqrt{2\mu(p)}}, \tag{1.36}$$

$${}^*\varphi_{m,\varkappa}(x,t) = (2\pi)^{-3/2}\,{}^*\!\!\int_{|p|\leq\varkappa}\left\{\left(\exp(-ipx)\right){}^*a^\dagger(p) + \left(\exp(ipx)\right){}^*a\,(p)\right\}\tfrac{d^3p}{\sqrt{2\mu(p)}}, \tag{1.37}$$

$${}^*\pi_{m,\varkappa}(x,t) = (2\pi)^{-3/2}\,{}^*\!\!\int_{|p|\leq\varkappa}\left\{\left(\exp(-ipx)\right){}^*a^\dagger(p) + \left(\exp(ipx)\right){}^*a\;(p)\right\}\tfrac{d^3p}{\sqrt{\mu(p)/2}}. \tag{1.38}$$

Theorem 1.5 Let $\Phi_m(x,t)$ and $\varphi_m(x,t)$, $\pi_m(x,t)$ be the free standard scalar field and the time zero fields respectively. Then for any $\varkappa \in {}^*\mathbb{R}_\infty$ the operator valued internal functions (1.35)-(1.37) gives internal

representatives for standard operator valued distributions $\Phi_m(x,t)$ and $\varphi_m(x,t)$, $\pi_m(x,t)$ respectively.
Definition 1.15 Let $\{X,\|\cdot\|\}$ be a standard Banach space. For $x \in {}^*X$ and $\varepsilon > 0, \varepsilon \approx 0$ we define the open $\approx$-ball about x of radius ε to be the set $B_\varepsilon(x) = \{y \in {}^*X | {}^*\|x-y\| < \varepsilon\}$.
Definition 1.16 Let $\{\{X,\|\cdot\|\}$ be a standard Banach space, $Y \subset X$, thus ${}^*Y \subset {}^*X$ and let $x \in {}^*X$.Then x is an $*$-accumu-lotion point of ${}^*Y$ if for any $\varepsilon \in {}^*\mathbb{R}_{\approx +}$ there is a hyper infinite sequence $\{x_n\}_{n=1}^{{}^*\infty}$in ${}^*Y$ such that $\{x_n\}_{n=1}^{{}^*\infty} \cap (B_\varepsilon(x) \backslash \{x\} \neq \emptyset)$.
Definition 1.17 Let $\{\{X,\|\cdot\|\}$ be a standard Banach space, let${}^*Y \subseteq {}^*X, {}^*Y$ is $*$ -closed if any $*$-accumulation point of ${}^*Y$ is an element of ${}^*Y$.
Definition 1.18 Let $\{\{X,\|\cdot\|\}$ be a standard Banach space. We shall say that internal hyper infinite sequence $\{x_n\}_{n=1}^{{}^*\infty}$in ${}^*X$ is $*$ -converges to $x \in {}^*X$ as $n \to {}^*\infty$ if for any $\varepsilon \in {}^*\mathbb{R}_{\approx +}$ there is $N \in {}^*\mathbb{N}$ such that for any $n > N$: ${}^*\|x-y\| < \varepsilon$.
Definition 1.19 Let $\{\{X,\|\cdot\|_X\}$, $\{\{Y,\|\cdot\|_Y\}$ be a standard Banach spaces. A linear internal operator $A: D(A) \subseteq {}^*X \to {}^*Y$ is $*$ -closed if for every internal hyper infinite sequence $\{x_n\}_{n=1}^{{}^*\infty}$ in $D(A)$ $*$ -converging to $x \in {}^*X$ such that $Ax_n \to y \in {}^*Y$ as $n \to {}^*\infty$ one has $x \in D(A)$ and $Ax = y$. Equivalently , A is $*$-closed if its graph is $*$ -closed in the direct sum ${}^*X \oplus {}^*Y$.
Definition 1.20 Let H be a standard external Hilbert space. The graph of the internal linear transformation $T: {}^*H \to {}^*H$ is the set of pairs $\{\langle\varphi, T\varphi\rangle | \varphi \in D(T)\}$. The graph of T, denoted by $\Gamma(\mathrm{T})$, is thus a subset of ${}^*H \times {}^*H$ which is internal Hilbert space with inner product $(\langle\varphi_1,\psi_1\rangle, \langle\varphi_2,\psi_2\rangle) = (\varphi_1,\varphi_2) + (\psi_1,\psi_2)$.The operator T is called a $*$-closed operator if $\Gamma(\mathrm{T})$ is a $*$ -closed subset of Cartesian product ${}^*H \times {}^*H$.
Definition 1.21 Let H be a standard Hilbert space. Let T_1 and T be internal operators on internal Hilbert space ${}^*H$. Note that if $\Gamma(T_1) \supset \Gamma(T)$, then T_1 is said to be an extension of T and we write $T_1 \supset T$. Equivalently, $T_1 \supset T$ if and only if $D(T_1) \supset D(T)$ and $T_1\varphi = T\varphi$ for all $\varphi \in D(T)$.
Definition 1.22 Any internal operator T on ${}^*H$ is $*$-closable if it has a $*$-closed extension. Every $*$-closable internal operator T has a smallest $*$-closed extension, called its $*$-closure, which we denote by $*$-$\bar{T}$.
Definition 1.23 Let H be a standard Hilbert space. Let T be a $*$-densely defined internal linear operator on internal Hilbert space ${}^*H$. Let $D(T^*)$ be the set of $\varphi \in {}^*H$ for which there is a vector $\xi \in {}^*H$ with $(T\psi,\varphi) = (\varphi,\xi)$ for all $\psi \in D(T)$, then for each $\varphi \in D(T^*)$, we define $T^*\varphi = \xi$. T^* is called the $*$-adjoint of T. Note that $S \subset T$ implies $T^* \subset S^*$.
Definition 1.24 Let H is a standard Hilbert space. A $*$-densely defined internal linear operator T on internal Hilbert space ${}^*H$ is called symmetric (or Hermitian) if $T \subset T^*$. Equivalently, T is symmetric if and only if $(T\varphi,\psi) = (\varphi, T\psi)$ for all $\varphi,\psi \in D(T)$.
Definition 1.25 Let H be a standard Hilbert space. A symmetric internal linear operator T on internal Hilbert space ${}^*H$ is called essentially self- $*$-adjoint if its $*$-closure $*$-$\bar{T}$ is self- $*$-adjoint. If T is $*$-closed, a subset $D \subset D(T)$ is called a $*$-core for T if $*$- $\left(\overline{T \upharpoonright D}\right) = T$. If T is essentially self- $*$-adjoint, then it has one and only one self -$*$-adjoint extension.
Theorem 1.6 Let $n_1, n_2 \in \mathbb{N}$ and suppose that $W(k_1, \dots k_{n_1}, p_1, \dots, p_{n_2}) \in {}^*L_2({}^*\mathbb{R}^{3(n_1+n_2)})$ where $W(k_1, \dots k_{n_1}, p_1, \dots, p_{n_2})$ is a ${}^*\mathbb{C}$ -valued internal function on ${}^*\mathbb{R}^{3(n_1+n_2)}$. Then there is a unique operator T_W on ${}^*\mathcal{F}({}^*L_2({}^*\mathbb{R}^3))$ so that ${}^*D_{*S} \subset D(T_W)$ is a $*$ - core for T_W and
(1) as ${}^*\mathbb{C}$-valued quadratic forms on ${}^*D_{*S} \times {}^*D_{*S}$

$$T_W = {}^*\!\!\int_{{}^*\mathbb{R}^{3(n_1+n_2)}} W(k_1, \dots k_{n_1}, p_1, \dots, p_{n_2}) \left(\prod_{i=1}^{n_1} {}^*a^\dagger(k_i)\right)\left(\prod_{i=1}^{n_2} {}^*a(p_i)\right) d^{n_1}k d^{n_2}p$$

(2) As ${}^*\mathbb{C}$-valued quadratic forms on $D_{*S} \times D_{*S}$

$$T_W^* = {}^*\!\!\int_{{}^*\mathbb{R}^{3(n_1+n_2)}} W(k_1, \dots k_{n_1}, p_1, \dots, p_{n_2}) \left(\prod_{i=1}^{n_1} {}^*a^\dagger(k_i)\right)\left(\prod_{i=1}^{n_2} {}^*a(p_i)\right) d^{n_1}k d^{n_2}p$$

(3) On vectors in ${}^*F_0$ the operators T_W and T_W^* are given by the explicit formulas

$$\left(T_W({}^*\psi)\right)^{(l-n_2+n_1)} =$$

$$K(l,n_1,n_2){}^*\mathbf{S}\left[{}^*\int_{|p_1|\le\varpi}\dots{}^*\int_{|p_{n_2}|\le\varpi}W\left(k_1,\dots k_{n_1},p_1,\dots,p_{n_2}\right){}^*\psi^{(l)}\left(p_1,\dots,p_{n_2},k_1,\dots k_{n_1}\right)d^{3n_2}\,p\right], \tag{1.39}$$

$\left(T_W^*({}^*\psi)\right)^n = 0$ if $n < n_1 - n_2$,

$$\left(T_W^*({}^*\psi)\right)^{(l-n_1+n_2)} =$$

$$K(l,n_2,n_1){}^*\mathbf{S}\left[{}^*\int_{|p_1|\le\varpi}\dots{}^*\int_{|p_{n_2}|\le\varpi}W\left(k_1,\dots k_{n_1},p_1,\dots,p_{n_2}\right){}^*\psi^{(l)}\left(p_1,\dots,p_{n_2},k_1,\dots k_{n_1}\right)d^{3n_1}\,k\right] \tag{1.40}$$

$\left(T_W^*({}^*\psi)\right)^n = 0$, if $n < n_2 - n_1$.

Here $\mathbf{S}$ is the symmetrization operator defined in [9] and $K(l,n_2,n_1) = \left[\frac{l!(l+n_1-n_2)!}{(l-n_2)^2}\right]^{1/2}$, $n_1,n_2 \in \mathbb{N}, l \in {}^*\mathbb{N}$.
Proof For vectors ${}^*\psi \in D_{{}^*S}$ we define $T_W({}^*\psi)$ by the formula (39). By the Schwarz inequality and the fact that ${}^*\mathbf{S}$ is a projection we get

$$\left({}^*\left\|\left(T_W({}^*\psi)\right)^{(l-n_2+n_1)}\right\|\right)^2 \le K(l,n_1,n_2){}^*\left\|\left({}^*\psi^{(l)}\right)\right\|^2{}^*\|W\|^2. \tag{1.41}$$

Let us now define the operator $T_W^*({}^*\psi)$ on $D_{{}^*S}$ by the formula (39), then for all ${}^*\varphi, {}^*\psi \in D_{{}^*S}$, then one obtains directly ${}^*({}^*\varphi, T_W{}^*\psi) = {}^*(T_W^*{}^*\varphi, {}^*\psi)$. Thus, T_W is $*$-closable and T_W^* is the restriction of the $*$-adjoint of T_W on $D_{{}^*S}$. We will use T_W to denote $*$-$\overline{T}_W$ and T_W^* to denote the $*$-adjoint of T_W. By the definition of T_W, $D_{{}^*S}$ is a $*$-core and further, since T_W is bounded on the l-particle vectors in $D_{{}^*S}$ we get ${}^*F_0 \subset D(T_W)$. Since the right-hand side of (39) is also bounded on the l-particle vectors, equation (38) represents T_W on all l-particle vectors. The proof of the statement (2) about T_W^* is the same.
Definition 1.26 [8] Define standard Q -space by $Q = \times_{k=1}^{\infty}\mathbb{R}$. Let σ be the σ-algebra generated by infinite products of measurable sets in $\mathbb{R}$ and set $\mu = \otimes_{k=1}^{\infty}\mu_k$ with $d\mu_k = \pi^{-1/2}\exp(-x_k^2/2)$. Denote the points of Q by $q = \langle q_1, q_2, \dots\rangle$. Then $\langle Q,\mu\rangle$ is a measure space and the set of the all functions of the form $P_n(q) = P(q_1,q_2,\dots,q_n)$, where $P_n(q)$ is a polynomial and $n \in \mathbb{N}$ is arbitrary, is dense in $L_2(Q,d\mu)$. Remind that there exists a unitary map $S{:}\,\mathcal{F}_s(H) \to L_2(Q,d\mu)$ of Fock space $\mathcal{F}_s(H)$ onto $L_2(Q,d\mu)$ so that $\mathrm{S}\varphi(f_k)\mathrm{S}^{-1} = q_k$ and $\mathrm{S}\Omega_0 = 1$. Here $\{f_k\}_{k=1}^{\infty}$ is an orthonormal basis for H. Then by transfer one obtains internal measure space ${}^*\langle Q,\mu\rangle = \langle {}^*Q, {}^*\mu\rangle$ and internal unitary map ${}^*\mathrm{S}{:}\,\mathcal{F}_s(H) \to {}^*L_2({}^*Q, d{}^*\mu)$ so that ${}^*\mathrm{S}\varphi(f_r){}^*\mathrm{S}^{-1} = q_r$, $r \in {}^*\mathbb{N}$ and ${}^*\mathrm{S}\Omega_0 = 1$. Here $\{f_r\}_{r=1}^{{}^*\infty}$ is an orthonormal basis for ${}^*H$.
Theorem 1.7 Let ${}^*\varphi_\varkappa(x,t)$ be internal free scalar boson field of mass m at time $t = 0$ with hyperfinite momentum cutoff $\varkappa$ in four-dimensional space-time. Let $g(x)$ be a real-valued internal function in ${}^*L_2({}^*\mathbb{R}^3) \cap {}^*L_1({}^*\mathbb{R}^3)$. Then the operator

$${}^*H_{I,\varkappa}(g) = \lambda(\varkappa)\,{}^*\int_{{}^*\mathbb{R}^3} g(x){:}\,{}^*\varphi_\varkappa^4(x){:}\,d^3x \tag{1.42}$$

is a well-defined internal symmetric operator on ${}^*D_{{}^*S_{\mathrm{fin}}}$. Here $:{}^*\varphi_\varkappa^4(x) := {}^*\varphi_\varkappa^4(x) + d_2(\varkappa)\left({}^*\varphi_\varkappa^2(x)\right) + d_1(\varkappa)$. where the coefficients $d_2(\varkappa)$ and $d_1(\varkappa)$ are independent of x. Let S denote the unitary map of $\mathcal{F}_s(H)$ onto $L_2(Q,d\mu)$ considered in [8]. Then $V = {}^*\mathrm{S}{}^*H_{I,\varkappa}(g){}^*\mathrm{S}^{-1}$ is multiplication by internal function $V_{I,\varkappa}(q)$ which satisfies:
(a) $V_{I,\varkappa}(q) \in {}^*L_p({}^*Q, d{}^*\mu)$ for all $p \in {}^*\mathbb{N}$, (b) $\exp\left(-tV_{I,\varkappa}(q)\right) \in {}^*L_1({}^*Q, d{}^*\mu)$ for all $t \in [0,{}^*\infty)$.

Proof Note that for each $x \in {}^*\mathbb{R}^3$, the operator ${}^*\mathrm{S}({}^*\varphi_\varkappa(x)){}^*\mathrm{S}^{-1}$ is just the operator on internal measurable space ${}^*L_2({}^*\mathrm{Q}, d^*\mu)$ on which this operator acts by multiplying by the function $\sum_{k=1}^{*\infty} c_k(x,\varkappa) q_k$, where $c_k(x,\varkappa) = (2\pi)^{3/2}\left(f_k, (\mu(p))^{1/2}\exp(ipx)\right)$. Furthermore, $\sum_{k=1}^{*\infty}|c_k(x,\varkappa)|^2 = (2\pi)^{3/2}\,{}^*\|\mu(p)^{1/2}\|_2^2$ so ${}^*\mathrm{S}\left({}^*\varphi_\varkappa^4(x)\right){}^*\mathrm{S}^{-1}$and ${}^*\mathrm{S}\left({}^*\varphi_\varkappa^2(x)\right){}^*\mathrm{S}^{-1}$ are in ${}^*L_2({}^*\mathrm{Q}, d^*\mu)$ and the corresponding ${}^*L_2({}^*\mathrm{Q}, d^*\mu)$-norms are uniformly bounded in x. Therefore, since $g \in {}^*L_1({}^*\mathbb{R}^3)$ the operator ${}^*\mathrm{S}\left({}^*H_{I,\varkappa}(g)\right){}^*\mathrm{S}^{-1}$ is just the operator on internal measurable space ${}^*L_2({}^*\Omega, d^*\mu)$ on which this operator acts by multiplying by the ${}^*L_2({}^*\mathrm{Q}, d^*\mu)$-function which we denote by $V_{\varkappa,\lambda}(q)$. Let us consider now the expression for ${}^*H_{I,\varkappa}(g){}^*\Omega$, obviously this is a vector $(0,0,0,0,\psi^4,0,\dots)$ with

$$\psi^4(p_1,p_2,p_3,p_4) = {}^*\!\!\int_{{}^*\mathbb{R}^3} \frac{\lambda(\varkappa) g(x) \prod_{i=1}^4[\chi(\varkappa,p_i)] \exp\left(-ix\sum_{i=1}^{i=4} p_i\right) d^3x}{(2\pi)^{3/2}\prod_{i=1}^4[2\mu(p_i)]^{1/2}}. \tag{1.43}$$

Here $\chi(\varkappa,p) \equiv 1$ if $|p| \le \varkappa$, $\chi(\varkappa,p) \equiv 0$ if $|p| > \varkappa$, $\varkappa \in {}^*\mathbb{R}_\infty$. We choose now the parameter $\lambda = \lambda(\varkappa) \approx 0$ such that ${}^*\|\psi^4\|_2^2 \in \mathbb{R}$ and therefore we obtain ${}^*\left\|{}^*H_{I,\varkappa,\lambda(\varkappa)}(g)\Omega_0\right\|_2^2 \in \mathbb{R}$, since ${}^*\left\|{}^*H_{I,\varkappa,\lambda(\varkappa)}(g)\Omega_0\right\|_2^2 = {}^*\|\psi^4\|_2^2$. But, since ${}^*\mathrm{S}^*\Omega_0 = 1$, we get the equalities

$${}^*\left\|{}^*H_{I,\varkappa,\lambda(\varkappa)}(g)\Omega_0\right\|_2 = \left\|{}^*\mathrm{S}\, H_{I,\varkappa,\lambda(\varkappa)}(g){}^*\mathrm{S}^{-1}\right\|_{{}^*L_2({}^*Q,d^*\mu)} = {}^*\left\|V_{I,\varkappa,\lambda(\varkappa)}(q)\right\|_{{}^*L_2({}^*Q,d^*\mu)}. \tag{1.44}$$

From (1.43) we get that ${}^*\left\|V_{I,\varkappa,\lambda(\varkappa)}(q)\right\|_{{}^*L_2({}^*Q,d^*\mu)} \in \mathbb{R}$ and it is easily verify, that each polynomial $P(q_1,q_2,\dots,q_n)$, is $n \in {}^*\mathbb{N}$ in the domain of the operator $V_{I,\varkappa,\lambda(\varkappa)}(q)$ and ${}^*\mathrm{S}\,{}^*H_{I,\varkappa,\lambda(\varkappa)}(g){}^*\mathrm{S}^{-1} \equiv V_{I,\varkappa,\lambda(\varkappa)}(q)$ on that domain. Since ${}^*\Omega_0$ is in the domain of ${}^*H^p{}_{I,\varkappa,\lambda(\varkappa)}(g)$, $p \in {}^*\mathbb{N}$, 1 is in the domain of the operator $V^p{}_{I,\varkappa,\lambda(\varkappa)}(q)$ for all $p \in {}^*\mathbb{N}$. Thus, for all $p \in {}^*\mathbb{N}$ $V_{I,\varkappa,\lambda(\varkappa)}(q) \in {}^*L_{2p}({}^*Q, d^*\mu)$, since ${}^*\mu\,({}^*Q\,)$ is finite, we conclude that $V_{I,\varkappa,\lambda(\varkappa)}(q) \in {}^*L_p({}^*Q, d^*\mu)$ for all $p \in {}^*\mathbb{N}$.

(b) Remind Wick's theorem asserts that $:{}^*\varphi_{m,\varkappa}^j(x) := \sum_{i=0}^{[j/2]}(-1)^i \frac{j!}{(j-2i)!\,i!} c_\varkappa^i\, {}^*\varphi_{m,\varkappa}^{(j-2i)}(x)$ with $c_\varkappa = {}^*\left\|{}^*\varphi_{m,\varkappa}(x){}^*\Omega_0\right\|_2^2$. For $j = 4$ we get $-O(c_\varkappa^2) \le :{}^*\varphi_{m,\varkappa}^4(x):$ and therefore $-\left({}^*\!\int_{{}^*\mathbb{R}^3} g(x)\, d^3x\right) O(c_\varkappa^2) \le {}^*H_{I,\varkappa,\lambda(\varkappa)}(g)$. Finally we obtain ${}^*\!\int_{{}^*Q} \exp\left(-t(:{}^*\varphi_{m,\varkappa}^4(x):)\right) d\,{}^*\mu \le \exp\left(O(c_\varkappa^2)\right)$ and this inequality finalized the proof.

Theorem 1.8 [8] Let $\langle M, \mu\rangle$ be a σ-measure standard space with $\mu(M) = 1$and let H_0 be the generator of a hyper-contractive semigroup on $L_2(M, d\mu)$. Let V be a $\mathbb{R}$-valued measurable function on $\langle M, \mu\rangle$ such that $V \in L_p(M, d\mu)$ for all $p \in [\![1,\infty)$ and $\exp(-tV) \in L_1(M, d\mu)$ for all $t > 0$. Then $H_0 + V$ is essentially self-adjoint on $C^\infty(H_0\,) \cap D(V)$ and is bounded below. Here $C^\infty(H_0\,) = \bigcap_{p\in\mathbb{N}} D\left(H_0^p\right)$.

Theorem 1.9 Let $\langle M, \mu\rangle$ be a σ-measure space with $\mu(M) = 1$and let H_0 be the generator of a hypercontractive semi-group on $L_2(M, d\mu)$. Let$V$ be a ${}^*\mathbb{R}$-valued internal measurable function on $\langle {}^*M, {}^*\mu\rangle$ such that $V \in {}^*L_p({}^*M, d^*\mu)$ for all $p \in [1, {}^*\infty)$ and ${}^*\exp(-tV) \in {}^*L_1({}^*M, d^*\mu)$ for all $t > 0$. Assume that a set $C^{*\infty}({}^*H_0\,) \cap D(V)$ is internal. Then operator ${}^*H_0 + V$ is essentially self-$*$ -adjoint internal operator on $C^{*\infty}({}^*H_0\,) \cap D(V)$ and it is hyper finitely bounded below. Here $C^{*\infty}({}^*H_0\,) = \bigcap_{p\in{}^*\mathbb{N}} D\left({}^*H_0^p\right)$.

Proof. It follows immediately by transfer from theorem 8.

Remark 1.4 Let $V_{I,\varkappa,\lambda}$ be operator on internal measurable space ${}^*L_2({}^*\Omega, d^*\mu)$ on which this operator acts by multiplying by the ${}^*L_2({}^*\mathrm{Q}, d^*\mu)$-function$V_{I,\varkappa,\lambda}$, see proof to Theorem 1.7. Note that for this operator a set $C^{*\infty}({}^*H_0\,) \cap D\left(V_{I,\varkappa,\lambda}\right)$ is not internal and therefore Theorem9 no longer holds. But without this theorem we cannot conclude that operator ${}^*H_0 + V_{I,\varkappa,\lambda}$ is essentially self-$*$ -adjoint internal operator on $C^{*\infty}({}^*H_0\,) \cap D\left(V_{I,\varkappa,\lambda}\right)$. Thus Robinson's transfer is of no help in the case corresponding to operator $V_{I,\varkappa,\lambda}$ considered above. In order to resolve this issue, we will use non conservative extension of the model theoretical nonstandard analysis, see [10]-[14].

§2. NON CONSERVATIVE EXTENSION OF THE MODEL THEORETICAL NONSTANDARD ANALYSIS

Remind that Robinson nonstandard analysis (RNA) many developed using set theoretical objects called super-structures [2]-[7]. A superstructure V(S) over a set S is defined in the following way: $\mathrm{V}_0(\mathrm{S}) = \mathrm{S}$, $\mathrm{V}_{n+1}(\mathrm{S}) = \mathrm{V}_n(\mathrm{S}) \cup \mathrm{P}(\mathrm{V}_n(\mathrm{S}))$, $\mathrm{V}(\mathrm{S}) = \bigcup_{n\in\mathbb{N}} \mathrm{V}_{n+1}(\mathrm{S})$. Making $\mathrm{S} = \mathbb{R}$ will suffice for virtually any construction necessary in analysis. Bounded formulas are formulas where all quantifiers occur in the form: $\forall x\,(x \in y \to \cdots)$, $\exists x\,(x \in y \to \cdots)$. A nonstandard embedding is a mapping $*\colon \mathrm{V}(X) \to \mathrm{V}(Y)$ from a superstructure $\mathrm{V}(X)$ called the standard universe, into another superstructure $\mathrm{V}(Y)$ called nonstandard universe, satisfying the following postulates:

1. $Y = {}^*X$
2. ***Transfer Principle*** For every bounded formula $\Phi(x_1, \dots, x_n)$ and elements $a_1, \dots, a_n \in V(X)$ the property $\Phi(a_1, \dots, a_n)$ is true for $a_1, \dots, a_n$ in the standard universe if and only if it is true for ${}^*a_1, \dots, {}^*a_n$ in the nonstandard universe $V(X) \models \Phi(x_1, \dots, x_n) \leftrightarrow V(Y) \models \Phi({}^*a_1, \dots, {}^*a_n)$.
3. ***Non-triviality*** For every infinite set A in the standard universe, the set $\{{}^*a \mid a \in A\}$ is a proper subset of ${}^*A$.

Definition 2.1 A set x is internal if and only if x is an element of ${}^*A$ for some $A \in \mathrm{V}(\mathbb{R})$. Let X be a set and $A = \{A_i\}_{i\in I}$ a family of subsets of X .Then the collection A has the infinite intersection property, if any infinite sub collection $J \subset I$ has non-empty intersection. Nonstandard universe is σ -saturated if whenever $\{A_i\}_{i\in I}$ is a collection of internal sets with the infinite intersection property and the cardinality of I is less than or equal to σ.

Remark 2.1 For each standard universe $U = V(X)$ there exists canonical language L_U and for each nonstandard universe $W = V(Y)$ there exists corresponding canonical nonstandard language ${}^*L = L_W$ [5],[7]

4.***The restricted rules of conclusion*** If Let A and B well formed, closed formulas so that $A, B \in {}^*L$. If $W \models A$, then $\neg A \nvdash_{RMP} B$. Thus, if a statement A holds in nonstandard universe, we cannot obtain from formula $\neg A$ any formula B whatsoever.

Definition 2.2 [10]-[14] A set $S \subset {}^*\mathbb{N}$ is a hyper inductive if the following statement holds in $V(Y)$:

$$\bigwedge_{\alpha\in{}^*\mathbb{N}} (\alpha \in S \to \alpha^+ \in S).$$

Here $\alpha^+ = \alpha + 1$.Obviously a set ${}^*\mathbb{N}$ is a hyper inductive.

5. ***Axiom of hyper infinite induction***

$$\forall S (S \subset {}^*\mathbb{N}) \{\forall \beta (\beta \subset {}^*\mathbb{N}) [\textstyle\bigwedge_{1\leq\alpha<\beta} (\alpha \in S \to \alpha^+ \in S)] \to S = {}^*\mathbb{N}\}.$$

Example 2.1 Remind the proof of the following statement: structure $(\mathbb{N}, <, =)$ is a well-ordered set.

Proof. Let X be a nonempty subset of $\mathbb{N}$. Suppose X does not have a <-least element. Then consider the set $\mathbb{N}\backslash X$.

Case1. $\mathbb{N}\backslash X = \emptyset$. Then $X = \mathbb{N}$ and so 0 is a $<$ -least element but this is a contradiction.

Case2. $\mathbb{N}\backslash X \neq \emptyset$. Then $1 \in \mathbb{N}\backslash X$ otherwise 1 is a $<$ -least element but this is a contradiction. Assume now that there exists some $n \in \mathbb{N}\backslash X$ such that $n \neq 1$, but since we have supposed that X does not have a $<$ -least element, thus $n + 1 \notin X$. Thus we see that for all n the statement $n \in \mathbb{N}\backslash X$ implies that $n + 1 \in \mathbb{N}\backslash X$. We can conclude by axiom of induction that $n \in \mathbb{N}\backslash X$ for all $n \in \mathbb{N}$. Thus $\mathbb{N}\backslash X = \mathbb{N}$ implies $X = \emptyset$. This is a contradiction to X being a non-empty subset of $\mathbb{N}$. Remind that structure $({}^*\mathbb{N}, <, =)$ is not a well-ordered set [5]-[7]. We set now $X_1 = {}^*\mathbb{N}\backslash\mathbb{N}$ and thus ${}^*\mathbb{N}\backslash X_1 = \mathbb{N}$. In contrast with a set X mentioned above the assumption $n \in {}^*\mathbb{N}\backslash X_1$ implies that $n + 1 \in {}^*\mathbb{N}\backslash X_1$ if and only if n is finite, since for any infinite $n \in {}^*\mathbb{N}\backslash\mathbb{N}$ the assumption $n \in {}^*\mathbb{N}\backslash X_1$ contradicts with a true statement $V(Y) \models n \notin {}^*\mathbb{N}\backslash X_1 = \mathbb{N}$ and therefore in accordance with postulate 4 we cannot obtain from $n \in {}^*\mathbb{N}\backslash X_1$ any closed formula B whatsoever.

Theorem 2.1[14] (Generalized Recursion Theorem) Let S be a set, $c \in S$ and $g\colon S \times {}^*\mathbb{N} \to S$ is any function with $\mathrm{dom}(g) = S \times {}^*\mathbb{N}$ and $\mathrm{range}(g) \subseteq S$, then there exists a function $\mathcal{F}\colon {}^*\mathbb{N} \to S$ such that: 1) $\mathrm{dom}(\mathcal{F}) = {}^*\mathbb{N}$ and $\mathrm{range}(\mathcal{F}) \subseteq S$; 2) $\mathcal{F}(1) = c$; 3) for all $x \in {}^*\mathbb{N}, \mathcal{F}(n + 1) = g(\mathcal{F}(n), n)$.

Definition 2.3 [12]-[14] (1) Suppose that S is a standard set on which a binary operations $(\cdot+\cdot)$ and $(\cdot\times\cdot)$ is defined and under which S is closed. Let $\{x_k\}_{k\in{}^*\mathbb{N}}$ be any hyper infinite sequence of terms of ${}^*S$. For every hyper natural $n\in{}^*\mathbb{N}$ we denote by $Ext\text{-}\sum_{k=1}^{n}x_k$ the element of ${}^*S$ uniquely determined by the following canonical conditions: (a) $Ext\text{-}\sum_{k=1}^{1}x_k=x_1$; (b) $Ext\text{-}\sum_{k=1}^{n+1}x_k=Ext\text{-}\sum_{k=1}^{n}x_k+x_{n+1}$ for all $n\in{}^*\mathbb{N}$.
(2) For every hyper natural $n\in{}^*\mathbb{N}_\infty$ we denote by $Ext\text{-}\prod_{k=1}^{n}x_k$ the element of ${}^*S$ uniquely determined by the following canonical conditions: (a) $Ext\text{-}\prod_{k=1}^{1}x_k=x_1$; (b) $Ext\text{-}\prod_{k=1}^{n+1}x_k=(Ext\text{-}\prod_{k=1}^{n}x_k)\times x_{n+1}$ for all $n\in{}^*\mathbb{N}$.
Theorem 2.2. [14] (1) suppose that S is a standard set on which a binary operation $(\cdot+\cdot)$ is defined and under which S is closed and that $(\cdot+\cdot)$ is associative on S. Let $\{x_k\}_{k\in{}^*\mathbb{N}}$ be any hyper infinite sequence of terms of ${}^*S$. Then for any $n,m\in{}^*\mathbb{N}$ we have: $Ext\text{-}\sum_{k=1}^{n+m}x_k=Ext\text{-}\sum_{k=1}^{n}x_k+Ext\text{-}\sum_{k=1}^{m}x_k$;
(2) suppose that S is a standard set on which a binary operation $(\cdot\times\cdot)$ is defined and under which S is closed and that $(\cdot\times\cdot)$ is associative on S. Let $\{x_k\}_{k\in{}^*\mathbb{N}}$ be any hyper infinite sequence of terms of ${}^*S$. Then for any $n,m\in{}^*\mathbb{N}$ we have: $Ext\text{-}\prod_{k=1}^{n+m}x_k=(Ext\text{-}\prod_{k=1}^{n}x_k)\times(Ext\text{-}\prod_{k=1}^{m}x_k)$; (3) for any $z\in{}^*S$ and for any $n\in{}^*\mathbb{N}_\infty$ we have: $z\times(Ext\text{-}\sum_{k=1}^{n}x_k)=Ext\text{-}\sum_{k=1}^{n}z\times x_k$.

§2.1. External non-Archimedean Field ${}^*\mathbb{R}_c^\#$ by Cauchy Completion of the Internal Non -Archimedean Field ${}^*\mathbb{R}$.

Definition 2.4 A hyper infinite sequence of hyperreal numbers from ${}^*\mathbb{R}$ is a function $a:{}^*\mathbb{N}\to{}^*\mathbb{R}$ from the hypernatural numbers ${}^*\mathbb{N}$ into the hyperreal numbers ${}^*\mathbb{R}$.We usually denote such a function by $n\mapsto a_n$, so the terms in the sequence are written as $\{a_1,a_2,\dots,a_n,\dots\}$.To refer to the whole hyper infinite sequence, we will write $\{a_n\}_{n=1}^{{}^*\infty}$ or $\{a_n\}_{n\in{}^*\mathbb{N}}$.
Abbreviation 2.1 For a standard set E we often write $E_{\rm st}$, let ${}^\sigma E_{\rm st}=\{{}^*x|x\in E_{\rm st}\}$.We identify z with ${}^\sigma z$ i.e., $z\equiv{}^\sigma z$ for all $z\in\mathbb{C}$. Hence, ${}^\sigma E_{\rm st}=E_{\rm st}$ if $E\subseteq\mathbb{C}$, e.g., ${}^\sigma\mathbb{C}=\mathbb{C}$, ${}^\sigma\mathbb{R}=\mathbb{R}$, etc.Let ${}^*\mathbb{R}_c^\#$, ${}^*\mathbb{R}_{c,\approx}^\#$, ${}^*\mathbb{R}_{c,\approx+}^\#$, ${}^*\mathbb{R}_{c,\rm fin}^\#$, ${}^*\mathbb{R}_{c,\infty}^\#$, ${}^*\mathbb{N}_\infty$ denote the sets of Cauchy hyper-real numbers, Cauchy infinitesimal hyper-real numbers, Cauchy positive infinitesimal hyperreal numbers, Cauchy finite hyper-real numbers, Cauchy infinite hyper-real numbers and infinite hypernatural numbers, respectively. Note that ${}^*\mathbb{R}_{c,\rm fin}^\#={}^*\mathbb{R}_c^\#\backslash{}^*\mathbb{R}_{c,\infty}^\#$.
Definition 2.5 Let $\{a_n\}_{n=1}^{{}^*\infty}$ be a hyper infinite ${}^*\mathbb{R}$- valued sequence mentioned above. We shall say that $\{a_n\}_{n=1}^{{}^*\infty}$ #-tends to 0 if, given any $\varepsilon\in{}^*\mathbb{R}_{\approx+}$, there is a hyper natural number $N\in{}^*\mathbb{N}$ such that for all $n>N$, $|a_n|\le\varepsilon$. We denote this symbolically by $a_n\to_\#0$.
Definition 2.6 Let $\{a_n\}_{n=1}^{{}^*\infty}$ be a hyper infinite ${}^*\mathbb{R}$-valued sequence mentioned above. We shall say that $\{a_n\}_{n=1}^{{}^*\infty}$ #-tends to $q\in{}^*\mathbb{R}$ if, given any $\varepsilon\in{}^*\mathbb{R}_{\approx+}$, there is a hyper natural number $N\in{}^*\mathbb{N}$ such that for all $n>N$, $|a_n-q|\le\varepsilon$ and we denote this symbolically by $a_n\to_\# q$ or by $\#\text{-}\lim_{n\to{}^*\infty}a_n=q$.
Definition 2.7 Let $\{a_n\}_{n=1}^{{}^*\infty}$ be a hyper infinite ${}^*\mathbb{R}$-valued sequence mentioned above. We shall say that sequence $\{a_n\}_{n=1}^{{}^*\infty}$ is bounded if there is a hyperreal $M\in{}^*\mathbb{R}$ such that for any $n\in{}^*\mathbb{N}$, $|a_n|\le M$.
Definition 2.8 Let $\{a_n\}_{n=1}^{{}^*\infty}$ be a hyper infinite ${}^*\mathbb{R}$-valued sequence mentioned above. We shall say that $\{a_n\}_{n=1}^{{}^*\infty}$ is a Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequence if , given any $\varepsilon\in{}^*\mathbb{R}_{\approx+}$, there is a hyper natural number $N(\varepsilon)\in{}^*\mathbb{N}$ such that for any $m,n>N$, $|a_n-a_m|<\varepsilon$.
Theorem 2.3 If $\{a_n\}_{n=1}^{{}^*\infty}$ is a #-convergent hyper infinite ${}^*\mathbb{R}$-valued sequence, i.e., that is, $a_n\to_\# q$ for some hyperreal number q, $q\in{}^*\mathbb{R}$ then $\{a_n\}_{n=1}^{{}^*\infty}$ is a Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequence.
Theorem 2.4 If $\{a_n\}_{n=1}^{{}^*\infty}$ is a Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequence, then it is finitely bounded or hyper finitely bounded; that is, there is some finite or hyperfinite $M\in{}^*\mathbb{R}_+$ such that $|a_n|\le M$ for all $n\in{}^*\mathbb{N}$.
Definition 2.8 Let S be a set, with an equivalence relation $(\cdot\sim\cdot)$ on pairs of elements. For$s\in S$, denote by $cl[s]$ the set of all elements in S that are related to s. Then for any $s,t\in S$, either $cl[s]=cl[t]$ or $cl[s]$ and $cl[t]$ are disjoint.

Remark 2.2 The hyperreal numbers ${}^{*}\mathbb{R}_{c}^{\#}$ will be constructed as equivalence classes of Cauchy hyper infinite ${}^{*}\mathbb{R}$-valued sequences. Let $\mathcal{F}\{{}^{*}\mathbb{R}\}$ denote the set of all Cauchy hyper infinite ${}^{*}\mathbb{R}$-valued sequences of hyperreal numbers. We define the equivalence relation on a set $\mathcal{F}\{{}^{*}\mathbb{R}\}$.

Definition 2.9 Let $\{a_n\}_{n=1}^{{}^{*}\infty}$ and $\{b_n\}_{n=1}^{{}^{*}\infty}$ be in $\mathcal{F}\{{}^{*}\mathbb{R}\}$. Say they are #-equivalent if $a_n - b_n \to_{\#} 0$ i.e., if and only if the hyper infinite ${}^{*}\mathbb{R}$-valued sequence $\{a_n - b_n\}_{n=1}^{{}^{*}\infty}$ #-tends to 0.

Theorem 2.5 [14] Definition above yields an equivalence relation on a set $\mathcal{F}\{{}^{*}\mathbb{R}\}$.

Definition 2.10 The external hyperreal numbers ${}^{*}\mathbb{R}_{c}^{\#}$ are the equivalence classes $cl[\{a_n\}]$ of Cauchy hyper infinite ${}^{*}\mathbb{R}$-valued sequences of hyperreal numbers, as per definition above. That is, each such equivalence class is an external hyperreal number.

Definition 2.11 Given any hyperreal number $q \in {}^{*}\mathbb{R}$**,** define a hyperreal number $q^{\#}$to be the equivalence class of the hyper infinite ${}^{*}\mathbb{R}$-valued sequence $\{a_n = q\}_{n=1}^{{}^{*}\infty}$consisting entirely of $q \in {}^{*}\mathbb{R}$. So we view ${}^{*}\mathbb{R}$ as being inside ${}^{*}\mathbb{R}_{c}^{\#}$ by thinking of each hyperreal number $q \in {}^{*}\mathbb{R}$ as its associated equivalence class $q^{\#}$. It is standard to abuse this notation, and simply refer to the equivalence class as q as well.

Definition 2.12 Let $s, t \in {}^{*}\mathbb{R}_{c}^{\#}$, so there are Cauchy hyper infinite ${}^{*}\mathbb{R}$-valued sequences $\{a_n\}_{n=1}^{{}^{*}\infty}$, $\{b_n\}_{n=1}^{{}^{*}\infty}$ of hyper-real numbers with $s = cl[\{a_n\}]$ and $t = cl[\{b_n\}]$.

(a) Define $s + t$ to be the equivalence class of the hyper infinite sequence $\{a_n + b_n\}_{n=1}^{{}^{*}\infty}$.

(b) Define $s \times t$ to be the equivalence class of the hyper infinite sequence $\{a_n + b_n\}_{n=1}^{{}^{*}\infty}$.

Theorem 2.6 [14] The operations $+,\times$ in definition above by the requirements (a) and (b) are well-defined.

Theorem 2.7 Given any hyperreal number $s \in {}^{*}\mathbb{R}_{c}^{\#}$, $s \neq 0$ there is a hyperreal number $t \in {}^{*}\mathbb{R}_{c}^{\#}$ such that $s \times t = 1$.

Theorem 2.8 If $\{a_n\}_{n=1}^{{}^{*}\infty}$ is a Cauchy hyper infinite sequence which does not #-tend to 0, then there is some $N \in {}^{*}\mathbb{N}$ such that, for all $n > N, a_n \neq 0$.

Definition 2.13 Let $s \in {}^{*}\mathbb{R}_{c}^{\#}$. Say that s is positive if $s \neq 0$, and if $s = cl[\{a_n\}]$ for some Cauchy hyper infinite sequence of hyperreal numbers such that for some $N \in {}^{*}\mathbb{N}, a_n > 0$ for all $n > N$. Then for a given two hyperreal numbers s, t, say that $s > t$ if $s - t$ is positive.

Theorem 2.9 Let $s, t \in {}^{*}\mathbb{R}_{c}^{\#}$ be hyperreal numbers such that $s > t$, and let $r \in {}^{*}\mathbb{R}_{c}^{\#}$, then $s + r > t + r$.

Theorem 2.10 Let $s, t \in {}^{*}\mathbb{R}_{c}^{\#}$ be hyperreal numbers such that $s, t > 0$. Then there is $m \in {}^{*}\mathbb{N}$ such that $m \times s > t$.

Theorem 2.11 Given any hyperreal number $r \in {}^{*}\mathbb{R}_{c}^{\#}$, and any hyperreal number $\varepsilon > 0, \varepsilon \approx 0$, there is a hyperreal number $q \in {}^{*}\mathbb{R}_{c}^{\#}$ such that $|r - q| < \varepsilon$.

Definition 2.14 Let $S \subsetneq {}^{*}\mathbb{R}_{c}^{\#}$ be a nonempty set of hyperreal numbers. A hyperreal number $x \in {}^{*}\mathbb{R}_{c}^{\#}$ is called an upper bound for S if $x \geq s$ for all $s \in S$. A hyperreal number x is the least upper bound (or supremum: sup$S$) for S if x is an upper bound for S and $x \leq y$ for every upper bound y of S.

Remark 2.3 The order $\leq$ given by definition above obviously is $\leq$-incomplete.

Definition 2.15 Let $S \subsetneq {}^{*}\mathbb{R}_{c}^{\#}$ be a nonempty set of hyperreal numbers. We will say that:

(1) S is $\leq$ -admissible above if the following conditions are satisfied:

(a) S is finitely bounded or hyper finitely bounded above;

(b) let $A(S)$ be a set such that $\forall x[x \in A(S) \Leftrightarrow x \geq S]$ then for any $\varepsilon > 0, \varepsilon \approx 0$ there are $\alpha \in S$ and $\beta \in A(S)$ such that $\beta - \alpha \leq \varepsilon \approx 0$. (2) S is $\leq$ -admissible belov if the following conditions are satisfied:

(a) S is finitely bounded or hyper finitely bounded below;

(b) let $L(S)$ be a set such that $\forall x[x \in L(S) \Leftrightarrow x \leq S]$ then for any $\varepsilon > 0, \varepsilon \approx 0$ there are $\alpha \in S$ and $\beta \in L(S)$ such that $\alpha - \beta \leq \varepsilon \approx 0$.

Theorem 2.12 [14] (a) Any $\leq$-admissible above subset $S \subset {}^{*}\mathbb{R}_{c}^{\#}$ has the least upper bound property.

(b) Any $\leq$-admissible above subset $S \subset {}^{*}\mathbb{R}_{c}^{\#}$ has the greatest lower bound property.

Theorem 2.13 [14] (Generalized Nested Intervals Theorem) Let $\{I_n\}_{n=1}^{{}^{*}\infty} = \{[a_n, b_n]\}_{n=1}^{{}^{*}\infty}$, $[a_n, b_n] \subset {}^{*}\mathbb{R}_{c}^{\#}$ be a hyper infinite sequence of #-closed intervals satisfying each of the following conditions: (a) $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \supseteq I_n \supseteq \cdots$ (b) $b_n - a_n \to_{\#} 0$ as $n \to {}^{*}\infty$, Then $\bigcap_{n=1}^{{}^{*}\infty} I_n$consists of exactly one hyperreal number $x \in {}^{*}\mathbb{R}_{c}^{\#}$.

Theorem 2.14 [14] (Generalized Squeeze Theorem) Let $\{a_n\}_{n=1}^{*\infty}$, $\{c_n\}_{n=1}^{*\infty}$ be two hyper infinite sequences #-converging to L, and $\{b_n\}_{n=1}^{*\infty}$ a hyper infinite sequence. If $\forall n > K, K \in {}^*\mathbb{N}$ we have $a_n \leq b_n \leq c_n$, then b_n also #-converges to L.
Theorem 2.15 [14] If $\#\text{-}\lim_{n\to{}^*\infty,}|\, a_n| = 0$, then $\#\text{-}\lim_{n\to{}^*\infty}, a_n = 0$.
Theorem 2.16 [14] (Generalized Bolzano -Weierstrass Theorem) Any finitely or hyper finitely bounded hyper infinite ${}^*\mathbb{R}_c^\#$ -valued sequence has #-convergent hyper infinite subsequence.
Definition 2.16 Let $\{a_n\}_{n=1}^{*\infty}$ be ${}^*\mathbb{R}_c^\#$-valued sequence. Say that a sequence $\{a_n\}_{n=1}^{*\infty}$ #-tends to 0 if, given any $\varepsilon > 0$, $\varepsilon \approx 0$, there is a hyper natural number $N \in {}^*\mathbb{N}_\infty, N = N(\varepsilon)$ such that, for all $n > N$, $|a_n| \leq \varepsilon$.
Definition 2.17 Let $\{a_n\}_{n=1}^{*\infty}$ be ${}^*\mathbb{R}_c^\#$-valued hyper infinite sequence. We call $\{a_n\}_{n=1}^{*\infty}$ a Cauchy hyper infinite sequence if given any hyperreal number $\varepsilon \in {}^*\mathbb{R}_{c,\approx+}^\#$, there is a hypernatural number $N = N(\varepsilon)$ such that for any $m, n > N$, $|a_n - a_m| < \varepsilon$.
Theorem 2.17 If $\{a_n\}_{n=1}^{*\infty}$is a #-convergent hyper infinite sequence i.e., $a_n \to_\# b$ for some hyperreal number $b \in {}^*\mathbb{R}_c^\#$, then $\{a_n\}_{n=1}^{*\infty}$ is a Cauchy hyper infinite sequence.
Theorem 2.18 If $\{a_n\}_{n=1}^{*\infty}$ is a Cauchy hyper infinite sequence, then it is bounded; that is, there is some $M \in {}^*\mathbb{R}_c^\#$ such that $|a_n| \leq M$ for all $n \in {}^*\mathbb{N}$.
Theorem 2.19 [14] Any Cauchy hyper infinite sequence $\{a_n\}_{n=1}^{*\infty}$ has a #-limit in ${}^*\mathbb{R}_c^\#$; that is, there exists $b \in {}^*\mathbb{R}_c^\#$ such that $a_n \to_\# b$.
Remark 2.4 Note that there exists canonical natural embedding ${}^*\mathbb{R} \hookrightarrow {}^*\mathbb{R}_c^\#$.
Remark 2.5 A nonempty set S of Cauchy hyperreal numbers ${}^*\mathbb{R}_c^\#$ is unbounded above if it has no hyperfinite upper bound, or unbounded below if it has no hyperfinite lower bound. It is convenient to adjoin to Cauchy hyperreal number system ${}^*\mathbb{R}_c^\#$ two points, $+\infty^\# = ({}^*{+\infty})^\#$ (which we also write more simply as $\infty^\#$) and $-\infty^\#$, and to define the order relationships between them and any Cauchy hyperreal number $x \in {}^*\mathbb{R}_c^\#$ by $-\infty^\# < x < \infty^\#$.
Definition 2.18 We will call $-\infty^\#$ and $\infty^\#$ are points at hyper infinity. If $S \subset {}^*\mathbb{R}_c^\#$ is a nonempty set of Cauchy hyperreals, we write $\sup(S) = \infty^\#$ to indicate that S is unbounded above, and $\inf(S) = -\infty^\#$ to indicate that S is unbounded below.
Definition 2.19 That is (ε, δ) definition of the #-limit of a function $f: D \to {}^*\mathbb{R}_c^\#$ is as follows: let $f(x)$ is a ${}^*\mathbb{R}_c^\#$- valued function defined on a subset $D \subset {}^*\mathbb{R}_c^\#$ of the Cauchy hyperreal numbers. Let c be a #-limit point of D and let $L \in {}^*\mathbb{R}_c^\#$ be Cauchy hyperreal number. We say that $\#\text{-}\lim_{x\to_\# c} f(x) = L$ if for every $\varepsilon \approx 0, \varepsilon > 0$ there exists a $\delta \approx 0, \delta > 0$ such that, for all $x \in D$, if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.
Definition 2.20 [13] The function $f: {}^*\mathbb{R}_c^\# \to {}^*\mathbb{R}_c^\#$ is a #-continuous (or micro continuous) at some point c of its domain if the #-limit of $f(x)$, as x #-approaches c through the domain of f, exists and is equal to $f(c)$: $\#\text{-}\lim_{x\to_\# c} f(x) = f(c)$.
Theorem 2.20 [14] Let $\{a_n\}_{n=1}^{*\infty}$ and $\{b_n\}_{n=1}^{*\infty}$ be ${}^*\mathbb{R}_c^\#$- valued hyper infinite sequences. Then the following equalities hold for any $n, k, l, j, m \in {}^*\mathbb{N}$:

$$b \times (Ext\text{-}\textstyle\sum_{i=1}^n a_i) = Ext\text{-}\sum_{i=1}^n b \times a_i \tag{2.1}$$

$$Ext\text{-}\textstyle\sum_{i=1}^n a_i \pm Ext\text{-}\sum_{i=1}^n b_i = Ext\text{-}\sum_{i=1}^n (a_i \pm b_i) \tag{2.2}$$

$$Ext\text{-}\textstyle\sum_{i=k_0}^{k_1}\left(Ext\text{-}\sum_{j=l_0}^{l_1} a_{ij}\right) = Ext\text{-}\sum_{j=l_0}^{l_1}\left(Ext\text{-}\sum_{i=k_0}^{k_1} a_{ij}\right) \tag{2.3}$$

$$(Ext\text{-}\textstyle\sum_{i=1}^n a_i) \times \left(Ext\text{-}\sum_{j=1}^n b_j\right) = Ext\text{-}\sum_{i=1}^n\left(Ext\text{-}\sum_{j=1}^n a_i \times b_j\right) \tag{2.4}$$

$$(Ext\text{-}\textstyle\prod_{i=1}^n a_i) \times (Ext\text{-}\prod_{i=1}^n b_i) = Ext\text{-}\prod_{i=1}^n a_i \times b_i \tag{2.5}$$

$$(Ext\text{-}\textstyle\prod_{i=1}^n a_i)^m = Ext\text{-}\prod_{i=1}^n a_i^m. \tag{2.6}$$

Theorem 2.21 [14] Let $\{a_n\}_{i=1}^{n}$ and $\{b_n\}_{i=1}^{n}$ be ${}^*\mathbb{R}_c^{\#}$- valued monotonically non-decreasing hyperfinite sequences. Suppose that $a_i \leq b_i, 1 \leq i \leq n$, then the following equalities hold for any $n \in {}^*\mathbb{N}$:

$$Ext\text{-}\prod_{i=1}^{n} a_i \leq Ext\text{-}\prod_{i=1}^{n} b_i. \tag{2.7}$$

Theorem 2.22 [14] Let $\{a_n\}_{i=1}^{n}$ and $\{b_n\}_{i=1}^{n}$ be ${}^*\mathbb{R}_c^{\#}$- valued hyperfinite sequences. Then the following inequalities hold for any $n \in {}^*\mathbb{N}$:

$$(Ext\text{-}\prod_{i=1}^{n} a_i \times b_i)^2 \leq (Ext\text{-}\prod_{i=1}^{n} a_i^2) \times (Ext\text{-}\prod_{i=1}^{n} b_i^2). \tag{2.8}$$

Definition 2.21 [13] Assume that $\{a_n\}_{n=1}^{{}^*\infty}$ is a ${}^*\mathbb{R}_c^{\#}$- valued hyper infinite sequence, the symbol $Ext\text{-}\sum_{n=1}^{{}^*\infty} a_n$ is a hyper infinite series, and a_n is the n-th term of the hyper infinite series.

Definition 2.22 [13] We shall say that a series $Ext\text{-}\sum_{n=1}^{{}^*\infty} a_n$ #-converges to the sum $A \in {}^*\mathbb{R}_c^{\#}$, and write $Ext\text{-}\sum_{n=1}^{{}^*\infty} a_n = A$ if the hyper infinite sequence $\{A_n\}_{n=1}^{{}^*\infty}$ defined by $A_m = Ext\text{-}\sum_{n=1}^{m} a_n$ #-converges to the sum A. The hyperfinite sum A_m is the n-th partial sum of $Ext\text{-}\sum_{n=1}^{{}^*\infty} a_n$. If $\#\text{-}\lim_{m \to {}^*\infty,} A_m = \infty^{\#}$ or $-\infty^{\#}$, we shall say that $Ext\text{-}\sum_{n=1}^{{}^*\infty} a_n$ #-diverges to $\infty^{\#}$ or to $-\infty^{\#}$.

Theorem 2.23 [13] The hyper infinite sum $Ext\text{-}\sum_{n=1}^{{}^*\infty} a_n$ of a #-convergent hyper infinite series is unique.

§2.2.Hyper infinite sequences and series of ${}^*\mathbb{R}_c^{\#}$- valued functions

Definition 2.23 [13] If $f_1, f_2, \ldots, f_k, f_{k+1}, \ldots, f_n, \ldots n \in {}^*\mathbb{N}$ are ${}^*\mathbb{R}_c^{\#}$- valued functions on a subset $D \subset {}^*\mathbb{R}_c^{\#}$ we say that $\{f_n\}_{n=1}^{{}^*\infty}$ is a hyper infinite sequence of ${}^*\mathbb{R}_c^{\#}$- valued functions on D.

Definition 2.24 [13] Suppose that $\{f_n\}_{n=1}^{{}^*\infty}$ is a hyper infinite sequence of ${}^*\mathbb{R}_c^{\#}$- valued functions on $D \subset {}^*\mathbb{R}_c^{\#}$ and the hyper infinite sequence of values $\{f_n(x)\}_{n=1}^{{}^*\infty}$ #-converges for each x in some subset S of D. Then we say that $\{f_n(x)\}_{n=1}^{{}^*\infty}$ #-converges pointwise on S to the #-limit function f, defined by $f(x) = \lim_{n \to {}^*\infty} f_n(x)$.

Definition 2.25 [13] If $\{f_n(x)\}_{n=1}^{{}^*\infty}$ is a hyper infinite sequence of ${}^*\mathbb{R}_c^{\#}$- valued functions on $D \subset {}^*\mathbb{R}_c^{\#}$, then

$$Ext\text{-}\sum_{n=1}^{{}^*\infty} f_n(x) \tag{2.9}$$

is a hyper infinite series of functions on D. The partial sums of (1), are defined by $F_n(x) = Ext\text{-}\sum_{k=1}^{n} f_n(x)$. If hyper infinite sequence $\{F_n(x)\}_{n=1}^{{}^*\infty}$#-converges pointwise to the #-limit function $F(x)$ on a subset $S \subset D$, we say that $\{F_n(x)\}_{n=1}^{{}^*\infty}$#-converges pointwise to the sum $F(x)$ on S, and write $F(x) = Ext\text{-}\sum_{n=1}^{{}^*\infty} f_n(x)$.

Definition 2.26 [13] A hyper infinite series of the form $Ext\text{-}\sum_{n=1}^{{}^*\infty}(x - x_0)^n, n \in {}^*\mathbb{N}$ is called a hyper infinite power series in $x - x_0$.

§2.3.The #-Derivatives and Riemann #-Integral of ${}^*\mathbb{R}_c^{\#}$-Valued Functions $f: D \to {}^*\mathbb{R}_c^{\#n}$

Definition 2.27 [13] A function $f: D \to {}^*\mathbb{R}_c^{\#}$ #-differentiable at an #-interior point $x \in D$ of its domain $D \subset {}^*\mathbb{R}_c^{\#}$ if the difference quotient $f(x) - f(x_0)/x - x_0$ has a #-limit: $\#\text{-}\lim_{x \to_{\#} x_0}(f(x) - f(x_0)/x - x_0)$. In this case the #-limit is called the #-derivative of f at interior point x_0, and is denoted by $f^{\#\prime}(x_0)$ or by $d^{\#}f\ (x_0)/d^{\#}x$.

Definition 2.28 If f is defined on an #-open set $S \subset {}^*\mathbb{R}_c^{\#}$**,** we say that f is #-differentiable on S if f is #-differentiable at every point of S. If f is #-differentiable on S, then $f^{\#\prime}(x)$ is a function on S.We say that f is #-continuously #-differentiable on S if $f^{\#\prime}(x)$ is #-continuous on S.

Definition 2.29 If f is #-differentiable on a #-neighbourhood of x_0, it is reasonable to ask if $f^{\#\prime}(x)$ is #-differentiable at x_0. If so, we denote the #-derivative of $f^{\#\prime}(x)$ at x_0 by $f^{\#\prime\prime}(x_0)$ or by $f^{\#(2)}(x_0)$ and this is the second #-derivative of f at x_0. Continuing inductively by hyper infinite induction, if $f^{\#(n-1)}(x)$ is defined on a #-neighbourhood of x_0, then the n-th #-derivative of f at x_0 denoted by $f^{\#(n)}(x_0)$ or by $d^{\#(n)}f\,(x_0)/d^{\#}x^n$, where $n \in {}^*\mathbb{N}$.
Theorem 2.24 [13] If f is #-differentiable at x_0 then f is #-continuous at x_0.
Theorem 2.25 [13] If f and g are #-differentiable at x_0, then so are $f \pm g$ and $f \times g$ with:
(a) $(f \pm g)^{\#\prime}(x_0) = f^{\#\prime}(x_0) \pm g^{\#\prime}(x_0)$, (b) $(f \times g)^{\#\prime}(x_0) = f^{\#\prime}(x_0)g(x_0) + g^{\#\prime}(x_0)f(x_0)$.
(c) The quotient f/g is #-differentiable at x_0 if $g(x_0) \neq 0$ with $(f/g)^{\#\prime} = \frac{f^{\#\prime}(x_0)g(x_0) - g^{\#\prime}(x_0)f(x_0)}{g(x_0)^2}$.
(d) If $n \in {}^*\mathbb{N}$ and $f_i, 1 \leq i \leq n$ are #-differentiable at x_0, then so are $Ext\text{-}\sum_{i=1}^{n} f_i$ with:

$$(Ext\text{-}\textstyle\sum_{i=1}^{n} f_i)^{\#\prime}(x_0) = Ext\text{-}\sum_{i=1}^{n} f_i^{\#\prime}(x_0).$$

(e) If $n \in {}^*\mathbb{N}$ and $f^{\#(n)}(x_0), g^{\#(n)}(x_0)$ exist, then so does $(f \times g)^{\#(n)}(x_0)$ and

$$(f \times g)^{\#(n)}(x_0) = Ext\text{-}\textstyle\sum_{i=0}^{n} \binom{n}{i} f^{\#(i)}(x_0) g^{\#(n-i)}(x_0)$$

Theorem 2.26 [13] (The Chain Rule) Suppose that g is #-differentiable at x_0 and f is #-differentiable at $g(x_0)$. Then the composite function $h = f \circ g$ defined by $h(x) = f(g(x))$ is #-differentiable at x_0 with $h^{\#\prime}(x_0) = f^{\#\prime}(g(x_0))g^{\#\prime}(x_0)$.
Theorem 2.27 [13] (Generalized Taylor's Theorem) Suppose that $f^{\#(n)}(x), n \in {}^*\mathbb{N}$ exists on an #-open interval I about x_0, and let $x \in I$. Let $P_n(x, x_0)$ be the n-th Taylor hyper polynomial of f about x_0, $P_n(x, x_0) = Ext\text{-}\sum_{r=0}^{n} \frac{f^{\#(r)}(x_0)(x-x_0)^r}{r!}$ Then the remainder $R(x, x_0) = f(x) - P_n(x, x_0)$ can be written as

$$R(x, x_0) = \frac{f^{\#(n+1)}(c)(x-x_0)^n}{(n+1)!}. \quad (2.10)$$

Here c depends upon x and is between x and x_0.
Definition 2.30 [13] Let $[a, b] \subset {}^*\mathbb{R}_c^{\#}$. A hyperfinite partition of $[a, b]$ is a hyperfinite set of subintervals $[x_0, x_1]$,…,$[x_{n-1}, x_n]$, with $n \in {}^*\mathbb{N}_\infty$, where $a = x_0 < x_1 \ldots < x_n = b$. A set of these points $x_0, x_1, \ldots, x_n$ defines a hyperfinite partition P of $[a, b]$, which we denote by $P = \{x_i\}_{i=0}^{n}$. The points $x_0, x_1, \ldots, x_n$ are the partition points of P.The largest of the lengths of the subintervals $[x_{i-1}, x_i]$, $0 \leq i \leq n$ is the norm of $P = \{x_i\}_{i=0}^{n}$ denoted by $\|P\|$; thus, $\|P\| = \max_{1\leq i\leq n}(x_i - x_{i-1})$.
Definition 2.31 Let P and P' are hyperfinite partitions of $[a, b]$, then P' is a refinement of P if every partition point of P is also a partition point of P'; that is, if P' is obtained by inserting additional points between those of P.
Definition 2.32 Let f be ${}^*\mathbb{R}_c^{\#}$- valued function $f\colon [a, b] \to {}^*\mathbb{R}_c^{\#}$, then we say that external hyperfinite sum σ^{Ext} defined by

$$\sigma^{Ext} = Ext\text{-}\textstyle\sum_{i=1}^{n} f(c_i)\,(x_i - x_{i-1}),\ x_{i-1} \leq c_i \leq x_i, \quad (2.11)$$

is a Riemann external hyperfinite sum of f over the hyperfinite partition $P = \{x_i\}_{i=0}^{n}$.
Definition 2.33 [13] Let f be ${}^*\mathbb{R}_c^{\#}$- valued function $f\colon [a, b] \to {}^*\mathbb{R}_c^{\#}$, then we say that f is Riemann #-integrable on $[a, b]$ if there is a number $L \in {}^*\mathbb{R}_c^{\#}$ with the following property: for every $\varepsilon \approx 0, \varepsilon > 0$, there is a $\delta \approx 0, \delta > 0$ such that $|L - \sigma^{Ext}| < \delta$ if σ^{Ext} is any Riemann external hyperfinite sum of f over a partition P of $[a, b]$ such that $\|P\| < \delta$. In this case, we say that L is the Riemann #-integral of f over $[a, b]$, and we shall write

$$L = Ext\text{-}\textstyle\int_a^b f(x)d^{\#}x. \quad (2.12)$$

Thus the Riemann #-integral of ${}^*\mathbb{R}_c^\#$- valued function $f\colon [a,b] \to {}^*\mathbb{R}_c^\#$ over $[a,b]$ is defined as #-limit of the external hyperfinite sums (55) with respect to partitions of the interval $[a,b]$:

$$Ext\text{-}\int_a^b f(x)d^\# x = \#\text{-}\lim_{n\to{}^*\infty}\left(Ext\text{-}\sum_{i=1}^{n} f(c_i)\,(x_i - x_{i-1})\right). \tag{2.13}$$

Definition 2.34 A coordinate rectangle R in ${}^*\mathbb{R}_c^{\#n}, n \in {}^*\mathbb{N}$ is the external finite or hyperfinite Cartesian product of n #-closed intervals; that is, $R = Ext\text{-}\times_{i=1}^{n}[a_i,b_i]$. The content of R is $V(R) = Ext\text{-}\prod_{i=1}^{n}(b_i - a_i)$. The hyperreal numbers $b_i - a_i$, $1 \le i \le n$ are the edge lengths of R. If they are equal, then R is finite or hyperfinite coordinate cube. If $a_l = b_l$ for some r, then $V(R) = 0$ and we say that R is degenerate; otherwise, R is nondegenerate.
Definition 2.35 If $R = Ext\text{-}\times_{i=1}^{n}[a_i,b_i]$ and $P_r = a_{r0} < a_{r1} <\dots< a_{rm_r}$ is an external hyperfinite partition of $[a_r,b_r], 1 \le r \le n$, then the set of all rectangles in ${}^*\mathbb{R}_c^{\#n}$ that can be written as $Ext\text{-}\times_{i=1}^{n}\left[a_{i,j_{i-1}}, a_{i,j_i}\right]$, $1 \le j_r \le m_r$, $1 \le r \le n$ is a partition of R. We denote this partition by $P = Ext\text{-}\times_{r=1}^{n} P_r$ and define its norm to be the maximum of the norms of $P_i, 1 \le i \le n$; thus, $\|P\| = \max_i\{P_i | 1 \le i \le n\}$.
Definition 2.36 If $P = Ext\text{-}\times_{i=1}^{n} P_i$ and $P' = Ext\text{-}\times_{i=1}^{n} P_i'$ are partitions of the same rectangle, then P' is a refinement of P if P_i' is a refinement of $P_i, 1 \le i \le n$ as defined above.
Definition 2.37 Suppose that f is a ${}^*\mathbb{R}_c^\#$- valued function defined on a rectangle R in ${}^*\mathbb{R}_c^{\#n}, n \in {}^*\mathbb{N}$, $P = \{P_i\}_{i=1}^{k}$ is a partition of R, and x_i is an arbitrary point in R_i, $1 \le j \le k$. Then a Riemann external hyperfinite sum σ^{Ext} of f over the partition P is defined by

$$\sigma^{Ext} = Ext\text{-}\sum_{i=1}^{k} f(x_i)\,V(R_i) \tag{2.14}$$

Definition 2.38 Let f be a ${}^*\mathbb{R}_c^\#$- valued function defined on a rectangle R in ${}^*\mathbb{R}_c^{\#n}, n \in {}^*\mathbb{N}$. We say that f is Riemann #-integrable on R if there is a number L with the following property: for every $\varepsilon \approx 0, \varepsilon > 0$, there is a $\delta \approx 0, \delta > 0$ such that $|L - \sigma^{Ext}| < \delta$ if σ^{Ext} is any Riemann external hyperfinite sum of f over a partition P of R such that $\|P\| < \delta$. In this case, we say that L is the Riemann #-integral of f over R, and write

$$L = Ext\text{-}\int_R f(x)d^{\#n}x. \tag{2.15}$$

Thus the Riemann #-integral of ${}^*\mathbb{R}_c^\#$- valued function f defined on a rectangle R in ${}^*\mathbb{R}_c^{\#n}$ is defined as #-limit of the external hyperfinite sums (58) with respect to partitions of the rectangle R:

$$Ext\text{-}\int_R f(x)d^{\#n}x = \#\text{-}\lim_{n\to{}^*\infty}\left(Ext\text{-}\sum_{i=1}^{k} f(x_i)\,V(R_i)\right). \tag{2.16}$$

§2.4.The ${}^*\mathbb{R}_c^\#$-Valued #-Exponential Function Ext-$\exp(x)$ and ${}^*\mathbb{R}_c^\#$-Valued Trigonometric Functions Ext-$\sin(x)$, Ext-$\cos(x)$

We define the #-exponential function $Ext\text{-}\exp(x)$ as the solution of the #-differential equation

$$f^{\#\prime}(x) = f(x), f(0) = 1. \tag{2.17}$$

We solve it by setting $f(x) = Ext\text{-}\sum_{n=0}^{{}^*\infty} x^n$, $f^{\#\prime}(x) = Ext\text{-}\sum_{n=0}^{{}^*\infty} nx^n$. Therefore

$$Ext\text{-}\exp(x) = Ext\text{-}\sum_{n=0}^{{}^*\infty}\frac{x^n}{n!}. \tag{2.18}$$

From (1) we get $\left(Ext\text{-}\exp(x)\right)\left(Ext\text{-}\exp(y)\right) = Ext\text{-}\exp(x+y)$ for any $x, y \in {}^*\mathbb{R}_c^\#$.

We define the #- trigonometric functions $Ext\text{-}\sin x$ and $Ext\text{-}\cos x$ by

$$Ext\text{-}\sin x \;=Ext\text{-}\textstyle\sum_{n=0}^{^*\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}, Ext\text{-}\cos x \;=Ext\text{-}\textstyle\sum_{n=0}^{^*\infty}(-1)^n \frac{x^{2n}}{(2n)!}. \quad (2.19)$$

It can be shown that the series (1) #-converges for all $x \in {}^*\mathbb{R}_c^\#$ #-differentiating yields

$$(Ext\text{-}\sin x\,)^{\#\prime} = Ext\text{-}\cos x,\,(Ext\text{-}\cos x\,)^{\#\prime} = -(Ext\text{-}\sin x\,). \quad (2.20)$$

§2.5. ${}^*\mathbb{R}_c^\#$ -Valued Schwartz Distributions

Definition 2.39 [13] Let U be an #- open subset of ${}^*\mathbb{R}_c^{\#n}$ and $f\colon U \to {}^*\mathbb{R}_c^\#$. The partial derivative of f at the point $x = (x_1, x_2, \dots, x_i, \dots, x_n)$ with respect to the i-th variable x_i is defined as

$$\frac{\partial^\# f}{\partial^\# x_i} = \#\text{-}\lim_{h\to_\# 0} \frac{f(x_1,x_2,\dots,x_i+h,\dots,x_n)-f(x_1,x_2,\dots,x_i,\dots,x_n)}{h}.$$

Definition 2.38 A multi-index of size $n \in {}^*\mathbb{N}$ is an element in ${}^*\mathbb{N}^n$, the length of a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in {}^*\mathbb{N}^n$ is defined as $Ext\text{-}\sum_{i=1}^n \alpha_i$ and denoted by $|\alpha|$. We introduce the following notations for a given multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in {}^*\mathbb{N}^n$: $x^\alpha = Ext\text{-}\prod_{i=1}^n x_i^{\alpha_i}$; $\partial^{\#\alpha} = Ext\text{-}\prod_{i=1}^n \frac{\partial^{\#\alpha_i}}{\partial^\# x_i^{\alpha_i}}$ or symbolically $\partial^{\#\alpha} = Ext\text{-}\frac{\partial^{\#\alpha}}{\partial^\# x_1^{\alpha_1}\dots\partial^\# x_n^{\alpha_n}}$..

Definition 2.40 The Schwartz space of rapidly decreasing ${}^*\mathbb{C}_c^\#$- valued test functions on ${}^*\mathbb{R}_c^{\#n}$, $n \in {}^*\mathbb{N}$ is the function space defined by

$$S^\#({}^*\mathbb{R}_c^{\#n}, {}^*\mathbb{C}_c^\#) = \{f \in C^{^*\infty}({}^*\mathbb{R}_c^{\#n}, {}^*\mathbb{C}_c^\#) | \forall(\alpha,\beta)(\alpha,\beta \in {}^*\mathbb{N}^n)\forall x(x \in {}^*\mathbb{R}_c^{\#n})[|x^\alpha\, D^{\#\beta} f(x)| < \infty^\#]\}.$$

Remark 2.6 Note that if $f \in S^\#({}^*\mathbb{R}_c^{\#n}, {}^*\mathbb{C}_c^\#)$ the integral of $x^\alpha |\, D^{\#\beta} f(x)|$ exists

$$Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}} |\, x^\alpha D^{\#\beta} f(x)| d^{\#n} < \infty^\#.$$

Definition 2.41 The Schwartz space of essentially rapidly decreasing ${}^*\mathbb{C}_c^\#$- valued test functions on ${}^*\mathbb{R}_c^{\#n}$, $n \in {}^*\mathbb{N}$ is the function space defined by

$$S^\#({}^*\mathbb{R}_c^{\#n}, {}^*\mathbb{C}_c^\#) = \{f \in C^{^*\infty}({}^*\mathbb{R}_c^{\#n}, {}^*\mathbb{C}_c^\#) | \forall\alpha(\alpha \in \mathbb{N}^n)\forall\beta(\beta \in {}^*\mathbb{N}^n)\forall x(x \in {}^*\mathbb{R}_c^{\#n})[|x^\alpha\, D^{\#\beta} f(x)| < \infty\,]\}.$$

Remark 2.7 Note that if $f \in S^\#({}^*\mathbb{R}_c^{\#n}, {}^*\mathbb{C}_c^\#)$ the integral of $x^\alpha |\, D^{\#\beta} f(x)|, \alpha \in \mathbb{N}^n, \beta \in {}^*\mathbb{N}^n$ exists and

$$Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}} |\, x^\alpha D^{\#\beta} f(x)| d^{\#n} < \infty\,.$$

Definition 2.42 The Schwartz space of rapidly decreasing ${}^*\mathbb{C}_c^\#$- valued test functions on ${}^*\mathbb{R}_{c,\mathrm{fin}}^{\#n}$, $n \in {}^*\mathbb{N}$ is the function space defined by

$$\breve{S}^\#({}^*\mathbb{R}_{c,\mathrm{fin}}^{\#n}, {}^*\mathbb{C}_c^\#) = \{f \in C^{^*\infty}({}^*\mathbb{R}_{c,\mathrm{fin}}^{\#n}, {}^*\mathbb{C}_c^\#) | \forall(\alpha,\beta)(\alpha,\beta \in {}^*\mathbb{N}^n)\forall x(x \in {}^*\mathbb{R}_{c,\mathrm{fin}}^{\#n})[|x^\alpha\, D^{\#\beta} f(x)| < \infty^\#]\}.$$

Remark 2.8 Note that if $f \in \breve{S}^\#({}^*\mathbb{R}_{c,\mathrm{fin}}^{\#n}, {}^*\mathbb{C}_c^\#)$ the integral of $x^\alpha |\, D^{\#\beta} f(x)|, \alpha \in {}^*\mathbb{N}^n, \beta \in {}^*\mathbb{N}^n$ exists and

$$Ext\text{-}\int_{{}^*\mathbb{R}^{\#n}_{c,\text{fin}}} \left| x^{\alpha} D^{\#\beta} f(x) \right| d^{\#n} < \infty^{\#}.$$

Definition 28.43 The Schwartz space of essentially rapidly decreasing ${}^*\mathbb{C}^{\#}_{c}$- valued test functions on ${}^*\mathbb{R}^{\#n}_{c,\text{fin}}$, $n \in {}^*\mathbb{N}$ is the function space defined by

$$\breve{S}^{\#}_{\text{fin}} \left({}^*\mathbb{R}^{\#n}_{c,\text{fin}}, {}^*\mathbb{C}^{\#}_{c} \right) =$$

$$\left\{ f \in C^{{}^*\infty}\left({}^*\mathbb{R}^{\#n}_{c,\text{fin}}, {}^*\mathbb{C}^{\#}_{c} \right) | \forall (\alpha, \beta)(\alpha \in \mathbb{N}^{n}, \beta \in \mathbb{N}^{n}) \exists c_{\alpha\beta} \left(c_{\alpha\beta} \in {}^*\mathbb{R}^{\#}_{c,\text{fin}} \right) \forall x \left(x \in {}^*\mathbb{R}^{\#n}_{c,\text{fin}} \right) \left[\left| x^{\alpha} \left(D^{\#\beta} f(x) \right) \right| < c_{\alpha\beta} \right] \right\}.$$

Remark 2.9 Note that if $f \in S^{\#}_{\text{fin}}({}^*\mathbb{R}^{\#n}_{c}, {}^*\mathbb{C}^{\#}_{c})$ the integral of $\left| {}^{x^{\alpha}} D^{\#\beta} f(x) \right|$ exists and finitely bounded above

$$Ext\text{-}\int_{{}^*\mathbb{R}^{\#n}_{c,\text{fin}}} \left| x^{\alpha} D^{\#\beta} f(x) \right| d^{\#n} < d_{\alpha\beta}, d_{\alpha\beta} \in {}^*\mathbb{R}^{\#}_{c,\text{fin}}.$$

Abbreviation 2.2 1) The Schwartz space of rapidly decreasing test functions on ${}^*\mathbb{R}^{\#n}_{c}$ we will be denoting by $S^{\#}({}^*\mathbb{R}^{\#n}_{c})$ and let $S^{\#}_{\text{fin}}({}^*\mathbb{R}^{\#n}_{c})$ denote the set of ${}^*\mathbb{C}^{\#}_{c}$-valued essentially rapidly decreasing test functions on ${}^*\mathbb{R}^{\#n}_{c}$.
2) The Schwartz space of rapidly decreasing ${}^*\mathbb{C}^{\#}_{c}$- valued test functions on ${}^*\mathbb{R}^{\#n}_{c,\text{fin}}$ we will be denoting by $\breve{S}^{\#}\left({}^*\mathbb{R}^{\#n}_{c,\text{fin}}\right)$ and let $\breve{S}^{\#}_{\text{fin}}\left({}^*\mathbb{R}^{\#n}_{c,\text{fin}}\right)$ denote the set of ${}^*\mathbb{C}^{\#}_{c}$-valued essentially rapidly decreasing test functions on ${}^*\mathbb{R}^{\#n}_{c,\text{fin}}$.

Definition 2.44 A linear functional $u: S^{\#}({}^*\mathbb{R}^{\#n}_{c}) \to {}^*\mathbb{C}^{\#}_{c}$ is a #-continuous if there exist $C, k \in {}^*\mathbb{N}$ and constants $c_{\alpha\beta}$ such that $|u(\varphi)| \leq C\left(Ext\text{-}\sum_{|\alpha| \leq k, |\beta| \leq k} c_{\alpha\beta}\right)$. Here $\forall x \left(x \in {}^*\mathbb{R}^{\#n}_{c}\right) \left[\left| x^{\alpha} \left(D^{\#\beta} \varphi(x) \right) \right| < c_{\alpha\beta} \right]$.

Definition 2.45 A linear functional $u: S^{\#}\left({}^*\mathbb{R}^{\#n}_{c,\text{fin}}\right) \to {}^*\mathbb{C}^{\#}_{c}$ is a strongly #-continuous if there exist $C, k \in {}^*\mathbb{N}$ and constants $c_{\alpha\beta}$ such that $|u(\varphi)| \leq C\left(Ext\text{-}\sum_{|\alpha| \leq k, |\beta| \leq k} c_{\alpha\beta}\right) \in {}^*\mathbb{R}^{\#}_{c,\text{fin}}$.

Definition 2.46 A generalized function $u \in S^{\#\prime}({}^*\mathbb{R}^{\#n}_{c})$ is defined as a #-continuous linear functional on vector space $S^{\#}({}^*\mathbb{R}^{\#n}_{c})$, symbolically it written as $u: \varphi \to (u, \varphi)$. Thus space $S^{\#\prime}({}^*\mathbb{R}^{\#n}_{c})$ of generalized functions is the space dual to $S^{\#}({}^*\mathbb{R}^{\#n}_{c})$.

Definition 2.47 A generalized function $u \in S^{\#\prime}\left({}^*\mathbb{R}^{\#n}_{c,\text{fin}}\right)$ is defined as a strongly #-continuous linear functional on vector space $S^{\#}\left({}^*\mathbb{R}^{\#n}_{c,\text{fin}}\right)$, symbolically it written as $u: \varphi \to (u, \varphi)$. Thus space $S^{\#\prime}\left({}^*\mathbb{R}^{\#n}_{c,\text{fin}}\right)$ of generalized functions is the space dual to $S^{\#}\left({}^*\mathbb{R}^{\#n}_{c,\text{fin}}\right)$.

Definition 2.48 Convergence of a hyper infinite sequence $\{u_n\}_{n=1}^{{}^*\infty}$ of generalized functions in $S^{\#\prime}({}^*\mathbb{R}^{\#n}_{c})$ is defined as weak #-convergence of the hyper infinite sequence of functionals in $S^{\#\prime}({}^*\mathbb{R}^{\#n}_{c})$ that is: $u_n \to_{\#} 0$, as $n \to {}^*\infty$, in $S^{\#\prime}({}^*\mathbb{R}^{\#n}_{c})$ means that $(u_n, \varphi) \to_{\#} 0$, as $n \to {}^*\infty$, for all $\varphi \in S^{\#}({}^*\mathbb{R}^{\#n}_{c})$.

Definition 2.49 Convergence of a hyper infinite sequence $\{u_n\}_{n=1}^{{}^*\infty}$ of generalized functions in $S^{\#\prime}\left({}^*\mathbb{R}^{\#n}_{c,\text{fin}}\right)$ is defined as weak #-convergence of functionals in $S^{\#\prime}\left({}^*\mathbb{R}^{\#n}_{c,\text{fin}}\right)$ that is: $u_n \to_{\#} 0$, as $n \to {}^*\infty$, in $S^{\#\prime}\left({}^*\mathbb{R}^{\#n}_{c,\text{fin}}\right)$ means that $(u_n, \varphi) \to_{\#} 0$, as $n \to {}^*\infty$, for all $\varphi \in S^{\#}\left({}^*\mathbb{R}^{\#n}_{c,\text{fin}}\right)$.

Definition 2.50 1) Let $u \in S^{\#'}({}^*\mathbb{R}^{\#n}_{c})$ and let $x = Ay + b$ be a linear transformation of ${}^*\mathbb{R}^{\#n}_{c}$ onto ${}^*\mathbb{R}^{\#n}_{c}$. The generalized function $u(Ay + b) \in S^{\#'}({}^*\mathbb{R}^{\#n}_{c})$ is defined by

$$(u(Ay + b), \varphi) = \left(u, \frac{\varphi[A^{-1}(x-b)]}{|\det A|} \right). \tag{2.21}$$

Formula (1) enables one to define generalized functions that are translation invariant, spherically symmetric, centrally symmetric, homogeneous, periodic, Lorentz invariant, etc.
2) Let the function $\alpha(x) \in C^{\#1}({}^*\mathbb{R}^{\#}_{c})$ have only simple zeros $x_k \in {}^*\mathbb{R}^{\#}_{c}, k \in {}^*\mathbb{N}$, the function $\delta(\alpha(x))$ is defined by

$$\delta\big(\alpha(x)\big) = Ext\text{-}\sum_{k=1}^{*\infty} \frac{\delta(x - x_k)}{|\alpha^{\#\prime}(x_k)|}. \tag{2.22}$$

3) Let $u \in S^{\#'}({}^*\mathbb{R}_c^{\#n})$, the generalized (weak) #-derivative $\partial^{\#\alpha} u$ of u of order α is defined as

$$(\partial^{\#\alpha} u, \varphi) = (-1)^{|\alpha|}(u, \partial^{\#\alpha}\varphi). \tag{2.23}$$

4) Let $u \in S^{\#'}({}^*\mathbb{R}_c^{\#n})$ and $g(x) \in C^{\#^*\infty}({}^*\mathbb{R}_c^{\#n})$, The product $gu = ug$ is defined by

$$(gu, \varphi) = (u, g\varphi). \tag{2.24}$$

5) Let $u_1 \in S^{\#'}({}^*\mathbb{R}_c^{\#n})$ and $u_2 \in S^{\#'}({}^*\mathbb{R}_c^{\#m})$ then their direct product is defined by the formula

$$(u_1 \times u_2, \varphi) = \big(u_1(x)(u_2(y), \varphi)\big),\ \ \varphi(x, y) \in S^{\#}({}^*\mathbb{R}_c^{\#n} \times {}^*\mathbb{R}_c^{\#m}). \tag{2.25}$$

6) The Fourier transform $\mathcal{F}[u]$ of a generalized function $u \in S^{\#'}({}^*\mathbb{R}_c^{\#n})$ is defined by the formula

$$(\mathcal{F}[u], \varphi) = (u, \mathcal{F}[\varphi]), \tag{2.26}$$

$$\mathcal{F}[\varphi] = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#n}} \varphi(x)(Ext\text{-}\exp[i(\xi, x)])d^{\#n}x. \tag{2.27}$$

Since the operation $\varphi(x) \to \mathcal{F}[\varphi](\xi)$ is an isomorphism of $S^{\#}({}^*\mathbb{R}_c^{\#n})$ onto $S^{\#}({}^*\mathbb{R}_c^{\#n})$, the operation $u \to \mathcal{F}[u]$ is an isomorphism of $S^{\#'}({}^*\mathbb{R}_c^{\#n})$ onto $S^{\#'}({}^*\mathbb{R}_c^{\#n})$ and the inverse of $\mathcal{F}[u]$ is given by: $\mathcal{F}^{-1}[u] = (2\pi)^{-n}\mathcal{F}[u(-\xi)]$. The following formulas hold for $u \in S^{\#'}({}^*\mathbb{R}_c^{\#n})$: (a) $\partial^{\#\alpha}\,\mathcal{F}[u] = \mathcal{F}[(ix)^{\alpha}u]$, (b) $\mathcal{F}[\,\partial^{\#\alpha}u] = (i\xi)^{\alpha}\mathcal{F}[u]$,(c) if the generalized function $u_1 \in S^{\#'}({}^*\mathbb{R}_c^{\#n})$ has #-com-pact support, then $\mathcal{F}[u_1 * u_2] = \mathcal{F}[u_1]\mathcal{F}[u_2]$.
7) If the generalized function u is periodic with n-period $T = (T_1, \dots, T_n)$, then $u \in S^{\#'}({}^*\mathbb{R}_c^{\#n})$, and it can be expanded in a hyper infinite trigonometric series

$$u(x) = Ext\text{-}\sum_{|k|=0}^{*\infty} c_k(u)(Ext\text{-}\exp[i(k\omega, x)]), |c_k(u)| \le A(1 + |k|)^m. \tag{2.28}$$

The series (1) #-converges to $u(x)$ in $S^{\#'}({}^*\mathbb{R}_c^{\#n})$, here $\omega = \left(\frac{2\pi}{T_1}, \dots, \frac{2\pi}{T_n}\right)$ and $k\omega = \left(\frac{2\pi k_1}{T_1}, \dots, \frac{2\pi k_n}{T_n}\right)$.

§3. A NON-ARCHIMEDEAN METRIC SPACES ENDOWED WITH ${}^*\mathbb{R}_{\mathbf{c}}^{\#}$ -VALUED METRIC

Definition 3.1 A non-Archimedean metric space is an ordered pair $(M, d^{\#})$ where M a set and $d^{\#}$ is a #-metric on M i.e., ${}^*\mathbb{R}_{\mathbf{c}+}^{\#}$- valued function $d^{\#}: M \times M \to {}^*\mathbb{R}_{\mathbf{c}+}^{\#}$such that for any triplet $x, y, z \in M$, the following holds:
1. $d^{\#}(x, y) = 0 \Longrightarrow x = y$. 2. $d^{\#}(x, y) = d^{\#}(y, x)$. 3. $d^{\#}(x, z) \le d^{\#}(x, y) + d^{\#}(y, z)$.
Definition 3.2 A hyper infinite sequence $\{x_n\}_{n=1}^{*\infty}$ of points in M is called #-Cauchy in $(M, d^{\#})$ if for every hyperreal $\varepsilon \in {}^*\mathbb{R}_{c+}^{\#}$ there exists some $N \in {}^*\mathbb{N}$ such that $d^{\#}(x_n, x_m) < \varepsilon$ if $n, m > N$.
Definition 3.3 A point x of the non-Archimedean metric space $(M, d^{\#})$ is the #-limit of the hyper infinite sequence $\{x_n\}_{n=1}^{*\infty}$ if for all $\varepsilon \in {}^*\mathbb{R}_{c+}^{\#}$, there exists some $N \in {}^*\mathbb{N}$ such that $d^{\#}(x_n, x) < \varepsilon$ if $n > N$.
Definition 3.4 A non-Archimedean metric space is #-complete if any of the following equivalent conditions are satisfied: 1.Every hyper infinite #-Cauchy sequence $\{x_n\}_{n=1}^{*\infty}$ of points in M has a #-limit that is also in M.
2.Every hyper infinite #-Cauchy sequence in M, #-converges in M that is, to some point of M.
For any non-Archimedean metric space $(M, d^{\#})$ one can construct a #-complete non-Archimedean metric space $(M', d^{\#})$ which is also denoted as $(\#\text{-}\bar{M}, d^{\#})$ and which contains M a #-dense subspace.

It has the following universal property: if K is any #-complete non-Archimedean metric space and $f: M \to K$ is any uniformly #-continuous function from M to K, then there exists a unique uniformly #-continuous function $f': M' \to K$ that extends f.The space #-$\bar{M}$ is determined up to #-isometry by this property (among all #-complete metric spaces #- isometrically containing non-Archimedean metric space (#-$\bar{M}, d^{\#}$), and is called the #-completion of $(M, d^{\#})$.

The #-completion of M can be constructed as a set of equivalence classes of Cauchy hyper infinite sequences in M. For any two hyper infinite Cauchy sequences $\{x_n\}_{n=1}^{^*\infty}$ and $\{y_n\}_{n=1}^{^*\infty}$ in M, we may define their distance as $d^{\#\prime} =$ #-$\lim_{n\to\infty^{\#}} d^{\#}(x_n, y_n)$. This #-limit exists because the hyperreal numbers $^*\mathbb{R}_{\mathbf{c}}^{\#}$ are #-complete. This is only a pseudo metric, not yet a metric, since two different hyper infinite Cauchy sequences may have the distance 0. But having distance 0 is an equivalence relation on the set of all hyper infinite Cauchy sequences, and the set of equivalence classes is a metric space, the #-completion of M. The original space is embedded in this space via the identification of an element x of M' with the equivalence class of hyper infinite sequences in M #-converging to x i.e., the equivalence class containing a hyper infinite sequence with constant value x. This defines a #-isometry onto a #-dense subspace, as required.

Example 3.1 Both $^*\mathbb{R}$ and $^*\mathbb{C}$ are internal metric spaces when endowed with the distance function $d(x, y) = |x - y|$.

Definition 3.5 About any point $x \in M$ we define the #-open ball of radius $r \in {^*\mathbb{R}_{c+}^{\#}}$ about x as the set $B_r(x) = \{y \in M | d^{\#}(x, y) < r\}$. These #-open balls form the base for a topology on M.

Definition 3.6 A non-Archimedean metric space $(M, d^{\#})$ is called hyper finitely bounded if there exists some $r \in {^*\mathbb{R}_{c,\mathrm{fin}+}}$ such that $d^{\#}(x, y) < r$ for all $x, y \in M$.

Definition 3.7 A non-Archimedean metric space $(M, d^{\#})$ is called finitely bounded if there exists some $r \in {^*\mathbb{R}_{c,\infty+}}$ such that $d^{\#}(x, y) < r$ for all $x, y \in M$.

Definition 3.8 A non-Archimedean metric space $(M, d^{\#})$ is called hyper finitely bounded if there exists some $r \in {^*\mathbb{R}_{c,\infty+}}$ such that $d^{\#}(x, y) < r$ for all $x, y \in M$.

Definition 3.9 Let $(M, d^{\#})$ be a non-Archimedean metric space. A set $A \subset X$ is called finitely bounded if there exists some $r \in {^*\mathbb{R}_{c,\mathrm{fin}+}}$ such that $A \subset B_r(a)$, $a \in X$.

Definition 3.10 A non-Archimedean metric space $(M, d^{\#})$ is called #-compact if every hyper infinite sequence $\{x_n\}_{n=1}^{^*\infty}$ in M has a hyper infinite subsequence that #-converges to a point in M. This sort of compactness is known as hyper sequential compactness and, in a non-Archimedean metric spaces is equivalent to the topological notions of hyper countable #-compactness.

Definition 3.11 A topological space X is called hyper countably #-compact if it satisfies any of the following equivalent conditions: (a) every hyper countable open cover U of X (i.e., $card(U) = card(^*\mathbb{N})$) has a finite or hyperfinite sub-cover.

For a function $f: M_1 \to M_2$ with a non-Archimedean metric spaces $(M_1, d_1^{\#})$ and $(M_2, d_2^{\#})$ the following definitions of uniform #-continuity and (ordinary) #-continuity hold.

Definition 3.12 A function f is called uniformly #-continuous if for every $\varepsilon \in {^*\mathbb{R}_{c\approx+}^{\#}}$ there exists $\delta \in {^*\mathbb{R}_{c\approx+}}$ such that for every $x, y \in M_1$ with $d_1^{\#}(x, y) < \delta$ we get $d_2^{\#}(f(x), f(y)) < \varepsilon$.

Definition 3.13 A function f is called #-continuous at $x \in M_1$ if for every $\varepsilon \in {^*\mathbb{R}_{c\approx+}^{\#}}$ there exists $\delta \in {^*\mathbb{R}_{c\approx+}^{\#}}$ such that for every $y \in M_1$ with $d_1^{\#}(x, y) < \delta$ we get $d_2^{\#}(f(x), f(y)) < \varepsilon$.

§4. LEBESGUE #-INTEGRATION OF $^*\mathbb{R}_{\mathbf{c}}^{\#}$ -VALUED FUNCTIONS

Let $C_0^{\#}(^*\mathbb{R}_c^{\#n})$ be the space of all $^*\mathbb{R}_c^{\#}$-valued #-compactly supported #-continuous functions of $^*\mathbb{R}_c^{\#n}$. Define a #-norm on $C_0^{\#}$ by the Riemann #-integral [13]:

$$\|f\|_{\#} = Ext\text{-}\int |f(x)| d^{\#n}x, \tag{4.1}$$

Note that the Riemann #-integral exists for any #-continuous function $f: {}^*\mathbb{R}_c^{\#n} \to {}^*\mathbb{R}_c^{\#}$, see [13]. Then $C_0^{\#}({}^*\mathbb{R}_c^{\#n})$ is a #-normed vector space and thus in particular, it is a non-Archimedean metric space. All non-Archimedean metric space, have a non-Archimedean #-completion $(\#\text{-}\overline{M}, d^{\#})$. Let $L_1^{\#}$ be this #-completion. This space $L_1^{\#}$ is isomorphic to the space of Lebesgue #-integrable functions modulo the subspace of functions with #-integral zero. Furthermore, the Riemann integral (1) is a uniformly #-continuous linear functional with respect to the #-norm on $C_0^{\#}({}^*\mathbb{R}_c^{\#n})$ which is #-dense in $L_1^{\#}$. Hence the Riemann #- integral $Ext\text{-}\int f(x)d^{\#n}x$ has a unique extension to all of $L_1^{\#}$. This integral is precisely the Lebesgue #-integral.

Definition 4.1 Suppose that $1 \le p < {}^*\infty$, and $[a,b]$ is an interval in ${}^*\mathbb{R}_c^{\#}$. We denote by $L_p^{\#}([a,b])$ the set of the all functions $f: [a,b] \to {}^*\mathbb{R}_c^{\#}$ such that $Ext\text{-}\int_a^b |f(x)|^p d^{\#}x < {}^*\infty.$ We define the $L_p^{\#}$ -#-norm of f by

$$\|f\|_{\#p} = \left(Ext\text{-}\int_a^b |f(x)|^p d^{\#}x\right)^{1/p}. \tag{4.2}$$

More generally, if E is a subset of ${}^*\mathbb{R}_c^{\#n}$, which could be equal to ${}^*\mathbb{R}_c^{\#n}$ itself, then $L_p^{\#}(E)$ is the set of Lebesgue #-measurable functions $f: E \to {}^*\mathbb{R}_c^{\#}$ whose p-th power is Lebesgue #-integrable, with the #-norm

$$\|f\|_{\#p} = \left(Ext\text{-}\int_E |f(x)|^p d^{\#n}x\right)^{1/p}. \tag{4.3}$$

Definition 4.2 A set $X \subset {}^*\mathbb{R}_c^{\#n}$ is #-measurable if there exists $Ext\text{-}\int 1_X\, d^{\#n}x$, where 1_X is the indicator function.

Definition 4.3 A ${}^*\mathbb{R}_c^{\#}$ -valued function f on ${}^*\mathbb{R}_c^{\#n}$ is a #-measurable if a set $\{x|f(x) > t\}$ is a #-measurable set for all $t \in {}^*\mathbb{R}_c^{\#n}$.

Remark 4.1 To assign a value to the Lebesgue #-integral of the indicator function 1_X of a #-measurable set X consistent with the given #-measure $\mu^{\#}$, the only reasonable choice is to set: $Ext\text{-}\int 1_X d\,\mu^{\#} = \mu^{\#}(X)$.

Definition 4.4 A hyperfinite linear combination of indicator functions $f = Ext\text{-}\sum_{k=1}^{n} \alpha_k\, 1_{X_k}$ where the coefficients $\alpha_k \in {}^*\mathbb{R}_c^{\#}$ and X_k are disjoint #-measurable sets, is called a #-measurable simple function.

Definition 4.5 When the coefficients α_k are positive, we set $Ext\text{-}\int f d\,\mu^{\#} = Ext\text{-}\sum_{k=1}^{n} \alpha_k\, \mu^{\#}(X_k)$. For a non-negative #-measurable function f, let $\{f_n(x)\}_{n=1}^{{}^*\infty}$be a hyper infinite sequence of the simple functions $f_n(x)$ whose values is $\frac{k}{2^n}$ whenever $\frac{k}{2^n} \le f(x) < \frac{k+1}{2^n}$ for k a non-negative hyperinteger less than 4^n. Then we set

$$Ext\text{-}\int f d\,\mu^{\#} = \#\text{-}\lim_{n \to {}^*\infty}\left(Ext\text{-}\int f_n d\,\mu^{\#}\right). \tag{4.4}$$

Definition 4.6 If f is a #-measurable function of the set E to the reals including $\pm\infty^{\#}$, then we can write $f = f^+ - f^-$, where: 1) $f^+(x) = f(x)$ if $f(x) > 0$ and $f^+(x) = 0$ if $f(x) \le 0$; 2) $f^-(x) = f(x)$ if $f(x) < 0$ and $f^-(x) = 0$ if $f(x) \ge 0$. Note that both f^+ and f^- are non-negative #-measurable functions and $|f| = f^+ + f^-$.

Definition 4.7 We say that the Lebesgue #-integral of the #-measurable function f exists, or is defined if at least one of $Ext\text{-}\int f^+ d\,\mu^{\#}$ and $Ext\text{-}\int f^- d\,\mu^{\#}$ is finite or hyperfinite. In this case we define

$$Ext\text{-}\int f d\,\mu^{\#} = \left(Ext\text{-}\int f^+ d\,\mu^{\#}\right) + \left(Ext\text{-}\int f^- d\,\mu^{\#}\right). \tag{4.5}$$

Theorem 4.1 Assuming that f is #-measurable and non-negative, the function $\check{f}(x) = \{x \in E|f(x) > t\}$ is monotonically non-increasing. The Lebesgue #-integral may then be defined as the improper Riemann #-integral of $\check{f}(x)$: $Ext\text{-}\int_E f d\,\mu^{\#} = Ext\text{-}\int_0^{{}^*\infty} \check{f}(x) d^{\#}x$.

Definition 4.8 Let X be any set. We denote by 2^X the set of all subsets of X.A family $\mathcal{F} \subset 2^X$ is called a #-σ-algebra on X (or $\sigma^{\#}$-algebra on X) if: 1) $\emptyset \in \mathcal{F}$. 2) A family $\mathcal{F}$ is closed under complements, i.e. $A \in \mathcal{F}$ implies $X \backslash A \in \mathcal{F}$. 3) A family $\mathcal{F}$ is closed under hyper infinite unions, i.e. if $\{A_n\}_{n \in {}^*\mathbb{N}}$ is a hyper infinite sequence in $\mathcal{F}$ then $\bigcup_{n \in {}^*\mathbb{N}} A_n \in \mathcal{F}$.

Theorem 4.2 If $\mathcal{F}$ is a #-σ-algebra on X then: (1) $\mathcal{F}$ is closed under hyper infinite intersections, i.e., if $\{A_n\}_{n \in {}^*\mathbb{N}}$ is a hyper infinite sequence in $\mathcal{F}$ then $\bigcap_{n\in{}^*\mathbb{N}} A_n \in \mathcal{F}$. (2) $X \in \mathcal{F}$.3) $\mathcal{F}$ is closed under hyperfinite unions and hyperfinite intersections.(4) $\mathcal{F}$ is closed under set differences. (5) $\mathcal{F}$ is closed under symmetric differences.

Theorem 4.3 If $\{A_\alpha\}_{\alpha\in I}$ is a collection of $\sigma^\#$-algebras on a set X, then $\bigcap_{\alpha\in I} A_\alpha$, is also an $\sigma^\#$-algebras on a set X.

Theorem 4.4 If $K \subset L$ then $\sigma^\#(K) \subset \sigma^\#(L)$.

Definition 4.9 (Borel $\sigma^\#$-algebra) Given a topological space X, the Borel $\sigma^\#$-algebra is the $\sigma^\#$-algebra generated by the #-open sets. It is denoted by $\mathcal{B}^\#(X)$. We call sets in $\mathcal{B}^\#(X)$ a Borel set. Specifically in the case $X = {}^*\mathbb{R}_c^{\#n}$ we have that $\mathcal{B}^\#({}^*\mathbb{R}_c^{\#n}) = \{U | U\ is\ \#\text{-open set}\}$. Note that the Borel $\sigma^\#$-algebra also contains all #-closed sets and is the smallest $\sigma^\#$-algebra with this property.

Definition 4.10 (#- Measures) A pair $(X, \mathcal{F})$ where $\mathcal{F}$ is an $\sigma^\#$-algebra on X is call a #- measurable space. Elements of $\mathcal{F}$ are called a #-measurable sets. Given a #-measurable space $(X, \mathcal{F})$, a function $\mu^\#: \mathcal{F} \to [0, {}^*\infty]$ is called a #-mea-sure on $(X, \mathcal{F})$ if: 1) $\mu^\#(\emptyset) = 0$. 2) For all hyper infinite sequences $\{A_n\}_{n\in{}^*\mathbb{N}}$ of pairwise disjoint sets in $\mathcal{F}$

$$\mu^\#\left(\bigcup_{n=1}^{{}^*\infty} A_n\right) = Ext\text{-}\sum_{n=1}^{{}^*\infty} \mu^\#(A_n). \tag{4.6}$$

§5. A NON-ARCHIMEDEAN BANACH SPACES ENDOWED WITH ${}^*\mathbb{R}_{\mathbf{c}}^\#$ -VALUED NORM

A non-Archimedean normed space with ${}^*\mathbb{R}_c^\#$ -valued norm (#-norm) is a pair $(X, \|\cdot\|_\#)$ consisting of a vector space X over a non-Archimedean scalar field ${}^*\mathbb{R}_c^\#$ or complex field ${}^*\mathbb{C}_c^\# = {}^*\mathbb{R}_c^\# + i{}^*\mathbb{R}_c^\#$ together with a norm $\|\cdot\|_\#: X \to {}^*\mathbb{R}_c^\#$. Like any norms, this norm induces a translation invariant distance function, called the norm induced non-Archimedean ${}^*\mathbb{R}_c^\#$ -valued metric $d^\#(x,y)$ for all vectors $x, y \in X$, defined by $d^\#(x,y) = \|x-y\|_\# = \|y-x\|_\#$. Thus $d^\#(x,y)$ makes X into a non-Archimedean metric space $(X, d^\#)$.

Definition 5.1 A hyper infinite sequence $\{x_n\}_{n=1}^{{}^*\infty}$ in X is called $d^\#$ - Cauchy or Cauchy in $(X, d^\#)$ or $\|\cdot\|_\#$ -Cauchy if for every hyperreal $\varepsilon \in {}^*\mathbb{R}_{c+}^\#$ there exists some $N \in {}^*\mathbb{N}$ such that $d^\#(x_n, y_m) = \|x_n - y_n\|_\# < \varepsilon$ if $n, m > N$.

Definition 5.2 The metric $d^\#$ is called a #-complete metric if the pair $(X, d^\#)$ is a #-complete metric space, which by definition means for every $d^\#$- Cauchy sequence $\{x_n\}_{n=1}^{{}^*\infty}$ in $(X, d^\#)$, there exists some $x \in X$ such that #-$\lim_{n\to{}^*\infty} \|x_n - x\|_\# = 0$.

§5.1. Semigroups on non-Archimedean Banach spaces and their generators

Definition 5.3 A family of bounded operators $\{T(t) | 0 < t < {}^*\infty\}$ on external hyper infinite dimensional non-Archimedean Banach space X endowed with ${}^*\mathbb{R}_c^\#$ -valued #-norm $\|\cdot\|_\#$ is called a strongly #-continuous semigroup if: (a) $T(0) = I$, (b) $T(s)T(t) = T(s+t)$ for all $s, t \in {}^*\mathbb{R}_{c,+}^\#$, (c) For each $\phi \in X, t \mapsto T(t)$ is #-continuous map-ping.

Definition 5.4 A family $\{T(t) | 0 < t < {}^*\infty\}$ of bounded or hyper bounded operators on external hyper infinite dimensional Banach space X is called a contraction semigroup if it is a strongly #-continuous semigroup and moreover $\|T(t)\|_\# < 1$ for all $t \in [0, {}^*\infty)$.

Theorem 5.1 Let $T(t)$ is a strongly #-continuous semigroup on a non-Archimedean Banach space X, let $A\varphi =$ #-$\lim_{r\to_\# 0} A_r\varphi$ where $A_r = r^{-1}(I - T(r))$ and let $D(A) = \{\varphi | \exists(\#\text{-}\lim_{r\to_\# 0} A_r\varphi)\}$, then the operator A is #-closed and #-densely defined. Operator A is called the infinitesimal generator of the semigroup $T(t)$.

Definition 5.5 We will also say that A generates the semigroup $T(t)$ and write $T(t) = Ext\text{-}\exp(-tA)$.

Theorem 5.2 (Generalized Hille -Yosida theorem) A necessary and sufficient condition that #-closed linear operator A on a non-Archimedean Banach space X generate a contraction semigroup is that: (a) $(-{}^*\infty, 0) \subset \rho(A)$,
(b) $\|(\lambda + A)^{-1}\|_\# \le \lambda^{-1}$ for all $\lambda > 0$.

Definition 5.6 Let X be a non-Archimedean Banach space, $\varphi \in X$.An element $l \in X^*$ that satisfies $\|l\|_\# = \|\varphi\|_\#$, and $l(\varphi) = \|\varphi\|_\#^2$ is called a normalized tangent functional to φ. By the generalized Hahn-Banach theorem, each $\varphi \in X$ has at least one normalized tangent functional.
Definition 5.7 A #-densely defined operator A on a non-Archimedean Banach space X is called accretive if for each $\varphi \in D(A)$, $\mathrm{Re}(l(A\varphi)) \geq 0$ for some normalized tangent functional to φ. Operator A is called maximal accretive if A is accretive and A has no proper accretive extension.
Remark 5.1 We remark that any accretive operator is #-closable. The #-closure of an accretive operator is again accretive, so every accretive operator has a smallest #-closed accretive extension.
Theorem 5.3 A #-closed operator A on a non-Archimedean Banach space X is the generator of a contraction semigroup if and only if A is accretive and $\mathrm{Ran}(\lambda_0 + A) = X$ for some $\lambda_0 > 0$.
Theorem 5.4 Let A be a #-closed operator on a non-Archimedean Banach space X. Then, if both A and it adjoint A^* are accretive, A generates a contraction semigroup.
Theorem 5.5 Let A be the generator of a contraction semigroup on a non-Archimedean Banach space X. Let D be a #-dense set, $D \subset D(A)$, so that $Ext\text{-}\exp(-tA): D \to D$. Then D is a #-core for A, i.e.,#-$\overline{A \upharpoonright D} = A$.

§5.2. Hypercontractive semigroups

In the previous section we discussed $L_\#^p$-contractive semigroups. In this section we give a self #- adjointness theorem for the operators of the form $A + V$, where V is a multiplication operator and A generates a $L_\#^p$-contractive semigroup that satisfies a strong additional property.
Definition 5.8 Let $\langle M.\mu^\#\rangle$ be a #-measure space with $\mu^\#(M) = 1$ and suppose that $A$is a positive self-adjoint operator on $L_\#^2(M, d^\#\mu^\#)$. We say that $Ext\text{-}\exp(-tA)$ is a hyper contractive semigroup if: (a) $Ext\text{-}\exp(-tA)$ is $L_\#^p$-contractive; (b) for some $b > 2$ and some constant C_b, there is a $T > 0$ so that $\|[Ext\text{-}\exp(-tA)]\varphi\|_{\#b} \leq \|\varphi\|_{\#2}$ for all $\varphi \in L_\#^2(M, d^\#\mu^\#)$.
Remark 5.2 Note that the condition (a) implies that $Ext\text{-}\exp(-tA)$ is a strongly #-continuous contraction semi-group for all $p < {}^*\infty$. Holder's inequality shows that $\|\cdot\|_{\#q} \leq \|\cdot\|_{\#p}$ if $p \geq q$. Thus the $L_\#^p$-spaces are a nested family of spaces which get smaller as p gets larger; this suggests that (b) is a very strong condition. The following proposition shows that constant b plays no special role.
Theorem 5.6 Let $Ext\text{-}\exp(-tA)$ be a hypercontractive semigroup on $L_\#^2(M, d^\#\mu^\#)$. Then for all $p, q \in (1, {}^*\infty)$ there is a constant $C_{p,q}$ and a $t_{p,q} > 0$ so that if $> t_{p,q}$, then $\|Ext\text{-}\exp(-tA)\varphi\|_{\#p} < C_{p,q}\|\varphi\|_{\#q}$, for all $\varphi \in L_q^\#$.
Theorem 5.7 Let $\langle M, \mu^\#\rangle$ be a $\sigma^\#$-measure space with $\mu^\#(M) = 1$and let H_0 be the generator of a hypercontractive semi-group on $L_2(M, d^\#\mu^\#)$. Let V be a ${}^*\mathbb{R}_c^\#$ -valued measurable function on $\langle M, \mu^\#\rangle$ such that $V \in L_p^\#(M, d^\#\mu^\#)$ for all $p \in [\![1, {}^*\infty)$ and $Ext\text{-}\exp(-tV) \in L_1^\#(M, d^\#\mu^\#)$ for all $t > 0$. Then $H_0 + V$ is essentially self #-adjoint on $C^{{}^*\infty}(H_0) \cap D(V)$ and is bounded below. Here $C^{{}^*\infty}(H_0) = \bigcap_{p\in{}^*\mathbb{N}} D(H_0^p)$.

§ 5.3. Strong #-convergence in the generalized sense

Let $\boldsymbol{X}$ be a non-Archimedean Banach space over field ${}^*\mathbb{C}_{\boldsymbol{c}}^\#$.

Let $T \in \mathcal{B}(\boldsymbol{X})$. A complex number $\lambda \in {}^*\mathbb{C}_{\boldsymbol{c}}^\#$ is called an eigenvalue (proper value, characteristic value) of T if there is a non-zero vector $u \in \boldsymbol{X}$ such that

$$Tu = \lambda u. \tag{5.3.1}$$

Such vector u is called an *eigenvector* (proper vector, characteristic vector) of T belonging to (associated with, etc.) the eigenvalue λ. The set N_λ of all $u \in \boldsymbol{X}$ such that $Tu = \lambda u$ is a linear sub manifold of $\boldsymbol{X}$; it is called the (geometric) eigenspace of T for the eigenvalue λ, and $\dim(N_\lambda)$ is called the (geometric) multiplicity of λ. N_λ is defined even when λ is not an eigenvalue; then we have $N_\lambda = \emptyset$. In this case it is often convenient to say that N_λ is the eigenspace for the eigenvalue λ with multiplicity zero, though this is not in strict accordance with the definition of an eigenvalue.

Remark 5.3.1 It can easily be proved that eigenvectors of T belonging to different eigenvalues are linearly independent.

Definition 5.3.1 The set of all eigenvalues of T is called the spectrum of T; we denote it by $\Sigma(T)$.

Let $T \in \mathcal{B}(\boldsymbol{X})$ and consider the inhomogeneous linear equation

$$(T-\xi)u=v, \tag{5.3.2}$$

where $\xi \in {}^*\mathbb{C}_c^\#$ is a given complex number, $v \in \boldsymbol{X}$ is given and $u \in \boldsymbol{X}$ is to be found. In order that this equation have a solution u for every v, it is necessary and sufficient that $T-\xi$ be non-singular, that is, ξ be different from any eigenvalue λ_T of T. Then' the inverse $(T-\xi)^{-1}$ exists and the solution u is given by

$$u=(T-\xi)^{-1}v. \tag{5.3.3}$$

Definition 5.3.2 The operator-valued function

$$R\,(\xi)=R\,(\xi,T)=(T-\xi)^{-1} \tag{5.3.4}$$

is called the resolvent of T.

Definition 5.3.3 The complementary set of the spectrum $\Sigma(T)$ (that is, the set of all complex numbers different from any of the eigenvalues of T) is called the resolvent set of, T and will be denoted by $\mathrm{P}(T)$. The resolvent $R\,(\xi,T)$ is thus defined for $\xi \in \mathrm{P}(T)$.

Let $T_n, n \in {}^*\mathbb{N}$ be a hyper infinite sequence of #-closed operators in a non-Archimedean Banach space $\boldsymbol{X}$. In the present section we are concerned with general considerations on strong #-convergence of the resolvents $R_n(\xi)=(T_n-\xi)^{-1}$.The fundamental result on the #-convergence in #-norm of the resolvents is given by theorem **5.3.**1.

Definition 5.3. (i) Let us define the *region of boundedness*, denoted by Δ_b, for the hyper infinite sequence $R_n(\xi), n \in {}^*\mathbb{N}$ as the set of all complex numbers $\xi \in {}^*\mathbb{C}_c^\#$ such that $\xi \in \mathrm{P}(T_n)$ for sufficiently large $n \in {}^*\mathbb{N}$ and the hyper infinite sequence $\|R_n(\xi)\|_\#, n \in {}^*\mathbb{N}$ is bounded for $n \in {}^*\mathbb{N}$ so large that the $R_n(\xi)$ are defined.

(ii) let Δ_s be the set of all $\xi \in {}^*\mathbb{C}_c^\#$ such that $s\text{-}\lim_{n\to{}^*\infty} R_n(\xi) = R'\,(\xi)$ exists. A set Δ_s will be called the region of strong #-convergence for $R_n(\xi), n \in {}^*\mathbb{N}$.

(iii) Similarly we define the region Δ_u of #-convergence in #-norm for $R_n(\xi), n \in {}^*\mathbb{N}$.

Remark 5.3. Note that obviously we have $\Delta_u \subset \Delta_s \subset \Delta_b$.

Theorem 5.3. If $R_n(\xi), n \in {}^*\mathbb{N}$ #-converges in #-norm to the resolvent $R\,(\xi)=(T-\xi)^{-1}$ of a #-closed operator T for some $\xi' \in P(T)$, then the same is true for every $\xi \in P(T)$.

§6. A NON-ARCHIMEDEAN HILBERT SPACES ENDOWED WITH ${}^{*}\mathbb{C}_{\boldsymbol{c}}^{\#}$ -VALUED INNER PRODUCT

Definition 6.1 Let H be external hyper infinite dimensional vector space over complex field ${}^{*}\mathbb{C}_{\boldsymbol{c}}^{\#} = {}^{*}\mathbb{R}_{\boldsymbol{c}}^{\#} + \mathrm{i}\,{}^{*}\mathbb{R}_{\boldsymbol{c}}^{\#}$. An inner product on H is a ${}^{*}\mathbb{C}_{\boldsymbol{c}}^{\#}$-valued function, $\langle\cdot,\cdot\rangle: H \times H \to {}^{*}\mathbb{C}_{\boldsymbol{c}}^{\#}$, such that (1) $\langle ax + by, z\rangle = \langle ax, z\rangle + \langle by, z\rangle$, (2) $\overline{\langle x, y\rangle} = \langle y, x\rangle$. (3) $\|x\|^2 \equiv \langle x, x\rangle \geq 0$ with equality $\langle x, x\rangle = 0$ if and only if $x = 0$.
Theorem 6.1 (Generalized Schwarz Inequality) Let $\{H, \langle\cdot,\cdot\rangle\}$be an inner product space, then for all $x, y \in H$: $|\langle x, y\rangle| \leq \|x\|\|y\|$ and equality holds if and only if x and y are linearly dependent.
Theorem 6.2 Let $\{H, \langle\cdot,\cdot\rangle\}$be an inner product space, and $\|x\|_{\#} = \sqrt{\langle x, x\rangle}$. Then $\|\cdot\|_{\#}$ is a ${}^{*}\mathbb{R}_{c}^{\#}$ -valued #-norm on a space H. Moreover $\langle x, x\rangle$ is #-continuous on Cartesian product $H \times H$, where H is viewed as the #-normed space $\{H, \|\cdot\|_{\#}\}$.
Definition 6.2 A non-Archimedean Hilbert space H is a #-complete inner product space.
Two elements x and y of non-Archimedean Hilbert space H are called orthogonal if $\langle x, y\rangle = 0$.

Example 6.1 The standard inner product on ${}^{*}\mathbb{C}_{\boldsymbol{c}}^{\#\boldsymbol{n}}, n \in {}^{*}\mathbb{N}_{\infty}$ is given by external hyperfinite sum

$$\langle x, y\rangle = Ext\text{-}\sum_{i=1}^{n} \overline{x_i}\, y_i. \tag{6.1}$$

Here $x = \{x_i\}_{i=1}^{n}, y = \{y_i\}_{i=1}^{n}$, with $x_i, y_i \in {}^{*}\mathbb{C}_{c}^{\#}, 1 \leq i \leq n$, see [14].
Example 6.2 The sequence space $l_2^{\#}$ consists of all hyper infinite sequences $z = \{z_i\}_{i=1}^{{}^{*}\infty}$ of complex numbers in ${}^{*}\mathbb{C}_{c}^{\#}$ such that the hyper infinite series Ext-$\sum_{i=1}^{n}|z_i|^2$ #-converges. The inner product on $l_2^{\#}$ is defined by

$$\langle z, w\rangle = Ext\text{-}\sum_{i=1}^{{}^{*}\infty} \overline{z_i}\, w_i. \tag{6.2}$$

Here $z = \{z_i\}_{i=1}^{{}^{*}\infty}, w = \{w_i\}_{i=1}^{{}^{*}\infty}$ and the latter hyper infinite series #-converging as a consequence of the generalized Schwarz inequality and the #-convergence of the previous hyper infinite series.
Example 6.3 Let $C^{\#}[a, b]$ be the space of the ${}^{*}\mathbb{C}_{\boldsymbol{c}}^{\#}$- valued #-continuous functions defined on the interval $[a, b] \subset {}^{*}\mathbb{R}_{c}^{\#}$, see [14]. We define an inner product on the space $C^{\#}[a, b]$ by the formula

$$\langle f, g\rangle = Ext\text{-}\int_a^b \overline{f(x)} g(x)\, d^{\#}x. \tag{6.3}$$

This space is not #-complete, so it is not a non-Archimedean Hilbert space. The #-complettion of $C^{\#}[a, b]$ with respect to the #-norm

$$\|f\|_{\#} = \left(Ext\text{-}\int_a^b |f(x)|^2\, d^{\#}x\right)^{1/2}, \tag{6.4}$$

is denoted by $L_2^{\#}[a, b]$.
Example 6.4 Let $C^{\#(k)}[a, b]$be the space of the ${}^{*}\mathbb{C}_{\boldsymbol{c}}^{\#}$- valued functions with $k \in {}^{*}\mathbb{N}$ #-continuous #-derivatives on $[a, b] \subset {}^{*}\mathbb{R}_{c}^{\#}$, see [14].We define an inner product on the space $C^{\#(k)}[a, b]$ by the formula

$$\langle f, g\rangle = Ext\text{-}\sum_{i=0}^{k} \left(Ext\text{-}\int_a^b \overline{f^{\#(i)}(x)} g^{\#(i)}(x)\, d^{\#}x\right). \tag{6.5}$$

Here $f^{\#(i)}$ and $g^{\#(i)}$ denotes the i-th #-derivatives of f and g respectively.The corresponding #-norm is

$$\|f\|_{\#} = \left(Ext\text{-}\sum_{i=1}^{k}\left(Ext\text{-}\int_{a}^{b}\left|f^{\#(i)}(x)\right|^{2} d^{\#}x\right)\right)^{1/2}. \quad (6.6)$$

This space is not #-complete, so it is not a non-Archimedean Hilbert space. The non-Archimedean Hilbert space obtained by #-complettion of $C^{\#(k)}[a,b]$ with respect to the #-norm (1) is non-Archimedean Sobolev space, denoted by $H^{\#k}[a,b]$.

Definition 6.3 The graph of the linear transformation $T: H \to H$ is the set of pairs $\{\langle\phi, T\phi\rangle | (\phi \in D(T))\}$. The graph of the operator T, denoted by $\Gamma(\mathrm{T})$, is thus a subset of $H \times H$ which is a non-Archimedean Hilbert space with the following inner product $(\langle\phi_1,\psi_1\rangle, \langle\phi_2,\psi_2\rangle)$. Operator T is called a #-closed operator if $\Gamma(\mathrm{T})$ is a #-closed subset of $H \times H$.

Definition 6.4 Let T_1 and T be operators on H. If $\Gamma(T_1) \supset \Gamma(\mathrm{T})$, then T_1 is said to be an extension of T and we write $T_1 \supset T$. Equivalently, $T_1 \supset T$ if and only if $D(T_1) \supset D(T)$ and $T_1\phi = T\phi$ for all $\phi \in D(T)$.

Definition 6.5 An operator T is #-closable if it has a #-closed extension. Every #-closable operator has a smallest #-closed extension, called its #-closure, which we denote by #-T.

Theorem 6.3 If T is #-closable, then $\Gamma(\#\text{-}\bar{T}) = \#\text{-}\overline{\Gamma(T)}$.

Definition 6.6 Let $D(T^*)$ be the set of $\varphi \in H$ for which there is an $\xi \in H$ with $(T\psi,\varphi) = (\psi,\xi)$ for all $\psi \in D(T)$. For each $\varphi \in D(T^*)$, we define $T^*\varphi = \xi$.The operator T^* is called the #-adjoint of T. Note that $\varphi \in D(T^*)$ if and only if $|(T\psi,\varphi)| \leq C\|\psi\|_{\#}$ for all $\psi \in D(T)$. Note that $S \subset T$ implies $T^* \subset S$.

Remark 6.1 Note that for ξ to be uniquely determined by the condition $(T\psi,\varphi) = (\psi,\xi)$ one need the fact that $D(T)$ is #-dense in H. If the domain $D(T^*)$ is #-dense in H, then we can define $T^{**} = (T^*)^*$.

Theorem 6.4 Let T be a #-densely defined operator on a non-Archimedean Hilbert space H. Then: (a) T^* is #-closed. (b) The operator T is #-closabie if and only if $D(T^*)$ is -dense in which case $T = T^{**}$. (c) If T is #-closable, then $(\#\text{-}\bar{T})^* = T^*$.

Definition 6.7 Let T be a #-closed operator on a non-Archimedean Hilbert space H. A complex number $\lambda \in {}^*\mathbb{C}_c^{\#}$ is in the resolvent set $\rho(T)$, if $\lambda I - T$ is a bijection of $D(T)$ onto H with a finitely or hyper finitely bounded inverse. If complex number $\lambda \in \rho(T)$, $R_\lambda = (\lambda I - T)^{-1}$ is called the resolvent of T at λ.

Definition 6.8 A #-densely defined operator T on a non-Archimedean Hilbert space is called symmetric or Hermitian if $T \subset T^*$, that is, $D(T) \subset D(T^*)$ and $T\varphi = T^*\varphi$ for all $\varphi \in D(T)$ and equivalently, T is symmetric if and only if $(T\varphi,\psi) = (\varphi,T\psi)$ for all $\varphi,\psi \in D(T)$.

Definition 6.9 A #-densely defined operator T is called self-#-adjoint if $T = T^*$, that is, if and only if T is symmetric and $D(T) = D(T^*)$.

Remark 6.2 A symmetric operator T is always #-closable, since $D(T)$ #-dense in H. If T is symmetric, T^* is a #-closed extension of T so the smallest #-closed extension T^{**} of T must be contained in T^*. Thus for symmetric operators, we have $T \subset T^{**} \subset T^*$, for #-closed symmetric operators we have $T = T^{**} \subset T^*$ and, for self-#-adjoint operators we have $T = T^{**} = T^*$. Thus a #-closed symmetric operator T is self-#-adjoint if and only if T^* is symmetric.

Definition 6.10 A symmetric operator T is called essentially self-#-adjoint if its #-closure #-$\bar{T}$ is self-#-adjoint. If T is #-closed, a subset $D \subset D(T)$ is called a core for T if #-$\overline{T \upharpoonright D} = T$.

Remark 6.3 If T is essentially self-#-adjoint, then it has one and only one self-#-adjoint extension.

Definition 6.11 Let A be an operator on a non-Archimedean Hilbert space $H^{\#}$.The set $C^{^*\infty}(A) = \bigcap_{n=1}^{^*\infty} D(A^n)$ is called the $C^{^*\infty}$-vectors for A. A vector $\varphi \in C^{^*\infty}(A)$ is called an #-analytic vector for A if

$$Ext\text{-}\sum_{n=0}^{^*\infty} \frac{\|A^n\|_{\#} t^n}{n!} < {}^*\infty \quad (6.7)$$

for some $t > 0$. If A is self-#-adjoint, then $C^{^*\infty}(A)$ will be #-dense in $D(A)$.

Theorem 6.5 (Generalized Nelson's analytic vector theorem) Let A be a symmetric operator on a non-Archimedean Hilbert space H. If $D(A)$ contains a #-total set of #-analytic vectors, then A is essentially self-#-adjoint.

Definition 6.12 [15] Operator A is relatively bounded with respect to operator T if $D(T) \subset D(A)$ and

$$\|Au\|_{\#} \leq a\|u\|_{\#} + b\|Tu\|_{\#}, u \in D(T). \quad (6.8)$$

Theorem 6.6 [15**]** Let $\boldsymbol{T}$ be self-#-adjoint. If $\boldsymbol{A}$ is symmetric and $\boldsymbol{T}$-bounded with $\boldsymbol{T}$-bound smaller than $\mathbf{1}$, then $\boldsymbol{T} + \boldsymbol{A}$ is also self-#-adjoint. In particular $\boldsymbol{T}+\boldsymbol{A}$ is self-#-adjoint if $\boldsymbol{A}$ is bounded and symmetric with $\boldsymbol{D}(\boldsymbol{T}) \subset \boldsymbol{D}(\boldsymbol{A})$.

Theorem 6.7 [15**] (**Generalized Kato perturbation theorem) Let $\boldsymbol{T}$ be self-#-adjoint. If $\boldsymbol{A}$ is symmetric and $\boldsymbol{T}$-bounded with $\boldsymbol{T}$-bound smaller than $\mathbf{1}$, then $\boldsymbol{T} + \boldsymbol{A}$ is also self-#-adjoint and its #-closure #-$(\overline{\boldsymbol{T}+\boldsymbol{A}})$ is equal to #-$\overline{\boldsymbol{T}}$ + #-$\overline{\boldsymbol{A}}$.In particular this is true if $\boldsymbol{A}$ is symmetric and bounded with $\boldsymbol{D}(\boldsymbol{T}) \subset \boldsymbol{D}(\boldsymbol{A})$.

Theorem 6.8 [15] Let $\boldsymbol{A}$ be essentially self -#-adjoint on the domain $\boldsymbol{D}(\boldsymbol{A})$ and let $\boldsymbol{B}$ be a symmetric operator on $\boldsymbol{D}(\boldsymbol{A})$. If there exists a constant $\boldsymbol{a} \in {}^*\mathbb{R}_c^{\#}$ such that for all $\boldsymbol{\psi} \in \boldsymbol{D}(\boldsymbol{A})$ and for all $\boldsymbol{\beta} \in {}^*\mathbb{R}_c^{\#}$ such that $\mathbf{0} \leq \boldsymbol{\beta} \leq \mathbf{1}$ and the inequality holds $\|\boldsymbol{B\psi}\|_{\#} \leq \boldsymbol{a}\|(\boldsymbol{A}+\boldsymbol{\beta B})\boldsymbol{\psi}\|_{\#}$, then $\boldsymbol{A} + \boldsymbol{B}$ is essentially self -#- adjoint on $\boldsymbol{D}(\boldsymbol{A})$ and its #-closure has domain $\boldsymbol{D}$(#-$\overline{\boldsymbol{A}}$).

Proof Let $\mathbf{0} \leq \boldsymbol{\gamma} \leq \mathbf{1}$ and $\boldsymbol{a_1} > \boldsymbol{a}$. Then $\boldsymbol{\gamma a_1 B}$ is a Kato perturbation of $\boldsymbol{A}$, for any vector $\boldsymbol{\psi} \in \boldsymbol{D}(\boldsymbol{A}), \boldsymbol{\beta} = \mathbf{0}$ we get the inequality

$$\|\gamma a_1^{-1} B\psi\|_{\#} \leq \delta, \delta < 1.$$

By Theorem 6.8, $A + \gamma a_1^{-1}B$ is essentially self- #-adjoint on $D(A)$ and the domain of its #-closure is D(#-$\bar{A}$). Thus by the inequality $\|B\psi\|_{\#} \leq a\|(A+\beta B)\psi\|_{\#}$ **)** with $\beta = a_1^{-1}$, we conclude that $\gamma a_1^{-1}B$ is a Kato perturbation of the operator $A + a_1^{-1}B$. Hence the operator $A + a_1^{-1}(1+\gamma)B$ is essentially self -#-adjoint on $D(A)$ and its #-closure has domain D(#-$\bar{A}$). Continuing inductively in this manner, for any integer $j \in {}^*\mathbb{N}$ satisfy$ja_1^{-1} \leq 1$, we obtain that $\gamma a_1^{-1}B$ is a Kato perturbation of the essentially self -#-adjoint operator $A + ja_1^{-1}B$, so that $A + a_1^{-1}(j+\gamma)B$ is essentially self -#-adjoint on $D(A)$ and the domain of its #-closure is D(#-$\bar{A}$). By choosing the largest such j, we obtain for some γ such that $0 \leq \gamma < 1, ja_1^{-1} = 1,$ and so we have proved the essential self -#-adjointness of the operator $A + B$.

Theorem 6.9 [15] Let A and B be the same as in Theorem 6.7. Then A and $A + B$ have the same #-cores. If A is bounded from below, then $A + B$ is bounded from below.

Proof If B is a Kato perturbation of A, the theorem holds. The proof of Theorem 6.9 exhibits $A + B$ as a finite or hyperfinite number of successive Kato perturbations, and yields the theorem.

Theorem 6.10 [15] Let $\alpha > 0$ and let $g_0 = [h_0]^2$, $g_1 = [h_1]^2, h_0, h_1 \in S_{\text{fin}}^{\#}({}^*\mathbb{R}_c^{\#3}), h_0 \geq 0, h_1 \geq 0$, then operator (see Definition 11.14)

$$M = \alpha H_{0,\varkappa} + T_0(g_0) + T_l(g_1)$$

is self -#-adjoint on $D(H_{0,\varkappa}) \cap D(T_l(g_1))$ and is essentially self -#-adjoint on $\mathrm{C}^{*\infty}(H_{0,\varkappa})$.

§7. GENERALIZED TROTTER PRODUCT FORMULA

Theorem 7.1 Let A and B be self-adjoint operators on non-Archimedean Hilbert space $H^{\#}$. Suppose that the operator $A + B$ is self-#-adjoint on $D = D(A) \cap D(B)$, then the following equality holds

$$\text{s-\#-}\lim_{n \to {}^*\infty} \left[\left(Ext\text{-}\exp\left(\frac{itA}{n}\right)\right)\left(Ext\text{-}\exp\left(\frac{itB}{n}\right)\right)\right]^n = Ext\text{-}\exp[it(A+B)]. \quad (7.1)$$

Theorem 7.2 Let A and B be self-adjoint operators on non-Archimedean Hilbert space $H^{\#}$. Suppose that the operator $A+B$ is essentially self-#-adjoint on $D = D(A) \cap D(B)$, then the following equality holds

$$\text{s-\#-}\lim_{n \to {}^*\infty} \left[\left(Ext\text{-}\exp\left(\frac{itA}{n}\right)\right)\left(Ext\text{-}\exp\left(\frac{itB}{n}\right)\right)\right]^n = Ext\text{-}\exp[it(A+B)]. \quad (7.2)$$

Theorem 7.3 Let A and B be the generators of contraction semigroups on non-Archimedean Banach space $B^{\#}$.Suppose that the #-closure of $(A+B) \upharpoonright D(A) \cap D(B)$ generates a contraction semigroup on $B^{\#}$. Then the following equality holds

$$\text{s-\#-}\lim_{n \to {}^*\infty} \left[\left(Ext\text{-}\exp\left(-\frac{tA}{n}\right)\right)\left(Ext\text{-}\exp\left(-\frac{tB}{n}\right)\right)\right]^n = Ext\text{-}\exp[-t(\#\text{-}\overline{A+B})]. \quad (7.3)$$

§8. FOCK SPACE OVER NONARCHIMEDEAN HILBERT SPACE

Definition 8.1 Let $H^{\#}$ be a complex hyper infinite-dimensional non-Archimedean Hilbert space over field ${}^*\mathbb{C}_c^{\#}$ and denote by $H^{\#(n)}$ the n-fold tensor product: $H^{\#(n)} = Ext\text{-}\otimes_{k=1}^{n} H^{\#}, n \in {}^*\mathbb{N}$. Set $H^{\#(0)} = {}^*\mathbb{C}_c^{\#}$ and define $\mathcal{F}(H^{\#}) = Ext\text{-}\oplus_{n \in {}^*\mathbb{N}}\left(H^{\#(n)}\right)$. $\mathcal{F}(H^{\#})$ is called the Fock space over non-Archimedean Hilbert space $H^{\#}$. Set $H^{\#} = L_2^{\#}\left({}^*\mathbb{R}_c^{\#3}\right)$, then an element $\psi \in \mathcal{F}(H^{\#})$ is a hyper infinite sequence of ${}^*\mathbb{C}_c^{\#}$-valued functions $\psi = \{\psi_0, \psi_1(x_1), \psi_2(x_1, x_2), \psi_2(x_1, x_2, x_3), \ldots, \psi_n(x_1, \ldots, x_n)\}, n \in {}^*\mathbb{N}$ and such that

$$\|\psi\|_{\#} = |\psi_0|^2 + Ext\text{-}\sum_{n \in {}^*\mathbb{N}}\left(Ext\text{-}\int |\psi_n(x_1, \ldots, x_n)|^2 d^{\#3n}x\right) < {}^*\infty.$$

Actually, it is not $\mathcal{F}(H^{\#})$ itself, but two of its subspaces which are used in quantum field theory. These two hyper infinite-dimensional subspaces are constructed as follows: Let P_n be the permutation group on $n \in {}^*\mathbb{N}$ elements and let $\{\varphi_k\}_{k=1}^{*\infty}$be a basis for a space $H^{\#}$. For each $\sigma \in P_n$ we define an operator (which we also denote by σ) on basis elements of $H^{\#(n)}$ by $\sigma\left(Ext\text{-}\otimes_{i=1}^{n}\varphi_{k_i}\right) = Ext\text{-}\otimes_{i=1}^{n}\varphi_{k_{\sigma(i)}}$. The operator extends by linearity to a bounded operator (of #-norm one) on $H^{\#}$ and we can define $\breve{\mathbf{S}}_n^{\#} = \left(\frac{1}{n!}\right)\left(Ext\text{-}\sum_{\sigma \in P_n}\sigma\right)$. It is easily to show by definitions that $\breve{\mathbf{S}}_n^{\#2} = \breve{\mathbf{S}}_n^{\#}$ and $\breve{\mathbf{S}}_n^{\#*} = \breve{\mathbf{S}}_n^{\#}$ so $\breve{\mathbf{S}}_n^{\#}$ is an orthogonal projection. The range of $\breve{\mathbf{S}}_n^{\#}$ is called the n-fold symmetric tensor product of $H^{\#}$. We now define $\mathcal{F}_s^{\#}(H^{\#}) = Ext\text{-}\oplus_{n \in {}^*\mathbb{N}}\breve{\mathbf{S}}_n^{\#}H^{\#(n)}$. Non-Archimedean Hilbert space $\mathcal{F}_s^{\#}(H^{\#})$ is called the symmetric Fock space over non-Archimedean Hilbert space $H^{\#}$ or the Boson Fock space over non-Archimedean Hilbert space $H^{\#}$.

§9. SEGAL QUANTIZATION OVER NONARCHIMEDEAN HILBERT SPACE

Let $H^{\#}$ be a complex non-Archimedean Hilbert space over field ${}^*\mathbb{C}_c^{\#}$ and let $\mathcal{F}(H^{\#}) = Ext\text{-}\oplus_{n \in {}^*\mathbb{N}}\left(H^{\#(n)}\right)$, where $H^{\#(n)} = Ext\text{-}\otimes_{k=1}^{n}H^{\#}$ be the Fock space over $H^{\#}$and let $\mathcal{F}_s(H^{\#})$ be the Boson subspace of $\mathcal{F}(H^{\#})$. Let $f \in H^{\#}$ be fixed. For vectors in $H^{\#(n)}$ of the form $\eta = Ext\text{-}\otimes_{i=1}^{n}\psi_i, n \in {}^*\mathbb{N}$ we define a map $b^-(f): H^{\#(n)} \to H^{\#(n-1)}$ by $b^-(f)\eta = (f, \psi_1)(Ext\text{-}\otimes_{i=2}^{n}\psi_i)$ and $b^-(f)$ extends by linearity to finite and hyperfinite linear combinations of such η, the extension is well defined, and $\|b^-(f)\eta\|_{\#} \leq \|f\|_{\#}\|\eta\|_{\#}$. Thus $b^-(f)$ extends to a bounded map (of #-norm $\|f\|_{\#}$) of $H^{\#(n)}$ into $H^{\#(n-1)}$. Since this holds for each $n \in {}^*\mathbb{N}$ (except for $n = 0$ in which case we define $b^-(f): H^{\#(0)} \to \{0\}$), $b^-(f)$ is a bounded operator of #-norm $\|f\|_{\#}$ from $\mathcal{F}(H^{\#})$ to $\mathcal{F}(H^{\#})$. It is easy to check that operator $b^+(f) = \left(b^-(f)\right)^*$takes each subspace $H^{\#(n)}$into $H^{\#(n+1)}$with the action $b^+(f)\eta = f \otimes Ext\text{-}\otimes_{i=1}^{n}\psi_i$ on product vectors. Note that the map $f \to b^+(f)$ is linear and the map $f \to b^-(f)$ is antilinear. Let S_n be the symmetrization operators introduced in previous section and then the operator $\breve{\boldsymbol{S}}^{\#} = Ext\text{-}\oplus_{n \in {}^*\mathbb{N}}\breve{\mathbf{S}}_n^{\#}$ is the projection onto the symmetric Fock space $\mathcal{F}_s(H^{\#}) = Ext\text{-}\oplus_{n \in {}^*\mathbb{N}}\breve{\mathbf{S}}_n^{\#}H^{\#(n)}$, we will write $\breve{\mathbf{S}}_n^{\#}H^{\#(n)} = H_s^{\#(n)}$and call $H_s^{\#(n)}$the n-particle subspace of $\mathcal{F}_s(H^{\#})$. Note that operator $b^-(f)$ takes space $\mathcal{F}_s(H^{\#})$ into itself, but the operator $b^+(f)$ does not. A vector $\psi = \{\psi^{(n)}\}_{n=1}^{^*\infty}$ with $\psi^{(n)} = 0$ for all except finite or hyperfinite set of number n is called a finite or hyperfinite particle vector correspondingly. We will denote the set of hyperfinite particle vectors by F_0. The vector $\Omega_0 = \langle 1,0,0,\dots\rangle$ is called the vacuum vector. Let A be any self-adjoint operator on $H^\#$ with domain of essential self-#-adjointness $D = D(A)$. Let $D_A = \{\psi \in F_0 | \psi^{(n)} \in Ext\text{-}\otimes_{i=1}^{n} D, n \in {}^*\mathbb{N}\}$ and define operator $d\Gamma^\#(A)$ on $D_A \cap H_s^{\#(n)}$ by $d\Gamma^\#(A)$=$A \otimes I \cdots \otimes I + I \otimes A \otimes \cdots \otimes I + \cdots + \otimes I \cdots \otimes I \otimes A$. Note that $d\Gamma^\#(A)$is essentially self-#-adjoint on D_A . Operator $d\Gamma^\#(A)$ is called the second quantization of the operator A. For example, let $A = I$, then its second quantization $N^\# = d\Gamma^\#(I)$ is essentially self-#-adjoint on F_0 and for $\psi \in H_s^{\#(n)}, N^\#\psi = n\psi$. $N^\#$ is called the number operator. If U is a unitary operator on space $H^\#$, we define $d\Gamma^\#(U)$ to be the unitary operator on $\mathcal{F}_s(H^\#)$ which equals $Ext\text{-}\otimes_{i=1}^{n} U$ when restricted to $H_s^{\#(n)}$for $n > 0$, and which equals the identity on $H_s^{\#(0)}$. If $Ext\text{-}\exp(itA)$ is a #-continuous unitary group on $H^\#$, then $\Gamma^\#(Ext\text{-}\exp(itA))$ is the group generated by $d\Gamma^\#(A)$, i.e., that expressed by the formula $\Gamma^\#(Ext\text{-}\exp(itA)) = Ext\text{-}\exp(itd\Gamma^\#(A))$.

Definition 9.1 We define the annihilation operator $a^-(f)$ on $\mathcal{F}_s(H^\#)$ with domain F_0 by the formula

$$a^-(f) = \sqrt{N+1}\,b^-(f). \tag{9.1}$$

Operator $a^-(f)$ is called an annihilation operator because it takes each $(n+1)$-particle subspace into the n-particle subspace. For each ψ and η in F_0, $(\sqrt{N+1}\,b^-(f)\psi, \eta) = (\psi, S^\# b^+(f)\sqrt{N+1})$, then we get

$$(a^-(f))^* \restriction F_0 = S^\# b^+(f)\sqrt{N+1}\,. \tag{9.2}$$

The operator $(a^-(f))^*$ is called a creation operator. Both $a^-(f)$ and $(a^-(f))^*$ #-closable; we denote their #-closures by $a^-(f)$ and $(a^-(f))^*$ also. The equation (1) implies that the Segal field operator $\Phi_S^\#(f)$ on F_0 defined by $\Phi_S^\#(f) = \frac{1}{\sqrt{2}}[a^-(f) + (a^-(f))^*]$ is symmetric and essentially self-#-adjoint. The mapping from $H^\#$ to the self-#-adjoint operators on $\mathcal{F}_s(H^\#)$ given by $f \to \Phi_S^\#(f)$ is called the Segal quantization over $H^\#$. Note that the Segal quantization is a real linear map.

Theorem 9.1 Let $H^\#$ be hyper infinite dimensional Hilbert space over complex field ${}^*\mathbb{C}_c^\# = {}^*\mathbb{R}_c^\# + \mathrm{i}\,{}^*\mathbb{R}_c^\#$ and $\Phi_S^\#(f)$ the corresponding Segal quantization. Then:

(a) (self-#-adjointness) for each $f \in H^\#$ the operator $\Phi_S^\#(f)$ is essentially self-#-adjoint on F_0, the hyperfinite particle vectors;

(b) (cyclicity of the vacuum) the vector Ω_0 is in the domain of all hyperfinite products $Ext\text{-}\prod_{i=1}^{n}\Phi_S^\#(f_i)\,, n \in {}^*\mathbb{N}$ and the set $\{Ext\text{-}\prod_{i=1}^{n}\Phi_S^\#(f_i)\,|f_i \in H^\#, n \in {}^*\mathbb{N}\}$ is #-total in $\mathcal{F}_s(H^\#)$;

(c) (commutation relations) for each $\psi \in F_0$ and $f, g \in H^\#$: $[\Phi_S^\#(f)\Phi_S^\#(g) - \Phi_S^\#(g)\Phi_S^\#(f)]\psi = i\mathrm{Im}(f,g)_{H^\#}\psi$;

(c') (generalized commutation relations) assuming that $(f,g)_{H^\#} \approx 0$ and $\psi \in F$ is a near standard vector we get $[\Phi_S^\#(f)\Phi_S^\#(g) - \Phi_S^\#(g)\Phi_S^\#(f)]\psi \approx 0$ and therefore $\mathrm{st}([\Phi_S^\#(f)\Phi_S^\#(g) - \Phi_S^\#(g)\Phi_S^\#(f)]\psi) = 0$;

(d) let $W(f)$ denotes the external unitary operator $Ext\text{-}\exp\left(i\Phi_S^\#(f)\right)$ then

$W(f+g) = \left[Ext\text{-}\exp\left(-\frac{i}{2}\mathrm{Im}(f,g)_{H^\#}\right)\right]W(f)W(g);$

(e) (#-continuity) if $\{f_n\}_{n=1}^{^*\infty}$ is hyper infinite sequence such as $\#\text{-}\lim_{n\to{}^*\infty} f_n = f$ in $H^\#$ then:

1) $\#\text{-}\lim_{n\to{}^*\infty} W(f_n)\psi$ exists for all $\psi \in \mathcal{F}_s(H^\#)$ and $\#\text{-}\lim_{n\to{}^*\infty} W(f_n)\psi = W(f)\psi$

2) $\#\text{-}\lim_{n\to{}^*\infty} \Phi_S^\#(f_n)\psi$ exists for all $\psi \in F_0$ and $\#\text{-}\lim_{n\to{}^*\infty} \Phi_S^\#(f_n)\psi = \Phi_S^\#(f)\psi$

(e) For every unitary operator U on $H^\#, \Gamma^\#(U): D(\#\text{-}\overline{\Phi_S^\#(f)}) \to D(\#\text{-}\overline{\Phi_S^\#(Uf)})$ and for all $\psi \in D(\#\text{-}\overline{\Phi_S^\#(Uf)})$, $\Gamma^\#(U)(\#\text{-}\overline{\Phi_S^\#(f)})\Gamma^{\#-1}(U)\psi = \#\text{-}\overline{\Phi_S^\#(Uf)}\psi$ for all $\psi \in F_0$ and $f \in H^\#$.

Remark 9.1 Henceforth we use $\Phi_S^\#(f)$ to denote the #-closure $\#\text{-}\overline{\Phi_S^\#(f)}$ of $\Phi_S^\#(f)$.

Definition 9.2 For each $m > 0, m \in \mathbb{R}$ let $H_m^\# = \{p \in {}^*\mathbb{R}_c^{\#4} | p \cdot \tilde{p} = m^2, p_0 > 0\}$, where $\tilde{p} = (p^0, -p^1, -p^2, -p^3)$, the sets $H_m^\#$, are called mass hyperboloids, are invariant under canonical Lorentz group${}^\sigma L_+^\uparrow$. Let j_m be the

#-homeomorphism of $H_m^\#$ onto ${}^*\mathbb{R}_c^{\#3}$ given by $j_m: \langle p_0, p_1, p_2, p_3\rangle \to \langle p_1, p_2, p_3\rangle = \boldsymbol{p}$. Define a #-measure $\Omega_m^\#$ on $H_m^\#$ for any #-measurable set $E \subset H_m^\#$ by

$$\Omega_m^\#(E) = Ext\text{-}\int_{j_m(E)} \frac{d^{\#3}\boldsymbol{p}}{\sqrt{|\boldsymbol{p}|^2+m^2}}. \tag{9.3}$$

Theorem 9.2 Let $\mu^\#$ be a polynomially bounded #-measure with support in #-$\bar{V}_+$. If $\mu^\#$ is ${}^\sigma L_+^\uparrow = L_+^\uparrow$- invariant, there exists a polynomially bounded #-measure $\rho^\#$ on $[0,\infty^\#)$ and a constant c so that for any $f \in S^\#({}^*\mathbb{R}_c^{\#4})$

$$Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#4}} f\, d^\#\mu^\# = cf(0) + Ext\text{-}\int_0^{{}^*\infty} d^\#\rho^\#(m)\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \frac{f\left(\sqrt{|\boldsymbol{p}|^2+m^2}, p_1, p_2, p_3\right) d^{\#3}\boldsymbol{p}}{\sqrt{|\boldsymbol{p}|^2+m^2}}\right). \tag{9.4}$$

Definition 9.3 Let $\boldsymbol{\mathcal{F}}(f)$ be a linear #-continuous functional $\mathcal{F}: S_{\text{fin}}^\#({}^*\mathbb{R}_c^{\#4}) \to {}^*\mathbb{R}_c^\#$. Functional $\mathcal{F}$ is $L_+^\uparrow$- $\approx$ - invariant if for any $\Lambda \in L_+^\uparrow$ the following property holds $\mathcal{F}\big(f(\Lambda x)\big) \approx \mathcal{F}(f)$ for all $f \in S_{\text{fin}}^\#({}^*\mathbb{R}_c^{\#4})$.

Theorem 9.3 Let $\mu^\#$ be a polynomially bounded $L_+^\uparrow$- invariant #-measure with support in #-$\bar{V}_+$. Let $\mathcal{F}(f)$ be a linear #-continuous functional $\mathcal{F}: S_{\text{fin}}^\#({}^*\mathbb{R}_c^{\#4}) \to {}^*\mathbb{R}_{c,\text{fin}}^\#$ defined by $Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#4}} f\, d^\#\mu^\#$ and there exists a polynomially bounded #-measure $\rho^\#$ on $[0,\infty^\#)$ such that $\int_0^{{}^*\infty} d^\#\rho^\#(m) \in {}^*\mathbb{R}_{c,\text{fin}}^\#$ and a constant $c \in {}^*\mathbb{R}_{c,\text{fin}}^\#$ so that (1) holds. Then for any $f \in S_{\text{fin}}^\#({}^*\mathbb{R}_c^{\#4})$ and for any $\varkappa \in {}^*\mathbb{R}_{c,\infty}^\#$ the following property holds

$$\mathcal{F}(f) \approx cf(0) + Ext\text{-}\int_0^{{}^*\infty} d^\#\rho^\#(m)\left(Ext\text{-}\int_{|p|\le\varkappa} \frac{f\left(\sqrt{|\boldsymbol{p}|^2+m^2}, p_1, p_2, p_3\right) d^{\#3}\boldsymbol{p}}{\sqrt{|\boldsymbol{p}|^2+m^2}}\right). \tag{9.5}$$

Definition 9.4 Let $\chi(\varkappa, \boldsymbol{p})$ be a function such that: $\chi(\varkappa, \boldsymbol{p}) \equiv 1$ if $|\boldsymbol{p}| \le \varkappa$, $\chi(\varkappa, \boldsymbol{p}) \equiv 0$ if $|\boldsymbol{p}| > \varkappa$. Define a #-measure $\Omega_{m,\varkappa}^\#$ on $H_m^\#$ by

$$\Omega_{m,\varkappa}^\#(E) = Ext\text{-}\int_{j_m(E)} \frac{\chi(\varkappa,\boldsymbol{p})\, d^{\#3}\boldsymbol{p}}{\sqrt{|\boldsymbol{p}|^2+m^2}}. \tag{9.6}$$

We use the Segal quantization to define the free Hermitian scalar field of mass m. We take $H^\# = L_2^\#\left(H_m^\#, d^\#\Omega_{m,\varkappa}^\#\right)$. For each $f \in S_{\text{fin}}^\#({}^*\mathbb{R}_c^{\#4})$ we define $Ef \in H^\#$ by $Ef = 2\pi\big(Ext\text{-}\hat{f}\big) \restriction H_m^\#$ where the Fourier transform is defined in terms of the Lorentz invariant inner product $p \cdot \tilde{x}$: $Ext\text{-}\hat{f} = \frac{1}{4\pi^2}\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#4}} Ext\text{-}\exp[i(p \cdot \tilde{x})] d^{\#4}x\right)$. If $\Phi_{S,\varkappa}^\#(\cdot)$ is the Segal quantization over $L_2^\#\left(H_m^\#, d^\#\Omega_{m,\varkappa}^\#\right)$, we define for each ${}^*\mathbb{R}_c^\#$- valued $f \in S^\#({}^*\mathbb{R}_c^{\#4})$: $\Phi_{m,\varkappa}^\#(f) = \Phi_{S,\varkappa}^\#(Ef)$ and for each ${}^*\mathbb{C}_c^\#$- valued $f \in S^\#({}^*\mathbb{R}_c^{\#4})$ we define $\Phi_{m,\varkappa}^\#(f) = \Phi_{m,\varkappa}^\#(\text{Re}f) + \text{i}\Phi_{m,\varkappa}^\#(\text{Im}f)$.

Definition 9.5 The mapping $f \to \Phi_{m,\varkappa}^\#(f)$ is called the free non-Archimedean Hermitian scalar field of mass m.

Definition 9.6 On $L_2^\#\left(H_m^\#, d^\#\Omega_{m,\varkappa}^\#\right)$ we define the following unitary representation of the restricted Poincare group $L_+^\uparrow$: $(U_m(a,\Lambda)\psi)(p) = (Ext\text{-}\exp[i(p \cdot \tilde{a})])\psi(\Lambda^{-1}p)$ where we are using Λ to denote both an element of the abstract restricted Lorentz group and the corresponding element in the standard representation on ${}^\sigma\mathbb{R}^4$.

Remark 9.2 Note that by Theorem 9.1(e) for all $\psi \in F_0$ and $f \in L_2^\#\left(H_m^\#, d^\#\Omega_{m,\varkappa}^\#\right)$ we get

$$\Gamma^\#\big(U_m(a,\Lambda)\big)\big(\#\text{-}\overline{\Phi_{m,\varkappa}^\#(f)}\big)\Gamma^{\#-1}\big(U_m(a,\Lambda)\big)\psi = \Gamma^\#\big(U_m(a,\Lambda)\big)\big(\#\text{-}\overline{\Phi_S^\#(Ef)}\big)\Gamma^{\#-1}\big(U_m(a,\Lambda)\big)\psi =$$

$$\#\text{-}\overline{\Phi_S^\#(U_m(a,\Lambda)Ef)}\psi.$$

A change of variables for all $f \in S_{\text{fin}}^\#({}^*\mathbb{R}_c^{\#4})$ gives that

$$U_m(a,\Lambda)Ef \approx EU_m(a,\Lambda)f.$$

Therefore for all $\psi \in D_{S_{\text{fin}}^\#} \subset F_0$ such that $\|\psi\|_\# \in {}^*\mathbb{R}_{c,\text{fin}}^\#$ and for ${}^*\mathbb{R}_{c,\text{fin}}^\#$-valued function f such that $f \in S_{\text{fin}}^\#({}^*\mathbb{R}_c^{\#4})$ we obtain that

$$\Gamma^{\#}\big(U_m(a,\Lambda)\big)\left(\#\text{-}\Phi^{\#}_{m,\varkappa}(f)\right)\Gamma^{\#-1}\big(U_m(a,\Lambda)\big)\psi \approx \#\text{-}\Phi^{\#}_{m,\varkappa}(U_m(a,\Lambda)f)\psi. \tag{9.7}$$

Definition 9.7 The #-conjugation on a non-Archimedean Hilbert space $H^{\#}$ is an antilinear #-isometry $\mathbf{C}^{\#}$ so that the following equality holds $\mathbf{C}^{\#2} = I$.

Definition 9.8 Let $H^{\#}$ be a non-Archimedean Hilbert space over field ${}^{*}\mathbb{C}^{\#}_{c}$, $\Phi^{\#}_{S}(\cdot)$ the associated Segal quantization. Let $H^{\#}_{\mathbf{C}^{\#}} = \{f|\mathbf{C}^{\#}f = f\}$. For each $f \in H^{\#}_{\mathbf{C}^{\#}}$ we define $\varphi^{\#}(f) = \Phi^{\#}_{S}(f)$ and $\pi^{\#}(f) = \Phi^{\#}_{S}(if)$, the map $f \to \varphi^{\#}(f)$ is called the canonical free field over the doublet $\langle \boldsymbol{H}^{\#}, \mathbf{C}^{\#}\rangle$ and the map $f \to \pi^{\#}(f)$ is called the canonical conjugate momentum.

Theorem 9.4 Let $H^{\#}$ be a non-Archimedean Hilbert space over field ${}^{*}\mathbb{C}^{\#}_{c}$ with #-conjugation $\mathbf{C}^{\#}$. Let$\varphi^{\#}(\cdot)$ and $\pi^{\#}(\cdot)$ be the corresponding canonical fields. Then: (a) For each $f \in H^{\#}_{\mathbf{C}^{\#}}, \varphi^{\#}(f)$ is essentially self-#-adjoint on F_0.
(b) $\{\varphi^{\#}(f)|f \in H^{\#}_{\mathbf{C}^{\#}}\}$ is a commuting family of self-#-adjoint operators. (c) Ω_0 is a #-cyclic vector for the family $\{\varphi^{\#}(f)|f \in H^{\#}_{\mathbf{C}^{\#}}\}$. (d) If $\{f_n\}_{n=1}^{{}^{*}\infty}$ is hyper infinite sequence such as $\#\text{-}\lim_{n\to{}^{*}\infty} f_n = f$ in $H^{\#}_{\mathbf{C}^{\#}}$, then $\#\text{-}\lim_{n\to{}^{*}\infty} \varphi^{\#}(f_n)\psi$ exists for all $\psi \in F_0$ and $\#\text{-}\lim_{n\to{}^{*}\infty} \varphi^{\#}(f_n)\psi = \varphi^{\#}(f)\psi$.
(e) $\#\text{-}\lim_{n\to{}^{*}\infty}(Ext\text{-}\exp[i\varphi^{\#}(f_n)]\psi) = Ext\text{-}\exp[i\varphi^{\#}(f)]\psi$ for all $\psi \in \mathcal{F}_s(H^{\#})$. (f) Properties (a)-(e) hold with $\varphi^{\#}(f)$ replaced by $\pi^{\#}(f)$. (g) If $f, g \in H^{\#}_{\mathbf{C}^{\#}}$, then $[\varphi^{\#}(f)\varphi^{\#}(g) - \varphi^{\#}(g)\varphi^{\#}(f)]\psi = i(f,g)$ for all $\psi \in \mathcal{F}_s(H^{\#})$ and $(Ext\text{-}\exp[i\varphi^{\#}(f)])(Ext\text{-}\exp[i\pi^{\#}(f)]) = (Ext\text{-}\exp[i(f,g)])(Ext\text{-}\exp[i\pi^{\#}(f)])(Ext\text{-}\exp[i\varphi^{\#}(f)])$.

Definition 9.9 We write now $f \in L^{\#}_{2}\big(H^{\#}_{m}, d^{\#}\Omega^{\#}_{m,\varkappa}\big)$ as $f(p_0,\boldsymbol{p})$ and define the #-conjugation $\mathbf{C}^{\#}$ by $\mathbf{C}^{\#}(f)(p_0,\boldsymbol{p}) = \overline{f(p_0,-\boldsymbol{p})}$. Note that $\mathbf{C}^{\#}$ is well-defined on $f \in L^{\#}_{2}\big(H^{\#}_{m}, d^{\#}\Omega^{\#}_{m,\varkappa}\big)$ since $\langle p_0,-\boldsymbol{p}\rangle \in H^{\#}_{m}$ if and only if $\langle p_0,\boldsymbol{p}\rangle \in H^{\#}_{m}$.

Definition 9.10 We denote the canonical fields corresponding to $\mathbf{C}^{\#}$ by $\varphi^{\#}(\cdot)$ and $\pi^{\#}(\cdot)$ and define $\varphi^{\#}_{m,\varkappa}(f) = \varphi^{\#}(Ef)$ and $\pi^{\#}_{m,\varkappa}(f) = \pi^{\#}(\mu(\boldsymbol{p})Ef), \mu(\boldsymbol{p}) = \sqrt{\boldsymbol{p}^2 + m^2}$ for ${}^{*}\mathbb{R}^{\#}_{c}$- valued $f \in L^{\#}_{2}({}^{*}\mathbb{R}^{\#4}_{c})$, extending to all of $L^{\#}_{2}({}^{*}\mathbb{R}^{\#4}_{c})$ by linearity. We let now $D_{S^{\#}_{\text{fin}}} = \{\psi|\psi \in F_0, \psi^{(n)} \in S^{\#}_{\text{fin}}({}^{*}\mathbb{R}^{\#3n}_{c})\}$ and for each $p \in {}^{*}\mathbb{R}^{\#3}_{c}$ we define the operator $a(p)$ on $\mathcal{F}_s\left(L^{\#}_{2}({}^{*}\mathbb{R}^{\#3}_{c})\right)$ with domain $D_{S^{\#}_{\text{fin}}}$ by $(a(p)\psi)^{(n)} = \sqrt{n+1}\,\psi^{(n+1)}(p,k_1,\dots k_n)$ and therefore the formal #-adjoint of the operator $a(p)$ reads $(a^{\dagger}(p)\psi)^{(n)} = \frac{1}{\sqrt{n}}\sum_{l=1}^{n}\delta^{(3)}(p-k_l)\psi^{(n-1)}(k_1,\dots,k_{l-1},k_{l+1},\dots,k_n)$. Note that the formulas

$$a(g) = Ext\text{-}\int_{{}^{*}\mathbb{R}^{\#3}_{c}} a(p)g(-p)d^{\#3}p, \tag{9.8}$$

$$a^{\dagger}(g) = Ext\text{-}\int_{{}^{*}\mathbb{R}^{\#3}_{c}} a^{\dagger}(p)g(p)d^{\#3}p \tag{9.9}$$

hold for all $g \in S^{\#}_{\text{fin}}({}^{*}\mathbb{R}^{\#3}_{c})$ if the equalities (9.8)-(9.9) are understood in the sense of quadratic forms. That is, (9.8) means that for $\psi_1,\psi_2 \in D_{S^{\#}_{\text{fin}}}$: $(\psi_1, a(g)\psi_2) = Ext\text{-}\int_{{}^{*}\mathbb{R}^{\#3}_{c}}(\psi_1, a(p)\psi_2)g(-p)d^{\#3}p$ and similarly (9.9) means that for $\psi_1,\psi_2 \in D_{S^{\#}_{\text{fin}}}$: $(\psi_1, a(g)\psi_2) = Ext\text{-}\int_{{}^{*}\mathbb{R}^{\#3}_{c}}(\psi_1, a^{\dagger}(p)\psi_2)g(p)d^{\#3}p$. The particles number operator reads

$$N_{0,\varkappa} = Ext\text{-}\int_{|p|\le\varkappa} a^{\dagger}(p)a(p)\,d^{\#3}p. \tag{9.10}$$

The generator of time translations in the free scalar field theory of mass m is given by

$$H_{0,\varkappa} = Ext\text{-}\int_{|p|\le\varkappa} \mu(p)a^{\dagger}(p)a(p)\,d^{\#3}p. \tag{9.11}$$

We express the free scalar field and the time zero fields in terms of $a^{\dagger}(p)$ and $a(p)$ as quadratic forms on $D_{S^{\#}_{\text{fin}}} \times D_{S^{\#}_{\text{fin}}}$ by

$$\Phi^{\#}_{0,m,\varkappa}(x,t) =$$

$$(2\pi)^{-3/2}Ext\text{-}\int_{|p|\leq\varkappa}\{(Ext\text{-}\exp(\mu(p)t-ipx))a^{\dagger}(p)+(Ext\text{-}\exp(\mu(p)t+ipx))a\,(p)\}\frac{d^{\#3}p}{\sqrt{2\mu(p)}}, \quad (9.12)$$

$$\Phi^{\#}_{0,m,\varkappa}(x)=$$

$$(2\pi)^{-3/2}Ext\text{-}\int_{|p|\leq\varkappa}\{(Ext\text{-}\exp(-ipx))a^{\dagger}(p)+(Ext\text{-}\exp(ipx))a\,(p)\}\frac{d^{\#3}p}{\sqrt{2\mu(p)}}, \quad (9.13)$$

$$\pi^{\#}_{0,m,\varkappa}(x)=$$

$$(2\pi)^{-3/2}Ext\text{-}\int_{|p|\leq\varkappa}\{(Ext\text{-}\exp(-ipx))a^{\dagger}(p)+(Ext\text{-}\exp(ipx))a\,(p)\}\frac{d^{\#3}p}{\sqrt{\mu(p)/2}}. \quad (9.14)$$

Abbreviation 9.1 We shall write for the sake of brevity through this paper $\Phi^{\#}_{0,\varkappa}(x,t)$, $\Phi^{\#}_{0,\varkappa}(x)$ and $\pi^{\#}_{0,\varkappa}(x)$ instead $\Phi^{\#}_{0,m,\varkappa}(x,t)$, $\Phi^{\#}_{0,m,\varkappa}(x)$ and $\pi^{\#}_{0,m,\varkappa}(x)$ correspondingly.

Theorem 9.5 Let $n_1,n_2\in\mathbb{N}$ and suppose that $W(k_1,\dots k_{n_1},p_1,\dots,p_{n_2})\in L_2^{\#}\left({}^*\mathbb{R}_c^{\#3(n_1+n_2)}\right)$ where $W(k_1,\dots k_{n_1},p_1,\dots,p_{n_2})$ is a ${}^*\mathbb{C}^{\#}_{c,\text{fin}}$ -valued function on ${}^*\mathbb{R}_c^{\#3(n_1+n_2)}$. Then there is a unique operator T_W on $\mathcal{F}_s\left(L_2^{\#}({}^*\mathbb{R}_c^{\#3})\right)$ so that $D_{S^{\#}_{\text{fin}}}\subset D(T_W)$ is a #- core for T_W .

(1) As ${}^*\mathbb{C}^{\#}_c$-valued quadratic forms on $D_{S^{\#}_{\text{fin}}}\times D_{S^{\#}_{\text{fin}}}$

$$T_W=$$

$$Ext\text{-}\int_{{}^*\mathbb{R}^{3(n_1+n_2)}}W(k_1,\dots k_{n_1},p_1,\dots,p_{n_2})\left(Ext\text{-}\prod_{i=1}^{n_1}a^{\dagger}(k_i)\right)\left(Ext\text{-}\prod_{i=1}^{n_2}a(p_i)\right)d^{\#3n_1}k\,d^{\#3n_2}p. \quad (9.15)$$

(2) As ${}^*\mathbb{C}^{\#}_c$-valued quadratic forms on $D_{S^{\#}_{\text{fin}}}\times D_{S^{\#}_{\text{fin}}}$

$$T_W^*=$$

$$Ext\text{-}\int_{{}^*\mathbb{R}^{3(n_1+n_2)}}\overline{W(k_1,\dots k_{n_1},p_1,\dots,p_{n_2})}\left(Ext\text{-}\prod_{i=1}^{n_1}a^{\dagger}(k_i)\right)\left(Ext\text{-}\prod_{i=1}^{n_2}a(p_i)\right)d^{\#3n_1}k\,d^{\#3n_2}p. \quad (9.16)$$

(3) If m_1 and m_2 are nonnegative integers so that $m_1+m_2=n_1+n_2$, then

$$\left\|(1+N^{\#})^{-m_1/2}T_W(1+N^{\#})^{-m_2/2}\right\|_{\#}^2\leq C(m_1,m_2)\|W\|_{L_2^{\#}}. \quad (9.17)$$

(4) On vectors in F_0 the operators T_W and T_W^* are given by the explicit formulas

$$(T_W\,\psi)^{l-n_2+n_1}=$$

$$K(l,n_1,n_2)\breve{\mathbf{S}}\left[Ext\text{-}\int_{|p_1|\leq\varkappa}\dots Ext\text{-}\int_{|p_{n_2}|\leq\varkappa}W(k_1,\dots k_{n_1},p_1,\dots,p_{n_2})\,\psi^{(l)}(p_1,\dots,p_{n_2},k_1,\dots k_{n_1})d^{\#3n_2}\,p\right], \quad (9.18)$$

$(T_W\,\psi)^n=0$ if $n<n_1-n_2$,

$$(T_W^*\,\psi)^{l-n_1+n_2}=$$

$$K(l,n_2,n_1)\breve{\mathbf{S}}\left[Ext\text{-}\int_{|k_1|\leq\varkappa}\dots Ext\text{-}\int_{|k_{n_1}|\leq\varkappa}\overline{W(k_1,\dots k_{n_1},p_1,\dots,p_{n_2})}\,\psi^{(l)}(p_1,\dots,p_{n_2},k_1,\dots k_{n_1})d^{\#3n_1}\,k\right] \quad (9.19)$$

$\left(T_W^*(\psi)\right)^n=0$ if and only if $n<n_2-n_1$. Here $\breve{\mathbf{S}}$ is the symmetrization operator.

(5) If $W_n \to_{\#} W$ in $L_2^{\#}\left({}^*\mathbb{R}_c^{\#3(n_1+n_2)}\right)$ then $T_{W_n} \to_{\#} T_W$ strongly on domain $D_{S_{\text{fin}}^{\#}}$.

Proof For vectors ψ in $D_{S_{\text{fin}}^{\#}}$, we define $T_W(\psi)$ by the formula (9.18). By the Schwarz inequality and the fact that $\breve{\mathbf{S}}$ is a projection we obtain

$$\|(T_W\,\psi)^{l-n_2+n_1}\|_{\#}^2 \leq K(l,n_1,n_2)\left\|\,\psi^{(l)}\right\|_{\#}^2\|W\|_{\#}^2 \tag{9.20}$$

If we now define an operator $T_W^*\,\psi$, on domain $D_{S_{\text{fin}}^{\#}}$ by using the formula (9.19) then for all $\varphi,\,\psi \in D_{S_{\text{fin}}^{\#}}$ we obtain that $\langle\varphi, T_W\,\psi\rangle = \langle T_W^*\varphi,\psi\rangle$. Thus, T_W is #-closable and T_W^* is the restriction of the adjoint of T_W on domain $D_{S_{\text{fin}}^{\#}}$. From now on we will use T_W to denote #- $\bar{T}_W$ and T_W^* to denote the #-adjoint of the operator T_W. By the definition of the operator T_W, $D_{S_{\text{fin}}^{\#}}$ is a #-core and further, since T_W is bounded on the l-particle vectors in $D_{S_{\text{fin}}^{\#}}$, we have $F_0 \subset D(T_W)$. Since the right-hand side of (9.18) is also bounded on the l-particle vectors, (9.18) represents T_W on all l-particle vectors. The proofs of the statements in (2) about T_W^* are the same. To prove (3), let $\psi \in D_{S_{\text{fin}}^{\#}}$. Then by the canonical computation we obtain

$$\left\|\left((1+N^{\#})^{-m_1/2}T_W(1+N^{\#})^{-m_2/2}\right)^{l-n_2+n_1}\right\|_{\#}^2 \leq \left[\frac{K(l,n_1,n_2)}{(1+l-n_2+n_1)^{\frac{m_1}{2}}(1+l)^{\frac{m_2}{2}}}\right]^2\left\|\,\psi^{(l)}\right\|_{\#}^2\|W\|_{\#}^2.$$

And therefore finally we get

$$\left\|\left((1+N^{\#})^{-m_1/2}T_W(1+N^{\#})^{-m_2/2}\right)^{l-n_2+n_1}\right\|_{\#}^2 \leq C(m_1,m_2)\|W\|_{L_2^{\#}}.$$

Here $C(m_1,m_2) = \sup_{l\in{}^*\mathbb{N}}\left(\frac{K(l,n_1,n_2)}{(1+l-n_2+n_1)^{\frac{m_1}{2}}(1+l)^{\frac{m_2}{2}}}\right) < {}^*\infty$ since $m_1+m_2 = n_1+n_2$. In all the sup's only l so that $l-n_2+n_1 > 0$ occur since the other terms are annihilated by the action of T_W. Thus, $(1+N^{\#})^{-m_1/2}T_W(1+N^{\#})^{-m_2/2}$ extends to a bounded operator on $\mathcal{F}_s(H^{\#})$ with #-norm less than or equal to $C(m_1,m_2)$. If $m_1 = n_1$ and $m_2 = n_2$, then $C(m_1,m_2) = 1$.

In order to prove (5) one needs only note that if $\psi = \left(0,\ldots,0,\psi^{(l)},0,\ldots\right) \in D_{S_{\text{fin}}^{\#}}$ and $W_n \to_{\#} W$ in $L_2^{\#}\left({}^*\mathbb{R}_c^{\#3(n_1+n_2)}\right)$, then $\left\|T_{W_n}-T_W\right\|_{\#} = \left\|T_{W_n-W}\right\|_{\#} \leq K(l,n_1,n_2)\|W_n-W\|_{\#}\|\psi\|_{\#} = \delta_n$, where #-$\lim_{n\to_{\#}{}^*\infty}\delta_n = 0$. Since $D_{S_{\text{fin}}^{\#}}$ consists of finite and hyperfinite linear combinations of such vectors, we have shown that T_{W_n} #-converges strongly on domain $D_{S_{\text{fin}}^{\#}}$ to operator T_W if $W_n \to_{\#} W$ in $L_2^{\#}\left({}^*\mathbb{R}_c^{\#3(n_1+n_2)}\right)$.

In order to prove (1) let $\psi_1,\psi_2 \in D_{S_{\text{fin}}^{\#}}$ where $\psi_1 = \left(0,\ldots,0,\psi^{(l-n_2+n_1)},0,\ldots\right)$ and $\psi_2 = \left(0,\ldots,0,\psi^{(l)},0,\ldots\right)$. Then, if $W = \left(Ext\text{-}\prod_{i=1}^{n_1}f(k_i)\right)\left(Ext\text{-}\prod_{i=1}^{n_2}g(p_i)\right)$ by the canonical definition of the form $\left(Ext\text{-}\prod_{i=1}^{n_1}a^{\dagger}(k_i)\right)\times\left(Ext\text{-}\prod_{i=1}^{n_2}a(p_i)\right)$ one obtains

$$\langle\psi_1, T_W\psi_2\rangle = Ext\text{-}\int_{{}^*\mathbb{R}^{3(n_1+n_2)}} W\left(k_1,\ldots k_{n_1},p_1,\ldots,p_{n_2}\right)\times \tag{9.21}$$

$$\langle\psi_1,\ \left(Ext\text{-}\prod_{i=1}^{n_1}a^{\dagger}(k_i)\right)\left(Ext\text{-}\prod_{i=1}^{n_2}a(p_i)\right)\psi_2\rangle d^{\#3n_1}k d^{\#3n_2}p.$$

Since both sides of (9.21) are linear in W, the relationship continues to hold for the all such W that are finite or hyperfinite linear combinations of such products. Since

$$\langle\psi_1,\ \left(Ext\text{-}\prod_{i=1}^{n_1}a^{\dagger}(k_i)\right)\left(Ext\text{-}\prod_{i=1}^{n_2}a(p_i)\right)\psi_2\rangle \in L_2^{\#}\left({}^*\mathbb{R}_c^{\#3(n_1+n_2)}\right)$$

and since statement (5) holds, both the right-hand sides and left-hand sides of (9.21) are #-continuous linear functionals on $L_2^{\#}\left({}^*\mathbb{R}_c^{\#3(n_1+n_2)}\right)$. Since they agree on a #-dense set, they agree everywhere. Finally, (9.21) extends by linearity to all of $D_{S_{\text{fin}}^{\#}} \times D_{S_{\text{fin}}^{\#}}$. This proves (1); the proof of (2) is similar.

Now we go to estimate monomials in creation and annihilation operators in terms of the operators $N_\tau^{\#}$ defined by

$$N_{\tau,\varkappa}^{\#} = Ext\text{-}\int_{|k|\leq\varkappa} a^{\dagger}(k)a\,(k)\,\mu(k)^{\tau} d^{\#3}k\,. \tag{9.22}$$

For first estimate we consider the following bilinear form, where the kernel $w(k,p)$ is #-measurable and $|w(k,p)|$ is symmetric.

$$Ext\text{-}\int_{|p|\leq\varkappa} Ext\text{-}\int_{|k|\leq\varkappa} a^{\dagger}(k)\,w(k,p)a\,(p)d^{\#3}k\,d^{\#3}p \tag{9.23}$$

Note that for $\tau \geq 1$

$$N_{\tau,\varkappa}^{\#} \leq H_{0,\varkappa}^{\tau} \text{ and } N_{\tau,\varkappa}^{\#2} \leq H_{0,\varkappa}^{2\tau}. \tag{9.24}$$

We introduce now the #-norms $M_{1,\varkappa}(\tau)$ and $M_{2,\varkappa}(\tau)$ on the kernel $w(k,p)$, which may be finite or hyperfinite

$$M_{1,\varkappa}(\tau) = \left(\sup_{|k|\leq\varkappa}\mu(k)^{-\tau}\right)\left(Ext\text{-}\int_{|p|\leq\varkappa}\{|w(k,p)|\}\,d^{\#3}p^{\#3}\right), \tag{9.25}$$

$$M_{2,\varkappa}(\tau) = \left(\sup_{|k|\leq\varkappa}\mu(k)^{-2\tau}\right)\left(Ext\text{-}\int_{|p|\leq\varkappa}\{|w(k,p)|\mu(p)^{\tau}\}\,d^{\#3}p^{\#3}\right). \tag{9.26}$$

Proposition 9.1 Assume that for some $\tau, M_{1,\varkappa}(\tau) < {}^*\infty$, then W is a bilinear form on the domain $\mathcal{D}\left(N_{\tau,\varkappa}^{\#1/2}\right) \times \mathcal{D}\left(N_{\tau,\varkappa}^{\#1/2}\right)$, and $N_{\tau,\varkappa}^{\#-1/2} W N_{\tau,\varkappa}^{\#-1/2}$ is a bounded operator on Fock space $\mathcal{F}^{\#}$ with

$$\left\|N_{\tau,\varkappa}^{\#-1/2} W N_{\tau,\varkappa}^{\#-1/2}\right\|_{\#} < M_{1,\varkappa}(\tau). \tag{9.27}$$

Note that: (a) the operator $N_{\tau,\varkappa}^{\#-1/2}$ is defined on the orthogonal complement of the no particle vector. Since W equals zero on the no particle vector, we define $W N_{\tau,\varkappa}^{\#-1/2}$ to be zero on the no particle vector; (b) if $\tau \geq 1$, then from (9.22) follows that $N_{\tau,\varkappa}^{\#-\tau/2} W N_{\tau,\varkappa}^{\#-\tau/2}$ is a bounded operator with #-norm less than $M_{1,\varkappa}(\tau)$.
Proof Since the bilinear form W commutes with the projection onto vectors with exactly n particles, it is sufficient to prove that for n particle vectors, it is sufficient to prove that for n particle vectors $\psi \in \mathcal{D}\left(N_{\tau,\varkappa}^{\#1/2}\right)$, the following inequality of forms holds

$$|\langle\psi, W\psi\rangle| \leq M_{1,\varkappa}(\tau)\left|\langle\psi, N_{\tau,\varkappa}^{\#}\psi\rangle\right|. \tag{9.28}$$

By definition one obtains

$$\langle\psi, W\psi\rangle = n\left(Ext\text{-}\int_{|p|\leq\varkappa} Ext\text{-}\int_{|k|\leq\varkappa} \overline{\psi(p,k_2,\ldots k_n)}\,w(p,q)\psi(q,k_2,\ldots k_n)d^{\#3}k\,d^{\#3}p\,d^{\#3}q\right).$$

By using the generalized Schwarz inequality in p and q, we obtain

$$|\langle\psi, W\psi\rangle| \leq n\left(Ext\text{-}\int_{|q|\leq\varkappa} Ext\text{-}\int_{|p|\leq\varkappa} Ext\text{-}\int_{|k|\leq\varkappa} |\psi^2(p,k)w(p,q)|\,d^{\#3}k\,d^{\#3}p\,d^{\#3}q\right)$$

and by (9.25) finally we get

$$|\langle\psi, W\psi\rangle| \leq n M_{1,\varkappa}(\tau)\left(Ext\text{-}\int_{|k|\leq\varkappa} Ext\text{-}\int_{|p|\leq\varkappa}|\psi^2(p,k)|\,\mu(p)^\tau d^{\#3}p\, d^{\#3}k\right).$$

The existence of a bounded operator satisfying (9.27) then follows by the generalized Riesz representation theorem.
Theorem 9.7 (Generalized Riesz Representation Theorem) If T is a bounded linear functional on a non-Archimedean Hilbert space H then there exists some $g \in H$ such that for every vector $f \in H$ we have that $T(f) = \langle f, g\rangle$ and $\| T \|_\# = \| g \|_\#$.

Proposition 9.2 Assume that for some $\tau, M_{1,\varkappa}(\tau)$ and $M_{2,\varkappa}(\tau)$ are finite or hyperfinite, then W determines an operator on $\mathcal{D}\left(N^{\#}_{\tau,\varkappa}\right)$ such that the operator $WN^{\#-1}_{\tau,\varkappa}$ is bounded with

$$\| WN^{\#-1}_{\tau,\varkappa} \|_\# \leq \left[M_{1,\varkappa}(\tau) + M_{2,\varkappa}(\tau)\right] \leq M_{3,\varkappa}(\tau). \tag{9.29}$$

Note that since $|w(p,q)|$ is symmetric, $N^{\#-1}_{\tau,\varkappa}W$ is also bounded with a #-norm less than $M_{3,\varkappa}(\tau)$. If $\tau \geq 1$, $WN^{\#-\tau}_{\tau,\varkappa}$ is bounded with a #-norm less than $M_{3,\varkappa}(\tau)$.
Proof As in Proposition 9.1, it is sufficient to prove that for n particle vectors $\psi \in \mathcal{D}\left(N^{\#}_{\tau,\varkappa}\right)$

$$\| W\psi \|_\# \leq M_{3,\varkappa}(\tau)\, \| N^{\#}_{\tau,\varkappa}\psi \|_\#. \tag{9.30}$$

We define now the quantity

$$C^{\#}_{j,l} = Ext\text{-}\int_{|k_1|\leq\varkappa} d^{\#3}k_1 \ldots Ext\text{-}\int_{|k_n|\leq\varkappa} d^{\#3}k_n \times \tag{9.31}$$

$$\left\{Ext\text{-}\int_{|p_j|\leq\varkappa} d^{\#3}p_j \bar{w}(k_j,p_j)\bar{\psi}(k_1,\ldots k_{j-1},p_j,k_{j+1},\ldots k_n) \times\right.$$

$$\left.Ext\text{-}\int_{|p_l|\leq\varkappa} d^{\#3}p_l \bar{w}(k_l,p_l)\bar{\psi}(k_1,\ldots k_{l-1},p_l,k_{l+1},\ldots k_n)\right\}.$$

Note that $\| W\psi \|^2_\# = Ext\text{-}\sum^n_{j,l} C^{\#}_{j,l}$ and for $j = l = 1$, and $\boldsymbol{k} = (q, k_2, \ldots k_n)d^{\#3}$, we get

$$C^{\#}_{j,l} = Ext\text{-}\int_{|k|\leq\varkappa} d^{\#3}k\; Ext\text{-}\int_{|q|\leq\varkappa} d^{\#3}q\left(\left|\int_{|p|\leq\varkappa} d^{\#3}p\,w(q,p)\psi(p,k)\right|\right)^2 \leq$$

$$Ext\text{-}\int_{|k|\leq\varkappa} d^{\#3}k Ext\text{-}\int_{|q|\leq\varkappa} d^{\#3}q\left(\int_{|p|\leq\varkappa} d^{\#3}p|w(q,p)\psi(p,k)|\right)^2. \tag{9.32}$$

By the generalized Schwarz inequality we get

$$C^{\#}_{j,j} \leq Ext\text{-}\int_{|k|\leq\varkappa} d^{\#3}k\left[Ext\text{-}\int_{|q|\leq\varkappa} d^{\#3}q\left(Ext\text{-}\int_{|p|\leq\varkappa}|w(q,p)|d^{\#3}p\left(Ext\text{-}\int_{|r|\leq\varkappa}|\psi^2(r,k)w(r,q)|d^{\#3}r\right)\right)\right]. \tag{9.33}$$

From (9.33) by (9.25) we get

$$C^{\#}_{j,j} \leq M_{1,\varkappa}(\tau)\left[Ext\text{-}\int_{|k|\leq\varkappa} d^{\#3}k\left[Ext\text{-}\int_{|q|\leq\varkappa} d^{\#3}q\left(Ext\text{-}\int_{|r|\leq\varkappa} d^{\#3}r\,\mu(q)^\tau|\psi\,(r,k)w(r,q)|\right)\right]\right]. \tag{9.34}$$

From (9.34) by (9.26) we get

$$C^{\#}_{j,j} \leq M_{1,\varkappa}(\tau)M_{2,\varkappa}(\tau)\left[Ext\text{-}\int_{|k|\leq\varkappa} d^{\#3}k\left[Ext\text{-}\int_{|p|\leq\varkappa} d^{\#3}p\mu(p)^{2\tau}|\psi\,(p,k)|^2\right]\right].$$

We estimate now $C^{\#}_{j,l}$ for $j \neq l$. Suppressing all but the essential variables k_j, p_j, k_l, and p_l we get

$$\left|C^{\#}_{j,l}\right| \leq Ext\text{-}\int_{|k_j|\leq\varkappa} d^{\#3}\,k_j \times \tag{9.35}$$

$$\left\{Ext\text{-}\int_{|k_l|\leq\varkappa} d^{\#3}k_l\left[Ext\text{-}\int_{|p_j|\leq\varkappa} d^{\#3}p_j\left(Ext\text{-}\int_{|p_l|\leq\varkappa} d^{\#3}p_l\right)\left|w(k_j,p_j)\psi\left(p_j,k_l\right)w(k_l,p_l)\psi\left(p_l,k_j\right)\right|\right]\right\}$$

By the generalized Schwarz inequality in p, and (9.25 we get

$$\left|C_{j,l}^{\#}\right| \leq M_{1,\varkappa}(\tau)\left\{Ext\text{-}\int_{|k_j|\leq\varkappa} d^{\#3}\,k_j\left[Ext\text{-}\int_{|k_l|\leq\varkappa} d^{\#3}k_l\mu(k_l)^{\tau/2}\right]\right\}\times \tag{9.36}$$

$$\left(Ext\text{-}\int_{|p_j|\leq\varkappa}\left|w(k_j,p_j)\psi^2\left(p_j,k_l\right)\right|d^{\#3}\,p_j\right)^{1/2}\left(Ext\text{-}\int_{|p_l|\leq\varkappa}\left|w(k_l,p_l)\psi^2\left(p_l,k_j\right)\right|d^{\#3}\,p_l\right)^{1/2}.$$

By the generalized Schwarz inequality in k from (9.36) we get

$$\left|C_{j,l}^{\#}\right| \leq M_{1,\varkappa}(\tau)\left\{Ext\text{-}\int_{|k_j|\leq\varkappa} d^{\#3}\,k_j\left[Ext\text{-}\int_{|k_l|\leq\varkappa} d^{\#3}k_l\left(\int_{|p_j|\leq\varkappa} d^{\#3}\,p_j\mu(k_l)^{\frac{\tau}{2}}\left|w(k_j,p_j)\psi^2\left(p_j,k_l\right)\right|\right)\right]\right\}. \tag{9.37}$$

From (9.37) by (9.25) we get

$$\left|C_{j,l}^{\#}\right| \leq M_{1,\varkappa}^2\,(\tau)\left[Ext\text{-}\int_{|k_l|\leq\varkappa} d^{\#3}\,k_l\left(Ext\text{-}\int_{|p_j|\leq\varkappa} d^{\#3}\,p_j\mu(k_l)^{\tau}\mu(p_j)^{\tau}\left|\psi^2\left(p_j,k_l\right)\right|\right)\right].$$

Finally by (9.35)-(9.37) we obtain

$$\parallel W\psi \parallel_{\#}^2 \leq M_{1,\varkappa}\left[M_{1,\varkappa}+M_{2,\varkappa}\right]\langle\psi, N_{\tau,\varkappa}^{\#2}\psi\rangle,$$

And therefore (9.30) is proved.

We now let

$$W = Ext\text{-}\int_{|p_1|\leq\varkappa} d^{\#3}\,p_1\ldots Ext\text{-}\int_{|p_s|\leq\varkappa} d^{\#3}\,p_s \times \tag{9.38}$$

$$\left[Ext\text{-}\int_{|k_1|\leq\varkappa} d^{\#3}k_1\ldots Ext\text{-}\int_{|k_r|\leq\varkappa} d^{\#3}\,p_1 a^{\dagger}(k_1)\cdots a^{\dagger}(k_r)w(k_1,\ldots,k_r;p_1,\ldots,p_s)a\,(p_1)\cdots a(p_s)\right].$$

Here $w(k_1,\ldots,k_r;p_1,\ldots,p_s)$ is a #-measurable kernel. Let $\alpha \leq r$, and define $E_C(k_1,\ldots,k_\alpha)$ by

$$E_C(k_1,\ldots,k_\alpha) = \mu(k_1)\cdots\mu(k_\alpha). \tag{9.39}$$

Let $\beta \leq s$ and define $E_A(p_1,\ldots,p_\beta)$ by

$$E_A(p_1,\ldots,p_\beta) = \mu(p_1)\cdots\mu(p_\beta). \tag{9.40}$$

Let $M_{4,\varkappa}\,(\tau)$ be

$$M_{4,\varkappa}\,(\tau) = \left\|\frac{w(k_1,\ldots,k_\alpha;p_1,\ldots,p_\beta)}{E_C(k_1,\ldots,k_\alpha)^{\tau/2}E_A(p_1,\ldots,p_\beta)^{\tau/2}}\right\|_{\#\mathrm{op}} \leq \tag{9.41}$$

$$\left\|\frac{w(k_1,\ldots,k_\alpha;p_1,\ldots,p_\beta)}{E_C(k_1,\ldots,k_\alpha)^{\tau/2}E_A(p_1,\ldots,p_\beta)^{\tau/2}}\right\|_{\#2}$$

where $\left\|v(k_1,\ldots,k_\alpha;p_1,\ldots,p_\beta)\right\|_{\#\mathrm{op}}$ denotes the operator #-norm of the kernel $v(k_1,\ldots,k_\alpha;p_1,\ldots,p_\beta)$ as an integral operator from $L_2^{\#}({}^*\mathbb{R}_c^{\#s})$ to $L_2^{\#}({}^*\mathbb{R}_c^{\#r})$. The #-norm $\|\cdot\|_{\#\mathrm{op}}$ op is dominated by the generalized Hubert Schmidt

#-norm $\|\cdot\|_{\#2}$.

Proposition 9.3 Assume that $M_{4,\varkappa}(\tau)$ is finite or hyperfinite for some α,β as above and for some τ, then W is a bilinear form on $\mathcal{D}\left(N_{\tau,\varkappa}^{\#\alpha/2}N_{,\varkappa}^{\#\delta/2}\right)\times\mathcal{D}\left(N_{\tau,\varkappa}^{\#\beta/2}N_{,\varkappa}^{\#\varepsilon/2}\right)$, where $\alpha\,+\,\delta\,=\,r,\beta+\varepsilon=s$. Also

$$\widetilde{W}_{\varkappa}=N_{\tau,\varkappa}^{\#-\alpha/2}N_{,\varkappa}^{\#-\delta/2}WN_{\tau,\varkappa}^{\#-\beta/2}N_{,\varkappa}^{\#-\varepsilon/2} \tag{9.42}$$

is a bounded operator and

$$\|\,\widetilde{W}_{\varkappa}\,\|_{\#}\leq M_{4,\varkappa}(\tau). \tag{9.43}$$

Proof Let Ω,ψ be vectors with a finite or hyperfinite number of particles and wave functions in Schwartz space $S_{\text{fin}}^{\#}$. Then if $A_C(k)\ =a(k_1)\cdots a(k_r)$ and $A_A(p)=a(p_1)\cdots a(p_s)$,

$$\langle\Omega,W\psi\rangle=Ext\text{-}\int_{|p|\leq\varkappa}d^{\#3}p\left[Ext\text{-}\int_{|k|\leq\varkappa}d^{\#3}k\langle A_C(k)\Omega,w(k,p)A_A(p)\psi\rangle\right].$$

By the generalized Schwarz inequality we get

$$|\Omega,W\psi|^2\leq\left(Ext\text{-}\int_{|p|\leq\varkappa}d^{\#3}p\left[Ext\text{-}\int_{|k|\leq\varkappa}d^{\#3}k\|A_C(k)\Omega\|_{\#}\cdot|w(k,p)|\cdot\|A_A(p)\psi\|_{\#}\right]\right)^2\leq \tag{9.44}$$

$$M_{4,\varkappa}^2\,(\tau)\left\{Ext\text{-}\int_{|p|\leq\varkappa}d^{\#3}p\left[Ext\text{-}\int_{|k|\leq\varkappa}d^{\#3}kE_C^{\tau}\,(k)\|A_C(k)\Omega\|_{\#}^{\,2}\cdot E_A^{\tau}(p)\cdot\|A_A(p)\psi\|_{\#}^2\right]\right\}\leq$$

$$M_{4,\varkappa}^2\,(\tau)\cdot\left\|N_{\tau,\varkappa}^{\#\alpha/2}N_{\varkappa}^{\#\delta/2}\Omega\right\|_{\#}^2\cdot\left\|N_{\tau,\varkappa}^{\#\beta/2}N_{\varkappa}^{\#\varepsilon/2}\psi\right\|_{\#}^2.$$

The last inequality (9.44) is proved as follows

$$Ext\text{-}\int_{|k|\leq\varkappa}d^{\#3}kE_C^{\tau}\,(k)\|A_C(k)\Omega\|_{\#}^{\,2}=$$

$$Ext\text{-}\sum_{n=0}^{*\infty}Ext\text{-}\int_{|k_1|\leq\varkappa}d^{\#3}k_1\ldots Ext\text{-}\int_{|k_1|\leq\varkappa}d^{\#3}k_{n+r}\,(n+1)\cdots(n+r)\mu(k_1)^{\tau}\cdots\mu(k_\alpha)^{\tau}\left|\Omega^{(n+r)}(k_1,\ldots,k_{n+r})\right|\leq$$

$$Ext\text{-}\sum_{n=0}^{*\infty}Ext\text{-}\int_{|k_1|\leq\varkappa}d^{\#3}k_1\ldots Ext\text{-}\int_{|k_1|\leq\varkappa}d^{\#3}k_{n+r}\left(Ext\text{-}\sum_{j=1}^{n+r}\mu(k_j)^{\tau}\right)^{\alpha}(n+r)^{\delta}\left|\Omega^{(n+r)}(k_1,\ldots,k_{n+r})\right|^2=$$

$\left\|N_{\tau,\varkappa}^{\#\alpha/2}N_{\varkappa}^{\#\delta/2}\Omega\right\|_{\#}^2$, since $\left|\Omega^{(n+r)}(k_1,\ldots,k_{n+r})\right|^2$ is symmetric and the product $\left(Ext\text{-}\sum_{j=1}^{n+r}\mu(k_j)^{\tau}\right)^{\alpha}\times\left(Ext\text{-}\sum_{j=1}^{n+r}\mu(k_j)^{0}\right)^{\delta}$ when expanded, has $Ext\text{-}\prod_{j=1}^{r}(n+j)$ terms with all variables distinct. The existence of the bounded operator W now follows by the generalized Riesz representation theorem, see Theorem 9.7.

Proposition 9.4 Assume that $\alpha\leq\,r,\beta\leq s$ and for some τ,σ

$$M_{5,\varkappa}(\tau,\sigma)=\left\|\frac{w(k,p)}{E_C(\alpha,\tau)E_A(\beta,\sigma)}\right\|_{\#op}\leq{}^*\infty. \tag{9.45}$$

Then W is a bilinear form on $\mathcal{D}\left(N_{\tau,\varkappa}^{\#\alpha/2}N_{,\varkappa}^{\#\delta/2}\right)\times\mathcal{D}\left(N_{\tau,\varkappa}^{\#\beta/2}N_{,\varkappa}^{\#\varepsilon/2}\right)$, for any δ,ε such that $\alpha\,+\,\beta\,+\,\delta\,+\,\varepsilon\,=\,r\,+\,s$. Furthermore

$$\widetilde{W}_{\varkappa}=(I+N_{\varkappa}^{\#})^{-\delta/2}N_{\tau,\varkappa}^{\#-\alpha/2}WN_{\sigma,\varkappa}^{\#-\beta/2}(I+N_{\varkappa}^{\#})^{-\varepsilon/2} \tag{9.46}$$

is a bounded operator with a #-norm such that

$$\| \widetilde{W}_{\varkappa} \|_{\#} \leq c M_{5,\varkappa}(\tau). \tag{9.47}$$

Where $c \in {}^{*}\mathbb{R}_{c}^{\#}$ is constant.

Proof Similarly as proof to proposition 9.3 above.

The energy-momentum density tensor $T_{\mu\upsilon,\varkappa}(x,t)$ for the $\lambda(\varphi_{\varkappa}^{4})_{4}$ theory with hyperfinite momentum cutoff $\varkappa$ is a bilinear form on non-Archimedean Fock space $\mathcal{F}_{s}(H^{\#})$. The energy momentum vector $P_{\mu,\varkappa}, \mu = 1,2,3$ is formally related to $T_{\mu\upsilon,\varkappa}(x,t)$ by the following formula

$$P_{\mu,\varkappa} = Ext\text{-}\int_{{}^{*}\mathbb{R}_{c}^{\#3}} T_{0\mu,\varkappa}(x,t)\, d^{\#3}x, \mu = 0,1,2,3. \tag{9.48}$$

The generators $M_{\varkappa}^{0k}$ of pure Lorentz transformations is formally related to $T_{\mu\upsilon,\varkappa}(x,t)$ by the following formula

$$M_{\varkappa}^{0k} = Ext\text{-}\int_{{}^{*}\mathbb{R}_{c}^{\#3}} T_{00,\varkappa}(x,0)\, x^{k} d^{\#3}x, k = 1,2,3. \tag{9.49}$$

The expression for the operator $T_{\mu\upsilon,\varkappa}(x,0)$ is a Wick ordered polynomial in the time zero canonical fields $\varphi_{\varkappa}$ and $\pi_{\varkappa}$. In this case the Hamiltonian $H = P_{0,\varkappa}$ defined by (9.48) is a bilinear form on Fock space $\mathcal{F}_{s}(H^{\#})$. In this section we show that for the $\lambda(\varphi_{\varkappa}^{4})_{4}$ theory the integration in (9.48) can be restricted to a bounded domain to yield a local energy or momentum operator on Fock space $\mathcal{F}_{s}(H^{\#})$. The local version of (9.49) can be handled similarly. It is customary to write the operator $T_{\mu\upsilon,\varkappa}(x,0)$ as the sum of a free field part and an interaction part. Explicitly, we write the energy density as

$$T_{00,\varkappa}(x,0) = T_{0,\varkappa}(x) + T_{I,\varkappa}(x). \tag{9.50}$$

Here

$$T_{0,\varkappa}(x) = H_{0,\varkappa}(x) = \tfrac{1}{2}\colon\!\left(\pi_{\varkappa}^{2}(x) + \left(\nabla^{\#}\varphi_{\varkappa}(x)\right)^{2} + m^{2}\varphi_{\varkappa}^{2}(x)\right)\!\colon, \tag{9.51}$$

$$T_{I,\varkappa}(x) = \lambda(\colon\varphi_{\varkappa}^{4}(x)\colon)_{4}. \tag{9.52}$$

For the momentum density vector $P_{\mu,\varkappa}, \mu = 1,2,3$ we set

$$P_{\mu,\varkappa}(x) = T_{0\mu,\varkappa}(x,0) = \tfrac{1}{2}\colon\!\left(\pi_{\varkappa}(x)\frac{\partial^{\#}}{\partial^{\#}x_{\mu}}\varphi_{\varkappa}(x) + \frac{\partial^{\#}}{\partial^{\#}x_{\mu}}\varphi_{\varkappa}(x)\pi_{\varkappa}(x)\right)\!\colon. \tag{9.53}$$

In order to avoid problems caused by sharp spatial boundaries, we consider

$$T_{\varkappa}(g) = Ext\text{-}\int_{{}^{*}\mathbb{R}_{c}^{\#3}} T_{\varkappa}(x)g(x)\, d^{\#3}x = T_{0,\varkappa}(g) + T_{I,\varkappa}(g), \tag{9.54}$$

$$P_{\mu,\varkappa}(g) = Ext\text{-}\int_{{}^{*}\mathbb{R}_{c}^{\#3}} P_{\mu,\varkappa}(x)g(x)\, d^{\#3}x, \mu = 1,2,3. \tag{9.55}$$

Remark 9.3 Here $g(x)$ is a ${}^{*}\mathbb{R}_{c}^{\#}$- valued function in $S_{\text{fin}}^{\#}({}^{*}\mathbb{R}_{c}^{\#3})$ i.e., $g(x)$ is rapidly #-decreasing.

For the local free field energy we set $T_{0,\varkappa}(g) = T_{0,\varkappa}^{1}(g) + T_{0,\varkappa}^{2}(g)$, where

$$T_{0,\varkappa}^{1}(g) = c_{1} Ext\text{-}\int_{|\boldsymbol{k}_{1}|\leq\varkappa} d^{\#3}\,\boldsymbol{k}_{1} Ext\text{-}\int_{|\boldsymbol{k}_{2}|\leq\varkappa} d^{\#3}\,\boldsymbol{k}_{2}\hat{g}(k_{1}^{1}-k_{2}^{1},k_{1}^{2}-k_{2}^{2},k_{1}^{3}-k_{2}^{3})\left\{\frac{\mu(\boldsymbol{k}_{1})\mu(\boldsymbol{k}_{2})+\langle\boldsymbol{k}_{1},\boldsymbol{k}_{2}\rangle+m^{2}}{\sqrt{\mu(\boldsymbol{k}_{1})\mu(\boldsymbol{k}_{2})}}\right\}\times \tag{9.56}$$

$a^{\dagger}(\boldsymbol{k}_{1})a(\boldsymbol{k}_{2})$,

$$T_{0,\varkappa}^{2}(g) = c_{2} Ext\text{-}\int_{|k_{1}|\leq\varkappa} d^{\#3}\,\boldsymbol{k}_{1} Ext\text{-}\int_{|k_{2}|\leq\varkappa} d^{\#3}\,\boldsymbol{k}_{2}\hat{g}(k_{1}^{1}-k_{2}^{1},k_{1}^{2}-k_{2}^{2},k_{1}^{3}-k_{2}^{3})\left\{\frac{-\mu(k_{1})\mu(k_{2})+\langle\boldsymbol{k}_{1},\boldsymbol{k}_{2}\rangle+m^{2}}{\sqrt{\mu(\boldsymbol{k}_{1})\mu(\boldsymbol{k}_{2})}}\right\}\times \tag{9.57}$$

$\times \{a^{\dagger}(\boldsymbol{k}_1)a^{\dagger}(-\boldsymbol{k}_2) + a(-\boldsymbol{k}_1)a(\boldsymbol{k}_2)\}.$

Here $\boldsymbol{k}_1 = (k_1^1, k_1^2, k_1^3)$, $\boldsymbol{k}_2 = (k_2^1, k_2^2, k_2^3)$, $\langle \boldsymbol{k_1}, \boldsymbol{k_2} \rangle = \sum_{i=1}^{3} k_1^i\, k_2^i$, $\hat{g}(p) = Ext\text{-}\int_{^*\mathbb{R}_c^{\#3}} (Ext\text{-}[i\langle p, x\rangle])g(x)\, d^{\#3}x.$

Similarly, for the components of the local momentum we set $P_{\mu,\varkappa}\ (g) = P_{\mu,\varkappa}^{(1)}\ (g) + P_{\mu,\varkappa}^{(2)}\ (g), \mu = 1,2,3$ where

$$P_{\mu,\varkappa}^{(1)}(g) = c_1 Ext\text{-}\int_{|k_1|\leq\varkappa} d^{\#3}\, k_1 Ext\text{-}\int_{|k_2|\leq\varkappa} d^{\#3}\, k_2 \hat{g}(k_1^1 - k_2^1, k_1^2 - k_2^2, k_1^3 - k_2^3) \times \tag{9.58}$$

$$\times \left\{\frac{(k_1^{\mu})\boldsymbol{\mu}(k_2) + (k_2^{\mu})\boldsymbol{\mu}(k_1)}{\sqrt{\boldsymbol{\mu}(k_1)\boldsymbol{\mu}(k_2)}}\right\} a^{\dagger}(k_1)a(k_2),$$

$$P_{\mu,\varkappa}^{(2)}(g) = c_2 Ext\text{-}\int_{|k_1|\leq\varkappa} d^{\#3}\, k_1 Ext\text{-}\int_{|k_2|\leq\varkappa} d^{\#3}\, k_2 \hat{g}(k_1^1 - k_2^1, k_1^2 - k_2^2, k_1^3 - k_2^3) \times \tag{9.59}$$

$$\times \left\{\frac{(k_1^{\mu})\boldsymbol{\mu}(k_2) - (k_2^{\mu})\boldsymbol{\mu}(k_1)}{\sqrt{\boldsymbol{\mu}(k_1)\boldsymbol{\mu}(k_2)}}\right\} \{-a^{\dagger}(k_1)a^{\dagger}(-k_2) + a(-k_1)a(k_2)\}.$$

Theorem 9.8 The bilinear forms $T_{0,\varkappa}(g)$ and $P_{\mu,\varkappa}(g)$ define symmetric operators on $\mathcal{D}\left(H_{0,\varkappa}^{\#}\right)$. The following operators are all bounded

$$T_{0,\varkappa}(g)\left(H_{0,\varkappa}^{\#} + I\right)^{-1}, P_{\mu,\varkappa}(g)\ \left(H_{0,\varkappa}^{\#} + I\right)^{-1}, \mu = 1,2,3, \tag{9.60}$$

$$\left(H_{0,\varkappa}^{\#} + I\right)^{-1/2} T_{0,\varkappa}(g)\left(H_{0,\varkappa}^{\#} + I\right)^{-1/2}, \tag{9.61}$$

$$\left(H_{0,\varkappa}^{\#} + I\right)^{-1/2} P_{\mu,\varkappa}(g)\left(H_{0,\varkappa}^{\#} + I\right)^{-1/2}, \mu = 1,2,3, \tag{9.62}$$

$$T_{0,\varkappa}^{(2)}(g)\left(N_{0,\varkappa}^{\#} + I\right)^{-1}, \text{ and } P_{\mu,\varkappa}^{(2)}(g)\left(N_{0,\varkappa}^{\#} + I\right)^{-1}, \mu = 1,2,3. \tag{9.63}$$

Proposition 9.5 The kernel of $\boldsymbol{T_{0,\varkappa}^{(2)}(g)}$ and the kernels of $\boldsymbol{P_{\mu,\varkappa}^{(2)}(g), \mu = 1,2,3}$ are $\boldsymbol{L_2^{\#}}$ functions even without hyperfinite momentum cutoff $|\boldsymbol{k_1}| \leq \boldsymbol{\varkappa}$, $|\boldsymbol{k_2}| \leq \boldsymbol{\varkappa}$.
Proof First notice that

$$\boldsymbol{\mu}(k_1)\boldsymbol{\mu}(k_2) - \langle k_1, k_2\rangle = \tfrac{1}{2}(k_1 - k_2)^2 - \tfrac{1}{2}[\boldsymbol{\mu}(k_1) - \boldsymbol{\mu}(k_2)]^2 + m^2 \leq \tfrac{1}{2}(k_1 - k_2)^2 + m^2,$$ so the following inequality holds

$$\boldsymbol{\mu}(k_1)\boldsymbol{\mu}(k_2) - \langle k_1, k_2\rangle \leq c[\boldsymbol{\mu}(k_1 - k_2)]^2. \tag{9.64}$$

Using now the inequality (9.64) we can estimate the kernel of the operator $T_{0,\varkappa}^{2}(g)$ in (9.57) by

$$\text{const}\left|\hat{g}(k_1^1 - k_2^1, k_1^2 - k_2^2, k_1^3 - k_2^3)\left\{\frac{\mu(\boldsymbol{k}_1)\mu(\boldsymbol{k}_2) + \langle \boldsymbol{k}_1, \boldsymbol{k}_2\rangle + m^2}{\sqrt{\mu(\boldsymbol{k}_1)\mu(\boldsymbol{k}_2)}}\right\}\right| \leq \tag{9.65}$$

$$\text{const}|\hat{g}(k_1^1 - k_2^1, k_1^2 - k_2^2, k_1^3 - k_2^3)|[\boldsymbol{\mu}(k_1 - k_2)]^2[\boldsymbol{\mu}(k_1)\boldsymbol{\mu}(k_2)]^{-1/2}.$$

Note that (9.65) is square #-integrable since $\hat{g} \in S_{\text{fin}}^{\#}(^*\mathbb{R}_c^{\#3})$ i.e., $\hat{g}$ is rapidly #-decreasing. Similarly, we bound the kernel of $P_{\mu,\varkappa}^{2}(g), \mu = 1,2,3$ by using the following inequalities:

$$\left|(k_1^{\mu})\boldsymbol{\mu}(k_2) - (k_2^{\mu})\boldsymbol{\mu}(k_1)\right| \leq 2\boldsymbol{\mu}(k_1)\boldsymbol{\mu}(k_2) \leq 2[\boldsymbol{\mu}(k_1 - k_2)]^2 \text{, where } k_1^{\mu}k_2^{\mu} < 0, \tag{9.66}$$

$$\left|(k_1^{\mu})\boldsymbol{\mu}(k_2) - (k_2^{\mu})\boldsymbol{\mu}(k_1)\right| \leq \boldsymbol{\mu}(k_1)\boldsymbol{\mu}(k_2) - k_1^{\mu}k_2^{\mu} \text{, where } k_1^{\mu}k_2^{\mu} \geq 0. \tag{9.67}$$

The inequality (9.66) is clear, while from $\left|\left(k_1^\mu\right)\boldsymbol{\mu}(k_2)\right| \leq \boldsymbol{\mu}(k_1)\boldsymbol{\mu}(k_2)$ and $\left|\left(k_2^\mu\right)\boldsymbol{\mu}(k_1)\right| \geq \left|k_1^\mu k_2^\mu\right|$ one obtains (9.67) when $\left|\left(k_1^\mu\right)\boldsymbol{\mu}(k_2)\right| > \left|\left(k_2^\mu\right)\boldsymbol{\mu}(k_1)\right|$, and by symmetry it is valid in general case. Thus by (9.64) and (9.66)-(9.66), we get

$$\left|\left(k_1^\mu\right)\boldsymbol{\mu}(k_2) - \left(k_2^\mu\right)\boldsymbol{\mu}(k_1)\right| \leq \text{const}[\boldsymbol{\mu}(k_1 - k_2)]^2. \tag{9.68}$$

Therefore the kernels of $P^2_{\mu,\varkappa}(g)$ in (9.59) are bounded above by the functions

$$\text{const}\left|\hat{g}(k_1^1 - k_2^1, k_1^2 - k_2^2, k_1^3 - k_2^3)\left\{\frac{\left(k_1^\mu\right)\boldsymbol{\mu}(k_2) - \left(k_2^\mu\right)\boldsymbol{\mu}(k_1)}{\sqrt{\boldsymbol{\mu}(\boldsymbol{k}_1)\boldsymbol{\mu}(\boldsymbol{k}_2)}}\right\}\right| \leq$$

$$\text{const}|\hat{g}(k_1^1 - k_2^1, k_1^2 - k_2^3, k_1^3 - k_2^3)|[\boldsymbol{\mu}(k_1 - k_2)]^2[\boldsymbol{\mu}(k_1)\boldsymbol{\mu}(k_2)]^{-1/2}.$$

These functions are square #-integrable.

Proposition 9.6 The kernel $T^1_{0,\varkappa}(g)$ and the kernels of $P^1_{\mu,\varkappa}(g), \mu = 1,2,3$ have finite $M_{1,\varkappa}(\tau)$ and finite $M_{2,\varkappa}(\tau)$ $\tau \geq 1$ defined above in (9.25)-(9.26).

Proof Both the kernel of $T^1_{0,\varkappa}(g)$ and the kernels of $P^1_{\mu,\varkappa}(g), \mu = 1,2,3$ are dominated by the function

$$\text{const}|\hat{g}(k_1^1 - k_2^1, k_1^2 - k_2^2, k_1^3 - k_2^3)|[\boldsymbol{\mu}(k_1)\boldsymbol{\mu}(k_2)]^{1/2}.$$

Therefore

$$M_{1,\varkappa}(1) \leq \text{const}\sup_{\mathrm{k}\in{}^*\mathbb{R}_c^{\#3}}[\boldsymbol{\mu}(\mathrm{k})]^{-\mathbf{1}}\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}}|\hat{g}(k^1 - p^1, k^2 - p^2, k^3 - p^3)|[\boldsymbol{\mu}(k)\boldsymbol{\mu}(p)]^{\frac{1}{2}}d^{\#3}p\right).$$

Since $[\boldsymbol{\mu}(p)]^{\frac{1}{2}} \leq [\boldsymbol{\mu}(k)\boldsymbol{\mu}(k-p)]^{\frac{1}{2}}$ and since $\hat{g} \in S^\#_{\text{fin}}({}^*\mathbb{R}_c^{\#3})$ i.e., $\hat{g}$ is rapidly #-decreasing, one obtains

$$M_{1,\varkappa}(1) \leq \text{const}\sup_{k\in{}^*\mathbb{R}_c^{\#3}}\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}}|\hat{g}(k^1 - p^1, k^2 - p^2, k^3 - p^3)|[\boldsymbol{\mu}(k-p)]^{\frac{1}{2}}d^{\#3}p\right) \leq \text{const}. \tag{9.69}$$

Similarly, $M_{2,\varkappa}(\tau)$ is finite for $\tau \geq 1$. This completes the proof of the proposition 9.6 and the proof of Theorem 9.8.

Definition 9.11 We define now a specified gyperfinite momentum cutoff operator $\tilde{T}_{0,\varkappa}$ and we establish properties of $\tilde{T}_{0,\varkappa}$ that will be useful later. We assume that

$$g(x) = h^2(x), h(x) \geq 0, h \in S^\#_{\text{fin}}({}^*\mathbb{R}_c^{\#3}) \tag{9.70}$$

And we use the specified cutoff function

$$G_\varkappa(k_1, k_2) = c\left(Ext\text{-}\int_{|p|\leq\varkappa}\hat{h}(p - k_1)\hat{h}(p - k_2)d^{\#3}p\right) \tag{9.71}$$

$c \in {}^*\mathbb{R}^\#_{c,\text{fin}}$. For $\varkappa < {}^*\infty$, $G_\varkappa(k_1, k_2) \in S^\#_{\text{fin}}({}^*\mathbb{R}_c^{\#6})$, and

$$G_{^*\infty}(k_1, k_2) = \hat{g}(k_1 - k_2). \tag{9.72}$$

Definition 9.12 We define now the operators

$$\tilde{T}_{0,\varkappa}(g) = \tilde{T}^{(1)}_{0,\varkappa}(g) + \tilde{T}^{(2)}_{0,\varkappa}(g) \tag{9.73}$$

using replacing $\hat{g}(k_1 - k_2)$ in the kernels of $T_{0,\varkappa}^{(i)}(g)$, i =1,2 defined above in (9.56)-((9.57) by $G_\varkappa(k_1, k_2)$. If $\varkappa < {}^*\infty$, then the operators $\tilde{T}_{0,\varkappa}^{(i)}(g), i = 1{,}2$ have $L_2^\#$ kernels and so $\tilde{T}_{0,\varkappa}(g)$ is essentially self #-adjoint on $\mathcal{D}(H_{0,\varkappa}^\#)$ since vectors with finite or hyperfinite number of particles are #-analytic vectors. We set now

$$\tilde{T}_0(g) = \tilde{T}_{0,\varkappa}(g) + \delta\tilde{T}_{0,\varkappa}(g) \tag{9.74}$$

Defining $\delta\tilde{T}_{0,\varkappa}(g)$ and similarly we define $\delta\tilde{T}_{0,\varkappa}^{(i)}(g)$, $i = 1{,}2$.

Theorem 9.9. 1) The bounded operators

$$\delta\tilde{T}_{0,\varkappa}^{(1)}(g)\left(I + H_{0,\varkappa}^\#\right)^{-1} \text{ and } \left(I + H_{0,\varkappa}^\#\right)^{-1/2}\delta\tilde{T}_{0,\varkappa}^{(1)}(g)\left(I + H_{0,\varkappa}^\#\right)^{-1/2} \tag{9.75}$$

#-converge strongly to zero as $\varkappa \to {}^*\infty$.

2) The kernel of $\delta\tilde{T}_{0,\varkappa}^{(2)}(g)$ has $L_2^\#$ #-norm that is $O(\varkappa^{-\varepsilon})$ for all $\varepsilon < 1/2$. Thus

$$\left\|\delta\tilde{T}_{0,\varkappa}^{(2)}(g)\left(I + N_{0,\varkappa}\right)^{-1}\right\|_\# \leq O(\varkappa^{-\varepsilon}), \varepsilon < 1/2. \tag{9.76}$$

3) As $\varkappa \to {}^*\infty$

$$\left\|\left(I + H_{0,\varkappa}^\#\right)^{-1}\delta\tilde{T}_{0,\varkappa}(g)\left(I + H_{0,\varkappa}^\#\right)^{-1}\right\|_\# \leq O(\varkappa^{-1}). \tag{9.77}$$

Proof 1) Note that the kernel of $\delta\tilde{T}_{0,\varkappa}^{(1)}(g)$ has bounded #-norms (9.25)-(9.26) for $\tau = 1$, and these bounds are uniform for $\varkappa \leq {}^*\infty$. Thus the operators (9.75) are uniformly bounded, and it is sufficient to prove #-convergence on a total set of vectors, namely vectors in $\mathcal{D}(H_{0,\varkappa}^\#)$ with exactly n particles. It is sufficient to prove the strong #-convergence of $\delta\tilde{T}_{0,\varkappa}^{(1)}(g)$ on this domain. For $\psi \in \mathcal{D}(H_{0,\varkappa}^\#)$ as $n \in {}^*\mathbb{N}$ particle vector in $\mathcal{D}(H_{0,\varkappa}^\#)$ we obtain

$$\left|\left(\delta\tilde{T}_{0,\varkappa}^{(1)}\psi\right)(k_1, \dots, k_n)\right|^2 =$$

$$\left|Ext\text{-}\sum_{j=1}^n Ext\text{-}\int_{|p|>\varkappa} d^{\#3}p Ext\text{-}\int_{|q|\leq\varkappa} d^{\#3}q \overline{\hat{h}(p - k_j)}\,\hat{h}(p - q)\left\{\frac{\mu(k_j)\mu(q) + \langle k_j, q\rangle + m^2}{\sqrt{\mu(k_j)\mu(q)}}\right\}\times\right. \tag{9.78}$$

$$\left.\times\,\psi(k_1, \dots, k_{j-1}, q, \dots, k_n)\right|^2 \leq$$

$$\leq \text{const}\left\{Ext\text{-}\sum_{j=1}^n Ext\text{-}\int_{|p|>\varkappa} d^{\#3}p Ext\text{-}\int_{|q|\leq\varkappa} d^{\#3}q \left|\hat{h}(p - k_j)\hat{h}(p - q)\right|\left|\sqrt{\mu(k_j)\mu(q)}\right|\psi(k_1, \dots, k_{j-1}, q, \dots, k_n)\right|\right\}^2.$$

The right side of (9.78) is monotonically decreasing as $\varkappa \to {}^*\infty$, since

$$\text{const}\sqrt{\mu(q)} \leq \sqrt{\mu(p - k)}\sqrt{\mu(p - q)}\sqrt{\mu(k)}$$

and since $\hat{h}$ is #-rapidly decreasing, i.e., $\hat{h} \in S_{\text{fin}}^\#({}^*\mathbb{R}_c^{\#3})$

$$w(k, q) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} d^{\#3}p\left|\hat{h}(p - k)\hat{h}(p - q)\right|\sqrt{\mu(k)}\sqrt{\mu(q)}$$

is a kernel with finite #-norms (9.25)-(9.26) for $\tau = 1$. But the right side of (9.78) has the form $(W|\psi|)^2$, where operator W is given by

$$W = Ext\text{-}\int_{|p_1|>\varkappa} d^{\#3}\, p_1 \ldots Ext\text{-}\int_{|p_s|>\varkappa} d^{\#3}\, p_s \times \tag{9.79}$$

$$\left[Ext\text{-}\int_{|k_1|\leq\varkappa} d^{\#3} k_1 \ldots Ext\text{-}\int_{|k_r|\leq\varkappa} d^{\#3}\, p_1 a^{\dagger}(k_1) \cdots a^{\dagger}(k_r) w(k_1,\ldots,k_r;p_1,\ldots,p_s) a\,(p_1)\cdots a(p_s)\right],$$

here $w(k_1,\ldots,k_r;p_1,\ldots,p_s)$ is a #-measurable kernel and $|\psi| \in \mathcal{D}(H^{\#}_{0,\varkappa})$ since $\psi \in \mathcal{D}(H^{\#}_{0,\varkappa})$. Hence by Proposition 9.2, the function $W|\psi| \in L^{\#}_2$ so that (9.78) is uniformly bounded by an $L^{\#}_1$ function. By the generalized dominated convergence theorem, the integral in RHS of (9.78) tends to zero as $\varkappa \to {}^*\infty$, which completes the proof of strong #-convergence.

2) Note that the kernel of $\delta\tilde{T}^2_{0,\varkappa}(g)$ is bounded above by

$$w(k,p) = \text{const}\mu^2(k-p)[\mu(k)\mu(p)]^{-1/2} Ext\text{-}\int_{|q|>\varkappa} \left|\hat{h}(q-k)\hat{h}(q-p)\right| d^{\#3}\, q. \tag{9.80}$$

By (9.67) we obtain

$$\begin{gathered}[\mu(k)]^{-\varepsilon} \leq \text{const}[\mu(p)]^{-\varepsilon}[\mu(k-p)]^{\varepsilon},\\ [\mu(k-p)]^{3+\varepsilon} \leq \text{const}[\mu(q-k)]^{3+\varepsilon}[\mu(q-p)]^{3+\varepsilon},\\ [\mu(k)]^{-\frac{1}{2}+\varepsilon} \leq \text{const}[\mu(q)]^{-\frac{1}{2}+\varepsilon}[\mu(q-k)]^{-\frac{1}{2}+\varepsilon}.\end{gathered} \tag{9.81}$$

From (9.80) and (9.81) we obtain

$$|w(k,p)| \leq \text{const}[\mu(k-p)]^{-1}[\mu(p)]^{-\frac{1}{2}-\varepsilon} \times \tag{9.82}$$

$$\times \left\{Ext\text{-}\int_{|q|>\varkappa} \left|\hat{h}(q-k)\hat{h}(q-p)\right| [\mu(q-k)]^{-\frac{7}{2}}[\mu(q-p)]^{3+\varepsilon}[\mu(q)]^{-\frac{1}{2}+\varepsilon} d^{\#3}\, q\right\} \leq$$

$$\leq \text{const}[\mu(k)]^{-\frac{1}{2}+\varepsilon}[\mu(k-p)]^{-1}[\mu(p)]^{-\frac{1}{2}-\varepsilon} \times$$

$$\times \left\{Ext\text{-}\int_{|q|>\varkappa} \left|\hat{h}(q-k)\hat{h}(q-p)\right| [\mu(q-k)]^{-\frac{7}{2}}[\mu(q-p)]^{3+\varepsilon} d^{\#3} q\right\}.$$

From (9.82) by using the generalizef Schwarz inequality in q and the rapid decrease of $\hat{h}$ to bound the integral over q by a constant we get

$$|w(k,p)| \leq \text{const}O\left(|\varkappa|^{-\frac{1}{2}+\varepsilon}\right)[\mu(k-p)]^{-1}[\mu(p)]^{-\frac{1}{2}-\varepsilon}. \tag{9.83}$$

Note that RHS of (9.83) for any $\varepsilon > 0$ obviously in $L^{\#}_2$ and has an $L^{\#}_2$ #-norm that is $O\left(|k|^{-\frac{1}{2}+\varepsilon}\right)$ for any $\varepsilon > 0$. This proves statement (2) of the theorem.

3) The proof of this estimate is carried out by estimates on the kernels of $\delta\tilde{T}^{(1)}_{0,\varkappa}(g)$ and $\delta\tilde{T}^{(2)}_{0,\varkappa}(g)$. The estimate on the kernel of $\delta\tilde{T}^{(2)}_{0,\varkappa}(g)$ is similar to the above. Now we estimate the $L^{\#}_2$ #-norm of $w(k,p)[\mu(k)]^{-\frac{1}{2}}[\mu(p)]^{-\frac{1}{2}}$ for the function $w(k,p)$ of (9.80). We then get an $L^{\#}_2$ #-norm that is $O\left(|k|^{-\frac{3}{2}+\varepsilon}\right)$, and by Proposition 9.3 in the case $\alpha = 2$, $\beta = 0, \tau = 1$ for the creation part or α = 0, β = 2, τ = 1 for the annihilation part we obtain

$$\left\|\left(I + H^{\#}_{0,\varkappa}\right)^{-1} \delta\tilde{T}^{(2)}_{0,\varkappa}(g)\left(I + H^{\#}_{0,\varkappa}\right)^{-1}\right\|_{\#} \leq O\left(|\varkappa|^{-\frac{3}{2}+\varepsilon}\right). \tag{9.84}$$

The estimate on the kernel of $\delta\tilde{T}^{(2)}_{0,\varkappa}(g)$ will be made with the #-norm (9.25). We find that $M_{1,\varkappa}(\tau=2)\sim O(|k|^{-1})$, so that by Proposition 9.3 following it

$$\left\|H^{\#-1}_{0,\varkappa}\delta\tilde{T}^{(2)}_{0,\varkappa}(g)H^{\#-1}_{0,\varkappa}\right\|_{\#}\leq O(|\varkappa|^{-1}). \quad (9.85)$$

We now prove this estimate on the kernel of $\delta\tilde{T}^{(1)}_{0,\varkappa}(g)$. The kernel of $\delta\tilde{T}^{(1)}_{0,\varkappa}(g)$ is dominated by the function

$$w_{>\varkappa}(k,p)=[\mu(k)\mu(q)]^{1/2}Ext\text{-}\int_{|p|>\varkappa}\left|\hat{h}(p-k)\hat{h}(p-q)\right|d^{\#3}p.$$

Note that

$$Ext\text{-}\int w_{>\varkappa}(k,p)d^{\#3}q\leq$$

$$\text{const}\mu^2(k)Ext\text{-}\int_{|p|>\varkappa}\left|\hat{h}(p-k)\hat{h}(p-q)\right|[\mu(p-q)]^{\frac{3}{2}}[\mu(p)]^{-1}d^{\#3}p\,d^{\#3}q\leq$$

$$\text{const}[\mu(k)]^{-1}\mu^2(k)Ext\text{-}\int_{|p|>\varkappa}\left|\hat{h}(p-k)\hat{h}(p-q)\right|[\mu(p-k)]^{\frac{3}{2}}[\mu(p-q)]^{-1}d^{\#3}p\,d^{\#3}q.$$

By the generalized Schwarz inequality we get

$$Ext\text{-}\int w_{>\varkappa}(k,p)d^{\#3}q\leq\text{const}[\mu(k)]^{-1}\mu^2(k).$$

Thus finally we obtain the inequality

$$\sup_{\varkappa}\left[\mu^2(k)Ext\text{-}\int w_{>\varkappa}(k,p)d^{\#3}q\right]\leq O(|\varkappa|^{-1})$$

which completes the proof of (9.85) and the proof of the theorem.

Definition 9.13 It is convenient to write $\tilde{T}_{0,\varkappa}(g)$ and $\delta\tilde{T}_{0,\varkappa}(g)$ in another form. We define the following operators with $L^{\#}_2$ kernels on the domain $\mathcal{D}\left(H^{\#}_{0,\varkappa}\right)$

$$B_1(p)=\frac{1}{(2\pi)^3}\left\{Ext\text{-}\int_{|k|\leq\varkappa}\hat{h}(p-k)[\mu(k)]^{\frac{1}{2}}a(k)d^{\#3}k\right\}, \quad (9.86)$$

$$B_2(p)=\frac{1}{(2\pi)^3}\left\{Ext\text{-}\int_{|k|\leq\varkappa}\hat{h}(p-k)|k|[\mu(k)]^{-\frac{1}{2}}a(k)d^{\#3}k\right\}, \quad (9.87)$$

$$B_3(p)=\frac{1}{(2\pi)^3}\left\{Ext\text{-}\int_{|k|\leq\varkappa}\hat{h}(p-k)m[\mu(k)]^{-\frac{1}{2}}a(k)d^{\#3}k\right\}. \quad (9.88)$$

Then for $g=h^2$, and $\varkappa\leq{}^*\infty$, on the domain $\mathcal{D}\left(H^{\#}_{0,\varkappa}\right)$ we get

$$\tilde{T}^{(1)}_{0,\varkappa}(g)=\frac{1}{2}Ext\text{-}\int_{|p|\leq\varkappa}\sum_{i=1}^{3}B_i^*(p)B_i(p)d^{\#3}p, \quad (9.89)$$

$$\delta\tilde{T}^{(1)}_{0,\varkappa}(g)=\frac{1}{2}Ext\text{-}\int_{|p|>\varkappa}\sum_{i=1}^{3}B_i^*(p)B_i(p)d^{\#3}p. \quad (9.90)$$

Definition 9.14 We also define now following operators $A_i(p), i=1,2,3$ on the domain $\mathcal{D}\left(N^{1/2}_{0,\varkappa}\right)$ by

$$A_i(p)=\frac{1}{2}\{B_i(p)+B_i^*(-p)\}. \quad (9.91)$$

Remark 9.4 Note that

$$[A_i(p), A_i^*(p)]\mathcal{D}(N_{0,\varkappa}) = 0. \tag{9.92}$$

The operators $A_i(p), i = 1,2,3$ are related to the operator $\tilde{T}_{0,\varkappa}(g)$ without Wick ordering.

Definition 9.15 For $\varkappa < {}^*\infty$ we define

$$\hat{T}_{0,\varkappa}(g) = \sum_{i=1}^{3}\left[Ext\text{-}\int_{|p|>\varkappa} A_i^*(p)A_i(p)d^{\#3}p\right] \geq 0. \tag{9.93}$$

Direct calculation shows that

$$\tilde{T}_{0,\varkappa}(g) = \hat{T}_{0,\varkappa}(g) - \langle\Omega_0, \hat{T}_{0,\varkappa}(g)\Omega_0\rangle. \tag{9.94}$$

Here Ω_0 is the no-particle vector. Since

$$\langle\Omega_0, \hat{T}_{0,\varkappa}(g)\Omega_0\rangle = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} G_\varkappa(p,p)\mu(p)d^{\#3}p. \tag{9.95}$$

Here $G_\varkappa(k_1,k_2)$ is defined in (9.71), we have for $\varkappa < {}^*\infty$ that $\tilde{T}_{0,\varkappa}(g)$ is bounded from below and

$$\tilde{T}_{0,\varkappa}(g) + Ext\text{-}\int G_\varkappa(p,p)\,\mu(p)d^{\#3}p \geq 0. \tag{9.96}$$

Theorem 9.10 Let ε > 0 and g, g_1 be positive as mentioned above in (9.70). Then there is a finite constant b such that on $\mathcal{D}(H_{0,\varkappa}^{\#}) \times \mathcal{D}(H_{0,\varkappa}^{\#})$

$$\delta\tilde{T}_{0,\varkappa}^{(1)}(g) \geq 0, \text{ for all } \ 0 \leq \varkappa \leq {}^*\infty \tag{9.97}$$

$$\varepsilon N_{0,\varkappa} + \tilde{T}_0(g) + b \geq 0, \tag{9.98}$$

$$\varepsilon N_{0,\varkappa} + \tilde{T}_{I,\varkappa}(g) + b \geq 0, \tag{9.99}$$

$$\varepsilon N_{0,\varkappa} + \tilde{T}_0(g) + \tilde{T}_{I,\varkappa}(g) + b \geq 0. \tag{9.100}$$

The inequalities (9.97)-(9.100) are also valid with $H_{0,\varkappa}^{\#}$ in place of $N_{0,\varkappa}$.

Proof The positivity of $\delta\tilde{T}_{0,\varkappa}^{(1)}(g)$ is a consequence of the representation (9.89). In order to prove (9.98) we let

$$\varepsilon N_{0,\varkappa} + \tilde{T}_0(g) = \varepsilon N_{0,\varkappa} + \delta\tilde{T}_{0,\varkappa}^{(2)}(g) + \delta\tilde{T}_{0,\varkappa}^{(1)}(g) + \tilde{T}_{0,\varkappa}(g).$$

Since $\tilde{T}_{0,\varkappa}^{(1)}(g)$ is positive by (9.97) and $\tilde{T}_{0,\varkappa}(g)$ is bounded from below by (9.96), we need only prove that $\varepsilon N_{0,\varkappa} + \delta\tilde{T}_{0,\varkappa}^{(2)}(g)$ is bounded from below. By Theorem 9.9 (2), $L_2^{\#}$ #-norm of the kernel of $\delta\tilde{T}_{0,\varkappa}^{(2)}(g)$ is $O(\varkappa^{-1/2})$ and therefore

$$\left\|\left(I + N_{0,\varkappa}\right)^{-1/2}\delta\tilde{T}_{0,\varkappa}^{(2)}(g)\left(I + N_{0,\varkappa}\right)^{-1/2}\right\|_{\#} \leq O\left(|\varkappa|^{-\frac{1}{2}}\right). \tag{9.101}$$

For sufficiently large $\varkappa$, (9.101) is less than ε. Hence $\varepsilon N_{0,\varkappa} + \delta\tilde{T}_{0,\varkappa}^{(2)}(g) + \varepsilon \geq 0$ and (9.98) is proved.

§10. SECOND ORDER ESTIMATES

In this section we consider a second order estimate on operators of the form

$$H_{0,\varkappa}^{\#} + T_{0,\varkappa}(g_0) + T_{I,\varkappa}(g_1). \tag{10.1}$$

Here g_0 and g_1 are spatial cutoffs satisfying (9.70). For $\lambda(\varphi^4)_2$ model such an estimate was proved in [18].

Theorem 10.1 Let $c > 1$. Then there is a constant $b < {}^*\infty$ such that for all $\beta,\ 0 \le \beta \le 1$,

$$\left(H_{0,\varkappa}^{\#} + I\right)^2 + \beta^2\left[T_{0,\varkappa}(g_0)\right]^2 + \left[T_{I,\varkappa}(g_1)\right]^2 \le c\left[H_{0,\varkappa}^{\#} + \beta T_{0,\varkappa}(g_0) + T_{I,\varkappa}(g_1) + b\right]^2, \tag{10.2}$$

as a bilinear form on $\mathcal{D}\left(H_{0,\varkappa}^{\#2}\right) \times \mathcal{D}\left(H_{0,\varkappa}^{\#2}\right)$.

Proposition 10.1 Let $c > 1$ and $\varepsilon > 0$. Then there is a constant $b < {}^*\infty$ such that

$$T_{0,\varkappa}(g_0)H_{0,\varkappa}^{\#} + H_{0,\varkappa}^{\#}T_{0,\varkappa}(g_0) \ge -\varepsilon H_{0,\varkappa}^{\#2} - b \tag{10.3}$$

And for all $\beta,\ 0 \le \beta \le 1$,

$$\left(H_{0,\varkappa}^{\#} + I\right)^2 + \beta^2\left[T_{0,\varkappa}(g_0)\right]^2 \le c\left[H_{0,\varkappa}^{\#} + \beta T_{0,\varkappa}(g_0) + b\right]^2 \tag{10.4}$$

as bilinear forms on $\mathcal{D}\left(H_{0,\varkappa}^{\#}\right) \times \mathcal{D}\left(H_{0,\varkappa}^{\#}\right)$.

Proof First notice that

$$\left[H_{0,\varkappa}^{\#} + \beta T_{0,\varkappa}(g_0) + b\right]^2 = \left(H_{0,\varkappa}^{\#} + I\right)^2 + \beta^2 T_{0,\varkappa}^2(g_0) + \tag{10.5}$$

$$+2(b-1)\left(H_{0,\varkappa}^{\#} + I + \beta_1 T_{0,\varkappa}(g_0) + \tfrac{1}{4}(b-1)\right) + \beta\left(H_{0,\varkappa}^{\#}T_{0,\varkappa}(g_0) + T_{0,\varkappa}(g_0)H_{0,\varkappa}^{\#}\right) + \tfrac{1}{2}(b-1)^2,$$

where $\beta_1 = \beta b(b-1)^{-1}$. For b sufficiently large, $H_{0,\varkappa}^{\#} + \beta_1 T_{0,\varkappa}(g_0) + \frac{b}{4}$, for the proof of Theorem 9.10 gives an estimate that is uniform for $0 \le \beta_1 \le 2$. Hence it is sufficient to prove (10.3) to establish (10.4), for if

$$H_{0,\varkappa}^{\#}T_{0,\varkappa}(g_0) + T_{0,\varkappa}(g_0)H_{0,\varkappa}^{\#} \ge -4\varepsilon H_{0,\varkappa}^{\#2} - \gamma, \tag{10.6}$$

we have choose ε and b such that $4\varepsilon \le 1$and $\frac{1}{2}b^2 \ge \gamma - 1$. We write now

$$T_0 = T_{0,\varkappa} + \delta T_{0,\varkappa}^{(2)} + \delta T_{I,\varkappa}^{(1)}. \tag{10.7}$$

We prove (10.6) separately for each term in (10.7). Using (9.91)-(9.96) we obtain

$$H_{0,\varkappa}^{\#}T_{0,\varkappa} + T_{0,\varkappa}H_{0,\varkappa}^{\#} = -2H_{0,\varkappa}^{\#}\left[Ext\text{-}\int G_{\varkappa}(k,k)\,\mu(k)d^{\#3}k\right] + H_{0,\varkappa}^{\#}\hat{T}_{0,\varkappa} + \hat{T}_{0,\varkappa}H_{0,\varkappa}^{\#} \ge \tag{10.8}$$

$$\ge -\varepsilon H_{0,\varkappa}^{\#2} - \text{const} + H_{0,\varkappa}^{\#}\hat{T}_{0,\varkappa} + \hat{T}_{0,\varkappa}H_{0,\varkappa}^{\#} =$$

$$= -\varepsilon H_{0,\varkappa}^{\#2} - \text{const} + 2\sum_{i=1}^{3}\left[Ext\text{-}\int_{|k|>\varkappa} A_i^*(p)H_{0,\varkappa}^{\#}A_i(p)d^{\#3}p\right] +$$

$$+\sum_{i=1}^{3}\left[Ext\text{-}\int_{|k|>\varkappa}\left[H_{0,\varkappa}^{\#}, A_i^*(p)\right] A_i(p)d^{\#3}p\right] + \sum_{i=1}^{3}\left[Ext\text{-}\int_{|k|>\varkappa} A_i^*(p)\left[H_{0,\varkappa}^{\#}, A_i(p)\right] d^{\#3}p\right].$$

Note that the kernels occurring in $A_i(p)$, $A_i^*(p)$, $[H_{0,\varkappa}^{\#}, A_i(p)]$, $[H_{0,\varkappa}^{\#}, A_i^*(p)]$ all belong to $S_{\text{fin}}^{\#}({}^*\mathbb{R}_c^{\#3})$ for fixed p. The $L_2^{\#}$ #-norms of these kernels are uniformly bounded on any #-compact set in $p \in {}^*\mathbb{R}_c^{\#3}$. Thus each of these operators is defined on domain $\mathcal{D}(N_{0,\varkappa}^{1/2})$ and maps $\mathcal{D}(H_{0,\varkappa}^{\#})$ into $\mathcal{D}(H_{0,\varkappa}^{\#1/2})$. As a consequence, each term in (10.8) is well defined. Since

$$Ext\text{-}\int_{|k|>\varkappa} A_i^*(p) H_{0,\varkappa}^{\#} A_i(p) d^{\#3}p \geq 0 \tag{10.9}$$

one needs only bound the commutator terms. By the above remarks on $L_2^{\#}$ #-norms of the kernels, the operators

$$\left(H_{0,\varkappa}^{\#}+I\right)^{-1/2}\left\{Ext\text{-}\int_{|k|>\varkappa}[H_{0,\varkappa}^{\#}, A_i^*(p)]\ A_i(p)d^{\#3}p + Ext\text{-}\int_{|k|>\varkappa} A_i^*(p)[H_{0,\varkappa}^{\#}, A_i(p)]\ d^{\#3}p\right\}\left(H_{0,\varkappa}^{\#}+I\right)^{-1/2}$$

are bounded for any $\varkappa < {}^*\infty$, so that

$$\sum_{i=1}^{3}\left[Ext\text{-}\int_{|k|>\varkappa}[H_{0,\varkappa}^{\#}, A_i^*(p)]\ A_i(p)d^{\#3}p\right] + \sum_{i=1}^{3}\left[Ext\text{-}\int_{|k|>\varkappa} A_i^*(p)[H_{0,\varkappa}^{\#}, A_i(p)]\ d^{\#3}p\right] \geq \tag{10.10}$$

$$\geq -\text{const}\left(H_{0,\varkappa}^{\#}+I\right) \geq -\varepsilon H_{0,\varkappa}^{\#} - \text{const}.$$

Thus by (10.8)-(10.10) we obtain

$$H_{0,\varkappa}^{\#} T_{0,\varkappa} + T_{0,\varkappa} H_{0,\varkappa}^{\#} \geq -\varepsilon H_{0,\varkappa}^{\#} - \text{const}\,, \tag{10.11}$$

which is the contribution of $T_{0,\varkappa}$ to (10.6). By Theorem 9.9 (2), the kernel of $\delta T_{0,\varkappa}^{(2)}$ has $L_2^{\#}$ #-norm that is $O(\varkappa^{-1/2})$ Hence

$$\left\|\left(H_{0,\varkappa}^{\#}+I\right)^{-1}\left(H_{0,\varkappa}^{\#}\delta T_{0,\varkappa}^{(2)} + \delta T_{0,\varkappa}^{(2)} H_{0,\varkappa}^{\#}\right)\left(H_{0,\varkappa}^{\#}+I\right)^{-1}\right\|_{\#} \leq O(\varkappa^{-1/2})$$

and for sufficiently large $\varkappa \in {}^*\mathbb{R}_c^{\#}$ we get

$$H_{0,\varkappa}^{\#}\delta T_{0,\varkappa}^{(2)} + \delta T_{0,\varkappa}^{(2)} H_{0,\varkappa}^{\#} \geq -\varepsilon\left(H_{0,\varkappa}^{\#2}+I\right),$$

which is the contribution of $\delta T_{0,\varkappa}^{(2)}$ to (10.6). Finally, for $\delta T_{0,\varkappa}^{(1)}$ we write

$$H_{0,\varkappa}^{\#}\delta T_{0,\varkappa}^{(1)} + \delta T_{0,\varkappa}^{(1)} H_{0,\varkappa}^{\#} = 2H_{0,\varkappa}^{\#1/2}\delta T_{0,\varkappa}^{(1)} H_{0,\varkappa}^{\#1/2} + \left[H_{0,\varkappa}^{\#1/2}, [H_{0,\varkappa}^{\#1/2}, \delta T_{0,\varkappa}^{(1)}]\right] \tag{10.12}$$

By Theorem 9.10, the first term on the right of (10.12) is positive, and we now study the double commutator. Since neither $\delta T_{0,\varkappa}^{(1)}$ nor $H_{0,\varkappa}^{\#1/2}$ changes the particle number, we restrict attention to vectors $\psi \in \mathcal{D}(H_{0,\varkappa}^{\#2})$ with exactly $n \in {}^*\mathbb{N}$ particles. Let $\delta t(k_1, k_2)$ be the kernel of $\delta T_{0,\varkappa}^{(1)}(g)$, then

$$\langle\psi, \left[H_{0,\varkappa}^{\#1/2}, [H_{0,\varkappa}^{\#1/2}, \delta T_{0,\varkappa}^{(1)}(g)]\right]\psi\rangle = \tag{10.13}$$

$$n\left(Ext\text{-}\int \bar{\psi}(k_1,\ldots,k_n)\,\psi(p,k_2,\ldots,k_n)\delta t(k_1,p)\lambda(p,k_1,\ldots,k_n)d^{\#3}p d^{\#3}k_1 \ldots d^{\#3}k_n\right),$$

where

$$\lambda(p,k_1,\ldots,k_n) = \tag{10.14}$$

$$\left[(Ext\text{-}\textstyle\sum_{i=1}^{n}\mu(k_i))^{1/2} - (\mu(p) + Ext\text{-}\sum_{i=2}^{n}\mu(k_i))^{1/2}\right]^2 =$$

$$= (Ext\text{-}\sum_{i=1}^{n}\mu(k_i))\left[\left(1+\frac{\mu(p)-\mu(k_1)}{(Ext\text{-}\sum_{i=1}^{n}\mu(k_i))^{1/2}}\right)^{1/2}-1\right]^2.$$

If $\mu(p)-\mu(k_1)\geq 0$, we use the inequality $(1+x)^{1/2}-1\leq\frac{1}{2}x$, for $x\geq 0$ to prove the inequality

$$\lambda(p,k_1,\dots,k_n)\leq\frac{1}{4}\big(\mu(p)-\mu(k_1)\big)^2. \tag{10.15}$$

Since $\lambda(p,k_1,\dots,k_n)=\lambda(k_1,p,\dots,k_n)$, the bound (10.15) is valid for all $p,k_1,\dots,k_n$. Since $|\mu(p)-\mu(k_1)|\leq$ const$\mu(p-k_1)$ we get the inequality

$$\lambda(p,k_1,\dots,k_n)\leq \mathrm{const}\times\mu^2(p-k_1). \tag{10.16}$$

Suppressing the variables $k_2,\dots,k_n$ in (10.13) we have by (19.16), the Schwarz inequality, and the symmetry of $|w(k,p)|$,

$$\langle\psi,\left[H_{0,\varkappa}^{\#1/2},\left[H_{0,\varkappa}^{\#1/2},\delta T_{0,\varkappa}^{(1)}(g)\right]\right]\psi\rangle\leq \mathrm{const}\times n\big(Ext\text{-}\textstyle\int|\psi^2(k_1)\delta t(k_1,p)|\mu^2(k_1-p)d^{\#3}pd^{\#3}k_1\big),$$

where the kernel $\delta t(k_1,p)$ is dominated by const $\times\,|\hat{g}(k_1-p)|[\mu(k_1)\mu(p)]^{1/2}$ and therefore we have the estimate

$$Ext\text{-}\textstyle\int|\delta t(k_1,p)|\mu^2(k_1-p)d^{\#3}p\leq\mathrm{const}\times\mu(k_1),$$

and so, by Proposition 9.1 we obtain

$$\langle\psi,\left[H_{0,\varkappa}^{\#1/2},\left[H_{0,\varkappa}^{\#1/2},\delta T_{0,\varkappa}^{(1)}(g)\right]\right]\psi\rangle\leq\mathrm{const}\times n\big(Ext\text{-}\textstyle\int|\psi^2(k_1)|\mu(k_1)\,d^{\#3}k_1\big)=\mathrm{const}\times\langle\psi,H_{0,\varkappa}^{\#}\psi,\rangle\leq$$

$$\leq\langle\psi,\big(\varepsilon H_{0,\varkappa}^{\#2}+\mathrm{const}\big)\psi,\rangle.$$

Thus for (10.12) we obtain

$$H_{0,\varkappa}^{\#}\delta T_{0,\varkappa}^{(1)}+\delta T_{0,\varkappa}^{(1)}H_{0,\varkappa}^{\#}\geq 2H_{0,\varkappa}^{\#1/2}\delta T_{0,\varkappa}^{(1)}(g)H_{0,\varkappa}^{\#1/2}-\varepsilon H_{0,\varkappa}^{\#2}-\mathrm{const}\geq-\varepsilon H_{0,\varkappa}^{\#2}-\mathrm{const}.$$

This establishes (10.6) as inequality on domain $\mathcal{D}\big(H_{0,\varkappa}^{\#2}\big)\times\mathcal{D}\big(H_{0,\varkappa}^{\#2}\big)$,it extends by #-closure to $\mathcal{D}\big(H_{0,\varkappa}^{\#}\big)\times\mathcal{D}\big(H_{0,\varkappa}^{\#}\big)$, and this completes the proof of the proposition.

Remark 10.1 Note that these methods can be used to prove that $W(\tau,n)=\big(adH_{0,\varkappa}^{\#\tau}\big)\left(\delta T_{0,\varkappa}^{(1)}(g)\right),\tau\leq 1,n\in{}^*\mathbb{N}$ is an operator on $\mathcal{D}\big(H_{0,\varkappa}^{\#}\big)$, and that $W(\tau,n)H_{0,\varkappa}^{\#-1}$ is bounded.

Proposition 10.2 Let $\varepsilon>0$ and $\varkappa<{}^*\infty$. Then there exists a constant $b<{}^*\infty$ such that on $\mathcal{D}\big(H_{0,\varkappa}^{\#2}\big)\times\mathcal{D}\big(H_{0,\varkappa}^{\#2}\big)$

$$T_{I,\varkappa}T_{0,\varkappa}+T_{I,\varkappa}T_{0,\varkappa}\geq-\varepsilon\left(H_{0,\varkappa}^{\#2}+T_{I,\varkappa}^2\right)-b. \tag{10.17}$$

Proof Using (9.91)-(9.95), we obtain the identity

$$T_{I,\varkappa}T_{0,\varkappa}+T_{I,\varkappa}T_{0,\varkappa}=-\mathrm{const}T_{I,\varkappa}+T_{I,\varkappa}\hat{T}_{0,\varkappa}+T_{I,\varkappa}\hat{T}_{0,\varkappa}= \tag{10.18}$$

$$=-\mathrm{const}T_{I,\varkappa}+\sum_{i=1}^{3}\left[Ext\text{-}\textstyle\int_{|p|\leq\varkappa}A_i^*(p)T_{I,\varkappa}A_i(p)d^{\#3}p+Ext\text{-}\int_{|p|\leq\varkappa}A_i(p)T_{I,\varkappa}A_i^*(p)d^{\#3}p\right]+$$

$$+\frac{1}{2}\sum_{i=1}^{3}\left[Ext\text{-}\textstyle\int_{|p|\leq\varkappa}\left[A_i(p),\left[A_i^*(p),T_{I,\varkappa}\right]\right]d^{\#3}p\right]+\frac{1}{2}\sum_{i=1}^{3}\left[Ext\text{-}\int_{|p|\leq\varkappa}\left[A_i^*(p),\left[A_i(p),T_{I,\varkappa}\right]\right]d^{\#3}p\right].$$

Note that (10.18) follows from the identity

$$B(A^*A+AA^*)+(A^*A+AA^*)B=2ABA^*+2A^*BA+\left[A,[A^*,B]\right]+\left[A^*,[A,B]\right].$$

We obtain a lower bound on each term on the right side of (10.18). Clearly for any $\varepsilon_1>0$, we have

$$-\text{const}T_{I,\varkappa}\geq-\varepsilon_1 T_{I,\varkappa}^2-\text{const}.$$

Furthermore, by (9.99), for $\varepsilon_2>0$,

$$A_i^*(p)T_{I,\varkappa}A_i(p)+A_i(p)T_{I,\varkappa}A_i^*(p)\geq$$

$$\geq-\text{const}A_i^*(p)A_i(p)-\varepsilon_2\left\{A_i^*(p)N_{0,\varkappa}A_i(p)+A_i(p)N_{0,\varkappa}A_i^*(p)\right\}. \quad (10.19)$$

By the remarks following (10.10) on the $L_2^{\#}$ nature of the kernels occurring in $A_i(p)$, we have for $|p|\leq\varkappa<{}^*\infty$, and any $\varepsilon_3>0$,

$$-\text{const}\left(A_i^*(p)A_i(p)\right)\geq-\text{const}\left(N_{0,\varkappa}+I\right)\geq-\varepsilon_3 H_{0,\varkappa}^{\#2}-\text{const}, \quad (10.20)$$

$$-\varepsilon_2\left\{A_i^*(p)N_{0,\varkappa}A_i(p)+A_i(p)N_{0,\varkappa}A_i^*(p)\right\}\geq-\varepsilon_2\text{const}\left(N_{0,\varkappa}+I\right)^2\geq-\varepsilon_2\text{const}\left(H_{0,\varkappa}^{\#}+I\right)^2. \quad (10.21)$$

Thus we can choose $\varepsilon_1,\varepsilon_2,\varepsilon_3$ sufficiently small so that after summing (10.19)-(10.21) over i and integrating over $|p|\leq\varkappa$ we obtain for (10.18),

$$T_{I,\varkappa}T_{0,\varkappa}+T_{I,\varkappa}T_{0,\varkappa}\geq-\tfrac{1}{2}\varepsilon\left(H_{0,\varkappa}^{\#2}+T_{I,\varkappa}^2\right)-\text{const}+$$

$$+\tfrac{1}{2}\sum_{i=1}^{3}\left[Ext\text{-}\int_{|p|\leq\varkappa}\left[A_i(p),\left[A_i^*(p),T_{I,\varkappa}\right]\right]d^{\#3}p\right]+\tfrac{1}{2}\sum_{i=1}^{3}\left[Ext\text{-}\int_{|p|\leq\varkappa}\left[A_i^*(p),\left[A_i(p),T_{I,\varkappa}\right]\right]d^{\#3}p\right]. \quad (10.22)$$

Note that $\left[A_i(p),\left[A_i^*(p),T_{I,\varkappa}\right]\right]$ and its #-adjoint are sums of second order monomials in creation and annihilation operators with $L_2^{\#}$ kernels that have uniformly bounded $L_2^{\#}$ #-norms for $|p|\leq\varkappa$, in this 3-dimensional region of p we get

$$\left[A_i(p),\left[A_i^*(p),T_{I,\varkappa}\right]\right]+\left[A_i^*(p),\left[A_i(p),T_{I,\varkappa}\right]\right]\geq-\text{const}\left(N_{0,\varkappa}+I\right)\geq-\varepsilon_1 N_{0,\varkappa}^2-\text{const}.$$

Thus by choosing βi sufficiently small, we obtain from (10.22) the following inequality

$$T_{I,\varkappa}T_{0,\varkappa}+T_{I,\varkappa}T_{0,\varkappa}\geq-\tfrac{1}{2}\varepsilon\left(H_{0,\varkappa}^{\#2}+T_{I,\varkappa}^2\right). \quad (10.23)$$

The inequality (10.23) is the desired inequality (10.17) and completes the proof.
Proposition 10.3 Given $\varepsilon>0$ there exists a hyperfinite constant $\varkappa_0$ such that for $\varkappa>\varkappa_0$

$$T_{I,\varkappa}\delta T_{0,\varkappa}^2+\delta T_{0,\varkappa}^2 T_{I,\varkappa}\geq-\varepsilon\left(H_{0,\varkappa}^{\#2}+T_{I,\varkappa}^2+I\right), \quad (10.24)$$

as bilinear forms on $\mathcal{D}\left(H_{0,\varkappa}^{\#}\right)\times\mathcal{D}\left(H_{0,\varkappa}^{\#}\right)$.
Proof For any $\varepsilon>0$ we have

$$\left|\psi,T_{I,\varkappa}\delta T_{0,\varkappa}^{(2)}(g)\psi\right|\leq\left\|T_{I,\varkappa}\psi\right\|_{\#}\left\|\delta T_{0,\varkappa}^{(2)}(g)\psi\right\|_{\#}\leq\tfrac{\varepsilon}{2}\left\|T_{I,\varkappa}\psi\right\|_{\#}^2+\tfrac{1}{2\varepsilon}\left\|\delta T_{0,\varkappa}^{(2)}(g)\psi\right\|_{\#}^2. \quad (10.25)$$

By Theorem **3.2.4** b, $\delta T_{0,\varkappa}^{(2)}$ has an $L_2^{\#}$ kernel with #-norm $O\left(\varkappa^{-1/2}\right)$ and therefore for given $\varepsilon>0$,

$$\frac{1}{2\varepsilon}\left\|\delta T_{0,\varkappa}^{(2)}\psi\right\|_{\#} \leq o(1)\left\|\left(N_{0,\varkappa}+I\right)\psi\right\|_{\#}^{2} = o(1)\left\|\left(H_{0,\varkappa}^{\#}+I\right)\psi\right\|_{\#}^{2} \leq \frac{\varepsilon}{2}\left(\left\|H_{0,\varkappa}^{\#}\psi\right\|_{\#}^{2}+\|\psi\|_{\#}^{2}\right)$$

for $\varkappa > \varkappa_0(\varepsilon)$. Thus for $\varkappa > \varkappa_0$ we get the inequality

$$T_{I,\varkappa}\delta T_{0,\varkappa}^{(2)} + \delta T_{0,\varkappa}^{(2)} T_{I,\varkappa} \geq -\varepsilon\left(H_{0,\varkappa}^{\#2}+T_{I,\varkappa}^{2}\right)-\varepsilon$$

which completes the proof.

Proposition 10.4 Given $\varepsilon > 0$ there exists a hyperfinite constant $\varkappa_0$ such that for $\varkappa > \varkappa_0$

$$T_{I,\varkappa}\delta T_{0,\varkappa}^{(1)} + \delta T_{0,\varkappa}^{(1)} T_{I,\varkappa} \geq -\varepsilon\left(H_{0,\varkappa}^{\#2}+I\right), \tag{10.26}$$

as bilinear forms on $\mathcal{D}\left(H_{0,\varkappa}^{\#2}\right)\times\mathcal{D}\left(H_{0,\varkappa}^{\#2}\right)$.

Proof We consider $\delta T_{0,\varkappa}^{(1)}$ as (**3.2.39**) and write

$$T_{I,\varkappa}\delta T_{0,\varkappa}^{(1)} + \delta T_{0,\varkappa}^{(1)} T_{I,\varkappa} = Ext\text{-}\int_{|p|>\varkappa} B_{i}^{*}(p) T_{I,\varkappa} B_i(p)\ d^{\#3}p + \tag{10.27}$$

$$+\frac{1}{2}\sum_{i=1}^{3} Ext\text{-}\int_{|p|>\varkappa}\left[T_{I,\varkappa}, B_{i}^{*}(p)\right] B_i(p)\ d^{\#3}p + \frac{1}{2}\sum_{i=1}^{3} Ext\text{-}\int_{|p|>\varkappa} B_{i}^{*}(p)\left[B_i(p), T_{I,\varkappa}\right] d^{\#3}p.$$

The integrals over p in (10.27) are absolutely #-convergent as weak integrals of bilinear forms on $\mathcal{D}\left(H_{0,\varkappa}^{\#2}\right)\times\mathcal{D}\left(H_{0,\varkappa}^{\#2}\right)$. Note that for any $\varepsilon_1 > 0$ by using (**3.2.48**) the inequality holds

$$\sum_{i=1}^{3} Ext\text{-}\int_{|p|>\varkappa} B_{i}^{*}(p) T_{I,\varkappa} B_i(p)\ d^{\#3}p \geq -\varepsilon_1 \sum_{i=1}^{3} Ext\text{-}\int_{|p|>\varkappa} B_{i}^{*}(p) N_{0,\varkappa} B_i(p)\ d^{\#3}p - b\delta T_{0,\varkappa}^{(1)}. \tag{10.28}$$

By Theorem 3.2.4c we get

$$-b\delta T_{0,\varkappa}^{(1)} \geq -O(\varkappa^{-1})\left(H_{0,\varkappa}^{\#}+I\right)^{2} \geq -\varepsilon_2\left(H_{0,\varkappa}^{\#}+I\right), \tag{10.29}$$

for $\varkappa$ sufficiently infinite large. Since the right side of (4.28) commutes with the projection onto vectors with n particles, it is sufficient to bound it below on such vectors. By Theorem **3.2.1**, or Lemma 3.2.3 we get

$$\sum_{i=1}^{3} Ext\text{-}\int_{|p|>\varkappa}\langle\psi, B_{i}^{*}(p) N_{0,\varkappa} B_i(p)\psi\rangle\ d^{\#3}p = 2(n-1)\,\langle\psi, \delta T_{0,\varkappa}^{(1)}\psi\rangle \leq \text{const}(n-1)\langle\psi, H_{0,\varkappa}^{\#}\psi\rangle \leq \tag{10.30}$$

$$\leq \text{const}\langle\psi, H_{0,\varkappa}^{\#2}\psi\rangle.$$

Inserting the bounds (10.29)-(10.30) into (10.28), we get for sufficiently small ε_1 and ε_2 ,

$$\sum_{i=1}^{3} Ext\text{-}\int_{|p|>\varkappa} B_{i}^{*}(p) T_{I,\varkappa} B_i(p)\ d^{\#3}p \geq -\frac{\varepsilon}{2}\left(H_{0,\varkappa}^{\#2}+I\right). \tag{10.31}$$

We now use Lemma **3.1.4** in order to obtain bound for the commutator terms in (10.27). We write out now

$$T_{I,\varkappa} = \sum_{r=0}^{4}\binom{4}{r} T_{I,\varkappa,r}, \tag{10.32}$$

$$T_{I,\varkappa,r} = Ext\text{-}\int b(k_1,\dots,k_4)\, a^{*}(k_1)\cdots a^{*}(k_r) a(-k_{r+1})\cdots a(-k_4) d^{\#3}k_1\cdots d^{\#3}k_4, \tag{10.33}$$

$$b(k_1,\dots,k_4) = c\,\frac{\hat{g}_1(k_1+\cdots+k_4)}{[\mu(k_1)\dots\mu(k_4)]^{1/2}} \tag{10.34}$$

for a constant c. Let us write $B_i(p)$ of (**3.2.35**)-(**3.2.37**) as

$$B_i(p) = Ext\text{-}\int \hat{h}(p-k)\, b_i(k) a(k) d^{\#3}k, \tag{10.35}$$

$$|b_i(k)| \le [\mu(k)]^{\frac{1}{2}}. \tag{10.36}$$

Let $W_{ir}(\varkappa)$ be the expression

$$W_{ir}(\varkappa) = \tfrac{1}{2}\binom{4}{r} Ext\text{-}\int_{|p|>\varkappa} B^{*}_{i}(p)\left[B_i(p), T_{I,\varkappa,r}\right] d^{\#3}p = \tag{10.37}$$

$$Ext\text{-}\int w_{i,r}(k_1,\dots,k_4;\varkappa)\, a^*(k_1)\cdots a^*(k_r) a(-k_{r+1})\cdots a(-k_4) d^{\#3}k_1\cdots d^{\#3}k_4.$$

Here $w_{i,r}(k_1,\dots,k_4;\varkappa)$ is the symmetrization in $k_1,\dots,k_r$ of

$$\tfrac{1}{2}\binom{4}{r} rc b_i(k_1)[\mu(k_1)\dots\mu(k_4)]^{-1/2} \times \tag{10.38}$$

$$Ext\text{-}\int_{|p|>\varkappa} d^{\#3}p\, Ext\text{-}\int d^{\#3}q b_i(q)\, [\mu(q)]^{-1/2}\overline{\hat{h}(p-k_1)}\hat{h}(p-q)\hat{g}_1(q+k_2+k_3+k_4).$$

Thus using (10.31) we write for (10.27)

$$T_{I,\varkappa}\delta T^{(1)}_{0,\varkappa} + \delta T^{(1)}_{0,\varkappa} T_{I,\varkappa} \ge -\tfrac{1}{2}\varepsilon\left(H^{\#2}_{0,\varkappa}+I\right) + \textstyle\sum_{i=1}^{3}\sum_{r=0}^{4}\left(W_{ir}(\varkappa) + \left(W_{ir}(\varkappa)\right)^*\right). \tag{10.39}$$

We will use now **Lemma 3.1.4** in the case of r creators, $(4-r)$ annihilators, $\alpha = \min(2,r)$, $\beta = \min(2, 4-r)$, $\tau = 1$ and $\sigma = 1$ to prove that

$$\left\| N^{\#-(2-\alpha)/2}_{\varkappa} H^{\#-\alpha/2}_{0,\varkappa} W_{ir}(\varkappa) H^{\#-\beta/2}_{0,\varkappa} N^{\#-(2-\beta)/2}_{\varkappa}\right\|_{\#} \le O\left(\varkappa^{-\delta}\right), \delta < \tfrac{1}{2}. \tag{10.40}$$

Assuming (10.40), we have for all i and r that, $\left\|\left(H^{\#}_{0,\varkappa}+I\right)^{-1} W_{ir}(\varkappa)\left(H^{\#}_{0,\varkappa}+I\right)^{-1}\right\|_{\#} \le O\left(\varkappa^{-\delta}\right), \delta < \tfrac{1}{2}$.

Exchanging α and β gives a similar bound for $\left(W_{ir}(\varkappa)\right)^*$. Thus for sufficiently infinite large $\varkappa$, we conclude from (10.39) that,

$$T_{I,\varkappa}\delta T^{(1)}_{0,\varkappa} + \delta T^{(1)}_{0,\varkappa} T_{I,\varkappa} \ge -\varepsilon\left(H^{\#2}_{0,\varkappa}+I\right), \tag{10.41}$$

which is the desired bound (10.26). We now estimate the kernel $w_{i,r}$ of (10.38). Note that by (10.36)

$$\left|Ext\text{-}\int_{|p|>\varkappa} d^{\#3}p\, Ext\text{-}\int d^{\#3}q b_i(q)\, [\mu(q)]^{-1/2}\overline{\hat{h}(p-k_1)}\hat{h}(p-q)\hat{g}_1(q+k_2+k_3+k_4)\right| \le \tag{10.42}$$

$$\le Ext\text{-}\int_{|p|>\varkappa} d^{\#3}p\, Ext\text{-}\int d^{\#3}q\left|\hat{h}(p-k_1)\hat{h}(p-q)\hat{g}_1(q+k_2+k_3+k_4)\right| \le$$

$$\le Ext\text{-}\int_{|p|>\varkappa} d^{\#3}p \left|\hat{h}(p-k_1) h_1(p+k_2+k_3+k_4)\right|.$$

Here $h_1(p) = Ext\text{-}\int\left|\hat{h}(p-q)\hat{g}_1(q)\right| d^{\#3}q$ is a rapidly decreasing function in $S^{\#}_{\text{fin}}({}^*\mathbb{R}^{\#3}_c)$. Since for $0 \le \varepsilon \le 1$, $1 \le \text{const} \times [\mu(p-k_1)]^{\varepsilon}[\mu(p)]^{-\varepsilon}[\mu(k_1)]^{\varepsilon}$ and therefore we have by (10.42)

$$\left|Ext\text{-}\int_{|p|>\varkappa} d^{\#3}p\, Ext\text{-}\int d^{\#3}q b_i(q)\, [\mu(q)]^{-1/2}\overline{\hat{h}(p-k_1)}\hat{h}(p-q)\hat{g}_1(q+k_2+k_3+k_4)\right| \le \tag{10.43}$$

$$\text{const} \times [\mu(k_1)]^{\varepsilon} Ext\text{-}\int_{|p|>\varkappa} d^{\#3}p [\mu(p)]^{-\varepsilon}[\mu(p-k_1)]^{\varepsilon}\ \left|\hat{h}(p-k_1)\hat{g}_1(q+k_2+k_3+k_4)\right| \le$$

$$\leq \text{const} \times [\mu(k_1)]^{\varepsilon}[\mu(\varkappa)]^{-\varepsilon} Ext\text{-}\int_{|p|>\varkappa} d^{\#3}p[\mu(p-k_1)]^{\varepsilon} \ \left|\hat{h}(p-k_1)\hat{g}_1(q+k_2+k_3+k_4)\right|$$

$$\leq \text{const} \times [\mu(k_1)]^{\varepsilon}[\mu(\varkappa)]^{-\varepsilon} g_2(k_1+k_2+k_3+k_4),$$

$$g_2(k) = Ext\text{-}\int d^{\#3}p[\mu(p)]^{\varepsilon} \left|\hat{h}(p)\right| h_1(p+k). \qquad (10.44)$$

Note that $g_2(k)$ is a rapidly decreasing function in $S^{\#}_{\text{fin}}({}^*\mathbb{R}^{\#3}_c)$ independent of $\varkappa$. Therefore for $w_{i,r}$, the symmetrization of (10.38), we have by (10.36) and (10.43) that

$$\left|w_{i,r}(k_1,\dots,k_4;\varkappa)\right| \leq \text{const} \times [\mu(\varkappa)]^{-\varepsilon}\left(\textstyle\sum_{j=1}^{4}\left[\mu\left(k_j\right)\right]^{1+\varepsilon}\right)\left(\mu(k_1)\cdots\mu(k_4)\right)^{-1/2} g_2(k_1+k_2+k_3+k_4) \qquad (10.45)$$

By applying **Lemma 3.1.4** with $\alpha = \min(2,r)$ and $\beta = \min(2,4-r)$, we have $2 \leq \alpha+\beta \leq 4$.

Since $E_{\,C}(\alpha,1)\ E_{\,A}(\beta,1)$ is a homogeneous polynomial of degree $\alpha+\beta$ in the $\mu(k_i)$'s, the most favorable bounds occur with α +β = 4 and the least favorable bounds occur with α − β = 2. In any case we get

$$E = \sup_{\substack{i\neq j \\ 1\leq i,j\leq 4}} \left[\mu(k_i)\mu\left(k_j\right)\right] \leq \text{const} \times E_{\,C}(\alpha,1)E_{\,A}(\beta,1). \qquad (10.46)$$

Note that

$$[\mu(k_i)]^2 \leq \text{const} \times E\mu(k_1+k_2+k_3+k_4) \leq \text{const} \times E_{\,C}(\alpha,1)E_{\,A}(\beta,1)\mu(k_1+k_2+k_3+k_4). \qquad (10.47)$$

Thus by (10.45) we obtain

$$\frac{\left|w_{i,r}(k_1,\dots,k_4;\varkappa)\right|}{\left(E_{\,C}(\alpha,1)E_{\,A}(\beta,1)\right)^{1/2}} \leq \text{const} \times [\mu(\varkappa)]^{-\varepsilon}\left(\textstyle\sum_{j=1}^{4}\left[\mu(k_j)\right]^{\varepsilon}\right)\left(\mu(k_1)\cdots\mu(k_4)\right)^{-1/2} \times \qquad (10.48)$$

$$\times\, \mu(k_1+k_2+k_3+k_4)g_2(k_1+k_2+k_3+k_4).$$

Since $g_2(k)$ is a rapidly decreasing function in $S^{\#}_{\text{fin}}({}^*\mathbb{R}^{\#3}_c)$, the right side of (10.48) is square #-integrable for $\varepsilon < 1/2$, and therefore

$$\left\|\frac{w_{i,r}(k_1,\dots,k_4;\varkappa)}{\left(E_{\,C}(\alpha,1)E_{\,A}(\beta,1)\right)^{\frac{1}{2}}}\right\|_{\#2} \leq O(\varkappa^{-\varepsilon}), \varepsilon < \tfrac{1}{2}.$$

Tus by Lemma 3.1.4, (10.40) is valid. This completes the proof of the proposition.
Proof of the Theorem 10.1.We expand now

$$\left[H^{\#}_{0,\varkappa}+\beta T_{0,\varkappa}(g_0)+T_{I,\varkappa}(g_1)+b\right]^2 = \left[H^{\#}_{0,\varkappa}+\beta T_{0,\varkappa}(g_0)+\frac{b}{2}\right]^2+\left[T_{I,\varkappa}(g_1)\right]^2+ \qquad (10.49)$$

$$b\left[H^{\#}_{0,\varkappa}+\beta T_{0,\varkappa}(g_0)+2T_{I,\varkappa}(g_1)+\frac{5b}{8}\right]+\frac{1}{8}b^2+T_{I,\varkappa}(g_1)\left[H^{\#}_{0,\varkappa}+\beta T_{0,\varkappa}(g_0)\right]+\left[H^{\#}_{0,\varkappa}+\beta T_{0,\varkappa}(g_0)\right]T_{I,\varkappa}(g_1).$$

Given $\varepsilon > 0$ and b sufficiently large, proposition 10.2 ensures that the first term on the right of (10.49) is greater than

$$(1-\varepsilon)\left[H^{\#2}_{0,\varkappa}+\beta^2\left(T_{0,\varkappa}(g_0)\right)^2\right]. \qquad (10.50)$$

Furthermore, for b sufficiently large, the proof of **Theorem 3.2.5** ensures that for $0 \leq \beta \leq 1$,

$$H^{\#}_{0,\varkappa} + \beta T_{0,\varkappa}(g_0) + 2T_{I,\varkappa}(g_1) + \frac{5b}{8} \geq 0. \tag{10.51}$$

Hence to prove the theorem it is sufficient to prove that for b sufficiently large, the last three terms of (10.49) satisfy

$$\frac{1}{8}b^2 + T_{I,\varkappa}(g_1)\left[H^{\#}_{0,\varkappa} + \beta T_{0,\varkappa}(g_0)\right] + \left[H^{\#}_{0,\varkappa} + \beta T_{0,\varkappa}(g_0)\right]T_{I,\varkappa}(g_1) \geq -\varepsilon\left[H^{\#2}_{0,\varkappa} + \beta^2\left(T_{0,\varkappa}(g_0)\right)^2\right]. \tag{10.52}$$

We set now $T_0 = T_{0,\varkappa} + \delta T^{(2)}_{0,\varkappa} + \delta T^{(1)}_{0,\varkappa}$. Then by propositions 10.3-10.5, for b sufficiently large we obtain

$$\frac{1}{16}b^2 + T_{I,\varkappa}(g_1)T_0(g_0) + T_0(g_0)T_{I,\varkappa}(g_1) \geq -\varepsilon\left[H^{\#2}_{0,\varkappa} + \beta^2\left(T_{0,\varkappa}(g_0)\right)^2\right]. \tag{10.53}$$

Hence we need only prove that for large b,

$$\frac{1}{16}b^2 + T_{I,\varkappa}(g_1)H^{\#}_{0,\varkappa} + H^{\#}_{0,\varkappa}T_{I,\varkappa}(g_1) \geq -\varepsilon H^{\#2}_{0,\varkappa}. \tag{10.54}$$

We expand now

$$T_{I,\varkappa}H^{\#}_{0,\varkappa} + H^{\#}_{0,\varkappa}T_{I,\varkappa} = 2H^{\#1/2}_{0,\varkappa}T_{I,\varkappa}H^{\#1/2}_{0,\varkappa} + \left[H^{\#1/2}_{0,\varkappa}, \left[H^{\#1/2}_{0,\varkappa}, T_{I,\varkappa}\right]\right].$$

Using **(3.2.48)** we get

$$T_{I,\varkappa}H^{\#}_{0,\varkappa} + H^{\#}_{0,\varkappa}T_{I,\varkappa} \geq -\varepsilon H^{\#2}_{0,\varkappa} - \text{const} + \left[H^{\#1/2}_{0,\varkappa}, \left[H^{\#1/2}_{0,\varkappa}, T_{I,\varkappa}\right]\right]. \tag{10.55}$$

Note that

$$\left[H^{\#1/2}_{0,\varkappa}, \left[H^{\#1/2}_{0,\varkappa}, T_{I,\varkappa}\right]\right] \geq -\varepsilon H^{\#2}_{0,\varkappa} - \text{const}. \tag{10.56}$$

Obviously from (10.55) and (10.56) one obtains (10.54).
Alternatively, a proof of (10.54) could be obtained by using the equality

$$T_{I,\varkappa}H^{\#}_{0,\varkappa} + H^{\#}_{0,\varkappa}T_{I,\varkappa} = 2Ext\text{-}\int a^*(k)T_{I,\varkappa}a(k)\mu(k)\,d^{\#3}k \; +$$

$$+Ext\text{-}\int\left\{\left[T_{I,\varkappa}, a^*(k)\right]a(k) + a^*(k)\left[T_{I,\varkappa}, a(k)\right]\right\}\mu(k)\,d^{\#3}k$$

and using the methods of the proof of Proposition 10.5

§11. FOURTH ORDER ESTIMATES

In this section we study the operator $M_\varkappa = \alpha H_{0,\varkappa} + T_{0,\varkappa}(\,g_0) + T_I(g_1)$.

Theorem 11.1 [15] Let $\alpha > 0$ and let $g_0 = [h_0]^2, g_1 = [h_1]^2, h_0, h_1 \in S^{\#}_{\text{fin}}({}^*\mathbb{R}^{\#3}_c), h_0 \geq 0, h_1 \geq 0$, then operator

$$M_\varkappa \; = \alpha H_{0,\varkappa} + T_{0,\varkappa}(\,g_0) + T_I(g_1) \tag{11.1}$$

is self -#-adjoint on $\mathcal{D}(H_{0,\varkappa}) \cap \mathcal{D}(T_I(g_1))$ and is essentially self -#-adjoint on $C^{*\infty}(H_{0,\varkappa})$.
Proof We let $\alpha = 1, A = H_{0,\varkappa} + T_I(g_1) + b$ and $B^{(k)} = T_0(\,g_0)$. We choose b sufficiently large so that $A \geq I$. Note that A is self -#-adjoint on $\mathcal{D}(H_{0,\varkappa}) \cap \mathcal{D}(T_I(g_1))$ and that A is essentially self -#-adjoint on $C^{*\infty}(H_{0,\varkappa})$. Let $\mathcal{D}(A) = C^{*\infty}(H_{0,\varkappa})$. The inequality $\|B\psi\|_\# \leq a\|(A + \beta B)\psi\|_\#$ is proved as follows: By Theorem 9.8, namely the

boundedness of (9.60), we have $\|T_0(g_0)\psi\|_\# \leq \text{const}\left\|\left(H^\#_{0,\varkappa}+I\right)\psi\right\|_\#$. By Theorem 10.1, if $c>1$ and b is sufficiently large, we obtain

$$\left\|\left(H^\#_{0,\varkappa}+I\right)\psi\right\|_\# \leq c\left\|\left[H^\#_{0,\varkappa}+\beta T_{0,\varkappa}(g_0)+T_{I,\varkappa}(g_1)+b\right]\psi\right\|_\#$$

for all β, $0\leq\beta\leq 1$. Thus for $\psi\in\mathcal{D}(A)=C^{*\infty}\left(H_{0,\varkappa}\right)$ we obtain

$$\|T_0(g_0)\psi\|_\# \leq c\left\|\left[H^\#_{0,\varkappa}+\beta T_{0,\varkappa}(g_0)+T_{I,\varkappa}(g_1)+b\right]\psi\right\|_\#.$$

By Theorem 6.8, M is essentially self #-adjoint on domain $C^{*\infty}\left(H_{0,\varkappa}\right)$ and M is self #-adjoint on domain $\mathcal{D}\left(H_{0,\varkappa}\right)\cap\mathcal{D}\left(T_I(g_1)\right)=\mathcal{D}(\#\text{-}\bar{A})$.

Theorem 11.2 [15] The operator $M\triangleq M_\varkappa$ defined by (11.1) has the same #-cores as the operator $H^\#_{0,\varkappa}+T_{I,\varkappa}(g_1)$.

Proof Directly from Theorem 11.2 and Theorem 6.9

Theorem 11.3 [15] Let M denote the self #-adjoint operator $M=\alpha H_{0,\varkappa}+T_0(g_0)+T_I(g_1)$ with α and g_0,g_1 as above. Then $\mathcal{D}(M^2)\subset\mathcal{D}\left(H^\#_{0,\varkappa}N^\#_\varkappa\right)$, and there are constants $b<{}^*\infty, c<{}^*\infty$ such that as forms on $\mathcal{D}(M^2)\times\mathcal{D}(M^2)$

$$H^{\#2}_{0,\varkappa}N^{\#2}_\varkappa\leq c(M+b)^4. \tag{11.2}$$

Proof We need to prove that $\mathcal{D}(N^\#_\varkappa M)\subset\mathcal{D}\left(N^\#_\varkappa H^\#_{0,\varkappa}\right)$ and that there are constants b,c such that, for $\psi\in\mathcal{D}(N^\#_\varkappa M)$

$$\left\|N^\#_\varkappa H^\#_{0,\varkappa}\psi\right\|_\#\leq c\|(I+N^\#_\varkappa)(M+b)\psi\|_\#. \tag{11.3}$$

The inequality (10.2), see Theorem 10.1 extends to $\mathcal{D}(M)\times\mathcal{D}(M)$ since by Theorem 11.1, $C^{*\infty}\left(H_{0,\varkappa}\right)$ is a #-core for M and the operators involved are #-closable. Hence $\mathcal{D}(M)\subset\mathcal{D}\left(H^\#_{0,\varkappa}\right)$, so

$$\mathcal{D}(M^2)\subset\mathcal{D}\left(H^\#_{0,\varkappa}M\right)\subset\mathcal{D}(N^\#_\varkappa M)\subset\mathcal{D}\left(N^\#_\varkappa H^\#_{0,\varkappa}\right)$$

and by (11.3) for new constants c_1,c_2,b_1 and $\psi\in\mathcal{D}(M^2)$

$$\left\|N^\#_\varkappa H^\#_{0,\varkappa}\psi\right\|_\#\leq c_1\|(I+N^\#_\varkappa)(M+b)\psi\|_\#\leq c_2(M+b_1)^4. \tag{11.4}$$

As a first step to prove (11.3), we prove that $C^{*\infty}\left(H_{0,\varkappa}\right)$ is a #-core for $\mathfrak{H}=(I+N^\#_\varkappa)(M+b)$, where b is sufficiently large so that $M+b$ is positive. It is sufficient to show that the range of $\mathfrak{H}\restriction C^{*\infty}\left(H_{0,\varkappa}\right)$ is #-dense, for this operator has a #-continuous inverse. Hence the #-closure of its inverse is the inverse of its #-closure. Let $\mathcal{D}^\#_0$ denote vectors in Fock space $\mathcal{F}^\#_s(H^\#)$ with a finite or hyperfinite number of particles.

Remark 11.1 Note that 1) $C^{*\infty}\left(H_{0,\varkappa}\right)\cap\mathcal{D}^\#_0$ is a #-core for $\alpha H^\#_{0,\varkappa}+T_{I,\varkappa}(g_1)$. Hence by Theorem 11.2, it is a #-core for operator M, so that $\mathcal{D}^\#_1=(M+b)\left(C^{*\infty}\left(H_{0,\varkappa}\right)\cap\mathcal{D}^\#_0\right)$ is #-dense. 2) Every vector in $\mathcal{D}^\#_1$ is an #-analytic vector for the operator $N^\#_\varkappa$, and hence $\mathcal{D}^\#_1$ is a #-core for the operator $N^\#_\varkappa$.

Thus we conclude that $(N^\#_\varkappa+I)\mathcal{D}^\#_1$ is #-dense; so $C^{*\infty}\left(H_{0,\varkappa}\right)$ is a #-core for $(I+N^\#_\varkappa)(M+b)$.

Note that it is sufficient to prove (11.3) for ψ belonging to a #-core for $(I+N^\#_\varkappa)(M+b)$, so we show that as forms on $\mathcal{D}\left(H^\#_{0,\varkappa}\right)\times\mathcal{D}\left(H^\#_{0,\varkappa}\right)$

$$H^\#_{0,\varkappa}N^{\#2}_\varkappa\leq c(M+b)(I+N^\#_\varkappa)^2(M+b). \tag{11.5}$$

Remark 11.2 Note that it is sufficient to prove (11.5) for $\alpha=1$, since the constant α may be absorbed into g_0, g_1, b and c.

Remark 11.3 Now we let $T_{\varkappa}^{\#} = T_{0,\varkappa} + T_{I,\varkappa}$, and note that (11.5) is equivalent to showing that the following operator is positive

$$H_{0,\varkappa}^{\#2}(I + N_{\varkappa}^{\#})^2 - c^{-1}H_{0,\varkappa}^{\#2}N_{\varkappa}^{\#2} + T_{\varkappa}^{\#}(I + N_{\varkappa}^{\#})^2\, T_{\varkappa}^{\#} + T_{\varkappa}^{\#}(I + N_{\varkappa}^{\#})^2 H_{0,\varkappa}^{\#} + \tag{11.6}$$

$$+H_{0,\varkappa}^{\#}(I + N_{\varkappa}^{\#})^2\, T_{\varkappa}^{\#} + b\left(H_{0,\varkappa}^{\#} + T_{\varkappa}^{\#}\right)(I + N_{\varkappa}^{\#})^2 + b(I + N_{\varkappa}^{\#})^2\left(H_{0,\varkappa}^{\#} + T_{\varkappa}^{\#}\right) + b^2(I + N_{\varkappa}^{\#})^2 =$$

$$H_{0,\varkappa}^{\#2}(I + N_{\varkappa}^{\#})^2 - c^{-1}H_{0,\varkappa}^{\#2}N_{\varkappa}^{\#2} + T_{\varkappa}^{\#}(I + N_{\varkappa}^{\#})^2\, T_{\varkappa}^{\#} + 2b(I + N_{\varkappa}^{\#})\left(H_{0,\varkappa}^{\#} + T_{\varkappa}^{\#} + \frac{b}{4}\right)(I + N_{\varkappa}^{\#}) +$$

$$+2b\left[N_{\varkappa}^{\#}, [N_{\varkappa}^{\#}, T_{\varkappa}^{\#}]\right] + T_{\varkappa}^{\#}(I + N_{\varkappa}^{\#})^2 H_{0,\varkappa}^{\#} + H_{0,\varkappa}^{\#}(I + N_{\varkappa}^{\#})^2\, T_{\varkappa}^{\#} + \frac{b^2}{2}(I + N_{\varkappa}^{\#})^2.$$

For sufficiently large b we get

$$T_{\varkappa}^{\#}(I + N_{\varkappa}^{\#})^2\, T_{\varkappa}^{\#} + 2b(I + N_{\varkappa}^{\#})\left(H_{0,\varkappa}^{\#} + T_{\varkappa}^{\#} + \frac{b}{4}\right)(I + N_{\varkappa}^{\#}) \geq 0$$

as a sum of positive terms and if $c > \frac{1}{2}$ we get

$$\frac{1}{2}H_{0,\varkappa}^{\#2}(I + N_{\varkappa}^{\#})^2 - c^{-1}H_{0,\varkappa}^{\#2}N_{\varkappa}^{\#2} \geq 0.$$

Thus (11.6) is positive for large b if the following inequalities hold:

$$\frac{1}{8}b(I + N_{\varkappa}^{\#})^2 + \left[N_{\varkappa}^{\#}, [N_{\varkappa}^{\#}, T_{\varkappa}^{\#}]\right] \geq 0, \tag{11.7}$$

$$\frac{1}{2}H_{0,\varkappa}^{\#2}(I + N_{\varkappa}^{\#})^2 + T_{\varkappa}^{\#}(I + N_{\varkappa}^{\#})^2 H_{0,\varkappa}^{\#} + H_{0,\varkappa}^{\#}(I + N_{\varkappa}^{\#})^2\, T_{\varkappa}^{\#} + \frac{1}{4}b^2(I + N_{\varkappa}^{\#})^2 \geq 0. \tag{11.8}$$

In order to prove (11.7), we note that $\left[N_{\varkappa}^{\#}, T_{0,\varkappa}^{1}\right] = 0$, therefore $\left[N_{\varkappa}^{\#}, [N_{\varkappa}^{\#}, T_{\varkappa}^{\#}]\right]$ is a sum of Wick ordered monomials of degree two or four with $L_2^{\#}$ kernels. Thus the operator $(I + N_{\varkappa}^{\#})^{-1}\left[N_{\varkappa}^{\#}, [N_{\varkappa}^{\#}, T_{\varkappa}^{\#}]\right](I + N_{\varkappa}^{\#})^{-1}$ is bounded and (11.7) is positive for large b. To prove (11.8), we note that

$$T_{\varkappa}^{\#}(I + N_{\varkappa}^{\#})^2 H_{0,\varkappa}^{\#} + H_{0,\varkappa}^{\#}(I + N_{\varkappa}^{\#})^2\, T_{\varkappa}^{\#} =$$

$$= (I + N_{\varkappa}^{\#})\left(T_{\varkappa}^{\#}H_{0,\varkappa}^{\#} + H_{0,\varkappa}^{\#}\, T_{\varkappa}^{\#}\right)(I + N_{\varkappa}^{\#}) + \left[[T_{\varkappa}^{\#}, N_{\varkappa}^{\#}], (I + N_{\varkappa}^{\#})^2 H_{0,\varkappa}^{\#}\right] =$$

$$= (I + N_{\varkappa}^{\#})\left(T_{\varkappa}^{\#}H_{0,\varkappa}^{\#} + H_{0,\varkappa}^{\#}\, T_{\varkappa}^{\#}\right)(I + N_{\varkappa}^{\#}) + 2(I + N_{\varkappa}^{\#})H_{0,\varkappa}^{\#1/2}\, T_{I,\varkappa}^{\#}H_{0,\varkappa}^{\#1/2}(I + N_{\varkappa}^{\#}) + \tag{11.9}$$

$$+(I + N_{\varkappa}^{\#})\left[H_{0,\varkappa}^{\#1/2}, \left[H_{0,\varkappa}^{\#1/2}, T_{I,\varkappa}^{\#}\right]\right](I + N_{\varkappa}^{\#}) + \left[[T_{\varkappa}^{\#}, N_{\varkappa}^{\#}], N_{\varkappa}^{\#}\right]H_{0,\varkappa}^{\#} + (I + N_{\varkappa}^{\#})\left[[T_{\varkappa}^{\#}, N_{\varkappa}^{\#}], H_{0,\varkappa}^{\#}\right].$$

By Proposition 10.1, we have for the first term in (11.9)

$$(I + N_{\varkappa}^{\#})\left(T_{\varkappa}^{\#}H_{0,\varkappa}^{\#} + H_{0,\varkappa}^{\#}\, T_{\varkappa}^{\#}\right)(I + N_{\varkappa}^{\#}) \geq -\frac{1}{7}H_{0,\varkappa}^{\#2}(I + N_{\varkappa}^{\#})^2 - b_1(I + N_{\varkappa}^{\#})^2 \tag{11.10}$$

for any $\varepsilon > 0$ and for some $b_1 < {}^{*}\infty$. The second term in (11.9) is bounded below since by using **(3.2.48)** we get

$$2(I + N_{\varkappa}^{\#})H_{0,\varkappa}^{\#1/2}\, T_{I,\varkappa}^{\#}H_{0,\varkappa}^{\#1/2}(I + N_{\varkappa}^{\#}) \geq -\varepsilon_1 H_{0,\varkappa}^{\#2}(I + N_{\varkappa}^{\#})^2 - b_1 H_{0,\varkappa}^{\#}(I + N_{\varkappa}^{\#})^2 \geq \tag{11.11}$$

$$\geq -\frac{1}{7}\varepsilon\, H_{0,\varkappa}^{\#2}(I + N_{\varkappa}^{\#})^2 - b\, H_{0,\varkappa}^{\#}(I + N_{\varkappa}^{\#})^2$$

for any $\varepsilon > 0$ and for some $b\ b(\varepsilon)$.

Remark 11.4 Note that for any $\varepsilon > 0$ there is a b such that

$$\left[H_{0,\varkappa}^{\#1/2},\left[H_{0,\varkappa}^{\#1/2},T_{I,\varkappa}^{\#}\right]\right] \geq -\frac{1}{7}\varepsilon\, H_{0,\varkappa}^{\#2} - b.$$

And therefore we obtain the inequality

$$(I+N_{\varkappa}^{\#})\left[H_{0,\varkappa}^{\#1/2},\left[H_{0,\varkappa}^{\#1/2},T_{I,\varkappa}^{\#}\right]\right](I+N_{\varkappa}^{\#}) \geq -\frac{1}{7}\varepsilon\, H_{0,\varkappa}^{\#2}(I+N_{\varkappa}^{\#})^2 - b\, H_{0,\varkappa}^{\#}(I+N_{\varkappa}^{\#})^2. \quad (11.12)$$

Since $[\,T_{\varkappa}^{\#},N_{\varkappa}^{\#}]$ contains second or fourth order Wick monomials with $L_2^{\#}$ kernels,

$$\mathfrak{I}_{\varkappa}^{\#} = (I+N_{\varkappa}^{\#})^{-1}\left[[\,T_{\varkappa}^{\#},N_{\varkappa}^{\#}],N_{\varkappa}^{\#}\right](I+N_{\varkappa}^{\#})^{-1}$$

is a bounded operator. Thus for any $\psi \in C^{*\infty}\left(H_{0,\varkappa}^{\#}\right)$ we obtain the inequality

$$\left|\langle\psi,\left[[\,T_{\varkappa}^{\#},N_{\varkappa}^{\#}],N_{\varkappa}^{\#}\right]H_{0,\varkappa}\psi\rangle\right| = \left|\langle(I+N_{\varkappa}^{\#})\psi,\mathfrak{I}_{\varkappa}^{\#}(I+N_{\varkappa}^{\#})H_{0,\varkappa}^{\#}\psi\rangle\right| \leq \quad (11.13)$$

$$\leq \text{const}\|(I+N_{\varkappa}^{\#})\psi\|_{\#}\left\|(I+N_{\varkappa}^{\#})H_{0,\varkappa}^{\#}\psi\right\|_{\#} \leq \frac{1}{7}\varepsilon\left\|H_{0,\varkappa}^{\#}(I+N_{\varkappa}^{\#})\psi\right\|_{\#}^{2} + \text{const}\|(I+N_{\varkappa}^{\#})\psi\|_{\#}^{2}.$$

Finally we consider the operator $(I+N_{\varkappa}^{\#})\left[[\,T_{\varkappa}^{\#},N_{\varkappa}^{\#}],H_{0,\varkappa}^{\#}\right]$. We write $T_{\varkappa}^{\#} = T_{0,\varkappa} + T_{I,\varkappa}$ and consider these two terms separately. Let

$$[\,T_{0,\varkappa}^{\#},N_{\varkappa}^{\#}] = \mathfrak{H}_1 + \mathfrak{H}_2.$$

Here $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are respectively terms of the form **(3.1.14)** with $r=2, s=0$ and with $r=0, s=2$ and each such term has $L_2^{\#}$ kernel. Applying **Lemma 3.1.3**, we have that $H_{0,\varkappa}^{\#-1}[\mathfrak{H}_1,H_{0,\varkappa}^{\#}]$ and $[\mathfrak{H}_2,H_{0,\varkappa}^{\#}]H_{0,\varkappa}^{\#-1}$ are bounded forms on $\mathcal{D}(H_{0,\varkappa}^{\#}) \times \mathcal{D}(H_{0,\varkappa}^{\#})$ and therefore we obtain the inequality

$$\left|\langle\psi,(I+N_{\varkappa}^{\#})\left[[\,T_{0,\varkappa}^{\#},N_{\varkappa}^{\#}],H_{0,\varkappa}^{\#}\right]\psi\rangle\right| \leq \left|\langle H_{0,\varkappa}^{\#}(I+N_{\varkappa}^{\#})\psi, H_{0,\varkappa}^{\#-1}[\mathfrak{H}_1,H_{0,\varkappa}^{\#}]\psi\rangle\right| + \quad (11.14)$$

$$+\left|\langle(I+N_{\varkappa}^{\#})\psi,[\mathfrak{H}_2,H_{0,\varkappa}^{\#}]H_{0,\varkappa}^{\#-1}H_{0,\varkappa}^{\#}\psi\rangle\right| \leq$$

$$\leq \text{const}\left(\left\|H_{0,\varkappa}(I+N_{\varkappa}^{\#})\psi\right\|_{\#}\|\psi\|_{\#} + \|(I+N_{\varkappa}^{\#})\psi\|_{\#}\left\|H_{0,\varkappa}\psi\right\|_{\#}\right) \leq$$

$$\leq \frac{1}{7}\varepsilon\left\|H_{0,\varkappa}(I+N_{\varkappa}^{\#})\psi\right\|_{\#}^{2} + \text{const}\|(I+N_{\varkappa}^{\#})\psi\|_{\#}^{2}.$$

The remaining part of the expression $(I+N_{\varkappa}^{\#})\left[[\,T_{\varkappa}^{\#},N_{\varkappa}^{\#}],H_{0,\varkappa}^{\#}\right]$ consists of the contribution from $\left[\,T_{I,\varkappa}^{\#},N_{\varkappa}^{\#}\right]$. Let $T_{\varkappa}^{\#} = T_{I,\varkappa}^{\#} + \delta\, T_{I,\varkappa}^{\#}$, where $T_{I,\varkappa}^{\#}$ is defined as in **(4.32)~(4.34),** but the kernel **(4.34)** is multiplied by the characteristic function of $\{k_i||k_i| \leq \varkappa, i=1,2,3,4\}$. Then $\left[[\,T_{I,\varkappa}^{\#},N_{\varkappa}^{\#}],H_{0,\varkappa}^{\#}\right]$ is consists of Wick monomials with $L_2^{\#}$ kernels. As in (11.13), we have

$$\left|\langle\psi,\left[[\,T_{\varkappa}^{\#},N_{\varkappa}^{\#}],N_{\varkappa}^{\#}\right]H_{0,\varkappa}\psi\rangle\right| = \left|\langle(I+N_{\varkappa}^{\#})\psi,\mathfrak{I}_{\varkappa}^{\#}(I+N_{\varkappa}^{\#})H_{0,\varkappa}^{\#}\psi\rangle\right| \leq \quad (11.15)$$

$$\leq \text{const}\|(I+N_{\varkappa}^{\#})\psi\|_{\#}\left\|(I+N_{\varkappa}^{\#})H_{0,\varkappa}^{\#}\psi\right\|_{\#} \leq \frac{1}{7}\varepsilon\left\|H_{0,\varkappa}^{\#}(I+N_{\varkappa}^{\#})\psi\right\|_{\#}^{2} + \text{const}\|(I+N_{\varkappa}^{\#})\psi\|_{\#}^{2}$$

Using **Lemma 3.1.4**, we analyze the high hyperfinite energy contribution, $\delta\, T_{I,\varkappa}^{\#}$. It is a sum of Wick monomials of degree four, and at least one variable kt is greater than K in magnitude. By Lemma **3.1.4**, and **(4.47),**

$$\wp_{\varkappa}^{\#}=\left(I+H_{0,\varkappa}^{\#}\right)^{-1}\left[\left[\delta\, T_{I,\varkappa}^{\#},N_{\varkappa}^{\#}\right],H_{0,\varkappa}^{\#}\right]\left(I+H_{0,\varkappa}^{\#}\right)^{-1}$$

is a bounded operator, and an estimate of the kernels of $\left[\left[\delta\, T_{I,\varkappa}^{\#},N_{\varkappa}^{\#}\right],H_{0,\varkappa}^{\#}\right]$ shows that $\|\wp_{\varkappa}^{\#}\|_{\#}\leq O(\varkappa^{-\tau}),\tau<1/2$. Thus for sufficiently infinite large $\varkappa$ we obtain the inequality

$$\left|\left(\psi,(I+N_{\varkappa}^{\#})\left[\left[\delta\, T_{I,\varkappa}^{\#},N_{\varkappa}^{\#}\right],H_{0,\varkappa}^{\#}\right]\psi\right)\right|\leq O(\varkappa^{-\tau})\left\|(I+N_{\varkappa}^{\#})\left(I+H_{0,\varkappa}^{\#}\right)\psi\right\|_{\#}\left\|\left(I+H_{0,\varkappa}^{\#}\right)\psi\right\|_{\#}\leq \tag{11.16}$$

$$\leq\frac{1}{7}\varepsilon\left\|H_{0,\varkappa}^{\#}(I+N_{\varkappa}^{\#})\psi\right\|_{\#}^{2}+\|(I+N_{\varkappa}^{\#})\psi\|_{\#}^{2}.$$

The inequalities (11.13)-(11.19) dominate the various terms in (11.12). Added together, they show that (11.12) is bounded by

$$T_{\varkappa}^{\#}\left(I+H_{0,\varkappa}^{\#}\right)^{2}H_{0,\varkappa}^{\#}+H_{0,\varkappa}^{\#}\left(I+H_{0,\varkappa}^{\#}\right)^{2}T_{\varkappa}^{\#}\geq-\varepsilon H_{0,\varkappa}^{\#2}\left(I+H_{0,\varkappa}^{\#}\right)^{2}-\mathrm{const}\left(I+H_{0,\varkappa}^{\#}\right)^{2}.$$

Thus (11.11) is valid for b sufficiently large and the proof of the theorem is complete.

§12. Q$^{\#}$-SPACE REPRESENTATION OF THE FOCK SPACE STRUCTURES

In this section the construction of a non-Archimedean $Q^{\#}$-space and $L_{2}^{\#}(Q^{\#},d^{\#}\mu^{\#})$, another representation of the Fock space structures are presented. In analogy with the one degree of freedom case where $\mathcal{F}^{\#}({}^{*}\mathbb{R}_{c}^{\#})$ is isomorphic to $L_{2}^{\#}({}^{*}\mathbb{R}_{c}^{\#},d^{\#}x)$ in such a way that $\Phi_{S}^{\#}(1)$ becomes multiplication by$x$, we will construct a $\sigma^{\#}$-measure space $\langle Q^{\#},\mu^{\#}\rangle$, with $\mu^{\#}(Q^{\#})=1$, and a unitary map $\mathrm{S}^{\#}:\mathcal{F}_{s}^{\#}(H^{\#})\rightarrow L_{2}^{\#}(Q^{\#},d^{\#}\mu^{\#})$ so that for each $f\in H_{C}^{\#}$, $\mathrm{S}^{\#}\phi_{\varkappa}^{\#}(f)\mathrm{S}^{\#-1}$ acts on $L_{2}^{\#}(Q^{\#},d^{\#}\mu^{\#})$ by multiplication by a $\mu^{\#}$-measurable function. We can then show that in the case of the free scalar field of mass m in 4-dimensional space-time $M_{4}^{\#}$, $V=\mathrm{S}^{\#}H_{I,\varkappa}^{\#}(g)\mathrm{S}^{\#-1}$ is just multiplication by a function $V(q)$ which is in $L_{2}^{\#}(Q^{\#},d^{\#}\mu^{\#})$ for each $p\in{}^{*}\mathbb{N}$. Let $\{g_{n}\}_{n=1}^{*\infty}$ be an orthonormal basis for $H^{\#}$ so that each $g\in H_{C}^{\#}$ and let $\{g_{n}\}_{n=1}^{N},N\in{}^{*}\mathbb{N}$ be a finite or hyperfinite subcollection of the set $\{f_{n}\}_{n=1}^{*\infty}$ .Let P_{N} be a set of the all external finite and hyperfinite polynomials $Ext\text{-}P[u_{1},\ldots,u_{N}]$ and $\mathcal{F}_{N}^{\#}$ be the #-closure of the set $\{Ext\text{-}P[\varphi_{\varkappa}^{\#}(g_{1}),\ldots,\varphi_{\varkappa}^{\#}(g_{N})]|P\in P_{N}\}$ in $\mathcal{F}_{s}^{\#}(H^{\#})$ and define a set $F_{0}^{N}=\mathcal{F}_{N}^{\#}\cap F_{0}$. From Theorem 55 it follows that $\varphi_{\varkappa}^{\#}(g_{k})$ and $\pi_{\varkappa}^{\#}(g_{k})$, for all $1\leq k,l\leq N$ are essentially self-#-adjoint on F_{0}^{N} and that

$$(Ext\text{-}\exp[it\varphi_{\varkappa}^{\#}(g_{k})])(Ext\text{-}\exp[it\pi_{\varkappa}^{\#}(g_{l})])=$$

$$\left(Ext\text{-}\exp\left[-ist\delta_{kl}\right]\right)(Ext\text{-}\exp[it\pi_{\varkappa}^{\#}(g_{l})])(Ext\text{-}\exp[it\varphi_{\varkappa}^{\#}(g_{k})])\ .$$

Therefore we have a representation of the generalized Weyl relations in which the vector Ω_{0} satisfies the equality $([\varphi_{\varkappa}^{\#}(g_{k})]^{2}+[\pi_{\varkappa}^{\#}(g_{l})]^{2}-1)\Omega_{0}=0$ and is cyclic for the operators $\{\varphi_{\varkappa}^{\#}(g_{k})\}_{k=1}^{N}$. Therefore there is a unitary map $\mathrm{S}^{\#(N)}:\mathcal{F}_{N}^{\#}\rightarrow L_{2}^{\#}({}^{*}\mathbb{R}_{c}^{\#N})$ such that: 1) $\mathrm{S}^{\#(N)}\varphi_{\varkappa}^{\#}(g_{k})\left(\mathrm{S}^{\#(N)}\right)^{-1}=x_{k}$, 2) $\mathrm{S}^{\#(N)}\pi_{\varkappa}^{\#}(g_{k})\left(\mathrm{S}^{\#(N)}\right)^{-1}=-\frac{1}{i}\frac{d^{\#}}{d^{\#}x_{k}}$ and 3) $\mathrm{S}^{\#(N)}\Omega_{0}=\pi^{-N/4}\left[Ext\text{-}\exp\left(-Ext\text{-}\sum_{k=1}^{N}\frac{x_{k}^{2}}{2}\right)\right]$. It is convenient to use the non-Archimedean Hilbert space $L_{2}^{\#}\left({}^{*}\mathbb{R}_{c}^{\#N},\pi^{-N/4}\left(Ext\text{-}\exp\left(-Ext\text{-}\sum_{k=1}^{N}\frac{x_{k}^{2}}{2}\right)\right)\right)d^{\#N}x$ instead of $L_{2}^{\#}({}^{*}\mathbb{R}_{c}^{\#N})$ so we let $d^{\#}\mu_{k}^{\#}=Ext\text{-}\exp\left(-\frac{x_{k}^{2}}{2}\right)d^{\#}x_{k}$ and define the operator $(Tf)(x)=\pi^{N/4}\left(Ext\text{-}\exp\left(Ext\text{-}\sum_{k=1}^{N}\frac{x_{k}^{2}}{2}\right)\right)$, Then T is a unitary map of $L_{2}^{\#}({}^{*}\mathbb{R}_{c}^{\#N})$ onto $L_{2}^{\#}\left({}^{*}\mathbb{R}_{c}^{\#N},Ext\text{-}\prod_{k=1}^{N}d^{\#}\mu_{k}^{\#}\right)$ and if we let $\mathrm{S}_{1}^{\#(N)}=T\mathrm{S}^{\#(N)}$ we get: 1) $\mathrm{S}_{1}^{\#(N)}:\mathscr{F}_{N}^{\#}\rightarrow L_{2}^{\#}\left({}^{*}\mathbb{R}_{c}^{\#N},Ext\text{-}\prod_{k=1}^{N}d^{\#}\mu_{k}^{\#}\right)$,

2) $\mathrm{S}_1^{\#(N)}\varphi_\varkappa^\#(g_k)\left(\mathrm{S}_1^{\#(N)}\right)^{-1} = x_k$, 3) $\mathrm{S}_1^{\#(N)}\pi_\varkappa^\#(g_k)\left(\mathrm{S}_1^{\#(N)}\right)^{-1} = -\frac{x_k}{i} + \frac{1}{i}\frac{d^\#}{d^\# x_k}$ and 4) $\mathrm{S}_1^{\#(N)}\Omega_0 = 1$, where 1 is the function identically one. Note that each #- measure $\mu_k^\#$ has mass one, which implies that

$$\langle\Omega_0, \left(Ext\text{-}\prod_{k=1}^N P_k\left(\varphi_\varkappa^\#(g_k)\right)\right)\Omega_0\rangle = \int_{{}^*\mathbb{R}_c^{\#N}}(Ext\text{-}\prod_{k=1}^N P_k(x_k))\left(Ext\text{-}\prod_{k=1}^N d^\#\mu_k^\#\right) = \tag{12.1}$$

$$= Ext\text{-}\prod_{k=1}^N \int_{{}^*\mathbb{R}_c^{\#N}} P_k(x_k)\, d^\#\mu_k^\# = Ext\text{-}\prod_{k=1}^N \int_{{}^*\mathbb{R}_c^{\#N}}\langle\Omega_0, P_k(\varphi_\varkappa^\#(g_k)\Omega_0)\rangle.$$

Here $P_1, \therefore, P_N$ are external finite and hyperfinite polynomials. Now we can to construct directly the $\sigma^\#$-measure space $\langle Q^\#, \mu^\#\rangle$. We define a space $Q^\# = \times_{k=1}^{*\infty} {}^*\mathbb{R}_c^\#$. Take the $\sigma^\#$-algebra generated by hyper infinite products of #-measurable sets in ${}^*\mathbb{R}_c^\#$ and set $\mu^\# = \bigotimes_{k=1}^{*\infty}\mu_k^\#$. We denote the points of $Q^\#$ symbolically by $q = \langle q_1, q_2, \ldots\rangle$, then $\langle Q^\#, \mu^\#\rangle$ is a $\sigma^\#$- measure space and the set of functions of the form $P(q_1, q_2, \ldots)$, where P is a polynomial and $n \in {}^*\mathbb{N}$ is arbitrary, is #-dense in $L_2^\#(Q^\#, d^\#\mu^\#)$. Let P be a polynomial in $N \in {}^*\mathbb{N}$ variables $P(x_1, x_2, \ldots, x_N) = Ext\text{-}\sum_{l_1,\ldots,l_N} c_{l_1,\ldots,l_N} x_{k_1}^{l_1}\cdots x_{k_N}^{l_N}$ and define $\mathbf{S}^\#: P\left(\varphi_\varkappa^\#(g_{k_1}), \ldots, \varphi_\varkappa^\#(g_{k_N})\right)\Omega_0 \to P(q_{k_1}, q_{k_2}, \ldots, q_{k_N})$. Then we get

$$\left(\varphi_\varkappa^\#(g_{k_1}), \ldots, \varphi_\varkappa^\#(g_{k_N})\right)\Omega_0 = Ext\text{-}\sum_{\boldsymbol{l},\boldsymbol{m}} c_{\boldsymbol{l}}\bar{c}_{\boldsymbol{m}}\left(\Omega_0, \varphi_\varkappa^\#(g_{k_1})^{l_1+m_1}, \ldots, \varphi_\varkappa^\#(g_{k_N})^{l_N+m_N}\Omega_0\right) =$$

$$Ext\text{-}\sum_{\boldsymbol{l},\boldsymbol{m}} c_{\boldsymbol{l}}\bar{c}_{\boldsymbol{m}}\int_{{}^*\mathbb{R}_c^{\#N}} q_{k_1}^{l_1+m_1}\times\ldots\times q_N^{l_N+m_N}\left(Ext\text{-}\prod_{i=1}^N d^\#\mu_{k_i}^\#\right) = Ext\text{-}\int_{Q^\#}\left|P(x_{k_1}, x_{k_2}, \ldots, x_{k_N})\right|^2 d^\#\mu^\#.$$

By the equation (99) and the fact that each measure $\mu_{k_i}^\#$ has mass one. Since Ω_0 is cyclic for polynomials in the fields, $\mathbf{S}^\#$extends to a unitary map of $\mathcal{F}_s^\#(H^\#)$ onto $L_2^\#(Q^\#, d^\#\mu^\#)$.

Theorem 12.1 [15] Let $\varphi_{m,\varkappa}^\#(x)$, $\varkappa \in {}^*\mathbb{R}_{c,\infty}^\#$be the free scalar field of mass m (in 4-dimensional space-time) at time zero. Let $g \in L_1^\#({}^*\mathbb{R}_c^{\#3}) \cap L_2^\#({}^*\mathbb{R}_c^{\#3})$ and define $H_{I,\varkappa,\lambda(\varkappa)}(g) = \lambda(\varkappa)\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} g(x){:}\ \varphi_{m,\varkappa}^{\#4}(x){:}\, d^{\#3}x\right)$, where $\lambda(\varkappa) \in {}^*\mathbb{R}_{c,\approx}^\#$. Let $\mathbf{S}^\#$ denote the unitary map $\mathbf{S}^\#: \mathcal{F}_s^\#(H^\#) \to L_2^\#(Q^\#, d^\#\mu^\#)$ constructed above. Then $V = \mathbf{S}^\# H_{I,\varkappa,\lambda}(g)\mathbf{S}^{\#-1}$is multiplication by a function $V_{\varkappa,\lambda}(q)$ which satisfies: (a) $V_{\varkappa,\lambda}(q) \in L_p^\#(Q^\#, d^\#\mu^\#)$ for all $p \in {}^*\mathbb{N}$. (b) $Ext\text{-}\exp\left(-tV_{\varkappa,\lambda}(q)\right) \in L_1^\#(Q^\#, d^\#\mu^\#)$ for all $t \in [0, {}^*\infty)$.

Proof (a) Note that $\varphi_{m,\varkappa}^\#(x)$ is a well-defined operator-valued function of $x \in {}^*\mathbb{R}_c^{\#3}$. We define now : $\varphi_{m,\varkappa}^{\#4}(x)$: by moving all the $a^\dagger$'s to the left in the formal expression for $\varphi_{m,\varkappa}^{\#4}(x)$. By Theorem 59 : $\varphi_{m,\varkappa}^{\#4}(x)$: is also a well-defined operator for each $x \in {}^*\mathbb{R}_c^{\#3}$. Notice that for each $x \in {}^*\mathbb{R}_c^{\#3}$ operator : $\varphi_{m,\varkappa}^{\#4}(x)$: takes F_0 into itself. Thus for each $x \in {}^*\mathbb{R}_c^{\#3}$ operator : $\varphi_{m,\varkappa}^{\#4}(x)$: reads : $\varphi_{m,\varkappa}^{\#4}(x) \coloneqq \varphi_{m,\varkappa}^{\#4}(x) + d_2(\varkappa)\,\varphi_{m,\varkappa}^{\#2}(x) + d_1(\varkappa)$ where the coefficients $d_1(\varkappa)$ and $d_2(\varkappa)$ are hyperfinite constant independent of x. For each $x \in {}^*\mathbb{R}_c^{\#3}$, $\mathbf{S}^\#\varphi_{m,\varkappa}^\#(x)(g)\mathbf{S}^{\#-1}$is the operator on #-measurable space $L_2^\#(Q^\#, d^\#\mu^\#)$ which acts by multiplying by the function $Ext\text{-}\sum_{k=1}^{*\infty} c_k(x,\varkappa)q_k$ where $c_k(x,\varkappa) = (2\pi)^{-3/2}\left(g_k, \left(Ext\text{-}\exp(ipx)\right)\chi(\varkappa,p)\mu(p)^{-1/2}\right)$ and $\chi(\varkappa,p) \equiv 1$ if $|p| \le \varkappa$, $\chi(\varkappa,p) \equiv 0$ if $|p| > \varkappa$. Note that

$$Ext\text{-}\sum_{k=1}^{*\infty}|c_k(x,\varkappa)|^2 = (2\pi)^{-3/2}\|\chi(\varkappa,p)\mu(p)\|_{\#2}^2, \tag{12.2}$$

so the functions $\mathbf{S}^\#\varphi_{m,\varkappa}^{\#4}(x)(g)\mathbf{S}^{\#-1}$ and $\mathbf{S}^\#\varphi_{m,\varkappa}^{\#2}(x)(g)\mathbf{S}^{\#-1}$ are in $L_2^\#(Q^\#, d^\#\mu^\#)$ and the $L_2^\#(Q^\#, d^\#\mu^\#)$ norms are uniformly bounded in x. Therefore, since $g \in L_1^\#({}^*\mathbb{R}_c^{\#3})$, $\mathbf{S}^\# H_{I,\varkappa,\lambda(\varkappa)}(g)\mathbf{S}^{\#-1}$operates on $L_2^\#(Q^\#, d^\#\mu^\#)$ by multiplication by some $L_2^\#(Q^\#, d^\#\mu^\#)$-function which we denote by $V_{I,\varkappa,\lambda(\varkappa)}(q)$. Consider now the expression for $H_{I,\varkappa,\lambda(\varkappa)}(g)\Omega_0$. This is a vector $(0,0,0,0,\psi^{\#4}, 0, \ldots)$ with

$$\psi^{\#4}(p_1,p_2,p_3,p_4) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}}\frac{\lambda(\varkappa)g(x)\chi(\varkappa,p)\left(Ext\text{-}\exp\left(-ix\sum_{i=1}^{i=4}p_i\right)\right)d^3x}{(2\pi)^{3/2}\prod_{i=1}^4[2\mu(p_i)]^{1/2}} = \frac{\lambda(\varkappa)\prod_{i=1}^4\chi(\varkappa,p_i)\left(Ext\text{-}\hat{g}\left(\sum_{i=1}^{i=4}p_i\right)\right)}{(2\pi)^{9/2}\prod_{i=1}^4[2\mu(p_i)]^{1/2}} \tag{12.3}$$

Here $|p_i| \leq \varkappa, 1 \leq i \leq 4$. We choose now the parameter $\lambda = \lambda(\varkappa) \approx 0$ such that $\|\psi^{\#4}\|_{\#2}^2 \in \mathbb{R}$ and therefore we obtain $\left\| H_{I,\varkappa,\lambda(\varkappa)}(g)\Omega_0 \right\|_{\#2}^2 \in \mathbb{R}$, since $\left\| H_{I,\varkappa,\lambda(\varkappa)}(g)\Omega_0 \right\|_{\#2}^2 = \|\psi^{\#4}\|_{\#2}^2$. But, since $\mathbf{S}^{\#}\Omega_0 = 1$, we get the equalities

$$\left\| H_{I,\varkappa,\lambda(\varkappa)}(g)\Omega_0 \right\|_{\#2} = \left\|\mathbf{S}^{\#} H_{I,\varkappa,\lambda(\varkappa)}(g)\mathbf{S}^{\#-1}\right\|_{L_2^{\#}(Q^{\#},d^{\#}\mu^{\#})} = \left\|V_{I,\varkappa,\lambda(\varkappa)}(q)\right\|_{L_2^{\#}(Q^{\#},d^{\#}\mu^{\#})}. \quad (12.4)$$

From (12.3)-(12.4) we get that $\left\|V_{I,\varkappa,\lambda(\varkappa)}(q)\right\|_{L_2^{\#}(Q^{\#},d^{\#}\mu^{\#})} \in \mathbb{R}$. It is easily verify that each polynomial $P(q_1, q_2, \ldots, q_n), n \in {}^*\mathbb{N}$ is in the domain of the operator $V_{I,\varkappa,\lambda(\varkappa)}(q)$ and $\mathbf{S}^{\#} H_{I,\varkappa,\lambda(\varkappa)}(g)\mathbf{S}^{\#-1} \equiv V_{I,\varkappa,\lambda(\varkappa)}(q)$ on that domain. Since Ω_0 is in the domain of $H^p{}_{I,\varkappa,\lambda(\varkappa)}(g), p \in {}^*\mathbb{N}$, 1 is in the domain of the operator $V^p{}_{I,\varkappa,\lambda(\varkappa)}(q)$ for all $p \in {}^*\mathbb{N}$. Thus, for all $p \in {}^*\mathbb{N}$ $V_{I,\varkappa,\lambda(\varkappa)}(q) \in L_{2p}^{\#}(Q^{\#}, d^{\#}\mu^{\#})$, since $\mu^{\#}(Q^{\#})$ is finite, we conclude that $V_{I,\varkappa,\lambda(\varkappa)}(q) \in L_p^{\#}(Q^{\#}, d^{\#}\mu^{\#})$ for all $p \in {}^*\mathbb{N}$. (b) Remind Wick's theorem asserts that $:\varphi_{m,\varkappa}^{\#j}(x) := \sum_{i=0}^{[j/2]}(-1)^i \frac{j!}{(j-2i)!i!} c_\varkappa^i \varphi_{m,\varkappa}^{\#(j-2i)}(x)$ with $c_\varkappa = \left\|\varphi_{m,\varkappa}^{\#}(x)\Omega_0\right\|_{\#2}^2$. For $j = 4$ we get $-O(c_\varkappa^2) \leq$ $:\varphi_{m,\varkappa}^{\#4}(x):$ and therefore $-\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} g(x)\, d^{\#3}x\right) O(c_\varkappa^2) \leq H_{I,\varkappa,\lambda(\varkappa)}(g)$.Finally we obtain $Ext\text{-}\int_{Q^{\#}} Ext\text{-}\exp\left(-t\left(:\varphi_{m,\varkappa}^{\#4}(x):\right)\right) d^{\#}\mu^{\#} \leq Ext\text{-}\exp\left(O(c_\varkappa^2)\right)$ and this inequality finalized the proof.

§13. GENERALIZED HAAG KASTLER AXIOMS

Definition 13.1 [15] A non- Archimedean Banach algebra $A_{\#}$ is a complex #-algebra over field ${}^*\mathbb{C}_c^{\#}$ (or ${}^*\mathbb{C}_{c,\text{fin}}^{\#} = {}^*\mathbb{R}_{c,\text{fin}}^{\#} + \mathrm{i}\,{}^*\mathbb{R}_{c,\text{fin}}^{\#}$) which is a non-Archimedean Banach space under a ${}^*\mathbb{R}_c^{\#}$ -valued -norm which is sub multiplicative, i.e., $\|xy\|_{\#} \leq \|x\|_{\#}\|y\|_{\#}$for all $x, y \in A_{\#}$. An involution on a non- Archimedean Banach algebra $A_{\#}$ is a conjugate-linear isometric antiautomorphism of order two denoted by $x \mapsto x^*$, i.e.,$(x + y)^* = x^* + y^*$, and for all $x, y \in A_{\#}$: $(xy)^* = y^*x^*$, $(\lambda x)^* = \bar{\lambda}x, (x^*)^* = x$, $\|x^*\|_{\#} = x$, $\lambda \in {}^*\mathbb{C}_c^{\#}$. A Banach #- algebra is a non-Archimedean Banach algebra with an involution.

Definition 13.2 An $\mathrm{C}_{\#}^*$-algebra is a Banach #-algebra $A_{\#}$ satisfying the $\mathrm{C}_{\#}^*$-axiom: for all $x \in A_{\#}$, $\|x^*x\|_{\#} = \|x\|_{\#}^2$.

Definition 13.3 1) A linear operator $a: H_{\#} \to H_{\#}$ on a non-Archimedean Hilbert space $H_{\#}$ is said to be bounded if there is a number $K \in {}^*\mathbb{R}_c^{\#}$ with $\|a\xi\|_{\#} \leq K\|\xi\|_{\#}$ for all $\xi \in H_{\#}$. 2) A linear operator $a: H_{\#} \to H_{\#}$ a non-Archimedean Hilbert space $H_{\#}$ is said to be finitely bounded if there is a number $K \in {}^*\mathbb{R}_{c,\text{fin}}^{\#}$ with $\|a\xi\|_{\#} \leq K\|\xi\|_{\#}$ for all $\xi \in H_{\#}$. The infimum of all such K if exists, is called the #-norm of a, written $\|a\|_{\#}$.

Abbreviation 13.1 The set of all finitely bounded operators $a: H_{\#} \to H_{\#}$ we will be denoting by $\mathcal{B}^{\#}(H_{\#})$.

Abbreviation 13.2 The set of all finitely bounded operators $a: H_{\#} \to H_{\#}$ we will be denoting by $\mathcal{B}_{\#}\,(H_{\#})$.

Remark 13.1 Note that $\mathcal{B}_{\#}\,(H_{\#})$ is a $\mathrm{C}_{\#}^*$-algebra over field ${}^*\mathbb{C}_{c,\text{fin}}^{\#}$.

Definition 13.4 If $S \subseteq \mathcal{B}^{\#}(H_{\#})$ (or $\mathcal{B}_{\#}\,(H_{\#})$) then the commutant S' of S is $S' = \{x \in \mathcal{B}^{\#}(H_{\#}) | \forall a \in S(xa = ax)\}$.

Remark 13.2 The algebra $\mathcal{B}^{\#}(H_{\#})$ of bounded linear operators on a non-Archimedean Hilbert space $H_{\#}$ is a $\mathrm{C}_{\#}^*$-algebra with involution $T \to T^*, T \in \mathcal{B}^{\#}(H_{\#})$. Clearly, any #-closed #-selfadjoint subalgebra of $\mathcal{B}^{\#}(H_{\#})$ is also a $\mathrm{C}_{\#}^*$-algebra.

Remark 13.3 We will be especially concerned with #-separable Hilbert Spaces where there is an orthonormal basis, i.e. a hyper infinite sequence , $\{\xi_i\}_{i=1}^{*\infty}$ of unit vectors with $\langle \xi_i, \xi_j \rangle = 0$ for $i \neq j$ and such that 0 is the only element of $H_{\#}$ orthogonal to all the ξ_i.

Definition 13.5 1) The topology on $\mathcal{B}^{\#}(H_{\#})$ (or $\mathcal{B}_{\#}\,(H_{\#})$ of pointwise #-convergence on $H_{\#}$ is called the strong operator topology. A basis of neighbourhoods of $a \in \mathcal{B}^{\#}(H_{\#})$ (or $a \in \mathcal{B}_{\#}\,(H_{\#})$ is formed by the following way

$$N(a, \{\xi_i\}_{i=1}^n, \varepsilon) = \{b | \|(b-a)\xi_i\|_{\#} < \varepsilon, \forall i(1 \leq i \leq n)\}.$$

2) The weak operator topology is formed by the basic neighbourhoods

$$N(a, \{\xi_i\}_{i=1}^n, \{\eta_i\}_{i=1}^n, \varepsilon) = \{b | \langle (b-a)\xi_i, \eta_i \rangle < \varepsilon, \forall i(1 \leq i \leq n)\}.$$

Theorem 13.1 If $M = M^*$ is subalgebra of $\mathcal{B}^{\#}(H_{\#})$ (or $\mathcal{B}_{\#}(H_{\#})$ with $1 \in M$, then the following statements are equivalent: 1) $M = M''$; 2) M is strongly #-closed; 3) M is weakly #-closed.
Definition 13.6 A subalgebra of $\mathcal{B}^{\#}(H_{\#})$ (or $\mathcal{B}_{\#}(H_{\#})$ satisfying the conditions of Theorem 61is called a von Neumann #-algebra.
Theorem 13.2 [15] (Generalized Gelfand-Naimark theorem) Let A be a $C^*_{\#}$-algebra with unit. Then there exist a non-Archimedean Hilbert space $H_{\#}$ and an #-isometric homomorphism U of A into $B(H_{\#})$ such that $Ux^* = Ux^*, x \in A$.
Abbreviation 13.3 We denote by $M_4^{\#} = \{{}^*\mathbb{R}_c^{\#4}, (\cdot,\cdot)\}$, the vector space ${}^*\mathbb{R}_c^{\#4}$ with the Minkowski product: $(x, y) = x_0 y_0 - x_i y_i$, $i = 1,2,3$.
Statement of the Axioms [15]. Let $M_4^{\#}$ be Minkowski space over field ${}^*\mathbb{R}_c^{\#}$ of four space-time dimensions.

1. ***Algebras of Local Observables.*** To each finitely bounded #-open set $O \subset M_4^{\#}$ we assign a unital $C^*_{\#}$ -algebra

$$O \to \mathcal{B}_{\#}(O)$$

2. ***Isotony***. If $O_1 \subset O_2$, then $\mathcal{B}(O_1)$ is the unital $C^*_{\#}$ -subalgebra of the unital $C^*_{\#}$-algebra $\mathcal{B}(O_2)$:

$$\mathcal{B}_{\#}(O_1) \subset \mathcal{B}_{\#}(O_2).$$

This axiom allow us to form the algebra of all local observables

$$\mathcal{B}_{\#\text{loc}} = \bigcup_{O \subset M_4^{\#}} \mathcal{B}_{\#}(O).$$

The algebra $\mathcal{B}_{\#\text{loc}}$ is a well-defined $C^*_{\#}$ -algebra because given any $O_1, O_2 \subset M_4^{\#}$, both $\mathcal{B}_{\#}(O_1)$ and $\mathcal{B}_{\#}(O_2)$ are subalgebras of the $C^*_{\#}$ -algebra$\mathcal{B}_{\#}(O_1 \cup O_2)$. From there one can take the #-norm completion to obtain

$$\mathcal{B}_{\#} = \#\text{-}\overline{\mathcal{B}_{\#\text{loc}}} \,,$$

called the algebra of quasi-local observables. This gives a $C^*_{\#}$ -algebra in which all the local observable $C^*_{\#}$ -algebras are embedded.

3. ***Poincare ≈ -Covariance.*** For each Poincare transformation $g \in {}^{\sigma}P_+^{\uparrow}$, there is a $C^*_{\#}$- isomorphism $\alpha_g : \mathcal{B}_{\#} \to \mathcal{B}_{\#}$ such that

$$\alpha_g(\mathcal{B}_{\#}(O)) \approx \mathcal{B}_{\#}(g(O)),$$

for all bounded #-open $O \subset M_4^{\#}$. For fixed $g \in \mathcal{B}_{\#}$, the map $g \to \alpha_g(A)$ is required to be #-continuous.

3′. For each Poincare transformation $g \in {}^{\sigma}P_+^{\uparrow}$, there is a $C^*_{\#}$- isomorphism $\alpha_g : \mathcal{B}_{\#} \to \mathcal{B}_{\#}$ such that

$$\text{st}\left(\alpha_g(\mathcal{B}_{\#}(O))\right) = \text{st}\left(\mathcal{B}_{\#}(g(O))\right),$$

for all bounded #-open $O \subset M_4^{\#}$. For fixed $g \in \mathcal{B}_{\#}$, the map $g \to \alpha_g(A)$ is required to be #-continuous.

4. ***≈-Causality.*** If O_1 and O_2 are spacelike separated, then all elements of $\mathcal{B}_{\#}(O_1) \approx$ -commute with all elements of a $C^*_{\#}$ -algebra $\mathcal{B}_{\#}(O_2)$

$$[\mathcal{B}_{\#}(O_1), \mathcal{B}_{\#}(O_2)] \approx 0.$$

4′. If O_1 and O_2 are space-like separated, then the standard part of the all elements of $C^*_{\#}$ -algebra $\mathcal{B}_{\#}(O_1)$ commute with the standard part of the all elements of $C^*_{\#}$ -algebra $\mathcal{B}_{\#}(O_2)$

$$\mathrm{st}(\mathcal{B}_\#(O_1),\mathcal{B}_\#(O_2))=0.$$

Definition 13.7 If $O\subset M_4^\#$, we say x belongs to the future causal shadow of O if every past directed time-like or light-like trajectory beginning at x intersects with O. Essentially, O separates the past light cone of x.Likewise, we say x belongs to the past causal shadow of O if every future-directed timelike or lightlike trajectory beginning at x inter-sects with O. The causal completion or causal envelope $\hat{O}$ of O is the union of its future and past directed causal shadows. This definition of the causal completion $\hat{O}$ can be reformulated in terms of "causal complements," which are computationally easier to deal with. If $O\subset M_4^\#$, we define the causal complement O' of O to be the set of all points with are spacelike to all points in O. Then $O''=\hat{O}$ is the causal completion of O. One expects the observables localized to $\hat{O}$ to be completely determined by the observables localized to O, carrying the same information.

5. ***Time Evolution.***

$$\mathcal{B}_\#(\hat{O})=\mathcal{B}_\#(O).$$

6. ***Vacuum state and positive spectrum.*** There exists a faithful irreducible representation $\pi_0:\mathcal{B}_\#\to\boldsymbol{B}(H_\#)$ with a unique (up to a factor) vector $\Omega\in H_\#$ such that Ω is cyclic and Poincaré invariant, and such that unitary representation of translations, given by

$$U(x)\pi_0(A)\Omega=\pi(\alpha_x(A))\Omega,$$

where $A\in\mathcal{B}_\#$ and $\alpha_x(\cdot)$ is the $\mathrm{C}_\#^*$-isomorphism from Axiom 3 associated with translation by $x\in M_4^\#$, has Hermitian generators $P^\mu,\mu=1,2,3$ whose joint spectrum lies in the forward light cone. The last phrase is the most physically important here; it simply states that we have energy-momentum operators whose spectrum satisfies $E^2-\mathbf{P}^2\gg 0$, i.e, or in other words, that the energy $E\geq 0$ and nothing can move faster than the speed of light. The vector Ω is the vacuum state This axiom does not appear to be purely algebraic; we have had to introduce an non-Archimedean Hilbert space $H_\#$. In fact, we can rewrite the axiom in a completely algebraic but less transparent way as follows. We postulate that there exists an vacuum state ω_0 on the $\mathrm{C}_\#^*$ -algebra (i.e., a normalized, positive, bounded linear functional) such that the following holds $\omega_0(Q^*Q)=0$ for all $Q\in\mathcal{B}_\#$ of the form

$$Q(f,A)=Ext\text{-}\int f(x)\alpha_x(A)\,d^{\#4}x$$

where $A\in\mathcal{B}_\#$ and $f(x)$ is a #-smooth function whose Fourier transform has bounded support disjoint from the forward light-cone centered at the origin in $M_4^\#$.
Remind that in a quantum system with a Hamiltonian H, the Heisenberg picture dynamics is given by the canonical formula

$$A(t)=\{Ext\text{-}\exp[itH]\}A(0)\{Ext\text{-}\exp[-itH]\}.$$

Then $A(t)$ is the observable at time t corresponding to the time zero observable $A(0)$. In our model we have hyper finitely locally correct Hamiltonians $H(g)$ but no hyper infinitely global Hamiltonian, and we construct the Heisenberg picture dynamics nonetheless. We do this by restricting the observables to lie in the local algebras $\mathcal{B}_\#(O)$ and by using the finite propagation speed implicit in axiom 3.
Definition 13.8 Let $\mathcal{F}_n^\#$ be the space of symmetric $L_2^\#({}^*\mathbb{R}_c^{\#3n})$ functions defined on ${}^*\mathbb{R}_c^{\#3n}$, $\mathcal{F}_0^\#={}^*\mathbb{C}_c^\#$ and let $\boldsymbol{\mathcal{F}}^\#=Ext\text{-}\bigoplus_{n=0}^{{}^*\infty}\mathcal{F}_n^\#$, $\Omega_0=1\in{}^*\mathbb{C}_c^\#\subset\mathcal{F}^\#$. Let S_n be the projection of $L_2^\#({}^*\mathbb{R}_c^{\#3n})$ onto $\mathcal{F}_n^\#$and let $D_\#$ be the #-dense domain in $\mathcal{F}^\#$ spanned algebraically by Ω_0 and vectors of the form $S_n(Ext\text{-}\prod_{k=1}^n f_k(k_n))$ where $f_k\in S_{\mathrm{fin}}^\#({}^*\mathbb{R}_c^{\#3},{}^*\mathbb{R}_c^{\#3}),n\in{}^*\mathbb{N}$.
Definition 13.9 We set now

$$H_{0,\varkappa} = Ext\text{-}\int \frac{1}{2} : \left(\pi_{\varkappa}^2(\boldsymbol{x}) + \nabla^{\#} \varphi_{\varkappa}^2(\boldsymbol{x}) + m^2 \varphi_{\varkappa}^2(\boldsymbol{x})\right) : d^{\#3}\boldsymbol{x}. \qquad (13.1)$$

Theorem 13.3 As the bilinear form on the domain $D_{\#} \times D_{\#}$

$$H_{0,\varkappa} = Ext\text{-}\int_{|\boldsymbol{k}| \leq \varkappa} \mu(k)\, a^{\dagger}(\boldsymbol{k}) a(\boldsymbol{k}) d^{\#3}\boldsymbol{k}. \qquad (13.2)$$

Theorem 13.4 (1) The operator $H_0 = H_{0,\varkappa}$ leaves each subdomain $D_{\#} \cap \mathcal{F}_n^{\#}$ invariant. (2) The operator $H_0 = H_{0,\varkappa}$ is essentially self-#-adjoint as an operator on the domain $D_{\#}$.
Definition 13.10 We set now

$$\varphi_{\varkappa,0}^{\#}(x,t) = Ext\text{-}\exp(itH_0) \varphi_{\varkappa}^{\#}(x) Ext\text{-}\exp(-itH_0) \qquad (13.3)$$

$$\pi_{\varkappa,0}^{\#}(x,t) = Ext\text{-}\exp(itH_0) \pi_{\varkappa}^{\#}(x) Ext\text{-}\exp(-itH_0) \qquad (13.4)$$

$$\varphi_{\varkappa,0}^{\#}(f,t) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \varphi_{\varkappa,0}^{\#}(x,t)\, f(x) d^{\#3}x \qquad (13.5)$$

$$\pi_{\varkappa,0}^{\#}(f,t) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \pi_{\varkappa,0}^{\#}(x,t)\, f(x) d^{\#3}x. \qquad (13.6)$$

Here $\varphi_{\varkappa}^{\#}(x)$ and $\pi_{\varkappa}^{\#}(x)$ is given by formulas (97) and (98) respectively.
Remark 13.4 Note that $\varphi_{\varkappa,0}^{\#}(x,t)$ and $\pi_{\varkappa,0}^{\#}(x,t)$ are bilinear forms defined on $D_{\#} \times D_{\#}$.
Theorem 13.5 As bilinear forms on $D_{\#} \times D_{\#}$.

$$\varphi_{\varkappa,0}^{\#}(x,t) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \Delta_{\#}(x-y,t)\, \pi_{\varkappa}^{\#}(x) d^{\#3}y + Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \frac{\partial^{\#}}{\partial^{\#}t} \Delta_{\#}(x-y,t)\, \varphi_{\varkappa}^{\#}(x) d^{\#3}y \qquad (13.7)$$

$$\pi_{\varkappa,0}^{\#}(x,t) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \frac{\partial^{\#}}{\partial^{\#}t} \Delta_{\#}(x-y,t)\, \pi_{\varkappa}^{\#}(x) d^{\#3}y + Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \frac{\partial^{\#2}}{\partial^{\#}t^2} \Delta_{\#}(x-y,t)\, \pi_{\varkappa}^{\#}(x) d^{\#3}y \qquad (13.8)$$

Remark 13.5 Here $\Delta_{\#}(x-y,t)$ is the solution of the generalized Klein-Gordon equation

$$\frac{\partial^{\#2}}{\partial^{\#}t^2} \Delta_{\#}(x,t) - \frac{\partial^{\#2}}{\partial^{\#}x_1^2} \Delta_{\#}(x,t) - \frac{\partial^{\#2}}{\partial^{\#}x_2^2} \Delta_{\#}(x,t) - \frac{\partial^{\#2}}{\partial^{\#}x_3^2} \Delta_{\#}(x,t) + m^2 \Delta_{\#}(x,t) = 0 \qquad (13.9)$$

with Cauchy data $\Delta_{\#}(x,0) = 0, \frac{\partial^{\#}}{\partial^{\#}t} \Delta_{\#}(x,0) = \delta(x)$.
Remark 13.6 Note the distribution $\Delta_{\#}(x,t)$ has support in the double light-cone $|x| \leq |t|$.
Theorem 13.6 Let $f_1, f_2 \in S^{\#}({}^*\mathbb{R}_c^{\#3}, {}^*\mathbb{R}_c^{\#3})$. The operator $\varphi_{\varkappa,0}^{\#}(f,t) + \pi_{\varkappa,0}^{\#}(f,t)$ is essentially self-#-adjoint on the domain $D_{\#}$.
Definition 13.11 We introduce now the class $\Im\left(S^{\#}({}^*\mathbb{R}_c^{\#3})\right)$ of bilinear forms on $D_{\#} \times D_{\#}$ expressible as a linear combination of the forms

$$V = \sum_{j=0}^{n} \binom{n}{j} Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3n}} v(k)\, a^{\dagger}(k_1) \cdots a^{\dagger}(k_j) a(k_{j+1}) \cdots a(k_n) d^{\#3n}k \qquad (13.10)$$

with symmetric kernels $v(k) \in S^{\#}({}^*\mathbb{R}_c^{\#3})$ having real Fourier transforms.
Theorem 13.7 Let $V \in \Im\left(S^{\#}({}^*\mathbb{R}_c^{\#3})\right)$. Then $V$is essentially self-#-adjoint on $D_{\#}$.
Theorem 13.8 Let O be a bounded #-open region of vector space${}^*\mathbb{R}_c^{\#3}$ and let $\mathcal{M}_{\#}(O)$ be the von Neumann algebra generated by the field operators Ext-$\exp[i\varphi_{\varkappa}^{\#}(f)]$ with $f \in S^{\#}({}^*\mathbb{R}_c^{\#3}, {}^*\mathbb{R}_c^{\#3})$ and supp$f \subset O$. Let $g(x) = 0$ on ${}^*\mathbb{R}_c^{\#3} \backslash O$. Then Ext-$\exp[itH_I(g)] \in \mathcal{M}_{\#}(O)$ for all $t \in {}^*\mathbb{R}_c^{\#}$.
Definition 13.12 Let O be a bounded #-open region of space and let $\mathcal{B}_{\#}(O)$ be the von Neumann algebra generated by the operators Ext-$\exp\left[i\left(\varphi_{\varkappa}^{\#}(f_1) + \pi_{\varkappa}^{\#}(f_2)\right)\right]$ with $f_1, f_2 \in S^{\#}({}^*\mathbb{R}_c^{\#3}, {}^*\mathbb{R}_c^{\#3})$ and suppf_1, supp$f_2 \subset O$. Let O_t be the set of points with distance less than $|t|$ to O for any instant of the time t.

Theorem 13.9 $Ext\text{-}\exp(itH_0)\mathcal{B}_\#(O)Ext\text{-}\exp(-itH_0) \subset \mathcal{B}_\#(O_t)$.
Theorem 13.10 If O_1 and O_2 are disjoint bounded open regions of vector space ${}^*\mathbb{R}_c^{\#3}$ then the standard part of the operators in $\mathcal{B}_\#(O_1)$ commute with the standard part of the operators in operators in $\mathcal{B}_\#(O_2)$.
Theorem 13.11 Let $g \in L_2^\#(({}^*\mathbb{R}_c^{\#3}))$, and let $g = 0$ on open region O, then $Ext\text{-}\exp[itH_I(g)] \in \mathcal{B}_\#(O)'$ for all $t \in {}^*\mathbb{R}_c^\#$.
Theorem 13.12 [15**]** (Free field ≈-Causality) Let $f_1, f_2 \in S_{\text{fin}}^{\#}({}^*\mathbb{R}_c^{\#4}, {}^*\mathbb{R}_c^{\#4})$ with $\text{supp}f_1 \subset O_1, \text{supp}f_2 \subset O_2$. We set now $\varphi_{\varkappa,0}^\#(f_1) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#4}} \varphi_{\varkappa,0}^\#(x,t)\, f_1(x,t) d^{\#4}x$ and $\varphi_{\varkappa,0}^\#(f_2) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#4}} \varphi_{\varkappa,0}^\#(x,t)\, f_2(x,t) d^{\#4}x$. If region O_1 and region O_2 are space-like separated, then $[\varphi_{\varkappa,0}^\#(f_1), \varphi_{\varkappa,0}^\#(f_2)]\psi \approx 0$ for all near standard vector $\psi \in H_\#$.
Proof. The commutator $[\varphi_{\varkappa,0}^\#(f_1), \varphi_{\varkappa,0}^\#(f_2)]$ reads

$$[\varphi_{\varkappa,0}^\#(f_1), \varphi_{\varkappa,0}^\#(f_2)] = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#4}} d^{\#3}x_1 d^\# t_1 Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#4}} d^{\#3}x_2 d^\# t_1 \Delta_\varkappa^\#(x_1 - x_2, t_1 - t_2) f_1(x_1,t_1) f_2(x_1,t_1),$$

$$\Delta_\varkappa^\#(x_1 - x_2, t_1 - t_2) = \Xi_1(x_1 - x_2, t_1 - t_2; \varkappa) - \Xi_2(x_1 - x_2, t_1 - t_2; \varkappa), \text{ where}$$

$$\Xi_1(x_1 - x_2, t_1 - t_2; \varkappa) = Ext\text{-}\int_{|\boldsymbol{p}|\leq\varkappa} \{\exp\{[i\boldsymbol{p}(x_1 - x_2)] - i\omega(\boldsymbol{p})(t_1 - t_2)\}\} \frac{d^{\#3}\boldsymbol{p}}{\sqrt{\boldsymbol{p}^2 + m^2}},$$

$$\Xi_2(x_1 - x_2, t_1 - t_2; \varkappa) = Ext\text{-}\int_{|\boldsymbol{p}|\leq\varkappa} \{-\exp[[i\boldsymbol{p}(x_1 - x_2)] + i\omega(\boldsymbol{p})(t_1 - t_2)]\} \frac{d^{\#3}\boldsymbol{p}}{\sqrt{\boldsymbol{p}^2 + m^2}}.$$

Here $\varkappa \in {}^*\mathbb{R}_{c,\infty}^\#$, $\omega(p) = \sqrt{\boldsymbol{p}^2 + m^2}$. Define $\breve{\Xi}_1(x_1 - x_2, t_1 - t_2; \varkappa)$ and $\breve{\Xi}_2(x_1 - x_2, t_1 - t_2; \varkappa)$ by

$$\breve{\Xi}_1(x_1 - x_2, t_1 - t_2; \varkappa) = Ext\text{-}\int_{|\boldsymbol{p}|>\varkappa} \{\exp\{[i\boldsymbol{p}(x_1 - x_2)] - i\omega(\boldsymbol{p})(t_1 - t_2)\}\} \frac{d^{\#3}\boldsymbol{p}}{\sqrt{\boldsymbol{p}^2 + m^2}},$$

$$\Xi_2(x_1 - x_2, t_1 - t_2; \varkappa) = Ext\text{-}\int_{|\boldsymbol{p}|>\varkappa} \{-\exp[[i\boldsymbol{p}(x_1 - x_2)] + i\omega(\boldsymbol{p})(t_1 - t_2)]\} \frac{d^{\#3}\boldsymbol{p}}{\sqrt{\boldsymbol{p}^2 + m^2}}.$$

Note that: (a) $\breve{\Xi}_1(x_1 - x_2, t_1 - t_2; \varkappa) \approx 0$ and $\Xi_2(x_1 - x_2, t_1 - t_2; \varkappa) \approx 0$, (b) $\Xi_1(x_1 - x_2, t_1 - t_2; \varkappa)$ and $\Xi_2(x_1 - x_2, t_1 - t_2; \varkappa)$ are Lorentz ≈-invariant tempered distribution (see definition 4), since the distributions $\Xi_1(x_1 - x_2, t_1 - t_2)$ and $\Xi_2(x_1 - x_2, t_1 - t_2)$ defined by

$$\Xi_1(x_1 - x_2, t_1 - t_2; \varkappa) + \breve{\Xi}_1(x_1 - x_2, t_1 - t_2; \varkappa) = Ext\text{-}\int \{\exp[[i\boldsymbol{p}(x_1 - x_2)] - i\omega(\boldsymbol{p})(t_1 - t_2)]\} \frac{d\#^3\boldsymbol{p}}{\sqrt{\boldsymbol{p}^2 + m^2}}$$

$$\Xi_2(x_1 - x_2, t_1 - t_2; \varkappa) + \breve{\Xi}_2(x_1 - x_2, t_1 - t_2; \varkappa) = Ext\text{-}\int \{\exp[[-i\boldsymbol{p}(x_1 - x_2)] + i\omega(\boldsymbol{p})(t_1 - t_2)]\} \frac{d\#^3\boldsymbol{p}}{\sqrt{\boldsymbol{p}^2 + m^2}}$$

are Lorentz invariant by Theorem 56. From expression of the distribution $\Xi_2(x_1 - x_2, t_1 - t_2; \varkappa)$ by replacement $\boldsymbol{p} \to -\boldsymbol{p}$ we obtain

$$\Xi_2(x_1 - x_2, t_1 - t_2; \varkappa) = -Ext\text{-}\int_{|\boldsymbol{p}|>\varkappa} \{\exp[[i\boldsymbol{p}(x_1 - x_2)] + i\omega(\boldsymbol{p})(t_1 - t_2)]\} \frac{d\#^3\boldsymbol{p}}{\sqrt{\boldsymbol{p}^2 + m^2}}.$$

And therefore finally we get

$$\Delta_\varkappa^\#(x_1 - x_2, t_1 - t_2) = Ext\text{-}\int_{|\boldsymbol{p}|\leq\varkappa} \sin[\omega(\boldsymbol{p})(t_1 - t_2)] \exp[i\boldsymbol{p}(x_1 - x_2)] \frac{d\#^3\boldsymbol{p}}{\sqrt{\boldsymbol{p}^2 + m^2}}.$$

Thus for any points (x_1, t_1) and (x_2, t_2) separated by space-like interval we obtain that $\Delta_\varkappa^\#(x_1 - x_2, t_1 - t_2) \approx 0$, since $\Delta_\varkappa^\#(x_1 - x_2, t_1 - t_2)$ is a Lorentz ≈-invariant tempered distribution.
Theorem 13.13 (Time zero free field ≈ -locality) Let $f_1, f_2 \in S_{\text{fin}}^{\#}({}^*\mathbb{R}_c^{\#3}, {}^*\mathbb{R}_c^{\#3})$ with $\text{supp}f_1 \subset O_1$, and $\text{supp}f_2 \subset$

O_2 are disjoint bounded open regions of vector space ${}^*\mathbb{R}_c^{\#3}$, then $\left[\varphi_{\varkappa,0}^{\#}(f_1,0),\varphi_{\varkappa,0}^{\#}(f_2,0)\right]\approx 0$.
Proof. It follows immediately from Theorem 11.12.
Theorem 13.14 Let O be a bounded #-open region of vector space ${}^*\mathbb{R}_c^{\#3}$, let $t\in{}^*\mathbb{R}_c^{\#}$, let g be a nonnegative function in $L_1^{\#}({}^*\mathbb{R}_c^{\#3})\cap L_2^{\#}({}^*\mathbb{R}_c^{\#3})$ and let g be identically equal to one on O_t.For $A\in\mathcal{B}_{\#}(O)$, then

$$\sigma_t(A)=\{Ext\text{-}\exp[itH(g)]\}A\{Ext\text{-}\exp[-itH(g)]\}$$

is independent of g and $\sigma_t(A)\in\mathcal{B}_{\#}(O_t)$.
Proof. Let $\sigma_t^0(A)=\{Ext\text{-}\exp[itH_0]\}A\{Ext\text{-}\exp[-itH_0]\}$ and $\sigma_t^I(A)=\{Ext\text{-}\exp[itH_I]\}A\{Ext\text{-}\exp[-itH_I]\}$. Notice that generalized Trotter's product formula is valid for the unitary group $Ext\text{-}\exp\left[it\left(H_0+H_I(g)\right)\right]$. Thus we get the following product formula for the associated automorphism group:

$$\sigma_t(A)=\#\text{-}\lim_{n\to{}^*\infty}\left[\left(\sigma_{t/n}^0\sigma_{t/n}^I\right)^n(A)\right].\tag{13.11}$$

Each automorphism σ_t^I maps each $\mathcal{B}_{\#}(O_s)$ into itself and is independent of g on $\mathcal{B}_{\#}(O_s)$ for $|s|\ll|t|$. To see this, let $\chi(O_s)$ be the characteristic function of a set O_s. We assert that

$$\sigma_{t/n}^I(C)=\left\{Ext\text{-}\exp\left[i(t/n)H_I\left(\chi(O_s)\right)\right]\right\}C\left\{Ext\text{-}\exp\left[-i(t/n)H_I\left(\chi(O_s)\right)\right]\right\}\tag{13.12}$$

for any $C\in\mathcal{B}_{\#}(O_s)$ and that $\sigma_t^I(C)\in\mathcal{B}_{\#}(O_s)$. In other words the interaction automorphism has propagation speed zero and is independent of g on $\mathcal{B}_{\#}(O_s)$ for $|s|\ll|t|$. The theorem follows from (13.11), (13.14) and Theorem 13.9. To prove (13.11), we rewrite $H_I(g)=H_I\left(\chi(O_s)\right)+H_I(g[1-\chi(O_s)])$ as a sum of commuting self-#-adjoint operators. By Theorem 13.15 $Ext\text{-}\exp\left[itH_I\left(\chi(O_s)\right)\right]\in\mathcal{B}_{\#}(O_s)$ and so the right side of (13.3) belongs to $\mathcal{B}_{\#}(O_s)$.
By **Theorem 70**,

$$Ext\text{-}\exp[itH_I(g[1-\chi(O_s)])]\in\mathcal{B}_{\#}(O_s)'$$

and (13.11) follows.
Definition 13.13 Let B be a bounded #-open region of spacetime $M_4^{\#}$ and for any time t, let $B(t)=\{x|\,x,t\in B\}$ be the time t time slice of B. We define $\mathcal{B}_{\#}(B)$ to be the von Neumann algebra generated by

$$\cup_s\sigma_s\left(\mathcal{B}_{\#}\left(B(t)\right)\right).\tag{13.13}$$

Theorem 13.15 The generalized Haag-Kastler axioms (1)-(5) are valid for all these local algebras $\mathcal{B}_{\#}(B)$.
Proof (Except Lorentz rotations) The axioms (1) and (2) are obvious, while (4) follows easily from the finite propagation speed, Theorem 11.10, together with the time zero ≈-locality, Theorem 11.12. Because the time zero fields coincide with the time zero free fields, and because the time zero fields generate $\mathcal{B}_{\#}$ by Theorem 11.12 and the definition of the local algebras, the free field result carries over to our scalar model with interaction $H_I\neq 0$. In the Poincaré covariance axiom (3), the time translation is given by σ_t. Let $B+t$ be the time translate of the space time region $B\subset M_4^{\#}$. Then $(B+t)(s)=B(s-t)$ and so

$$\sigma_t\left[\cup_s\sigma_s\left(\mathcal{B}_{\#}\left(B(s)\right)\right)\right]=\cup_s\sigma_{s+t}\left(\mathcal{B}_{\#}\left(B(s)\right)\right)=\cup_s\sigma_s\left(\mathcal{B}_{\#}\left(B(s-t)\right)\right)=\cup_s\sigma_{s+t}\left(\mathcal{B}_{\#}\left(B(s+t)\right)\right)\tag{13.14}$$

Thus $\sigma_t\left(\mathcal{B}_{\#}(B)\right)=\mathcal{B}_{\#}(B+t)$ and axiom (3) is verified for time translations. Since the local algebras are #-norm dense in $\mathcal{B}_{\#}$ and since automorphisms of $C_{\#}^*$-algebras preserve the #-norm, σ_t extends to an automorphism of algebra $\mathcal{B}_{\#}$.
Definition 13.14 To define the space translation automorphism σ_s, we set now

$$P^\mu=Ext\text{-}\int_{\|p\|\ll\varkappa}p^\mu a^\dagger(p)a(p)\,d^{\#4}p,\mu=1,2,3;\ \sigma_t(A)=\{Ext\text{-}\exp[-ixP]\}A\{Ext\text{-}\exp[ixP]\}.\tag{13.15}$$

Then we get $\{Ext\text{-}\exp[-ixP]\}\varphi_{\varkappa}(x)\{Ext\text{-}\exp[ixP]\} = \varphi_{\varkappa}(x+y)$, $\{Ext\text{-}\exp[-ixP]\}\pi_{\varkappa}(x)\{Ext\text{-}\exp[ixP]\} = \varphi(x+y)$.

The following theorem completes the proof of Theorem 11.16 except for Lorentz rotations.

Theorem 13.16 The automorphism $\sigma_x(\mathcal{B}_{\#}(B)) = \mathcal{B}_{\#}(B+x), \mathrm{st}(\sigma_x)$ extends up to $C^*_{\#}$-automorphism of $\mathcal{B}_{\#}$, and $\langle x,t\rangle \to \mathrm{st}(\sigma_x)\mathrm{st}(\sigma_t) == \mathrm{st}(\sigma_t)\mathrm{st}(\sigma_x)$ defines a 4-parameter abelian automorphism group of $\mathcal{B}_{\#}$.

Theorem 13.17 Let O be a bounded #-open region of space and let $\mathcal{B}_{\#}(O)$ be the von Neumann algebra generated by the operators $Ext\text{-}\exp\left[i\left(\varphi_{\varkappa}(f_1)+\pi_{\varkappa}(f_2)\right)\right]$ where $f_1, f_2 \in \mathcal{E}^{\#}_{\mathrm{fin}}({}^*\mathbb{R}^{\#}_c)$ and $\mathrm{supp} f_1 \subset B, \mathrm{supp} f_2 \subset B$. Then

$$Ext\text{-}\exp(itH_0)\mathcal{B}_{\#}(O)Ext\text{-}\exp(-itH_0) \subset \mathcal{B}_{\#}(O_t).$$

Remark 13.7 We reformulate the theorem by saying that H_0 has propagation speed at most one.

In order to obtain automorphisms for the full Lorentz group and to complete the proof of Theorem 11.16, there are four separate steps.

1. The first step is to construct a self-#-adjoint locally correct generator for Lorentz rotations. This generator then defines a locally correct unitary group and automorphism group.
2. The second step is to prove this statement for the fields, by showing that the field $\varphi_{\varkappa}(x,t)$, considered as a non-standard operator valued function on a suitable domain, and is transformed locally correctly by our unitary group.
3. The third step is to show that the local algebras $\mathcal{B}_{\#}(B)$ are also transformed correctly.
4. The fourth final step is to reconstruct the Lorentz group automorphisms from the locally correct pieces given by the first three steps. This final step is not difficult as in in the case of the two dimensional spacetime $d=2$, see [16], [17],[18].

Let $H_{0,\varkappa}(x)$ denote the integrand in (13.1), where

$$H_{0,\varkappa} = Ext\text{-}\int H_{0,\varkappa}(\boldsymbol{x})d^{\#3}\boldsymbol{x} = Ext\text{-}\int \tfrac{1}{2}\colon\left(\pi^2_{\varkappa}(\boldsymbol{x}) + \nabla^{\#}\varphi^2_{\varkappa}(\boldsymbol{x}) + m^2\varphi^2_{\varkappa}(\boldsymbol{x})\right)\colon d^{\#3}\boldsymbol{x}\,. \tag{13.16}$$

The formal generator of classical Lorentz rotations is

$$M^{0k}_{\varkappa} = M^{0k}_{0,\varkappa} + M^{0k}_{I,\varkappa} = Ext\text{-}\int x^k H_{0\varkappa,}(\boldsymbol{x})d^{\#3}\boldsymbol{x} + Ext\text{-}\int x^k\colon P\,(\varphi_{\varkappa}(\boldsymbol{x}))\colon d^{\#3}\boldsymbol{x},\ k=1,2,3. \tag{13.17}$$

The local Lorentzian rotations are

$$M^{0k}_{\varkappa}\left(g^{(k)}_1, g^{(k)}_2\right) = \varepsilon H_{0,\varkappa} + H_{0,\varkappa}\left(g^{(k)}_1\right) + H_{I,\varkappa}\left(, g^{(k)}_2\right), H_{0,\varkappa}\left(g^{(k)}_1\right) = Ext\text{-}\int H_{0,\varkappa}(\boldsymbol{x})g^{(k)}_1(\boldsymbol{x})d^{\#3}\boldsymbol{x}. \tag{13.18}$$

We require that $0<\varepsilon$ and that: $g^{(k)}_1(x_1,x_2,x_3), g^{(k)}_2(x_1,x_2,x_3)$, $k=1,2,3$ be nonnegative $C^{*\infty}_0$ functions. In the second step we require more, for example that $\varepsilon + g^{(k)}_1(x_1,x_2,x_3) = x_k$ and $g^{(k)}_2(x_1,x_2,x_3) = x_k$, $k=1,2,3$ in some local space region. This region is contained in the Cartesian product $[\varepsilon, {}^*\infty)\times[\varepsilon, {}^*\infty)\times[\varepsilon, {}^*\infty)$. By using decomposing $H_{0,\varkappa}\left(g^{(k)}_1\right)$ into a sum of a diagonal and an off-diagonal term we obtain $H_{0,\varkappa}\left(g^{(k)}_1\right) =$

$$Ext\text{-}\int v^{(k)}_{D,\varkappa}(\boldsymbol{k},\boldsymbol{l})\,a^*(\boldsymbol{k})a(\boldsymbol{l})d^{\#3}\boldsymbol{k}d^{\#3}\boldsymbol{l} + Ext\text{-}\int v^{(k)}_{0D,\varkappa}(\boldsymbol{k},\boldsymbol{l})\,[a^*(\boldsymbol{k})a^*(\boldsymbol{l}) + a(-\boldsymbol{k})a(-\boldsymbol{l})]d^{\#3}\boldsymbol{k}d^{\#3}\boldsymbol{l} =$$

$$= H^D_{0,\varkappa}\left(g^{(k)}_1\right) + H^{0D}_{0,\varkappa}\left(g^{(k)}_1\right),$$

where

$$v^{(k)}_{D,\varkappa}(\boldsymbol{k},\boldsymbol{l}) = c_1\chi(\boldsymbol{k},\boldsymbol{l},\varkappa)(\mu(\boldsymbol{k})\mu(\boldsymbol{l}) + \langle\boldsymbol{k},\boldsymbol{l}\rangle + m^2)[\mu(\boldsymbol{k})\mu(\boldsymbol{l})]^{-1/2}\hat{g}^{(k)}_1(-k_1+l_1, -k_2+l_2, -k_3+l_3),$$

$$v_{0D,\varkappa}^{(k)}(\boldsymbol{k},\boldsymbol{l}) = c_2\chi(\boldsymbol{k},\boldsymbol{l},\varkappa)(-\mu(\boldsymbol{k})\mu(\boldsymbol{l}) - \langle\boldsymbol{k},\boldsymbol{l}\rangle + m^2)[\mu(\boldsymbol{k})\mu(\boldsymbol{l})]^{-1/2}\hat{g}_1^{(1)}(-k_1 - l_1, -k_2 - l_2, -k_3 - l_3),$$

and where $\boldsymbol{k} = (k_1, k_2, k_3), \boldsymbol{l} = (l_1, l_2, l_3), \langle\boldsymbol{k},\boldsymbol{l}\rangle = \sum_{i=1}^{3} k_i l_i$, $\chi(\boldsymbol{k},\boldsymbol{l},\varkappa) = 1$ if $|\boldsymbol{k}| \leq \varkappa$ and $|\boldsymbol{l}| \leq \varkappa$, otherwise $\chi(\boldsymbol{k},\boldsymbol{l},\varkappa) = 0$.

Theorem 13.18 (a) $v_{0D,\varkappa}^{(k)} \in L_2^{\#}({}^*\mathbb{R}_c^{\#3})$. (b) Function $v_{D,\varkappa}^{(k)}$ is the kernel of a nonnegative operator and $\varepsilon\mu(\boldsymbol{k})\delta(\boldsymbol{k} - \boldsymbol{l}) + \beta v_{D,\varkappa}^{(k)}$ is the kernel of a positive self-#-adjoint operator, for $\beta \geq 0$, these operators are real in configuration space.

Proof. The statement (a) is obvious. The statement (b) is proved by using a finite sequence of Kato perturbations. Let$v_\beta^{(k)} = \varepsilon\mu(\boldsymbol{k})\delta(\boldsymbol{k} - \boldsymbol{l}) + \beta v_{D,\varkappa}^{(k)}$ and let V_β and V_D denote the operators with kernels $v_\beta^{(k)}$ and $v_{D,\varkappa}^{(k)}$ correspondingly. The operator V_D is a sum of three terms of the form $A^* M_{g_1} A$ in configuration space, where $M_{g_1}$is multiplication by $g_1 \geq 0$. Thus $0 \leq V_D$. Moreover for γ sufficiently small, but chosen independently of β, we obtain $\gamma V_D \leq \frac{1}{2}V_0 \leq \frac{1}{2}(V_0 + \beta V_D) = \frac{1}{2}V_\beta$ and therefore $V_{\beta+\gamma} = V_\beta + \gamma V_D$ is a Kato perturbation, in the sense of bilinear forms. Consequently if the operator V_β is self-#-adjoint, so is $V_{\beta+\gamma}$ and $D\left(V_{\beta+\gamma}^{1/2}\right) = D\left(V_\gamma^{1/2}\right)$. Thus canonical finite induction starting from $V_0 = V_0^*$ shows that V_β is self-adjoint, for all $\beta \geq 0$.

Theorem 13.19 The operator $H_0^D\left(g_1^{(k)}\right)$ is nonnegative and $\varepsilon H_0 + \beta H_0^D\left(g_1^{(k)}\right)$ is self-#-adjoint, for all $\beta > 0$.

The main purpose of the third step is to give a covariant definition of the local algebras $\mathcal{B}_{\#}(B)$. Le $f \in \mathcal{E}_{\text{fin}}^{\#}(B)$ be the ${}^*\mathbb{R}_c^{\#3}$-valued function with support in B. Let $\{\alpha_i\}_{i=1}^{n}, n \in {}^*\mathbb{N}$ be finite hyperreal numbers and consider the expressions

$$\varphi_\varkappa^{\#}(f) = Ext\text{-}\int \varphi_\varkappa^{\#}(x,t)\, f(x,t) d^{\#3}x d^{\#}t \tag{13.19}$$

$$\varphi_\varkappa^{\#}(f,t) = Ext\text{-}\int \varphi_\varkappa^{\#}(x,t)\, f(x,t) d^{\#3}x \tag{13.20}$$

$$\Re(f) = Ext\text{-}\textstyle\sum_{i=1}^{n} \alpha_i \varphi_\varkappa^{\#}(f, t_i) \tag{13.21}$$

$$\pi_\varkappa^{\#}(f,t) = Ext\text{-}\int \pi_\varkappa^{\#}(x,t)\, f(x,t) d^{\#3}x. \tag{13.22}$$

For $g \equiv 1$ on a sufficiently large set (the domain of dependence of the region B), the time integration in (1) #-converges strongly, and all four operators above are symmetric and defined on $D\big(H(g)\big)$.

Theorem 13.20 The operators (13.19)-(13.22) are essentially self-#-adjoint on any #-core for $H(g)^{1/2}$.

Theorem 13.21 The algebra $\mathcal{B}_{\#}(B)$ is the von Neumann algebra generated by finitely bounded functions of operators of the form (13.19).

Proof. Note that if a hyper infinite sequence $\{A_n\}$ of self-#-adjoins operators #-converges strongly to a self #-adjoint #-limit A on a core for A then the unitary operators $Ext\text{-}\exp(itA_n)$ #-converge strongly to $Ext\text{-}\exp(itA)$. Using this fact, one can easily show that the operators (1) and (4) generate the same von Neumann algebra, $\mathcal{B}_{\#1}(B)$ and that $\mathcal{B}_{\#1}(B) \supset \mathcal{B}_{\#}(B)$. To show that $\mathcal{B}_{\#1}(B) \subset \mathcal{B}_{\#}(B)$, recall that a self- #-adjoint operator A commutes with a finitely bounded operator C provided $CD \subset D(A)$ and $CA = AC$ on D, for some core D of A. Equivalently is the condition that the operator C commutes with all finitely bounded functions of A. Also equivalent is the relation $CA = AC$ on $D(A)$. We choose $D = D(H(g))$. If the operator C commutes with all operators of the form (13.20), it also commutes on $D(H(g))$ with all operators of the form (13.21). Hence we get $\mathcal{B}_{\#}(B)' \subset \mathcal{B}_{\#1}(B)'$ and so $\mathcal{B}_{\#1}(B) = \mathcal{B}_{\#1}(B)'' \subset \mathcal{B}_{\#}(B)'' = \mathcal{B}_{\#}(B)''$.

Remark 13.8 The Poincare group ${}^\sigma P_+^\uparrow$ is the semidirect product of the space-time translations group $\mathbb{R}^{1,3}$ with the Lorentz group $O(1,3)$ such that $\{a_1 + \Lambda_1\}\{a_2 + \Lambda_2\} = \{a_1 + \Lambda_1 a_2, \Lambda_1\Lambda_2\}$. Here $a \in \mathbb{R}^{1,3}$ and $\Lambda(\beta)\colon (x_i, t) \to \big(x_i \times \cosh(\beta) + t \times \sinh(\beta), x_i \times \sinh(\beta) + t \times \cosh(\beta)\big), i = 1,2,3$. We prove that there exists a representation $\sigma(a, \Lambda)$ of the Poincare group ${}^\sigma P_+^\uparrow$ by $*$ - automorphisms of $\mathcal{B}_{\#}$, such that $\sigma(a,\Lambda)\big(\mathcal{B}_{\#}(O)\big) = \mathcal{B}_{\#}(\{a,\Lambda\}O)$ for all bounded open sets O and all $\{a, \Lambda\} \in {}^\sigma P_+^\uparrow$. The Lorentz group composition law gives $\sigma(a,\Lambda) = \sigma(a, I)\sigma(0,\Lambda)$.

Obviously the existence of the automorphism representation $\sigma(a,\Lambda)$ follows directly from the construction of the pure Lorentz transformation $\sigma(0,\Lambda)=\sigma(\Lambda)$. One obtains $\sigma(\Lambda)$ by constructing locally correct infinitesimal generators. Formally, the operators,

$$M_{\varkappa}^{0k}=M_{0,\varkappa}^{0k}+M_{I,\varkappa}^{0k}=Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}}\frac{1}{2}\left\{:\pi_{\varkappa}(x)^2:+:\left(\nabla\varphi_{\varkappa}(x)\right)^2:+m^2:\varphi_{\varkappa}(x)^2:\right\}x^k d^{\#3}x+H_{I,\varkappa}(x^k g) \qquad (13.23)$$

$k=1,2,3$ are infinitesimal generators of Lorentz transformations in a region O if the cutoff function g equals one on a sufficiently large interval. We consider now the regions O_1 contained in the sets $\{x\in{}^*\mathbb{R}_c^{\#3}|\,x_1,x_2,x_3>|t|+1\}$. Thus for such regions O_1 we may replace (1) by $M^{0k}=Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}}H(x)\,x^k g(x)d^{\#3}x$, with a nonnegative functions $x^k g(x)$, $k=1,2,3$. Here $H(x)$ is the formally positive energy density:

$$H(x)=\frac{1}{2}\left\{:\pi_{\varkappa}(x)^2:+:\left(\nabla^{\#}\varphi_{\varkappa}(x)\right)^2:+m^2:\varphi_{\varkappa}(x)^2:\right\}+H_{I,\varkappa}(x)=H_{0,\varkappa}(x)+H_{I,\varkappa}(x).$$

Therefore M^{0k} is formally positive. In fact it is technically convenient to use different spatial cutoffs in the free and the interaction part of M^{0k}, $k=1,2,3$. Final formulas for $M_{\varkappa}^{0k}$ reads

$$M_{\varkappa}^{0k}=M_{\varkappa}^{0k}\left(g_0^k,g^k\right)=\alpha H_{0,\varkappa}+H_{0,\varkappa}(x^k g_0^k)+H_{I,\varkappa}(x^k g). \qquad (13.24)$$

Here

$$0<\alpha \text{ and } 0\le x^k g_0^k(x),0\le x^k g(x),k=1,2,3$$

and in order that (13.24) be formally correct, we assume that:

$$\alpha+x^k g_0^k=x^k=x^k g \qquad (13.24')$$

on $[1,R]^3=[1,R]\times[1,R]\times[1,R]$ with R sufficiently large.

For technical reasons we assume that:

$$\alpha+x^k g_0^k(x)=x^k,k=1,2,3 \text{ on } \operatorname{supp}(g).$$

By above restrictions on g_0^k and g we have that $\operatorname{supp}(g_0^k),\operatorname{supp}(g)\subset\{x|\alpha\le x^k,k=1,2,3\}$ and we show that the operator $M_{\varkappa}^{0k}$ is essentially self- #-adjoint and it generates Lorentz rotations in an algebra $\mathcal{B}_{\#}(O_1)$

$$Ext\text{-}\exp(i\beta M_{\varkappa}^{0k})\mathcal{B}_{\#}(O_1)Ext\text{-}\exp(-i\beta M_{\varkappa}^{0k})\subset\mathcal{B}_{\#}(\{a,\Lambda(\beta)\}O_1) \qquad (13.25)$$

provided that O_1 and $\{a,\Lambda(\beta)\}O_1$ are contained in the region

$$\{x\in{}^*\mathbb{R}_c^{\#3},t\in{}^*\mathbb{R}_c^{\#}|\,|t|+1<x_k<R-|t|,k=1,2,3\}, \qquad (13.26)$$

where M^{0k} is formally correct. These results permit us to define the Lorentz rotation automorphism $\sigma(\Lambda)$ on an arbitrary local algebra $\mathcal{B}_{\#}(O)$. Using a space time translation $\sigma(a)$, $a\in{}^*\mathbb{R}_c^{\#4}$ we can translate O into a region $O+a=O_1\subset\{x\in{}^*\mathbb{R}_c^{\#3},t\in{}^*\mathbb{R}_c^{\#}|\,x_1>|t|+1\}$ and for $R\in{}^*\mathbb{R}_c^{\#}$ large enough, O_1 and $\{a,\Lambda(\beta)\}O_1$ are contained in the region (1) we define $\sigma\left(0,\Lambda(\beta)\right)=\sigma\left(\Lambda(\beta)\right)$ by

$$\sigma\left(\Lambda(\beta)\right)\upharpoonright\mathcal{B}_{\#}(O)=\sigma(\{-\Lambda(\beta)a,I\})\sigma(\{0,\Lambda(\beta)\})\sigma(\{a,I\})\upharpoonright\mathcal{B}_{\#}(O).$$

Theorem 13.22 Let $M^{0k}(g_0,g)$, $k=1,2,3$ be given by (126), with $\alpha,g_0(x),g(x)$ restricted as mentioned above. Then $M^{0k}(g_0,g)$ is essentially self #-adjoint on $C^{*\infty}(H\cap H_0)$.

Theorem 13.23 Let O_1 and $\{0,\Lambda(\beta)\}O_1$ be contained in the set (1). Then the following identity holds between self-#-adjoint operators:

$$Ext\text{-}\exp(i\beta M^{0k})\varphi_{\varkappa}^{\#}(f)Ext\text{-}\exp(i\beta M^{0k}) \approx \varphi_{\varkappa}^{\#}\big(f(\{0,\Lambda(\beta)\}x)\big) =$$

$$\int_{{}^*\mathbb{R}_c^{\#4}} \varphi_{\varkappa}^{\#}\Big(f\big(\{0,\Lambda(\beta)\}(x,t)\big)\Big)\, d^{\#3}x d^{\#}t. \qquad (13.27)$$

Here provided $\operatorname{supp}(f) \subset O_1$.

The proof of the Theorem 13.23 is reduced to the verification of the following equations

$$\left\{x_k \frac{\partial^{\#}}{\partial^{\#}t} + t\frac{\partial^{\#}}{\partial^{\#}x_k}\right\}\varphi_{\varkappa}^{\#}(x,t) = [iM^{0k},\varphi_{\varkappa}^{\#}(x,t)], k = 1,2,3. \qquad (13.28)$$

Here (13.28) that is equation for bilinear forms on an appropriate domain. Since M^{0k} is self #-adjoint, we can integrate (13.28), thus we compute formally for $H = H_{0,\varkappa} + H_{I,\varkappa}(g)$,

$$[iM^{0k},\varphi_{\varkappa}^{\#}(x,t)] = [iM^{0k}, Ext\text{-}\exp(itH)\varphi_{\varkappa}^{\#}(x,t)Ext\text{-}\exp(-itH)] =$$

$$Ext\text{-}\exp(itH)[iM^{0k}(-t),\varphi_{\varkappa}^{\#}(x,0)]Ext\text{-}\exp(-itH). \qquad (13.29)$$

Here $M^{0k}(-t) = Ext\text{-}\exp(-itH)M^{0k}Ext\text{-}\exp(itH)$. Formally one obtains that

$$M^{0k}(-t) = Ext\text{-}\sum_{n=0}^{{}^*\infty} \frac{(-t)^n}{n!} ad^n(iH)(M^{0k}), k = 1,2,3.$$

Note that if M^{0k} and H were the correct global Lorentzian generators and Hamiltonian they would satisfy

$$[iH,M^{0k}] = ad\,(iH)(M^{0k}) = P^k, \big[iH,[iH,\ M^{0k}]\big] = 0,\ M^{0k}(-t) = \ M^{0k} - P^k t. \qquad (13.30)$$

Here $P^k, k = 1,2,3$ are the generators of space translations. Thus from (131) we get

$$[\ iM^{0k},\varphi_{\varkappa}^{\#}(x,0)] = [iM_0^{0k}] = x\pi_{\varkappa}^{\#}(x,0), [iP^k,\varphi_{\varkappa}^{\#}(x,0)] = -\nabla^{\#}(\varphi_{\varkappa}^{\#})(x,0).$$

Formally we have (130).However the difficulty with this formal argument is that H and M^{0k} do not obey (132) exactly, since they are correct only in O_1. We have instead (13.30) the equations

$$[iH,M^{0k}] = P_{loc}^k, \big[iH,[iH,\ M^{0k}]\big] = R_k^{loc}, k = 1,2,3. \qquad (13.31)$$

Here P_{loc}^k acts like the momentum operators only in the region O_1, i.e.

$$\big[P_{loc}^k,\varphi_{\varkappa}^{\#}(x,t)\big] = [P^k,\varphi_{\varkappa}^{\#}(x,t)], (x,t) \in O_1.$$

Hence $\big[iH,P_{loc}^k\big] = R_k^{loc}, k = 1,2,3$ is not identically zero, but commutes with $\mathcal{B}_{\#}(O_1)$. Formally, further commutators of $R_k^{loc}, k = 1,2,3$ with H are localized outside region O_1, and (13.28) follows formally even for our approximate, but locally correct H and M^{0k}. In order to convert this formal argument into a rigorous mathematical result, we apply now generalized Taylor series expansion [13] for the quantities

$$E_k(-t) = \langle\Omega,[\ iM^{0k}(-t),\varphi_{\varkappa}^{\#}(x,0)]\Omega\rangle, k = 1,2,3. \qquad (13.32)$$

Here $\Omega \in C^{{}^*\infty}(H)$ and thus we obtain

$$E_k(-t) = E_k(0) - t\frac{d^{\#}E_k(0)}{d^{\#}t} + \frac{t^2}{2}\frac{d^{\#2}E_k(\xi)}{d^{\#}t^2}, \text{ where } \xi \in [-t,t].$$

From (13.31) we obtain

$$\frac{d^{\#2}E_k(-\xi)}{d^{\#}t^2} = \langle Ext\text{-}\exp(i\xi H)\Omega, \left[iR_k^{loc}, \varphi_\varkappa^{\#}(x,\xi)\right] Ext\text{-}\exp(i\xi H)\Omega\rangle.$$

Note that $(x,t) \in O_1$, so that with $\xi \in [-t,t]$, $(x,\xi) \in O_1$ and therefore

$$\left[R_k^{loc}, \varphi_\varkappa^{\#}(x,\xi)\right] \equiv 0. \quad (13.33)$$

After integration over $x \in {}^*\mathbb{R}_c^{\#3}$ with a function $f \in S_{\text{fin}}^{\#}({}^*\mathbb{R}_c^{\#3})$ we obtain the operator identity:

$$Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \left[R_k^{loc}, \varphi_\varkappa^{\#}(x,\xi)\right] f(x) d^{\#3}x \equiv 0, k = 1,2,3. \quad (13.34)$$

Therefore $\frac{d^{\#2}E_k(\xi)}{d^{\#}t^2} \equiv 0$ if $|\xi| \le |t|$ and

$$E_k(-t) = E_k(0) - t\frac{d^{\#}E_k(0)}{d^{\#}t} = \langle \Omega, \{[\, iM^{0k}, \varphi_\varkappa^{\#}(x,0)] - t\left[P_{loc}^k, \varphi_\varkappa^{\#}(x,0)\right]\}\Omega\rangle =$$

$$= \langle \Omega, \{x\pi_\varkappa^{\#}(x,0) + t\nabla^{\#}(\varphi_\varkappa^{\#})(x,0)\}\, \Omega\,\rangle.$$

Thus we get

$$[\, iM^{0k}(-t), \varphi_\varkappa^{\#}(x,0)] = x\pi_\varkappa^{\#}(x,0) + t\nabla^{\#}\varphi_\varkappa^{\#}(x,0) \quad (13.35)$$

Inserting the relation (13.35) in (131) finally we obtain (13.28).This completes the proof of Lorentz covariance.

Definition 13.14 For the local free field energy we set $T_0(g) = T_0^1(g) + T_0^2(g)$, where

$$T_0^1(g) = c_1 Ext\text{-}\int_{|\boldsymbol{k}_1|\le\varkappa} d^{\#3}\,\boldsymbol{k}_1 Ext\text{-}\int_{|\boldsymbol{k}_2|\le\varkappa} d^{\#3}\,\boldsymbol{k}_2 \hat{g}(k_1^1 - k_2^1, k_1^2 - k_2^3, k_1^3 - k_2^3)\left\{\frac{\mu(\boldsymbol{k}_1)\mu(\boldsymbol{k}_2)+\langle \boldsymbol{k}_1,\boldsymbol{k}_2\rangle+m^2}{\sqrt{\mu(\boldsymbol{k}_1)\mu(\boldsymbol{k}_2)}}\right\}\times \quad (13.36)$$

$$a^{\dagger}(\boldsymbol{k}_1)a(\boldsymbol{k}_2) =$$

$$\sum_{i=1}^{i=3} c_1 Ext\text{-}\int_{|\boldsymbol{k}_1|\le\varkappa} d^{\#3}\,\boldsymbol{k}_1 Ext\text{-}\int_{|\boldsymbol{k}_2|\le\varkappa} d^{\#3}\,\boldsymbol{k}_2 \hat{g}(k_1^1 - k_2^1, k_1^2 - k_2^3, k_1^3 - k_2^3)\left\{\frac{\mu(\boldsymbol{k}_1)\mu(\boldsymbol{k}_2)+k_1^i k_2^i+m^2}{\sqrt{\mu(\boldsymbol{k}_1)\mu(\boldsymbol{k}_2)}}\right\} a^{\dagger}(\boldsymbol{k}_1)a(\boldsymbol{k}_2) =$$

$$= \sum_{i=1}^{3} T_{0,i}^1(g),$$

$$T_0^2(g) = c_2 Ext\text{-}\int_{|\boldsymbol{k}_1|\le\varkappa} d^{\#3}\,\boldsymbol{k}_1 Ext\text{-}\int_{|\boldsymbol{k}_2|\le\varkappa} d^{\#3}\,\boldsymbol{k}_2 \hat{g}(k_1^1 - k_2^1, k_1^2 - k_2^3, k_1^3 - k_2^3)\left\{\frac{-\mu(\boldsymbol{k}_1)\mu(\boldsymbol{k}_2)+\langle \boldsymbol{k}_1,\boldsymbol{k}_2\rangle+m^2}{\sqrt{\mu(\boldsymbol{k}_1)\mu(\boldsymbol{k}_2)}}\right\}\times \quad (13.37)$$

$$\times \{a^{\dagger}(\boldsymbol{k}_1)a^{\dagger}(-\boldsymbol{k}_2) + a(-\boldsymbol{k}_1)a(\boldsymbol{k}_2)\} =$$

$$\sum_{i=1}^{i=3} c_2 Ext\text{-}\int_{|\boldsymbol{k}_1|\le\varkappa} d^{\#3}\,\boldsymbol{k}_1 Ext\text{-}\int_{|\boldsymbol{k}_2|\le\varkappa} d^{\#3}\,\boldsymbol{k}_2 \hat{g}(k_1^1 - k_2^1, k_1^2 - k_2^3, k_1^3 - k_2^3)\left\{\frac{-\mu(\boldsymbol{k}_1)\mu(\boldsymbol{k}_2)+k_1^i k_2^i+m^2}{\sqrt{\mu(\boldsymbol{k}_1)\mu(\boldsymbol{k}_2)}}\right\}\times$$

$$\times \{a^{\dagger}(\boldsymbol{k}_1)a^{\dagger}(-\boldsymbol{k}_2) + a(-\boldsymbol{k}_1)a(\boldsymbol{k}_2)\} = \sum_{i=1}^{3} T_{0,i}^1(g).$$

Here $\boldsymbol{k}_1 = (k_1^1, k_1^2, k_1^3)$, $\boldsymbol{k}_2 = (k_2^1, k_2^2, k_2^3)$, $\langle \boldsymbol{k}_1, \boldsymbol{k}_2\rangle = \sum_{i=1}^{3} k_1^i\, k_2^i$, $\hat{g}(\boldsymbol{p}) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} (Ext\text{-}[i\langle \boldsymbol{p}, \boldsymbol{x}\rangle])g(x)\, d^{\#3}\boldsymbol{x}$.

Similarly, for the components of the local momentum we set $P^i(g) = P^{i(1)}(g) + P^{i(2)}(g), i = 1,2,3$ where

$$P^{i(1)}(g) = c_1 Ext\text{-}\int_{|\boldsymbol{k}_1|\leq\varkappa} d^{\#3}\,\boldsymbol{k}_1 Ext\text{-}\int_{|\boldsymbol{k}_2|\leq\varkappa} d^{\#3}\,\boldsymbol{k}_2 \hat{g}(k_1^1 - k_2^1, k_1^2 - k_2^2, k_1^3 - k_2^3) \times \tag{13.38}$$

$$\times \left\{\frac{k_1^i \mu(\boldsymbol{k}_2) + k_2^i \mu(\boldsymbol{k}_1)}{\sqrt{\mu(\boldsymbol{k}_1)\mu(\boldsymbol{k}_2)}}\right\} a^\dagger(\boldsymbol{k}_1) a(\boldsymbol{k}_2),$$

$$P^{i(2)}(g) = c_2 Ext\text{-}\int_{|\boldsymbol{k}_1|\leq\varkappa} d^{\#3}\,\boldsymbol{k}_1 Ext\text{-}\int_{|\boldsymbol{k}_2|\leq\varkappa} d^{\#3}\,\boldsymbol{k}_2 \hat{g}(k_1^1 - k_2^1, k_1^2 - k_2^3, k_1^3 - k_2^3) \times \tag{13.39}$$

$$\times \left\{\frac{k_1^i \mu(\boldsymbol{k}_2) - k_2^i \mu(\boldsymbol{k}_1)}{\sqrt{\mu(\boldsymbol{k}_1)\mu(\boldsymbol{k}_2)}}\right\} \{-a^\dagger(\boldsymbol{k}_1) a^\dagger(-\boldsymbol{k}_2) + a(-\boldsymbol{k}_1) a(\boldsymbol{k}_2)\}.$$

Definition 13.15 Let $\breve{P}_\varkappa(f)$ be the local operator, defined for $f \in S_{\mathrm{fin}}^{\#}({}^*\mathbb{R}_c^{\#3})$ by

$$\breve{P}_\varkappa(f) = H_{0\varkappa,}(f) - m^2 \int_{{}^*\mathbb{R}_c^{\#3}} :\varphi_\varkappa^{\#2}(x): f(x) d^{\#3}x \tag{13.40}$$

Theorem 13.24 Suppose that the operators $M^{0k}, k = 1,2,3$ and H are given by $M^{0k} = \alpha H_0 + T_0\left(x_k g_0^{(k)}\right) + T_I(x_k g_1),\ H \triangleq H_{0,\varkappa} + H_{I,\varkappa}$, where $H_0 \triangleq H_{0,\varkappa}$ and $T_I \triangleq H_{I,\varkappa}$. Then the following statements hold.
(1) For $k = 1,2,3$, $D((M^{0k})^2) \subset D(H\ ), D(H^2) \subset D(M^{0k})$.
(2) For $k = 1,2,3$, $D(M^{0k}) \subset D\left((H\ + b)^{\frac{1}{2}}\right), D(H\ ) \subset D\left((M^{0k}\ + b)^{\frac{1}{2}}\right)$, where b is a constant sufficiently large so that $H + b$ and $M +\ b$ are positive.
Proof By **Theorem 11.3**, $D((M^{0k})^2) \subset D(H_0 N_\varkappa)$ and $D(H^2) \subset D(H_0 N_\varkappa)$. Elementary estimates show that

$$D(N_\varkappa^2) \subset D\big(T_I(x_k g_1)\big) \cap D\big(T_I(g_1)\big)$$

and by Theorem **3.2.1**, we get $D(H_0) \subset D\left(T_0\left(x_k g_0^{(k)}\right)\right)$ and therefore $D(H_0 N_\varkappa) \subset D(M^{0k}) \cap D(H)$. This proves inclusions (1). Note that

$$D(H_0) \subset D\left((H\ + b)^{\frac{1}{2}}\right) \tag{13.40′}$$

By **Theorem 3.2.1**, the proof of (13.40′) extends to show that $D(H_0) \subset D\left((M + b)^{\frac{1}{2}}\right)$, since $D(M^{0k}) \cup D(H) \subset (H_0)$ the inclusions (2) hold.
Theorem 13.25 Let the operators $M^{0k}, k = 1,2,3$ are given by $M^{0k} = \alpha H_0 + T_0\left(x_k g_0^{(k)}\right) + T_I(x_k g_1)$, where $H_0 \triangleq H_{0,\varkappa}$ and $T_I \triangleq H_{I,\varkappa}$. Then the following statements hold.
(1) For $l = 2,3,4, k = 1,2,3$

$$M^{0k}: D(H^l) \to D(H^{l-2}). \tag{13.41}$$

(2) As operator equalities on $D(H^3)$ for $k = 1,2,3$,

$$[iH, M^{0k}] = \sum_{i=1}^{i=3} P^i \left(\frac{d^\#\left(x_k g_0^{(k)}\right)}{d^\# x_i}\right). \tag{13.42}$$

(3) As operator equalities on $D(H^4)$, for $k = 1,2,3$,

$$\left[iH, [iH, M^{0k}]\right] = \sum_{i=1}^{i=3} \breve{P}_\varkappa \left(\frac{d^{\#2}\left(x_k g_0^{(k)}\right)}{d^\# x_i^2}\right) - \sum_{i=1}^{i=3} T_I\left(\frac{d^\# g_1}{d^\# x_i}\right). \tag{13.43}$$

(4) The roles of H and M^{0k} can be interchanged in the following sense: for $l = 2,3,4$ and for $k = 1,2,3$,

$$H: D((M^{0k})^l) \to D((M^{0k})^{l-2}). \tag{13.41′}$$

The equalities (13.42) hold on the domain $D((M^{0k})^3)$, and on the domain $D((M^{0k})^4)$, for $k = 1,2,3$,

$$[iM^{0k},[iM^{0k},H]] =$$

$$\sum_{i=1}^{i=3} T_0\left(\left(\frac{d^\#}{d^\# x_i}\left(x_k g_0^{(k)}\right)\right)^2\right) + \sum_{i=1}^{i=3} T_I\left(\left(\frac{d^\#}{d^\# x_i}\left(x_k g_1\right)\right)\right) - \sum_{i=1}^{i=3} \breve{P}_\varkappa\left(\left(\alpha + x_k g_0^{(k)}\right)\frac{d^{\#2}}{d^\# x_k^2}\left(x_k g_0^{(k)}\right)\right) \qquad (13.43')$$

Remark 13.9 If condition (13.24′) also holds, then the double commutators (13.43) is formally localized outside a neighbourhood of the region $\mathfrak{R}^4_{[a,b]}$. It is this localization, made precise in the following sense: that these results in M^{0k} generating Lorentz transformations in the region $\mathfrak{R}^4_{[a,b]}$, see Definition 11.16.

Proof The case of (13.41) for $l = 2$ is covered by Theorem 13.24, which also defines M^{0k}, $k = 1,2,3$ as a bilinear forms on $D(H^2) \times D(H^2)$. From this and the fact that P, $\breve{P}_\varkappa$, and T_I are operators defined on $D(H_0 N_\varkappa) \supset D(H^2)$ it follows that the terms involved in (13.42) and (13.43) are defined as bilinear forms on $D(H^2) \times D(H^2)$. In **Lemma 6.6** we will prove that (13.42)-(13.43) hold as bilinear forms on $D(H^2) \times D(H^2)$. Assuming this, we now prove parts (1)-(3) of the theorem. Let $\chi, \psi \in D(H^3)$. We have for $k = 1,2,3$,

$$\langle H\chi, M^{0k}\psi\rangle_\# = \langle \chi, M^{0k}H\psi\rangle_\# - i\,\langle \chi, \sum_{i=1}^{i=3} P^i\left(\frac{d^\#\left(x_k g_0^{(k)}\right)}{d^\# x_i}\right)\psi\rangle_\#. \qquad (13.44)$$

Since, by Theorem 10.1 and Theorem 11.1

$$\|(H_0 + I)\Omega\|_\# \leq \text{const}\ \|(H + b)\Omega\|_\#. \qquad (13.45)$$

for all $\Omega \in D(H)$, it follows from Theorem 11.3 that

$$\|M^{0k}\Omega\|_\# \leq \|(H_0 + I)\Omega\|_\# + \text{const}\ \|N_\varkappa^2\Omega\|_\# \leq \text{const}\ \|(H + b)^2\Omega\|_\#, \qquad (13.46)$$

for all $\Omega \in D(H^2)$. Let $\Omega = H\psi$, then by (13.46) we obtain the inequality

$$|\langle \chi, M^{0k}H\psi\rangle_\#| \leq [\text{const}\|(H + b)^3\psi\|_\#]\|\chi\|_\#. \qquad (13.47)$$

Since by Theorem 9.8 and (13.45) we have the inequality

$$\left|\langle \chi, \sum_{i=1}^{i=3} P^i\left(\frac{d^\#\left(x_k g_0^{(k)}\right)}{d^\# x_i}\right)\psi\rangle_\#\right| \leq [\text{const}\|(H_0 + I)\,\psi\|_\#]\|\chi\|_\# \leq [\text{const}\|(H + b)\,\psi\|_\#]\|\chi\|_\#,$$

we get by (13.44) and (13.47) that

$$|\langle H\chi, M^{0k}\psi\rangle_\#| \leq [\text{const}\|(H + b)^3\psi\|_\#]\|\chi\|_\#. \qquad (13.48)$$

Hence $M^{0k}\psi \in D\left(\left(H \upharpoonright D(H^3)\right)^*\right)$ since H is essentially self #-adjoint on $D(H^3)$. This proves part (1) for $l = 3$. As a consequence, $i[H, M^{0k}], k = 1,2,3$, is an operators on $D(H^3)$ and by (13.44), we obtain

$$\langle \chi, [H, M^{0k}]\psi\rangle_\# = \langle \chi, \sum_{i=1}^{i=3} P^i\left(\frac{d^\#\left(x_k g_0^{(k)}\right)}{d^\# x_i}\right)\psi\rangle_\#.$$

for all $\chi, \psi \in D(H^3)$. This proves (13.42), since the χ's are #-dense.

The proof of (13.41) for the case $l = 4$ and the proof of (13.43) are similar. Let $\chi, \psi \in D(H^4)$. From (13.41) with $l = 2,3$, and the assumption that (13.43) is valid as a bilinear form, we have for $k = 1,2,3$

$$\langle H^2\chi, M^{0k}\psi\rangle_\# = -\langle\chi, M^{0k}H^2\psi\rangle_\# + 2\langle H\,\chi, M^{0k}H\psi\rangle_\# - \langle\chi, [iH,[iH,M^{0k}]]\psi\rangle_\# = \tag{13.49}$$

$$= -\langle\chi, M^{0k}H^2\psi\rangle_\# + 2\langle\chi, HM^{0k}H\,\psi\rangle_\# - \langle\chi, \left\{\sum_{i=1}^{i=3}\breve{P}_\varkappa\left(\frac{d^{\#2}\left(x_k g_0^{(k)}\right)}{d^\# x_i^2}\right) - \sum_{i=1}^{i=3}T_I\left(\frac{d^\# g_1^{(k)}}{d^\# x_i}\right)\right\}\psi\rangle_\#.$$

By (13.48), (13.46) and the inequality

$$\left|\langle\chi, \left\{\sum_{i=1}^{i=3}\breve{P}_\varkappa\left(\frac{d^{\#2}\left(x_k g_0^{(k)}\right)}{d^\# x_i^2}\right) - \sum_{i=1}^{i=3}T_I\left(\frac{d^\# g_1^{(k)}}{d^\# x_i}\right)\right\}\psi\rangle_\#\right| \leq \text{const}[\|(H_0+I)\,\psi\|_\# + \|(N^2+I)\,\psi\|_\#]\|\chi\|_\#$$

which follows directly from Theorem 11.3, we have from (13.49) the inequality

$$\langle H^2\chi, M^{0k}\psi\rangle_\# \leq [\text{const}\|(H\ +b)^4\psi\|_\#]\|\chi\|_\#.$$

Hence $M^{0k}\psi \in D\left(\left(H^2 \upharpoonright D(H^4)\right)^*\right) = D(H^2)$, proving (13.41) for the case $l = 4$. Thus $[iH,[iH,M^{0k}]]$ is an operators defined on $D(H^4)$, and we find from (13.49) that (13.43) holds.

The proof of parts (1)-(3) of the theorem is thus completed when we establish the equalities (13.42)-(13.43) in the sense of bilinear forms on $D(H^3)\times D(H^3)$ and $D(H^4)\times D(H^4)$ respectively. The proof of part (4) of the theorem is similar. For example, we replace the inequality (13.45) by the inequalities

$$\|(H_0+I)\Omega\|_\# \leq \text{const}\ \|(M^{0k}\ +b)\Omega\|_\#. \tag{13. 50}$$

for all $\Omega \in D(M^{0k})$. This also follows from Theorem 10.1 and Theorem 11.1. By Theorem 11.3, we replace (13.46) with

$$\|H\Omega\|_\# \leq \text{const}\ \|(M^{0k}\ \ +b)^2\Omega\|_\#, \tag{13.51}$$

To complete the proof of part (4) of the theorem, we need to establish **(**13.42) as a bilinear form on $D((M^{0k})^3)\times D((M^{0k})^3)$ and (13.43′) as a form on $D((M^{0k})^4)\times D((M^{0k})^4)$.

Theorem 13.26 As bilinear forms on $D(H_0)\times D(H_0)$ for $f,g \in S_{\text{fin}}^\#({}^*\mathbb{R}_c^{\#3})$

$$[iT_0(f), T_0(g)] = \sum_{i=1}^{i=3}P^i\left(f\frac{d^\# g}{d^\# x_i} - g\frac{d^\# f}{d^\# x_i}\right), \tag{13.52}$$

$$[iT_0(f), P^i(g)] = \breve{P}\left(f\frac{d^\# g}{d^\# x_i}\right) - T_0\left(g\frac{d^\# f}{d^\# x_i}\right). \tag{13.53}$$

The equalities (13.52)-(13.53) also hold if $f = 1$ or $g = 1$. In particular from (13.53) we get

$$[iH_0, P^i(g)] = \breve{P}\left(\frac{d^\# g}{d^\# x_i}\right). \tag{13.54}$$

Since $D(H_0) \supset D(H\,) \cup D(M^{0k})$, these equalities hold as forms on $D(H\,)\times D(H\,)$ and on $D(M^{0k})\times D(M^{0k})$.

Proof The operators $T_0, P, \breve{P}$ are #-closable (symmetric), defined on $D(H_0)$ and bounded as operators relative to $H_0 + I$. Therefore (13. 52)-(13. 53) are defined as bilinear forms on $D(H_0)\times D(H_0)$ and it suffices to establish equality on a core for H_0, e.g. on $D^\# = \left\{\psi \in \mathcal{F}^\# | \psi^{(n)} \in S_{\text{fin}}^\#({}^*\mathbb{R}_c^{\#3n}), \psi^{(m)} = 0 \text{ for all sufficiently large m}\right\}$. By direct calculations on $D^\#\times D^\#$ one obtains the equalities (13.44)-(13.46). For example

$$[iH_0, T_0^1(g)] = c_1 Ext\text{-}\int_{|\boldsymbol{k}_1|\leq\varkappa} d^{\#3}\,\boldsymbol{k}\ Ext\text{-}\int_{|\boldsymbol{k}_2|\leq\varkappa} d^{\#3}\,\boldsymbol{p}\hat{g}(k_1-p_1, k_2-2, k_3-p_3)\left\{\frac{\mu(\boldsymbol{k})\mu(\boldsymbol{p})+\langle\boldsymbol{k},\boldsymbol{p}\rangle+m^2}{\sqrt{\mu(\boldsymbol{k})\mu(\boldsymbol{p})}}\right\}\times$$
$$[H_0, a^\dagger(\boldsymbol{k})a(\boldsymbol{p})] =$$

$$ic_1 Ext\text{-}\int_{|\boldsymbol{k}_1|\le\varkappa} d^{\#3}\boldsymbol{k}\, Ext\text{-}\int_{|\boldsymbol{k}_2|\le\varkappa} d^{\#3}\boldsymbol{p}\,\hat{g}(k_1-p_1,k_2-2,k_3-p_3)\big(\mu(\boldsymbol{k})-\mu(\boldsymbol{p})\big)\left\{\frac{\mu(\boldsymbol{k})\mu(\boldsymbol{p})+\langle\boldsymbol{k},\boldsymbol{p}\rangle+m^2}{\sqrt{\mu(\boldsymbol{k})\mu(\boldsymbol{p})}}\right\}a^{\dagger}(\boldsymbol{k})a(\boldsymbol{p})$$

$$=\sum_{i=1}^{i=3}\left\{c_1 Ext\text{-}\int_{|k_1|\le\varkappa} d^{\#3}\boldsymbol{k}\, Ext\text{-}\int_{|k_2|\le\varkappa} d^{\#3}\boldsymbol{p}[(k_i-p_i)\hat{g}(k_1-p_1,k_2-2,k_3-p_3)]\left\{\frac{k_i\mu(\boldsymbol{p})+p_i\mu(\boldsymbol{k})}{\sqrt{\mu(\boldsymbol{k})\mu(\boldsymbol{p})}}\right\}\right\}=$$

$$=\sum_{i=1}^{i=3} P^{i(1)}\left(\frac{d^{\#}g}{d^{\#}x_i}\right), \qquad (13.55)$$

since the following equality holds

$$[\mu(\boldsymbol{k})-\mu(\boldsymbol{p})][\mu(\boldsymbol{k})\mu(\boldsymbol{p})+\langle\boldsymbol{k},\boldsymbol{p}\rangle+m^2]=\left\{\sum_{i=1}^{i=3}(k_i-p_i)\,[k_i\mu(\boldsymbol{p})+p_i\mu(\boldsymbol{k})]\right\}.$$

By a similar calculation on $D^{\#}\times D^{\#}$ one obtains

$$\left[iT_0^{(1)}(f),T_0^{(1)}(g)\right]+\left[iT_0^{(2)}(f),T_0^{(2)}(g)\right]=\sum_{i=1}^{i=3}P^{i(1)}\left(f\frac{d^{\#}g}{d^{\#}x_i}-g\frac{d^{\#}f}{d^{\#}x_i}\right).$$

The remaining calculations are similar.

Theorem 13.27 As bilinear forms on $D\big(H_{0,\varkappa}N_{\varkappa}\big)\times D\big(H_{0,\varkappa}N_{\varkappa}\big)$

$$[iT_I(h),T_0(f)]=-4\lambda Ext\text{-}\int_{^*\mathbb{R}_c^{\#3}} f(x)\,h(x){:}\,\varphi_{\varkappa}^{\#3}(x)\pi_{\varkappa}^{\#}(x){:}\,d^{\#3}x, \qquad (13.56)$$

$$[iT_I(h),P^i(f)]=-T_I\left(\frac{d^{\#}(fh)}{d^{\#}x_i}\right). \qquad (13.57)$$

Proof. The operators T_0,T_I,P are #-closable, defined on $D\big(H_{0,\varkappa}N_{\varkappa}\big)$, and are bounded as operators relative to $\big(H_{0,\varkappa}N_{\varkappa}+I\big)$. Note that the right hand side of (13.56) is a bilinear form on $D\big(H_{0,\varkappa}N_{\varkappa}\big)\times D\big(H_{0,\varkappa}N_{\varkappa}\big)$, and that $\big(H_{0,\varkappa}N_{\varkappa}+I\big)^{-1}\left[Ext\text{-}\int_{^*\mathbb{R}_c^{\#3}} f(x)\,h(x){:}\,\varphi_{\varkappa}^{\#3}(x)\pi_{\varkappa}^{\#}(x){:}\,d^{\#3}x\right]\big(H_{0,\varkappa}N_{\varkappa}+I\big)^{-1}$ is a bounded operator. Hence each term in (150)-(151) is a bilinear form on $D\big(H_{0,\varkappa}N_{\varkappa}\big)\times D\big(H_{0,\varkappa}N_{\varkappa}\big)$. It suffices to establish equality on $D^{\#}\times D^{\#}$, as in the proof of the **Theorem 84**, since $D^{\#}$ is a #-core for $H_{0,\varkappa}N_{\varkappa}$. Note that on the domain $D^{\#}\times D^{\#}$, the equalities (150)-(151) are seen to hold by direct computation in momentum space similarly to proof of the Theorem 11.27.

Remark 13.10 We assume now the relations:

$$0<\alpha,\ x_k g_i^{(k)}(x_1,x_2,x_3)=\left[h_i^{(k)}(x_1,x_2,x_3)\right]^2, k=1,2,3; i=0,1; h_i^{(k)}\in S_{\text{fin}}^{\#}(^*\mathbb{R}_c^{\#3}). \qquad (13.58)$$

On a neighbourhood of a polyhedron $[a,b]^3\subset {}^*\mathbb{R}_c^{\#}$, we assume for $k=1,2,3$

$$\alpha+x_k g_0^{(k)}(x_1,x_2,x_3)=x_k=x_k g_1(x_1,x_2,x_3). \qquad (13.59)$$

For all $x_k\in {}^*\mathbb{R}_c^{\#3}, k=1,2,3$, we assume

$$x_k g_1(x_1,x_2,x_3)=\left(\alpha+x_k g_0^{(k)}(x_1,x_2,x_3)\right)g_1(x_1,x_2,x_3). \qquad (13.60)$$

The conditions (13.60) are satisfied if $\alpha+x_k g_0^{(k)}(x_1,x_2,x_3)=x_k$ is valid on the support of g_1 for $k=1,2,3$. The condition (13.60) makes the required commutators densely defined operators, rather than bilinear forms.

Definition 13.16 Let $\mathfrak{R}_{[a,b]}^4$ be a set

$$\mathfrak{R}_{[a,b]}^4=\{(x_1,x_2,x_3,t)\in {}^*\mathbb{R}_c^{\#4}\,|\,a+|t|<x_k<b-|t| \text{ for all } k=1,2,3\}. \qquad (13.61)$$

Remark 13.10 Note that the operators M^{0k}, $k = 1,2,3$ are formally a Lorentz generators for the space-time region $\mathfrak{R}^4_{[a,b]}$, also note that (13.58) implies that interval $I = [a,b]$ lies in the positive half line. Of course, we can also consider the operators $\widetilde{M}^{0k} = -\alpha H_0 + T_0\left(x_k \tilde{g}_0^{(k)}\right) + T_I\left(x_k \tilde{g}_1^{(k)}\right)$ with $\tilde{g}_i^{(k)}(x) = g_i^{(k)}(-x)$ and therefore the operators $\widetilde{M}^{0k}$, $k = 1,2,3$ are locally correct generators for $\widetilde{\mathfrak{R}}^4_{[a,b]} = \mathfrak{R}^4_{[-a,-b]}$.

Definition 13.17 We also write $\mathfrak{R}^4_I$ instead $\mathfrak{R}^4_{[a,b]}$ for $I = [a,b]$ and we write I^3_s for $I^3 = [a-s, b+s]^3$. The conditions (13.58)-(13.60) are satisfied since we can choose $g_i^{(k)}$ so that for some $\varepsilon, 0 < \varepsilon < a/3$,

$$\operatorname{supp} g_1 \subset I^3_{2\varepsilon}; \operatorname{supp} g_0^{(k)} \subset I^3_{3\varepsilon},\ k = 1,2,3 \tag{13.62}$$

and $\alpha + x_k g_0^{(k)}(x_1,x_2,x_3) = x_k,\ x_k \in I^3_{2\varepsilon}$. Hence the conditions (154) hold. We can also let$g_1 = 1, x_k \in I^3_\varepsilon$; so the conditions (13.59) hold on I^3_ε. The Hamiltonian

$$H = H_{0\varkappa} + T_I(g_1) \tag{13.63}$$

is correct in the region $\mathfrak{R}^4_I$. We shall work as above with this particular choice of the Hamiltonian.

Theorem 13.28 For the operators M^{0k} in Theorem 11.25 and H in (13.63) the following hold:

(1) $D((M^{0k})^2) \subset D(H), D(H^2) \subset D(M^{0k}), k = 1,2,3$

(2) $D(M^{0k}) \subset D\left((H+b)^{\frac{1}{2}}\right), D(H) \subset D\left((M^{0k}+b)^{\frac{1}{2}}\right), k = 1,2,3$

where b is an constant sufficiently large so that the operators $H + b$ and $M^{0k} + b$ are positive.

Theorem 13.29 Ander the conditions (13.59) and (13.60) the equalities (13.42)-(13.43) and (13.43′) hold as bilinear forms on $D(H^2) \times D(H^2)$ and on $D((M^{0k})^2) \times D((M^{0k})^2)$.

Proof As bilinear forms on $D(H^2) \times D(H^2)$ or $D((M^{0k})^2) \times D((M^{0k})^2)$ for $k = 1,2,3$ the following equalities hold $[iH, M^{0k}] = \left[iH_0, T_0\left(x_k g_0^{(k)}\right)\right] + \left\{[iH_0, T_I(x_k g_1)] + [iT_I(g_1), \alpha H_0] + \left[iT_I(g_1), T_0\left(x_k g_0^{(k)}\right)\right]\right\}$. In order to compute these commutators we apply Theorem 11.27 and Theorem 11.28.

$$[iH, M^{0k}] = \sum_{i=1}^{i=3} P^i\left(\frac{d^\#\left(x_k g_0^{(k)}\right)}{d^\# x_i}\right) + 4\lambda Ext\text{-}\int_{{}^*\mathbb{R}^{\#3}_c}\left\{x_k g_1(x) - \alpha g_1(x) - x_k g_1(x) g_0^{(k)}(x)\right\}: \varphi^{\#3}_\varkappa(x)\pi^\#_\varkappa(x): d^{\#3}x = \sum_{i=1}^{i=3} P^i\left(\frac{d^\#\left(x_k g_0^{(k)}\right)}{d^\# x_i}\right). \tag{13.64}$$

By condition (13.60),

$$\frac{d^\#}{d^\# x_i}\left[\left(x_k - \alpha - x_k g_0^{(k)}(x_1,x_2,x_3)\right) g_1(x_1,x_2,x_3)\right] = 0.$$

And $x_k - \alpha - x_k g_0^{(k)}(x_1,x_2,x_3) = 0$ for $x_k \in \operatorname{supp}(g_1)$. Therefore for $i = k$ we get

$$g_1(x_1,x_2,x_3) = g_1(x_1,x_2,x_3)\frac{d^\#}{d^\# x_k}\left(x_k g_0^{(k)}(x_1,x_2,x_3)\right). \tag{13.65}$$

And for $i \neq k$, $x_i, x_k \in \operatorname{supp}(g_1)$ we get

$$\frac{d^\#}{d^\# x_i}\left(x_k g_0^{(k)}(x_1,x_2,x_3)\right) = 0. \tag{13.66}$$

From (13.64)-(13.66) we get for $i = k, k = 1,2,3$

$$[iH, M^{0k}] = P^k\left(\frac{d^{\#}\left(x_k g_0^{(k)}\right)}{d^{\#}x_k}\right). \tag{13.67}$$

And for $i \neq k$,

$$P^i\left(\frac{d^{\#}\left(x_k g_0^{(k)}\right)}{d^{\#}x_i}\right) = 0. \tag{13.68}$$

These equalities (13.67)-(13.68) hold by the conditions (13.60). Hence the equality (13.42) holds on $D(H^2) \times D(H^2)$ and on the domain $D((M^{0k})^2) \times D((M^{0k})^2)$. This proves (13.42).
Similarly, using Theorem 11.27 and Theorem 11.28, we compute in the sense of bilinear forms on $D(H^2) \times D(H^2)$ or on $D((M^{0k})^2) \times D((M^{0k})^2)$

$$\left[iH, P^i\left(\frac{d^{\#}\left(x_k g_0^{(k)}\right)}{d^{\#}x_i}\right)\right] = \left[iH_{0\varkappa}, P^i\left(\frac{d^{\#}\left(x_k g_0^{(k)}\right)}{d^{\#}x_i}\right)\right] + \left[iT_I(g_1),\ P^i\left(\frac{d^{\#}\left(x_k g_0^{(k)}\right)}{d^{\#}x_i}\right)\right] = \tag{13.69}$$

$$= \breve{P}_{\varkappa}\left(\frac{d^{\#2}\left(x_k g_0^{(k)}\right)}{d^{\#}x_i^2}\right) - T_I\left(\frac{d^{\#}x_k g_0^{(k)}}{d^{\#}x_i}\right).$$

From (13.65)-(13.66) and (13.68) we get for $i = k$, $k = 1,2,3$

$$\left[iH, P^k\left(\frac{d^{\#}\left(x_k g_0^{(k)}\right)}{d^{\#}x_k}\right)\right] = \breve{P}_{\varkappa}\left(\frac{d^{\#2}\left(x_k g_0^{(k)}\right)}{d^{\#}x_k^2}\right) - T_I\left(\frac{d^{\#}x_k g_0^{(k)}}{d^{\#}x_k}\right). \tag{13.70}$$

And for $i \neq k$,

$$\breve{P}_{\varkappa}\left(\frac{d^{\#2}\left(x_k g_0^{(k)}\right)}{d^{\#}x_i^2}\right) - T_I\left(\frac{d^{\#}x_k g_0^{(k)}}{d^{\#}x_i}\right) = 0. \tag{13.71}$$

These equalities (13.70)-(13.71) prove (13.43).
Similarly for $k = 1,2,3$

$$-\left[iM^{0k}, P^i\left(\frac{d^{\#}\left(x_k g_0^{(k)}\right)}{d^{\#}x_i}\right)\right] = -\alpha\left[iH_0, P^i\left(\frac{d^{\#}\left(x_k g_0^{(k)}\right)}{d^{\#}x_i}\right)\right] - \tag{13.72}$$

$$-\left[iT_0\left(x_k g_0^{(k)}\right), P^i\left(\frac{d^{\#}\left(x_k g_0^{(k)}\right)}{d^{\#}x_i}\right)\right] - \left[iT_I(x_k g_1), P^i\left(\frac{d^{\#}\left(x_k g_0^{(k)}\right)}{d^{\#}x_i}\right)\right] =$$

$$= -\alpha\breve{P}_{\varkappa}\left(\frac{d^{\#2}\left(x_k g_0^{(k)}\right)}{d^{\#}x_i^2}\right) - \breve{P}_{\varkappa}\left(x_k g_0^{(k)}\frac{d^{\#2}\left(x_k g_0^{(k)}\right)}{d^{\#}x_i^2}\right) + T_0\left(\left(\frac{d^{\#}\left(x_k g_0^{(k)}\right)}{d^{\#}x_i}\right)^2\right) + T_I\left(\frac{d^{\#}}{d^{\#}x_i}\left(x_k g_1\frac{d^{\#}\left(x_k g_0^{(k)}\right)}{d^{\#}x_i}\right)\right).$$

From (13.65)-(13.66) and (13.72) we get for $i = k$, $k = 1,2,3$

$$-\left[iM^{0k}, P^k\left(\frac{d^{\#}\left(x_k g_0^{(k)}\right)}{d^{\#}x_k}\right)\right] = -\alpha\left[iH_0, P^k\left(\frac{d^{\#}\left(x_k g_0^{(k)}\right)}{d^{\#}x_k}\right)\right] - \tag{13.73}$$

$$-\left[iT_0\left(x_k g_0^{(k)}\right), P^k\left(\frac{d^{\#}\left(x_k g_0^{(k)}\right)}{d^{\#}x_k}\right)\right] - \left[iT_I(x_k g_1), P^k\left(\frac{d^{\#}\left(x_k g_0^{(k)}\right)}{d^{\#}x_k}\right)\right] =$$

$$= -\alpha\breve{P}_{\varkappa}\left(\frac{d^{\#2}\left(x_k g_0^{(k)}\right)}{d^{\#}x_k^2}\right) - \breve{P}_{\varkappa}\left(x_k g_0^{(k)}\frac{d^{\#2}\left(x_k g_0^{(k)}\right)}{d^{\#}x_k^2}\right) + T_0\left(\left(\frac{d^{\#}\left(x_k g_0^{(k)}\right)}{d^{\#}x_k}\right)^2\right) + T_I\left(\frac{d^{\#}}{d^{\#}x_k}\left(x_k g_1\frac{d^{\#}\left(x_k g_0^{(k)}\right)}{d^{\#}x_k}\right)\right),$$

which simplifies to (13.43′) by condition (13.60).

Again let the operators $M^{0k}, k = 1,2,3$ and H are given by $M^{0k} = \alpha H_0 + T_0\left(x_k g_0^{(k)}\right) + T_I(x_k g_1),\ H \triangleq H_{0,\varkappa} + H_{I,\varkappa}$, where $H_0 \triangleq H_{0,\varkappa}$ and $T_I \triangleq H_{I,\varkappa}$ and assume that (13.58) and (13.60)) hold:

Theorem 13.30 If $n \geq 2$, $D(H^n)$ is a #-core for M^{0k} and $D((M^{0k})^n)$ is a #-core for H.

Proof $D(H^2) \subset D(M^{0k}),\ k = 1,2,3$ by Theorem 13.24. We prove first that $D(H^2)$ is a #-core for M^{0k}. Since $D((M^{0k})^2)$ is a #-core for M^{0k}, it suffices to show that

$$D\left(M^{0k} \upharpoonright D(H^2)\right) \supset D(H^2) \tag{13.74}$$

We use the smoothing operator, for $j = 1, 2, 3, \ldots$,

$$E_j = \left[1 + \tfrac{1}{j}(H + b)\right]^{-1}, \tag{13.75}$$

which has the following properties

$$E_j\colon D(H^l) \to D(H^{l+1}), \tag{13.76}$$

$$\left\|E_j\right\|_{\#} \leq 1, \tag{13.77}$$

$$\text{st.}\#\text{-}\lim_{j\to{}^*\infty} E_j, \tag{13.78}$$

and on $D(H)$, $\left[E_j, H\right] = 0$. Let $\psi \in D((M^{0k})^2)$. Since $D((M^{0k})^2) \subset D(H)$, $E_j\psi \in D(H^2)$, by (13.76). Since $E_j\psi \to \psi$ the desired inclusion (13.74) would follow from

$$M^{0k}E_j\psi \to M^{0k}\psi. \tag{13.79}$$

We now prove (13.79) for all $\psi \in D((M^{0k})^2)$. First we show that for $\Omega \in D(H^2), k = 1,2,3$,

$$M^{0k}E_j\Omega = E_j M^{0k}\Omega - \tfrac{i}{j}E_j P^k\left(\frac{d^{\#}\left(x_k g_0^{(k)}\right)}{d^{\#}x_i}\right)E_j\Omega. \tag{13.80}$$

Each term in (13.80) is defined since $D(H^2) \subset D(M^{0k}), k = 1,2,3,$ and P^k is defined on $D(H) \subset D(H_0)$. We now compute $\left[E_j, M^{0k}\right]$ on $D(H^2)$. If $\Omega \in D(H^2)$

$$\left[E_j, M^{0k}\right]\Omega = E_j E_j^{-1}\left[E_j, M^{0k}\right]E_j^{-1}E_j\Omega = E_j\left[M^{0k}, E_j^{-1}\right]E_j\Omega = \tfrac{1}{j}E_j[M^{0k}, H]E_j\Omega = \tfrac{i}{j}E_j P^k\left(\frac{d^{\#}\left(x_k g_0^{(k)}\right)}{d^{\#}x_i}\right)E_j\Omega.$$

Here we have used Theorem 13.25, part (1) and (6.17). Hence we have established (13.80) on the domain $D(H^2)$. Let $\psi \in D((M^{0k})^2), \Omega \in D(H^2)$. Since $M^{0k}, k = 1,2,3$ is self #-adjoint on $D(M^{0k})$,

$$\langle E_j M^{0k}\Omega, \psi\rangle_{\#} = \langle M^{0k}\Omega, E_j\psi\rangle_{\#} = \langle\Omega, M^{0k}E_j\psi\rangle_{\#}.$$

And

$$\langle M^{0k}E_j\Omega, \psi\rangle_{\#} = \langle\Omega, E_j M^{0k}\psi\rangle_{\#}.$$

Thus one obtains

$$\langle\Omega,[M^{0k}E_j]\psi\rangle_{\#}=\langle[E_jM^{0k}]\Omega,\psi\rangle_{\#}=\langle\frac{i}{j}\ E_jP^k\left(\frac{d^{\#}\left(x_kg_0^{(k)}\right)}{d^{\#}x_i}\right)E_j\Omega,\psi\rangle_{\#}=\langle\Omega,-\frac{i}{j}\ E_jP^k\left(\frac{d^{\#}\left(x_kg_0^{(k)}\right)}{d^{\#}x_i}\right)E_j\psi\rangle_{\#}.$$

Since $D(H^2)$ is #-dense,

$$M^{0k}E_j\psi=E_jM^{0k}\psi-\frac{i}{j}\,E_jP^k\left(\frac{d^{\#}\left(x_kg_0^{(k)}\right)}{d^{\#}x_i}\right)E_j\psi. \tag{13.81}$$

And therefore (13.80) holds on $D((M^{0k})^2)$.The strong #-convergence (13.78) now follows. By (13.77),

$$E_jM^{0k}E_j\psi\to E_jM^{0k}\psi.$$

And

$$\frac{1}{j}\left\|E_jP^k\left(\frac{d^{\#}\left(x_kg_0^{(k)}\right)}{d^{\#}x_i}\right)E_j\psi\right\|_{\#}\leq\frac{1}{j}\left\|P^k\left(\frac{d^{\#}\left(x_kg_0^{(k)}\right)}{d^{\#}x_i}\right)E_j\psi\right\|_{\#}\leq\text{const}\frac{1}{j}\left\|(H_0+I)\,E_j\psi\right\|_{\#}\leq$$

$$\leq\text{const}\frac{1}{j}\left\|(H_0+b)\,E_j\psi\right\|_{\#}=\text{const}\frac{1}{j}\left\|E_j(H_0+b)\,\psi\right\|_{\#}\leq\text{const}\frac{1}{j}\|(H_0+b)\,\psi\|_{\#}\to_{\#}0\text{ if }j\to{}^*\infty.$$

We have used the fact that $\psi\in D((M^{0k})^2)\subset D(H\,)\subset D(H_{\,0})$. Hence by (13.81),

$$M^{0k}E_j\psi\to M^{0k}\psi$$

which proves (13.79) and establishes that $D(H^2)$ is a #-core for M^{0k}. The inequality (13.46) and the fact that $D(H^n)$ for $n\geq 2$, is a #-core for H^2 shows that

$$D\left(M^{0k}\restriction D(H^2)\right)\supset D(H^2),.$$

Since $D(H^2)$ is a #-core, it follows that $D(H^n)$ is also a #-core for M^{0k}. The proof that $D((M^{0k})^n)$ is a #-core for H is similar, and follows the above proof by interchanging H with M^{0k}. In the following, we assume that M^{0k} and H are given by by

$$M^{0k}=\alpha H_0+T_0\left(x_kg_0^{(k)}\right)+T_I(x_kg_1),\ H\triangleq H_{0,\varkappa}+H_{I,\varkappa}\,,$$

where $H_0\triangleq H_{0,\varkappa}$ and $T_I\triangleq H_{I,\varkappa}$ and assume that (13.58-(13.60)) hold.

Theorem 13.31 Let $f\in S^{\#}_{\text{fin}}({}^*\mathbb{R}^{\#3}_c)$ and $\text{supp}f\subset\mathfrak{R}^4_{[a,b]}$, then the operator $\varphi^{\#}(f)$ is defined on $D((M^{0k})^2)$, $\varphi^{\#}(f)\colon D((M^{0k})^2)\to D(M^{0k}),k=1,2,3$ and, as the operator equalities on $D(M^{0k}),k=1,2,3$

$$[iM^{0k},\varphi^{\#}_{\varkappa}(f)]=-\varphi^{\#}_{\varkappa}\left(t\frac{\partial^{\#}f}{\partial^{\#}x_k}+x_k\frac{\partial^{\#}f}{\partial^{\#}t}\right). \tag{13.82}$$

Remark 13.11 Note that for f real, the operator $\varphi^{\#}_{\varkappa}(f)$ is essentially self #-adjoint on $D(H^n)$ for any $n\geq 1/2$ and

$$\varphi^{\#}_{\varkappa}(f)\colon D((H+b)^n)\to D\left((H+b)^{n-\frac{1}{2}}\right). \tag{13.83}$$

Proof The terms in (13.82) are operators on $D(H^3)$ since $\varphi^{\#}_{\varkappa}(f)D(H^3)\subset D(H^2)\subset D(M^{0k}),k=1,2,3$ and $M^{0k}D(H^3)\subset D(H\,)\subset D\left(\varphi^{\#}_{\varkappa}(f)\right)$ by (13.83) and Theorem 13.25. Note that by **Theorem 11.40** (13.82) holds on the domain $D(H^5)$. Assuming this, we now can to prove the theorem. Let $\psi \in D((M^{0k})^2), k = 1,2,3$. By Theorem **11.29**, $D((M^{0k})^2) \subset D(H)$ and by (13.83) we get $\psi \in D(\varphi_{\varkappa}^{\#}(f))$. Let us prove now that

$$\varphi_{\varkappa}^{\#}(f)\psi \in D(M^{0k}), k = 1,2,3. \tag{13.84}$$

Note that $M^{0k}\psi \in D(M^{0k}) \subset D\left((H+b)^{\frac{1}{2}}\right) \subset D(\varphi_{\varkappa}^{\#}(f))$ by Theorem 11.29 and (159), also for $k = 1,2,3$

$$\psi \in D\left(\varphi_{\varkappa}^{\#}\left(t\frac{\partial^{\#} f}{\partial^{\#} x_k} + x_k\frac{\partial^{\#} f}{\partial^{\#} t}\right)\right).$$

Therefore by the assumption mentioned above that (158) holds on domain $D(H^5)$, we get for all $k = 1,2,3$ and for all $\chi \in D(H^5)$ that

$$\langle M^{0k}\chi, \varphi_{\varkappa}^{\#}(f)M^{0k}\psi\rangle = \langle \chi, \varphi_{\varkappa}^{\#}(f)M^{0k}\psi\rangle + i\,\langle \chi, \varphi_{\varkappa}^{\#}\left(t\frac{\partial^{\#} f}{\partial^{\#} x_k} + x_k\frac{\partial^{\#} f}{\partial^{\#} t}\right)\psi\rangle. \tag{13.85}$$

So $\varphi_{\varkappa}^{\#}(f)\psi \in D\left(\left(M^{0k} \upharpoonright D(H^5)\right)^*\right)$ for $k = 1,2,3$. By Theorem **11.31**, $D(H^5)$ is a #-core for the $M^{0k}, k = 1,2,3$ and therefore we get inclusion (13.84). By using (13.84) we can rewrite (13.85) in the following equivalent form

$$\langle \chi, [M^{0k}, \varphi_{\varkappa}^{\#}(f)]\psi\rangle = \langle \chi, i\varphi_{\varkappa}^{\#}\left(t\frac{\partial^{\#} f}{\partial^{\#} x_k} + x_k\frac{\partial^{\#} f}{\partial^{\#} t}\right)\psi\rangle. \tag{13.86}$$

Since $D(H^5)$ is #-dense, we get$[M^{0k}, \varphi_{\varkappa}^{\#}(f)]\psi = i\varphi_{\varkappa}^{\#}\left(t\frac{\partial^{\#} f}{\partial^{\#} x_k} + x_k\frac{\partial^{\#} f}{\partial^{\#} t}\right)\psi$, proving (13.82) on the stated domains $D(M^{0k}), k = 1,2,3$.

Remark 13.12 Let us consider the self #-adjoint operators

$$M^{0k}(t) = Ext\text{-}\exp(-itH)M^{0k}Ext\text{-}\exp(itH), k = 1,2,3.$$

Since the operator $Ext\text{-}\exp(itH)$ leaves $D(H^n)$ invariant, we have by Theorem **11.29** and Theorem **11.26** that $D(H^2) \subset D(M^{0k}(t))$, $k = 1,2,3$ and for $l = 2, 3, 4$ we have that

$$M^{0k}(t): D(H^l) \to D(H^{l-2}), k = 1,2,3. \tag{13.87}$$

Let $f \in S_{\text{fin}}^{\#}({}^*\mathbb{R}_c^{\#4})$ with $\operatorname{supp} f \subset \mathfrak{R}_I^4$ for $I = [a,b]$. By (13.83) and (13.87) we can to conclude that $\varphi^{\#}(f)D(H^3) \subset D(H^2) \subset D(M^{0k}(t)), k = 1,2,3$ and $M^{0k}(t)D(H^3) \subset D(H) \subset D(\varphi_{\varkappa}^{\#}(f))$ or more generally, we can replace the operator $\varphi_{\varkappa}^{\#}(f)$ by $Ext\text{-}\exp(itH)\varphi_{\varkappa}^{\#}(f)Ext\text{-}\exp(-itH)$. Thus for $\psi \in D(H^3)$ and $f \in S_{\text{fin}}^{\#}({}^*\mathbb{R}_c^{\#4})$ **with** $\operatorname{supp} f \subset \mathfrak{R}_\Delta^4$, we can to define the functions

$$F_k(t) = \langle \psi, [iM^{0k}(t), \varphi_{\varkappa}^{\#}(f)]\psi\rangle = \langle \psi(t), [iM^{0k}, Ext\text{-}\exp(itH)\varphi_{\varkappa}^{\#}(f)Ext\text{-}\exp(-itH)]\psi(t)\rangle. \tag{13.88}$$

Where

$$\psi(t) = Ext\text{-}\exp(itH)\psi. \tag{13.89}$$

Let $I = [a,b]$, $I_\delta = [a-\delta, b+\delta]$ and let $\mathfrak{R}_{\Delta_\delta}$ be the causal shadow of $\Delta_\delta = I_\delta \times I_\delta \times I_\delta$. Let $\mathfrak{R}_s^4$ be a set

$$\mathfrak{R}_s^4 = \mathfrak{R}_{\Delta_{-|s|}} \cap \left\{(x,t)\,||t| < \tfrac{1}{2}\varepsilon\right\} = \left\{(x,t)\,||t| < \tfrac{1}{2}\varepsilon, a + |s| + |t| < b - |s| - |t|\right\}. \tag{13.90}$$

Note that the points of $\mathfrak{R}_s^4$ have small times, and $\mathfrak{R}_s^4$ translated by times less than $|s|$ lies in $\mathfrak{R}_\Delta^4$.

Theorem 13.32 Let $\psi \in D(H^5)$, then $F_k(t), k = 1,2,3$ in (13.88) is twice #-continuously #-differentiable. If function f has #-compact support in $\Re_s$, then for $|t| \leq |s|$, $\frac{d^{\#2} F_k(t)}{d^\# t^2} \equiv 0$.

Proof First we prove the #-differentiability of $F_k(t), k = 1,2,3$. Let Δ_n be the difference quotient for the n-derivative of $Ext\text{-}\exp(itH)$ at $t = 0$. For instance,

$$\Delta_1(\varepsilon) = \varepsilon^{-1}(Ext\text{-}\exp(i\varepsilon H) - I\ ).$$

Note that for a given vector $\psi \in D(H^n)$, and $m + j \leq n$, as $\varepsilon \to_\# 0$, we get

$$\left\| H^m\{\Delta_j(\varepsilon) - (iH)^j\}\psi \right\|_\# = \left\| \{\Delta_j(\varepsilon) - (iH)^j\} H^m \psi \right\|_\# \to_\# 0.$$

Hence, for $\psi \in D(H^n)$, the operator valued functions $M^{0k}(Ext\text{-}\exp(itH)\ )$ is $n - 2$ times #-differentiable, since for $j \leq n - 2$ we get $\left\| M^{0k}(Ext\text{-}\exp(itH)\ )\{\Delta_j(\varepsilon) - (iH)^j\}\psi \right\|_\# \leq \left\| \{\Delta_j(\varepsilon) - (iH)^j\}(H + b)^2 \psi \right\|_\# \to_\# 0$. All these functions $F_k(t)$ has the following form

$$F_k(t) = i\langle M^{0k}(Ext\text{-}\exp(itH)\ )\psi, Ext\text{-}\exp(itH)\varphi_\varkappa^\#(f)\psi\rangle - i\langle Ext\text{-}\exp(itH)\varphi_\varkappa^\#(f)\psi, M^{0k}(Ext\text{-}\exp(itH)\ )\psi\rangle.$$

For a given vector $\psi \in D(H^5)$, $\varphi_\varkappa^\#(f)\psi \in D(H^4)$ and $F_k(t)$ is three times #-continuously #-differentiable. Note that

$$\frac{d^\# F_k(t)}{d^\# t} = \langle M^{0k} H\psi(t), Ext\text{-}\exp(itH)\varphi_\varkappa^\#(f)\psi\rangle - \langle M^{0k}\psi(t), H(Ext\text{-}\exp(itH)\ )\psi\rangle - \tag{13.91}$$

$$-\langle Ext\text{-}\exp(itH)\varphi_\varkappa^\#(f)\psi, HM^{0k}\psi(t)\rangle + \langle Ext\text{-}\exp(itH)\varphi_\varkappa^\#(f)\psi, M^{0k}H\psi(t)\rangle.$$

By rearranging the terms in (13.91) and using the domain relations of Theorem **11.26.1** we obtain by **(143**) that

$$\frac{d^\# F_k(t)}{d^\# t} = \langle \psi, [H, M^{0k}(t)]\varphi_\varkappa^\#(f)\psi\rangle - \langle \varphi_\varkappa^\#(f)\psi, [H, M^{0k}(t)]\psi\rangle = \tag{13.92}$$

$$-i\,\langle \psi, (Ext\text{-}\exp(-itH)\ )P\left(\frac{d^\#\left(x_k g_0^{(k)}\right)}{d^\# x_k}\right)(Ext\text{-}\exp(itH)\ )\varphi_\varkappa^\#(f)\psi\rangle +$$

$$i\,\langle \varphi_\varkappa^\#(f)\psi, (Ext\text{-}\exp(-itH)\ )P\left(\frac{d^\#\left(x_k g_0^{(k)}\right)}{d^\# x_k}\right)(Ext\text{-}\exp(itH)\ )\psi\rangle.$$

By #-differentiating (13.92) and writing P_k for the operator $P\left(\frac{d^\#\left(x_k g_0^{(k)}\right)}{d^\# x_k}\right)$ we obtain

$$\frac{d^{\#2} F_k(t)}{d^\# t^2} = -\langle \psi, (Ext\text{-}\exp(-itH)\ )[H, P_k](Ext\text{-}\exp(itH)\ )\psi\rangle + \tag{13.93}$$

$$\langle \varphi_\varkappa^\#(f)\psi, (Ext\text{-}\exp(-itH)\ )[H, P_k](Ext\text{-}\exp(itH)\ )\psi\rangle =$$

$$i\,\langle \psi(t), \left[\check{P}\left(\frac{d^{\#2}\left(x_k g_0^{(k)}\right)}{d^\# x_k^2}\right) - T_I\left(\frac{d^\#(g_1)}{d^\# x_k}\right), (Ext\text{-}\exp(itH)\ )\varphi_\varkappa^\#(f)(Ext\text{-}\exp(-itH)\ )\psi\right]\rangle.$$

Note that the all terms in (13.93) are well defined. For instance, $HP_k(Ext\text{-}\exp(itH)\ )\varphi_\varkappa^\#(f)\psi$ is well defined since, for a given vector $\psi \in D(H^5)$, $(Ext\text{-}\exp(itH)\ )\varphi_\varkappa^\#(f)\psi \in D(H^4)$, and by **Theorem 11.26** for all $k = 1,2,3$ we obtain

$$P_k(Ext\text{-}\exp(itH)\ )\varphi_\varkappa^\#(f)\psi = [iH, M^{0k}](Ext\text{-}\exp(itH)\ )\varphi_\varkappa^\#(f)\psi.$$

Note that $HM^{0k}(D(H^4)) \subset D(H)$ and $M^{0k}H(D(H^4)) \subset D(H)$, so $HP_k(Ext\text{-}\exp(itH)\,)\varphi_{\varkappa}^{\#}(f)\psi$ is well defined. Now, assuming that $\mathrm{supp} f \subset \mathfrak{R}_s^4, |t| \leq |s|$ we can to show that $\frac{d^{\#2}F_k(t)}{d^{\#}t^2} \equiv 0, k = 1,2,3,$ this proof is based on the locality of the operators $S_k, k = 1,2,3$

$$S_k = \breve{P}_{\varkappa}\left(\frac{d^{\#2}\left(x_k g_0^{(k)}\right)}{d^{\#}x_k^2}\right) - T_I\left(\frac{d^{\#}g_1}{d^{\#}x_k}\right). \tag{13.94}$$

The operators S_k are symmetric on $D(H_0 N)$ and by (153) for $k = 1,2,3$ and $i = 1,2,3$

$$\frac{d^{\#2}\left(x_k g_0^{(k)}\right)}{d^{\#}x_k^2} = 0 = \frac{d^{\#}g_1}{d^{\#}x_k}$$

in a neighbourhood of $\Delta = [a,b]^3$. We prove that $S_k, k = 1,2,3$ commutes with the von Neumann algebra $W(I) = \left\{Ext\text{-}\exp\left(i\varphi_{\varkappa}^{\#}(h_1) + i\pi_{\varkappa}^{\#}(h_2)\right) | h_i = \overline{h_i} \in S_{\mathrm{fin}}^{\#}({}^{*}\mathbb{R}_c^{\#3}), \mathrm{supp} h_i \subset \mathfrak{R}_I\right\}''$ generated by the spectral projections of the time zero fields $Ext\text{-}\int_{{}^{*}\mathbb{R}_c^{\#3}} \varphi_{\varkappa}^{\#}(x)\, h_1(x) d^{\#3}x$ and $Ext\text{-}\int_{{}^{*}\mathbb{R}_c^{\#3}} \pi_{\varkappa}^{\#}(x)\, h_2(x) d^{\#3}x$, $h_i = \overline{h_i} \in S_{\mathrm{fin}}^{\#}({}^{*}\mathbb{R}_c^{\#3}), \mathrm{supp} h_i \subset \mathfrak{R}_I$.

Theorem 13.33 On the domain $D(H^2)$ for $k = 1,2,3$ the equalities hold

$$[S_k, W(I)]D(H^2) = 0. \tag{13.95}$$

Proof Let $D^{\#}$ be the domain of well-behaved vectors.

$$D^{\#} = \left\{\psi \in \mathcal{F}^{\#} | \psi^{(n)} \in S_{\mathrm{fin}}^{\#}({}^{*}\mathbb{R}_c^{\#3n}), \psi^{(m)} = 0 \text{ for all sufficiently large m}\right\}. \tag{13.96}$$

For $\chi_1, \chi_2 \in D^{\#}$, direct momentum space computation gives for all $n \in {}^{*}\mathbb{N}$

$$\langle S_k\chi_1, \left(\varphi_{\varkappa}^{\#}(h_1) + \pi_{\varkappa}^{\#}(h_2)\right)^n \chi_2\rangle = \langle\left(\varphi_{\varkappa}^{\#}(h_1) + \pi_{\varkappa}^{\#}(h_2)\right)^n \chi_1, S_k\chi_2\rangle \tag{13.97}$$

By easy computation we get the inequality $\left\|\left(\varphi_{\varkappa}^{\#}(h_1) + \pi_{\varkappa}^{\#}(h_2)\right)^n \chi\right\| \leq c_1 c_2^n (n!)^{\frac{1}{2}}$ for constants c_1 and c_2 depending on vector $\chi \in D^{\#}$. Therefore $\chi \in D^{\#}$ are entire vectors for the operator $\left(\varphi_{\varkappa}^{\#}(h_1) + \pi_{\varkappa}^{\#}(h_2)\right)$, and the sum

$$U\chi = Ext\text{-}\sum_{n=0}^{{}^{*}\infty} \frac{\left(i\varphi_{\varkappa}^{\#}(h_1) + i\pi_{\varkappa}^{\#}(h_2)\right)^n}{n!}\chi = Ext\text{-}\exp\left[i\left(\varphi_{\varkappa}^{\#}(h_1) + \pi_{\varkappa}^{\#}(h_2)\right)\right]\chi \tag{13.98}$$

#-converges strongly. Now, we multiply (13.97) by $i^n (n!)^{-1}$ and by summation over n using the #-convergence of the hyper infinite series (13.98) we get for all $k = 1,2,3$ that

$$\langle S_k\chi_1, U\chi_2\rangle_{\#} = \langle U^{*}\chi_1, S_k\chi_2\rangle_{\#} = \langle\chi_1, US_k\chi_2\rangle_{\#}$$

for $\chi_i \in D^{\#}, i = 1,2$. Note that this equality extends to $\chi_i \in D(H_{0\varkappa}N), i = 1,2$ since $D^{\#}$ is a core for operators $H_{0\varkappa}N$ and S_k and

$$\|S_k\chi\|_{\#} \leq \mu\|(H_{0\varkappa}N + I)\chi\|_{\#},$$

where μ is finite constant. Therefore for $\chi \in D(H_{0\varkappa}N)$, we have proved that $U\chi \in D(S_k^{*})$ and

$$S_k^{*}U\chi = US_k\chi, k = 1,2,3.$$

For the next step we now prove that $\chi \in D(H_{0\varkappa}N) \Rightarrow U\chi \in D(H_{0\varkappa}N)$, so that

$$S_k U\chi = US_k\chi, k = 1,2,3, \quad (13.99)$$

since the operators S_k are symmetric on $D(H_{0\varkappa}N)$. We define on $D(H_{0\varkappa}N)$ a #-norm by

$$\|\chi\|_{\#} = \|(H_{0\varkappa}N + I)\chi\|_{\#1}.$$

Note that the corresponding scalar product makes $D(H_{0\varkappa}N)$ a non-Archimedean Hubert space, say $H_{\#1}$. For the next step we now prove that the operator $\mathcal{B} = \varphi_{\varkappa}^{\#}(h_1) + \pi_{\varkappa}^{\#}(h_2)$ generates a one parameter group

$$U(\alpha) = Ext\text{-}\exp(i\alpha\mathcal{B}) = Ext\text{-}\exp\left[i\alpha\left(\mathcal{B} = \varphi_{\varkappa}^{\#}(h_1) + \pi_{\varkappa}^{\#}(h_2)\right)\right]$$

on $H_{\#1}$ and therefore we need to prove that the operator

$$\widehat{\mathcal{B}} = (H_{0\varkappa}N + I)\mathcal{B}(H_{0\varkappa}N + I)^{-1} \quad (13.100)$$

is a generator to one parameter group on a corresponding Fock space. Since $\widehat{\mathcal{B}}$ is essentially self #-adjoint on $D^{\#}$, and on this domain we have that

$$\widehat{\mathcal{B}} = \mathcal{B} + [H_{0\varkappa}N, \mathscr{B}](H_{0\varkappa}N + I)^{-1} = \mathscr{B} + [N, \mathscr{B}]H_{0\varkappa}(H_{0\varkappa}N + I)^{-1} + N[H_{0\varkappa}, \mathscr{B}](H_{0\varkappa}N + I)^{-1} = \mathscr{B} + A.$$

Hear A is bounded operator. Note that $\widehat{\mathcal{B}} \upharpoonright D^{\#}$ is a bounded perturbation of an essentially self #-adjoint operator. Hence it #- closure #-$\overline{(\widehat{\mathcal{B}} \upharpoonright D^{\#})}$ generates a one parameter group on Fock space $\mathcal{F}^{\#}$, and operator $\mathcal{B} \upharpoonright (H_{0\varkappa}N + I)D^{\#}$ has a #- closure in $H_{\#1}$ that generates a one parameter group on $H_{\#1}$. Since the topology of $H_{\#1}$ is stronger than that of $\mathcal{F}^{\#}$, the #-closure of $\mathcal{B} \upharpoonright (H_{0\varkappa}N + I)D^{\#}$ in $H_{\#1}$ is a restriction of #- $\overline{\mathcal{B}}$ in $\mathcal{F}^{\#}$ and the one parameter group in $H_{\#1}$ is a restriction of the one parameter group generated by #- $\overline{\mathcal{B}}$ in $\mathcal{F}^{\#}$. This proves that

$$U: D(H_{0\varkappa}N) \to D(H_{0\varkappa}N) \quad (13.101)$$

And (13.99).Therefore we have proved that $S_k U\chi = US_k\chi, k = 1,2,3$. Now by passing to strong limits of linear combinations of such operators U we obtain (13.95) on restricting to the domain $D(H^2) \subset D(H_{0\varkappa}N)$. This makes precise the statement that operators $S_k, k = 1,2,3$ are localized outside $\Delta = [a, b]^3$.

Remark 13.13 Note that for each t_1, $|t_1| \leq |s_1|$, the spectral projections of $Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \varphi_{\varkappa}^{\#}(x) f(x, t_1) d^{\#3}x$ belong to $W\left(\#\text{-int}\left(\Delta_{-|s|}\right)\right)$, where $\#\text{-int}\left(\Delta_{-|s|}\right)$ is the #-interior of $\Delta_{-|s|} = \{x | (x, t_1) \in \mathfrak{R}_s^4\} = \{(x_1, x_2, x_3) | a + |s| < x_k < b - |s|\}$. Note that $\operatorname{supp} f \subset \mathfrak{R}_s^4$, hence the spectral projections of

$$Ext\text{-}\exp[iH(t + t_1)]\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \varphi_{\varkappa}^{\#}(x) f(x, t_1) d^{\#3}x\right) Ext\text{-}\exp[-iH(t + t_1)] \quad (13.102)$$

belong to $W\left(\#\text{-int}\left(\Delta_{|t|-|s|}\right)\right)$. For $|t| \leq |s|$, $\#\text{-}int\left(\Delta_{|t|-|s|}\right) \subset \Delta$; so the spectral projections of (13.102) belong to $W(\Delta)$. Now we use the locality property (13.95) of the operators $S_k, k = 1,2,3$. Note that for vector $\chi \in D(H^2)$, $\psi \in D(H^3)$ we have that

$$\psi \in D\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \varphi_{\varkappa}^{\#}(x, 0) f(x, t_1) d^{\#3}x\right),$$

and for $\varphi_{\varkappa}^{\#}(f) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \varphi_{\varkappa}^{\#}(x, t) f(x, t) d^{\#3}x\, d^{\#}t$, by (159) it follows

$$Ext\text{-}\exp[itH]\varphi_{\varkappa}^{\#}(f) Ext\text{-}\exp[itH]\psi \in D(H^2). \quad (13.103)$$

Therefore by (13.95) and the localization of (13.102) for all $k = 1,2,3$ we get

$$\langle S_k\chi, Ext\text{-}\exp[iH(t+t_1)]\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}}\varphi_\varkappa^\#(x)f(x,t_1)d^{\#3}x\right)Ext\text{-}\exp[-iH(t+t_1)]\psi\rangle = \quad (13.104)$$

$$\langle Ext\text{-}\exp[iH(t+t_1)]\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}}\varphi_\varkappa^\#(x)f(x,t_1)d^{\#3}x\right)Ext\text{-}\exp[-iH(t+t_1)]\chi, S_k\psi\rangle.$$

Note that for $|t| \leq |s|$ and $f \in S_{\text{fin}}^\#({}^*\mathbb{R}_c^{\#4})$ with $\text{supp}f \subset \mathfrak{R}_s^4$ we can integrate the equality (13.104) over t_1 to obtain

$$\langle S_k\chi, Ext\text{-}\exp[iH(t)]\varphi_\varkappa^\#(f)Ext\text{-}\exp[-iH(t)]\psi\rangle = \langle Ext\text{-}\exp[iH(t)]\varphi_\varkappa^\#(f)Ext\text{-}\exp[-iH(t)]\chi, S_k\psi\rangle =$$

$$\langle \chi, S_k Ext\text{-}\exp[iH(t)]\varphi_\varkappa^\#(f)Ext\text{-}\exp[-iH(t)]\psi\rangle. \quad (13.105)$$

Here the last equality in (13.105) follows by (13.103) and the fact that S_k is a symmetric operator on $D(H_{0\varkappa}N) \supset D(H^2)$. From (13.105) we obtain that $S_k\psi \in D\left(\left((Ext\text{-}\exp[iH(t)]\varphi_\varkappa^\#(f)Ext\text{-}\exp[-iH(t)]) \upharpoonright D(H^2)\right)^*\right)$ and therefore that

$$S_k\psi \in D(Ext\text{-}\exp[iH(t)]\varphi_\varkappa^\#(f)Ext\text{-}\exp[-iH(t)]),$$

since $D(H^2)$ is a #-core for $\varphi_\varkappa^\#(f)$. Finally from (13.105) we get for $|t| \leq |s|$ and $f \in S_{\text{fin}}^\#({}^*\mathbb{R}_c^{\#4})$ with $\text{supp}f \subset \mathfrak{R}_s^4$ for all $k = 1,2,3$ that

$$S_k Ext\text{-}\exp[iH(t)]\varphi_\varkappa^\#(f)Ext\text{-}\exp[-iH(t)]\psi = Ext\text{-}\exp[iH(t)]\varphi_\varkappa^\#(f)Ext\text{-}\exp[-iH(t)]S_k\psi. \quad (13.106)$$

We apply the relations (13.106) to (13.93). In that case $\psi(t) \in D(H^5) \subset D(H^3)$, so

$$\frac{d^{\#2}F_k(t)}{d^\#t^2} \equiv 0, \text{ for } |t| \leq |s|.$$

Theorem 13.34 [15] Let $f \in S_{\text{fin}}^\#({}^*\mathbb{R}_c^{\#4})$ and $\text{supp}f \subset \mathfrak{R}_s^4$, then on domain $D(H^5)$ the operator equalities hold for all $k = 1,2,3$

$$[iM^{0k}(s), \varphi_\varkappa^\#(f)\,] = [iM^{0k}, \varphi_\varkappa^\#(f)\,] - s\left[P^k\left(\frac{d^\#\left(x_k g_0^{(k)}\right)}{d^\#x_k}\right), \varphi_\varkappa^\#(f)\right]. \quad (13.107)$$

Proof Each of the six terms in (13.107) is an operator defined on $D(H^5)$, since $\varphi_\varkappa^\#(f)\colon D(H^l) \to D(H^{l-1})$, $M^{0k}(s)\colon D(H^l) \to D(H^{l-1})$ for $l = 2, 3, 4$, $k = 1,2,3$, and (by Theorem **6.3**)

$$P^k\left(\frac{d^\#\left(x_k g_0^{(k)}\right)}{d^\#x_k}\right)\colon D(H^3) \to D(H\,).$$

Let $\psi \in D(H^5)$. Then we get

$$\langle\psi, [iM^{0k}(s), \varphi_\varkappa^\#(f)\,]\psi\rangle_\# = F_k(s)$$

for $F_k, k = 1,2,3$ defined in (6.45). By Theorem 13.32, any F_k has two #-derivatives. Hence by generalized Taylor's theorem with remainder **[**13**],**

$$F_k(s) = F_k(0) + sF_k^{\#\prime}(0) + \frac{s^2}{2}F_k^{\#\prime\prime}(t)$$

for some $t, |t| \leq |s|$. Furthermore, by Theorem 13.32 for $k = 1,2,3$,

$$F_k(s) = F_k(0) + sF_k^{\#\prime}(0)$$

By definition, for $k = 1,2,3$,

$$F_k(0) = \langle\psi, [iM^{0k}, \varphi_\varkappa^\#(f)\ ]\psi\rangle_\#$$

and by (13.92),

$$F_k^{\#\prime}(0) = -i\,\langle\psi, \left[P^k\left(\frac{d^\#\left(x_k g_0^{(k)}\right)}{d^\# x_k}\right), \varphi_\varkappa^\#(f)\ \right]\psi\rangle_\#.$$

This proves the equality

$$\langle\psi, [M^{0k}(s), \varphi_\varkappa^\#(f)\ ]\psi\rangle_\# = \langle\psi, [iM^{0k}, \varphi_\varkappa^\#(f)\ ]\psi\rangle_\# - s\,\langle\psi, \left[iP^k\left(\frac{d^\#\left(x_k g_0^{(k)}\right)}{d^\# x_k}\right), \varphi_\varkappa^\#(f)\ \right]\psi\rangle_\#$$

which proving (13.107) by polarization and the #-density of $D(H^5)$.
The next step in the proof of Theorem 13.31 is to pass to the sharp time #-limit of Theorem **11.35**, thus we need to choose a hyper infinite sequence of functions $f_n \in S_{\text{fin}}^\#({}^*\mathbb{R}_c^{\#4}), n \in {}^*\mathbb{N}$ which pick out a time zero contribution in the #-limit. Let us define now

$$A_\varkappa(f,t) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \varphi_\varkappa^\#(x) f(x,t) d^{\#3}x, \qquad (13.108)$$

$$B_\varkappa(f,t) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \pi_\varkappa^\#(x) f(x,t) d^{\#3}x. \qquad (13.109)$$

Where $\varphi_\varkappa^\#(x)$ and $\pi_\varkappa^\#(x)$ the canonical time-zero fields. For real $f \in S_{\text{fin}}^\#({}^*\mathbb{R}_c^{\#4})$, with #-compact support, $A_\varkappa(f,t)$ and $B_\varkappa(f,t)$ are essentially self-#-adjoint on $D\left((H+b)^{\frac{1}{2}}\right)$. Let $f \in C_0^{*\infty}(\mathfrak{R}_I^4)$ and let $f_n(x,t) \in S_{\text{fin}}^\#({}^*\mathbb{R}_c^{\#4}), n \in {}^*\mathbb{N}$ be a hyper infinite sequence of functions of the following form $f_n(x,t) = f_n(x,s)\delta_n(t)$ with support in $\mathfrak{R}_s^4$ and #-converging in the weak sense to $f_n(x,s)\delta\,(t)$ as $n \to {}^*\infty$. For the vector $\psi \in D(H^5)$, the vectors $M^{0k}(s)\psi, k = 1,2,3$, and the vectors

$$M^{0k}(s)\psi, M^{0k}\psi,\ P\left(\frac{d^\#\left(x_k g_0^{(k)}\right)}{d^\# x_k}\right)\psi$$

the same as in the proof of Theorem **11.35**. Note that the bilinear form $\varphi_\varkappa^\#(x,t)$ for $(x,t) \in \mathfrak{R}_I^4$ determines a bounded operator

$$G(x,t) = (H+b)^{\frac{1}{2}}\,\varphi_\varkappa^\#(x,t)(H+b)^{-\frac{1}{2}}. \qquad (13.110)$$

Note that the operator valued function $G(x,t)$ is #-continuous in variable (x,t).
Theorem 13.35 Let $f \in S_{\text{fin}}^\#({}^*\mathbb{R}_c^{\#4})$ and $\text{supp} f \subset \mathfrak{R}_\Delta^4$. Then, in the sense of bilinear forms on $D(H^5) \times D(H^5)$, for all $k = 1,2,3$

$$[iM^{0k}(s), A_\varkappa(f,s)] = [iM^{0k}, A_\varkappa(f,s)] - s[iP_k, A_\varkappa(f,s)] \qquad (13.111)$$

Here $P_k = P^k\left(\frac{d^\#\left(x_k g_0^{(k)}\right)}{d^\# x_k}\right)$.

Proof Choose a w^*-#-convergent sequence of #-measures $f_n(x,t) \in S_{\text{fin}}^\#({}^*\mathbb{R}_c^{\#4}), n \in {}^*\mathbb{N}$ as above. Consider, for example, the first term in (13.107) as a bilinear form on $D(H^5) \times D(H^5)$. Let $\psi, \chi \in D(H^5)$

$$\langle\psi, [iM^{0k}(s), \varphi_\varkappa^\#(f_n)\ ]\chi\rangle_\# = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#4}} \langle -iM^{0k}(s)\chi, \varphi_\varkappa^\#(x,t)\psi\rangle_\# f(x,s)\delta_n(t) d^{\#3}x\, d^\# t +$$

$$+Ext\text{-}\int_{^*\mathbb{R}_c^{\#4}}\langle\varphi_{\varkappa}^{\#}(x,t)\chi, iM^{0k}(s)\psi\rangle_{\#}f(x,s)\delta_n(t)d^{\#3}x\, d^{\#}t, \quad (13.112)$$

where on the right hand side $\varphi_{\varkappa}^{\#}(x,t)$ is considered as a bilinear form on $D\left((H\ +b)^{\frac{1}{2}}\right)\times D\left((H\ +b)^{\frac{1}{2}}\right)$ #-continuous in (x,t) by (13.110). Thus, by the #-convergence of the f_n the terms on the right hand side of (13.112) #-converge as $n\to {}^*\infty$ to

$$Ext\text{-}\int_{^*\mathbb{R}_c^{\#4}}\langle -iM^{0k}(s)\chi,\varphi_{\varkappa}^{\#}(x)\psi\rangle_{\#}f(x,s)d^{\#3}x + Ext\text{-}\int_{^*\mathbb{R}_c^{\#4}}\langle\varphi_{\varkappa}^{\#}(x)\chi, iM^{0k}(s)\psi\rangle_{\#}f(x,s)d^{\#3}x.$$

This is the left side of (13.111), evaluated on $\chi\ \times\ \psi$. The other terms of (13.111) are similarly obtained by passing to the same #-limit in (13.107).

Theorem 13.36 [15] Let $f\in C_0^{^*\infty}(\mathfrak{R}_{\Delta}^4)$. As an equality of bilinear forms on $D(H\,)\times D(H\,)$

$$[i\,P_k, A_{\varkappa}(f,s)] = A_{\varkappa}\left(\frac{d^{\#}f}{d^{\#}x_k},s\right). \quad (13.113)$$

$$P_k = P^k\left(\frac{d^{\#}\left(x_k g_0^{(k)}\right)}{d^{\#}x_k}\right). \quad (13.114)$$

Proof Let $D^{\#}$ is the domain $D^{\#}=\left\{\psi\in\mathcal{F}^{\#}|\psi^{(n)}\in S_{\mathrm{fin}}^{\#}({}^*\mathbb{R}_c^{\#3n}),\psi^{(m)}=0 \text{ for all sufficiently large m}\right\}$ of #-smooth vectors. We prove (13.113) in the sense of bilinear forms on $D^{\#}\times D^{\#}$ by direct computation in momentum space for $k=1,2,3$ (e.g. as in the proof of Theorem 13.26):

$$[i\,P_k, A_{\varkappa}(f,s)] = A_{\varkappa}\left(\frac{d^{\#}}{d^{\#}x_k}\left(f\frac{d^{\#}x_k g_0^{(k)}}{d^{\#}x_k}\right),s\right)$$

which agrees with (13.113) because $x_k g_0^{(k)} = x_k-\alpha$ on a #-neighbourhood of $\Delta = I^3$, while $f(x,t)$ vanishes for $x\notin\Delta$. Note that $D^{\#}$ is a #-core for $H_{0\varkappa}$ and

$$|\langle\, P_k\psi, A_{\varkappa}(f,s)\psi\rangle_{\#}|\leq \mathrm{const}\|(H_{0\varkappa}+I)\|_{\#},$$

for all $\psi\in D(H_{0\varkappa})$. Hence the equality (13.113) extends from $D^{\#}\times D^{\#}$ to $D(H_{0\varkappa})\times D(H_{0\varkappa})$, since the operators involved are closable. Since $D(H_{\varkappa})\subset D(H_{0\varkappa})$, the theorem is proved.

Theorem 11.37 Let $f\in C_0^{^*\infty}(\mathfrak{R}_{\Delta}^4)$. As the equalities of bilinear forms on $D(H^2)\times D(H^2)$ for all $k=1,2,3$

$$[iM^{0k}, A_{\varkappa}(f,s)] = [iH, A_{\varkappa}(x_k f,s)] = B_{\varkappa}(x_k f,s). \quad (13.115)$$

Proof The proof is similar to the proof of Theorem 13.36.

Theorem 11.38 [15] Let $|f|_{\#1}$ be the #-norm $|f|_{\#1} = c\left(Ext\text{-}\int_{^*\mathbb{R}_c^{\#3}}\left\{\|f(\cdot,t)\|_{\#2}+\sum_{i=1}^{3}\left\|\partial_{x_i}^{\#}f(\cdot,t)\right\|_{\#2}\right\}d^{\#}t\right)$.

Let $|f|_{\#1}$ is finite. Then on the domain $D\left((H+b)^{\frac{3}{2}}\right)$,), the field $\varphi_{\varkappa}^{\#}(f)$ satisfies the following equation

$$(\partial_t^{\#}\varphi_{\varkappa}^{\#})(f) = -\varphi_{\varkappa}^{\#}(\partial_t^{\#}f) = \pi_{\varkappa}^{\#}(f) = [iH,\varphi_{\varkappa}^{\#}(f)]. \quad (13.116)$$

Proof Note that the first equality in (13.116) is the definition of a distribution #-derivative. The out the difference quotient $\Delta_{\varepsilon}f(x,t)$ to #-derivative $\partial_t^{\#}f$ reads $\Delta_{\varepsilon}f(x,t)=\frac{[f(x+\varepsilon,t)-f(x,t)]}{\varepsilon}$, note that $\#\text{-}\lim_{\varepsilon\to_{\#}0}\Delta_{\varepsilon}f(x,t) = \partial_t^{\#}f(x,t)$. Note that for any vector ψ such that $\psi\in D\left((H+b)^{\frac{1}{2}}\right)$ by canonical consideration we get

$$\#\text{-}\lim_{\varepsilon\to_{\#}0}\left\|\varphi_{\varkappa}^{\#}(\partial_t^{\#}f)\psi-\varphi_{\varkappa}^{\#}\left(\Delta_{\varepsilon}f(x,t)\right)\psi\right\|_{\#}=0.$$

We have for $\psi\in D\left((H+b)^{\frac{3}{2}}\right)$ that

$$\varphi_{\varkappa}^{\#}\big(\Delta_{\varepsilon}f(x,t)\big)\psi = \varepsilon^{-1}(I - Ext\text{-}\exp[i\varepsilon H])\left\{Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \varphi_{\varkappa}^{\#}(x,t-\varepsilon)f(x,t)d^{\#3}x\psi d^{\#}t\right\}+$$

$$\varepsilon^{-1}\left\{Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} A_{\varkappa}(f,t)(Ext\text{-}\exp[i\varepsilon H] - I)\psi d^{\#}t\right\}.$$

Here the last term #-converges as $\varepsilon \to_{\#} 0$ and it #-limit is: $i\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} A_{\varkappa}(f,t)H\psi d^{\#}t\right)$. Since $\varphi_{\varkappa}^{\#}\big(\Delta_{\varepsilon}f(x,t)\big)\psi$ #-converges as $\varepsilon \to_{\#} 0$, the remaining term in expression for $\varphi_{\varkappa}^{\#}\big(\Delta_{\varepsilon}f(x,t)\big)\psi$ #-converges also to a #-limit ψ_1. For $\chi \in D(H)$ we obtain that

$$\langle \chi, \psi_1\rangle = \#\text{-}\lim_{\varepsilon \to_{\#} 0} \langle \chi, \varepsilon^{-1}(I - Ext\text{-}\exp[i\varepsilon H])\left\{Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \varphi_{\varkappa}^{\#}(x,t-\varepsilon)f(x,t)d^{\#3}x\psi d^{\#}t\right\}\rangle = \langle iH\chi, \varphi_{\varkappa}^{\#}(f)\psi\rangle.$$

Since $H = H^*$, it follows that $\varphi_{\varkappa}^{\#}(f)\psi \in D(H)$ and $\psi_1 = iH\varphi_{\varkappa}^{\#}(f)\psi$ and therefore: $-\varphi_{\varkappa}^{\#}(\partial_t^{\#}f)\psi = [iH, \varphi_{\varkappa}^{\#}(f)]\psi$. From the above equation we obtain

$$\langle \psi, \varphi_{\varkappa}^{\#}(\partial_t^{\#}f)\psi\rangle = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#}} \langle H\psi(t), Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \varphi_{\varkappa}^{\#}(x,0)f(x,t)d^{\#3}x\psi(t)\,\rangle\, d^{\#}t -$$

$$Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#}} \langle Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \varphi_{\varkappa}^{\#}(x,0)f(x,t)d^{\#3}x\psi(t),\ H\psi(t)\rangle\, d^{\#}t.$$

Here $\psi(t) = Ext\text{-}\exp[itH]\psi$. Note that $\psi(t) \in D(H_{0\varkappa}) \cap D(H_{I,\varkappa})$, and $\left\|H_{I,\varkappa}\big(\psi(t) - \psi(s)\big)\right\|_{\#} \le a\left\|(H + b)\big(\psi(t) - \psi(s)\big)\right\|_{\#} \to_{\#} 0$, as $|t - s| \to_{\#} 0$. Therefore we may substitute $H_{0\varkappa} + H_{I,\varkappa}$ for H and consider each term separately. Note that the operators $H_{I,\varkappa}$ and $Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \varphi_{\varkappa}^{\#}(x,0)f(x,t)d^{\#3}x$ commute and therefore $H_{I,\varkappa}$ contribute zero to equality above. The following identity by canonical computation holds for any $\psi \in D(H_{0\varkappa})$, in particular for $\psi(t) = Ext\text{-}\exp[itH]\psi \in D(H_{0\varkappa})$

$$\langle H_{0\varkappa}\psi, Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \varphi_{\varkappa}^{\#}(x,0)f(x,t)d^{\#3}x\psi\,\rangle - \langle Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \varphi_{\varkappa}^{\#}(x,0)f(x,t)d^{\#3}x\psi\ , H_{0\varkappa}\psi\rangle =$$

$$\langle \psi, -iExt\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \pi_{\varkappa}^{\#}(x,0)f(x,t)d^{\#3}x\psi\,\rangle.$$

Therefore finally we get

$$i\langle \psi, \varphi_{\varkappa}^{\#}(\partial_t^{\#}f)\psi\rangle = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#}} \langle \psi(t), -iExt\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \pi_{\varkappa}^{\#}(x,0)f(x,t)d^{\#3}x\psi\,\rangle\, d^{\#}t = \langle \psi, -i\pi_{\varkappa}^{\#}(f)\psi\rangle.$$

This equality finalized the proof.

Theorem 13.39 As the operator equalities on $D(H^5)$ for all $k = 1,2,3$

$$[iM^{0k}, \varphi_{\varkappa}^{\#}(f)] = -\varphi_{\varkappa}^{\#}\left(t\frac{\partial^{\#}f}{\partial^{\#}x_k} + x_k\frac{\partial^{\#}f}{\partial^{\#}t}\right). \tag{13.117}$$

Proof We first prove (13.117) as equalities of bilinear forms on $D(H^5) \times D(H^5)$. Let ψ is a near standard vector and $\psi \in D(H^5)$. By Theorems 13.37-13.39, for all $k = 1,2,3$ we get

$$\langle \psi, iM^{0k}(s), A_{\varkappa}(f,s)\psi\rangle_{\#} = \langle \psi, B_{\varkappa}(x_k f, s)\psi,\rangle_{\#} - \langle \psi, A_{\varkappa}\left(\frac{d^{\#}f}{d^{\#}x_k}, s\right)\psi\rangle_{\#}.$$

Substituting $Ext\text{-}\exp(iHs)$ for ψ, we obtain for all $k = 1,2,3$ that

$$\langle \psi, [iM^{0k}, Ext\text{-}\exp(iHs)A_{\varkappa}(f,s)Ext\text{-}\exp(-iHs)]\psi\rangle_{\#} = \tag{13.118}$$

$$\langle \psi, Ext\text{-}\exp(iHs)\left\{B_{\varkappa}(x_k f, s) - A_{\varkappa}\left(s\frac{d^{\#}f}{d^{\#}x_k}, s\right)\right\}Ext\text{-}\exp(-iHs)\psi\rangle_{\#}.$$

From (13.116) we get

$$Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#4}} Ext\text{-}\exp(iHt)\,\pi_\varkappa^\#(x) Ext\text{-}\exp(iHt) f(x,t) d^{\#3}x d^\# t = -\varphi_\varkappa^\#\left(\frac{\partial^\# f}{\partial^\# t}\right). \tag{13.119}$$

Using (13.119) we integrate (13.118) over s to obtain for all $k = 1,2,3$ the equalities of bilinear forms

$$\langle\psi, iM^{0k}, \varphi_\varkappa^\#(f)\psi\rangle_\# = -\langle\psi, \varphi_\varkappa^\#\left(t\frac{\partial^\# f}{\partial^\# x_k} + x_k\frac{\partial^\# f}{\partial^\# t}\right)\psi\rangle_\#. \tag{13.120}$$

Since $M^{0k}\varphi_\varkappa^\#(f), \varphi_\varkappa^\#(f)M^{0k}$, and $\varphi_\varkappa^\#\left(t\frac{\partial^\# f}{\partial^\# x_k} + x_k\frac{\partial^\# f}{\partial^\# t}\right)$ are operators on $D(H^5)$ for all $k = 1,2,3$, the operator equalities (13.117) follows by polarization and the #-density of $D(H^5)$. This final remark completes the proof of the theorem and hence it completes the proof of Theorem 13.39.

Theorem 13.40 [15] Let $\mathfrak{R} \subset {}^*\mathbb{R}_{c,\text{fin}}^{\#4}$ be an bounded region in ${}^*\mathbb{R}_{c,\text{fin}}^{\#4}$ and let $F_k(\beta,x,t), k = 1,2,3$ be a functions such that $F_k(\beta,x,t), \beta \in {}^*\mathbb{R}_{c,\text{fin}}^\#$ and $\frac{\partial^\# F_k(\beta,x,t)}{\partial^\#\beta}$ are #- continuous in (β,x,t), where the partial #-derivative exists for each point $(x,t) \in {}^*\mathbb{R}_{c,\text{fin}}^{\#4}$. Assume that for all $f(x,t) \in C_{0,\text{fin}}^{*\infty}(\mathfrak{R})$ the following equalities hold for all $k = 1,2,3$,

$$Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \frac{\partial^\# F_k(\beta,x,t)}{\partial^\#\beta} f(x,t) d^{\#3}x d^\# t = -Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} F_k(\beta,x,t)\left[x_k\frac{\partial^\# f}{\partial^\# t} + t\frac{\partial^\# f}{\partial^\# x_k}\right] d^{\#3}x d^\# t. \tag{13.121}$$

Then for all (β,x,t) such that $\Lambda_{\gamma\beta}(x,t) \in \mathfrak{R}$ for $0 \le \gamma \le 1, k = 1,2,3$

$$F_k(\beta,x,t) = F_k\left(0,\Lambda_{\gamma\beta}(x,t)\right) + \delta(\beta,x,t) = \tag{13.122}$$

$$F_k(0, x_k\cosh\beta + t\sinh\beta, x_k\sinh\beta + t\cosh\beta) + \delta(\beta,x,t).$$

Here $\delta(\beta,x,t)$ is a nonzero function such that $\delta(\beta,x,t) \neq 0$ and $\delta(\beta,x,t)$ is #-differentiable with zero partial #-derivatives $\delta_\beta^{\#'}(\beta,x,t) \equiv 0, \delta_{x_k}^{\#'}(\beta,x,t) \equiv 0, \delta_t^{\#'}(\beta,x,t) \equiv 0$.

Proof Obviously (13.122) is a solution to the equations (13.121). Thus we need prove uniqueness (13.122) for a given function $\delta(\beta,x,t)$ and for all $k = 1,2,3$ and it is sufficient to prove uniqueness for the case $F_k(0,x,t) = \delta(0,x,t)$. Let A_k be the operator $A_k = x_k\frac{\partial^\#}{\partial^\# t} + t\frac{\partial^\#}{\partial^\# x_k}$. Note that by (177), provided $\operatorname{supp} f\left(\Lambda_{\gamma\beta'}(x,t)\right) \subset \mathfrak{R}$ we get

$$\frac{\partial^\#}{\partial^\#\beta'}\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} F_k(\beta',x,t) f\left(\Lambda_{\gamma\beta'}(x,t)\right) d^{\#3}x d^\# t\right) = \tag{13.123}$$

$$Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}}\left\{\frac{\partial^\# F_k(\beta',x,t)}{\partial^\#\beta'} f\left(\Lambda_{\gamma\beta'}(x,t)\right) + F_k(\beta',x,t) A_k f\left(\Lambda_{\gamma\beta'}(x,t)\right)\right\} d^{\#3}x d^\# t = 0.$$

Let $\breve{\mathfrak{R}} = \bigcap_{0\le\gamma\le 1}\Lambda_{\gamma\beta}\,\mathfrak{R}$ and $f(x,t) \in C_{0,\text{fin}}^{*\infty}(\breve{\mathfrak{R}})$, then (13.123) holds for all β' such that $0 \le \beta' \le \beta$. Note that for all functions $f(x,t) \in C_{0,\text{fin}}^{*\infty}(\mathfrak{R})$ the following equalities (13.124) hold for all $k = 1,2,3$,

$$Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} F_k(\beta,x,t) f\left(\Lambda_{\gamma\beta'}(x,t)\right) d^{\#3}x d^\# t = 0. \tag{13.124}$$

Thus, in the sense of distributions we obtain for all $k = 1,2,3$ that

$$F_k(\beta,x,t) = 0, (x,t) \in \breve{\mathfrak{R}}. \tag{13.125}$$

Since $F_k(\beta,x,t)$ is #-continuous, (13.125) hold in usual sense everywhere in $\breve{\mathfrak{R}}$. This establishes required uniqueness, and completes the proof of the theorem.

Definition 13.18 (1) Let $(H_{\#},\|\cdot\|_{\#})$ be a linear normed space over field ${}^{*}\mathbb{C}_{c}^{\#}$. An element $x\in H_{\#}$ is called finite or norm finite if $\|x\|_{\#}\in{}^{*}\mathbb{R}_{c,\text{fin}}^{\#}$ and we let $\text{Fin}(H_{\#})$ denote the set of the all finite elements of $H_{\#}$; the element $x\in H_{\#}$ is called infinitesimal if $\|x\|_{\#}\approx 0$ and we write $x\approx y$ for $\|x-y\|_{\#}\approx 0$. (2)Let $(H_{\#},\langle\cdot,\cdot\rangle_{\#})$ be a non-Archimedean Hilbert space over field ${}^{*}\mathbb{C}_{c}^{\#}$ endowed with a canonical #-norm $\|x\|_{\#}=\sqrt{\langle x,x\rangle_{\#}}$, then we apply the same definition as in (1).
Definition 13.19 Let A be a linear operator $A:H_{\#}\to H_{\#}$ with domain $D(A)$. Let $D_{\text{fin}}(A)\subset D(A)$ be a subdomain such that for all $\psi\in D(A)$: $\psi\in D_{\text{fin}}(A)\Leftrightarrow\|x\|_{\#}\in{}^{*}\mathbb{R}_{c,\text{fin}}^{\#}$ and let $D_{\text{fin}}^{\#}(A)$ be a subdomain $D_{\text{fin}}^{\#}(A)\subset D_{\text{fin}}(A)$ such that for all $\psi\in D_{\text{fin}}(A)$: $\psi\in D_{\text{fin}}^{\#}(A)\Leftrightarrow\|Ax\|_{\#}\in{}^{*}\mathbb{R}_{c,\text{fin}}^{\#}$.
Definition 13.20 Let $q(\cdot,\cdot)$ be a bilinear form with domain $D(q)\times D(q)$ on $H_{\#}$ such that $D(q)\times D(q)\subsetneq H_{\#}\times H_{\#}$ and $D(q)\times D(q)\to{}^{*}\mathbb{C}_{c}^{\#}$. Let $D_{\text{fin}}(q)\times D_{\text{fin}}(q)\subset D(q)\times D(q)$ be a subdomain such that for all $\{\psi_1,\psi_2\}\in D_{\text{fin}}(q)\times D_{\text{fin}}(q)\Leftrightarrow|\langle\psi_1,\psi_2\rangle_{\#}|\in{}^{*}\mathbb{R}_{c,\text{fin}}^{\#}$. Let $D_{\text{fin}}^{\#}(q)\times D_{\text{fin}}^{\#}(q)\subset D_{\text{fin}}(q)\times D_{\text{fin}}(q)$ be a subdomain such that for all $\{\psi_1,\psi_2\}\in D_{\text{fin}}(q)\times D_{\text{fin}}(q)$: $\{\psi_1,\psi_2\}\in D_{\text{fin}}^{\#}(q)\times D_{\text{fin}}^{\#}(q)\Leftrightarrow q(\psi_1,\psi_2)\in{}^{*}\mathbb{C}_{c,\text{fin}}^{\#}$.
Theorem 13.41[15] Assume that the operators $M^{0k}=M_{\varkappa}^{0k}=M_{0,\varkappa}^{0k}+M_{I,\varkappa}^{0k}$, $k=1,2,3$ satisfy conditions (**152**)-(**154**) and where the operators $M_{0,\varkappa}^{0k}$ are defined by (**125**). We set now $\delta(\beta,x,t)\approx 0$.
(1) If $f\in S_{\text{fin}}^{\#}({}^{*}\mathbb{R}_{c}^{\#4})$, $\text{supp}f\subset\#\text{-int}(\mathfrak{R}_{\Delta}^{4})$, $\Delta=[a,b]^3$ and $\text{supp}f_{\Lambda(\beta)}\subseteq\#\text{-int}(\mathfrak{R}_{\Delta}^{4})=\wp_{\Delta}^{4}$, then for all $k=1,2,3$ on domains $D_{\text{fin}}((M^{0k})^2)$

$$Ext\text{-}\exp(iM^{0k}\beta)\varphi_{\varkappa}^{\#}(f)Ext\text{-}\exp(-iM^{0k}\beta)\approx\varphi_{\varkappa}^{\#}(f_{\Lambda(\beta)}).\quad(13.126)$$

Here the $\approx$ - equalities (198) hold as $\approx$ -equalites for self #-adjoint operators.
(2) If $(x,t)\in\mathfrak{R}_{\Delta}^{4}$ and $\Lambda_{\beta}(x,t)\in\mathfrak{R}_{\Delta}^{4}$, then for all $k=1,2,3$

$$Ext\text{-}\exp(iM^{0k}\beta)\varphi_{\varkappa}^{\#}(x,t)Ext\text{-}\exp(-iM^{0k}\beta)\approx\varphi_{\varkappa}^{\#}\left(\Lambda_{\beta}(x,t)\right)\quad(13.127)$$

Here the $\approx$ - equalities (13.127) hold in the sense of ${}^{*}\mathbb{R}_{c,\text{fin}}^{\#}$- valued bilinear forms on domains $D_{\text{fin}}^{\#}(M^{0k})\times D_{\text{fin}}^{\#}(M^{0k})$ and on domains $D_{\text{fin}}^{\#}(M^{0k})\times D_{\text{fin}}^{\#}(M^{0k})$.
Remark 13.15 Note that: (1) for real-valued $f\in S_{\text{fin}}^{\#}({}^{*}\mathbb{R}_{c}^{\#4})$ is a self-#-adjoint operator $\varphi_{\varkappa}^{\#}(f)$, essentially self -#-adjoint operator on a variety of appropriate domains. It is for this self #-adjoint operator that (13.126) is valid; (2) on the subdomains $D_{\text{fin}}^{\#}((M^{0k})^2)$ $\approx$ -equalites (13.126) entail for all $k=1,2,3$ the equalities

$$\text{st}\big(Ext\text{-}\exp(iM^{0k}\beta)\varphi_{\varkappa}^{\#}(x,t)Ext\text{-}\exp(-iM^{0k}\beta)\big)=\text{st}\left(\varphi_{\varkappa}^{\#}\left(\Lambda_{\beta}(x,t)\right)\right);$$

(3) on the subdomains $D_{\text{fin}}^{\#}((M^{0k})^2)$ the $\approx$ -equalites (13.126) entail for all $k=1,2,3$ the equalities

$$\text{st}\big(Ext\text{-}\exp(iM^{0k}\beta)\varphi_{\varkappa}^{\#}(f)Ext\text{-}\exp(-iM^{0k}\beta)\big)=\text{st}\left(\varphi_{\varkappa}^{\#}(f_{\Lambda(\beta)})\right).$$

Proof Let $\psi\in D(M^{0k})$ and let $F_k(\beta,x,t)$, $k=1,2,3$ be the functions is defined by

$$F_k(\beta,x,t)=\langle Ext\text{-}\exp(-iM^{0k}\beta)\psi,\varphi_{\varkappa}^{\#}(x,t)(Ext\text{-}\exp(-iM^{0k}\beta)\psi)\rangle_{\#}.\quad(13.128)$$

For all $(\beta,x,t)\in{}^{*}\mathbb{R}_{c,\text{fin}}^{\#}\times{}^{*}\mathbb{R}_{c,\text{fin}}^{\#4}$ and for $f\in S_{\text{fin}}^{\#}({}^{*}\mathbb{R}_{c}^{\#4})$, let $F_k(\beta,f)$ be the function is defined by

$$F_k(\beta,f)=\langle Ext\text{-}\exp(-iM^{0k}\beta)\psi,\varphi_{\varkappa}^{\#}(f)(Ext\text{-}\exp(-iM^{0k}\beta)\psi)\rangle_{\#}=$$

$$Ext\text{-}\int_{\wp_{\Delta}^{4}}F_k(\beta,x,t)f(x,t)d^{\#3}xd^{\#}t.\quad(13.129)$$

Note that $\varphi_{\varkappa}^{\#}(x,t)$ is a bilinear form defined on $D\left((H+b)^{\frac{3}{2}}\right)\times D\left((H+b)^{\frac{3}{2}}\right)$, #-continuous in $(x,t)\in{}^{*}\mathbb{R}_{c,\text{fin}}^{\#4}$. By

Theorem 13.28 $D(M^{0k}) \subset D\left((H+b)^{\frac{1}{2}}\right)$ and therefore $F_k(\beta,x,t)$ is well defined and #-continuous in (x,t). Note that a function $F_k(\beta,x,t)$ is #-continuously #-differentiable in $\beta \in {}^*\mathbb{R}^\#_{c,\mathrm{fin}}$ and for all $k=1,2,3$

$$\frac{\partial^\# F_k(\beta,x,t)}{\partial^\# \beta} = -\langle Ext\text{-}\exp(-iM^{0k}\beta)iM^{0k}\psi, \varphi^\#_\varkappa(f)(Ext\text{-}\exp(-iM^{0k}\beta)\psi)\rangle_\# \tag{13.130}$$

$$-\langle Ext\text{-}\exp(-iM^{0k}\beta)\psi, \varphi^\#_\varkappa(f)(Ext\text{-}\exp(-iM^{0k}\beta)iM^{0k}\psi)\rangle_\#.$$

By the canonical argument, we have for all $k=1,2,3$ that

$$\frac{\partial^\# F_k(\beta,f)}{\partial^\# \beta} = \langle Ext\text{-}\exp(-iM^{0k}\beta)\psi, [iM^{0k}, \varphi^\#_\varkappa(f)](Ext\text{-}\exp(-iM^{0k}\beta)\psi)\rangle_\# = \tag{13.131}$$

$$Ext\text{-}\int_{\wp^4_\Delta} F_k(\beta,x,t)f(x,t)d^{\#3}xd^\#t.$$

By Theorem 13.39 under the condition $\mathrm{supp}f \subset \#\text{-int}(\mathfrak{R}^4_\Delta)$ we have for all $k=1,2,3$ that

$$\frac{\partial^\# F_k(\beta,f)}{\partial^\# \beta} = -\langle Ext\text{-}\exp(-iM^{0k}\beta)\psi, \varphi^\#_\varkappa\left(x_k\frac{\partial^\# f}{\partial^\# t} + t\frac{\partial^\# f}{\partial^\# x_k}\right)Ext\text{-}\exp(-iM^{0k}\beta)\psi\rangle_\# =$$

$$-Ext\text{-}\int_{{}^*\mathbb{R}^{\#3}_c} F_k(\beta,x,t)\left(x_k\frac{\partial^\# f}{\partial^\# t} + t\frac{\partial^\# f}{\partial^\# x_k}\right)f(x,t)d^{\#3}xd^\#t. \tag{13.132}$$

Therefore by Theorem 13.40 under the condition

$$\bigcup_{0\le\gamma\le 1}\Lambda_{\gamma\beta}(x,t) \in \mathfrak{R}^4_\Delta \tag{13.133}$$

we have for all $k=1,2,3$ that

$$F_k(\beta,x,t) = F_k\left(0,\ \Lambda_{\gamma\beta}(x,t)\right) + \delta(\beta,x,t) \tag{13.134}$$

That is, if (13.133) holds, then (13.134) also holds for all $k=1,2,3$ and finally we get

$$Ext\text{-}\exp(iM^{0k}\beta)\varphi^\#_\varkappa(x,t)Ext\text{-}\exp(-iM^{0k}\beta) = \varphi^\#_\varkappa\left(\Lambda_\beta(x,t)\right) + \delta(\beta,x,t). \tag{13.135}$$

Here the equations (13.135) hold in the sense of bilinear forms on $D((M^{0k})^2)\times D((M^{0k})^2)$, i.e.

$$\langle\psi_1, Ext\text{-}\exp(iM^{0k}\beta)\varphi^\#_\varkappa(x,t)Ext\text{-}\exp(-iM^{0k}\beta)\psi_2\rangle_\# = \langle\psi_1, \varphi^\#_\varkappa\left(\Lambda_\beta(x,t)\right)\psi_2\rangle_\# + \delta(\beta,x,t)\langle\psi_1,\psi_2\rangle_\#. \tag{13.136}$$

From (13.136) on the domain $D^\#_{\mathrm{fin}}((M^{0k})^2)\times D^\#_{\mathrm{fin}}((M^{0k})^2) \subset D_{\mathrm{fin}}((M^{0k})^2)\times D_{\mathrm{fin}}((M^{0k})^2) \subset D((M^{0k})^2)\times D((M^{0k})^2)$ we get the $\approx$ -equality

$$\langle\psi_1, Ext\text{-}\exp(iM^{0k}\beta)\varphi^\#_\varkappa(x,t)Ext\text{-}\exp(-iM^{0k}\beta)\psi_2\rangle_\# \approx \langle\psi_1, \varphi^\#_\varkappa\left(\Lambda_\beta(x,t)\right)\psi_2\rangle_\#, \tag{13.137}$$

since $\langle\psi_1,\psi_2\rangle$ is finite and therefore $\delta(\beta,x,t)\langle\psi_1,\psi_2\rangle_\# \approx 0$.

Note that in the #-limit $\lambda \to_\# 0$ by (**125**) we get

$$\#\text{-}\lim_{\lambda\to_\# 0} M^{0k} = M^{0k}_\varkappa. \tag{13.138}$$

Therefore in the #-limit $\lambda \to_\# 0$ from (13.136) and (13.138) we obtain that

$$\lim_{\lambda \to_{\#} 0} \langle \psi_1, Ext\text{-}\exp(iM^{0k}\beta)\varphi^{\#}_{\varkappa}(x,t)Ext\text{-}\exp(-iM^{0k}\beta)\psi_2\rangle_{\#} = \tag{211}$$

$$\langle \psi_1, Ext\text{-}\exp(iM^{0k}_{\varkappa}\beta)\varphi^{\#}_{0,\varkappa}(x,t)Ext\text{-}\exp(-iM^{0k}_{\varkappa}\beta)\psi_2\rangle_{\#} =$$

$$\mathrm{Lim}_{\lambda \to_{\#} 0} \langle \psi_1, \varphi^{\#}_{\varkappa}\left(\Lambda_\beta(x,t)\right)\psi_2\rangle_{\#} + \delta(\beta,x,t)\langle \psi_1,\psi_2\rangle_{\#} = \langle \psi_1, \varphi^{\#}_{0,\varkappa}\left(\Lambda_\beta(x,t)\right)\psi\rangle_{\#} + \delta(\beta,x,t)\langle \psi_1,\psi_2\rangle_{\#}.$$

From (211) on the domain $D^{\#}_{\mathrm{fin}}\left((M^{0k})^2\right) \times D^{\#}_{\mathrm{fin}}\left((M^{0k})^2\right) \subset D_{\mathrm{fin}}\left((M^{0k})^2\right) \times D_{\mathrm{fin}}\left((M^{0k})^2\right) \subset D\left((M^{0k})^2\right) \times D\left((M^{0k})^2\right)$ we get the $\approx$ -equality for free quantum field $\varphi^{\#}_{0,\varkappa}(x,t)$

$$\langle \psi_1, Ext\text{-}\exp(iM^{0k}_{\varkappa}\beta)\varphi^{\#}_{0,\varkappa}(x,t)Ext\text{-}\exp(-iM^{0k}_{\varkappa}\beta)\psi_2\rangle_{\#} \approx \langle \psi_1, \varphi^{\#}_{0,\varkappa}\left(\Lambda_\beta(x,t)\right)\psi_2\rangle_{\#}. \tag{212}$$

Remark 11.16 Note that the $\approx$ -equality required by (212) is necessary, see Remark 9.2.
The $\approx$ -equality (209) extends by #-closure to $D^{\#}_{\mathrm{fin}}(M) \times D^{\#}_{\mathrm{fin}}(M)$, since $D^{\#}_{\mathrm{fin}}(M) \subset D^{\#}_{\mathrm{fin}}\left((H+b)^{1/2}\right)$ by Theorem 11.29, and the estimate

$$|\langle \psi, Ext\text{-}\exp(iM^{0k}\beta)\varphi^{\#}_{\varkappa}(x,t)Ext\text{-}\exp(-iM^{0k}\beta)\psi\rangle_{\#}| \approx \tag{213}$$

$$\left|\langle \psi, \varphi^{\#}_{\varkappa}\left(\Lambda_\beta(x,t)\right)\psi\rangle_{\#}\right| \leq c\left\|(H+b)^{1/2}\psi\right\|^2_{\#}.$$

Here c is finite constant. Furthermore $D\left((M^{0k})^2\right)$ for any $k = 1,2,3$ is a #-core for H, by Theorem 11.31, and therefore a #-core for $(H+b)^{\frac{1}{2}}$. Thus (208) extends to $D\left((M^{0k})^2\right) \times D\left((M^{0k})^2\right)$ and on this domain we also have #-continuity of the form in $(x,t) \in {}^*\mathbb{R}^{\#4}_{c,\mathrm{fin}}$. Note that it is necessary to assume that $\bigcup_{0\leq\gamma\leq 1} \Lambda_{\gamma\beta}(x,t) \in \mathfrak{R}^4_\Delta$. However for the regions $\mathfrak{R}^4_\Delta$ this statement follows from the condition $(x,t) \in \mathfrak{R}^4_\Delta \Rightarrow \Lambda_\beta(x,t) \in \mathfrak{R}^4_\Delta$. This final remark completes the proof of this theorem part (2). Now we go to prove the operator $\approx$ -equality (198) for the case $f \in S^{\#}_{\mathrm{fin}}({}^*\mathbb{R}^{\#4}_c)$, $\mathrm{supp} f \cup \mathrm{supp} f_{\Lambda_\beta}$. By Theorem 11.29, the operators $\varphi^{\#}_{\varkappa}(f)$ and $\varphi^{\#}_{\varkappa}\left(f_{\Lambda_\beta}\right)$ are defined on domain $D\left((M^{0k})^2\right)$. Integrating (207) against $f(x,t)$, we get the equalities

$$Ext\text{-}\exp(iM^{0k}\beta)\varphi^{\#}_{\varkappa}(f)Ext\text{-}\exp(-iM^{0k}\beta) = \varphi^{\#}_{\varkappa}\left(f_{\Lambda_\beta}\right) + Ext\text{-}\int_{\mathfrak{R}^4_\Delta}\delta(\beta,x,t)f(x,t)d^{\#3}xd^{\#}t. \tag{214}$$

Obviously the equalities (213) hold on the domains $D\left((M^{0k})^2\right)$ with $k = 1,2,3$ correspondingly. For any vector ψ such that $\psi \in D\left((M^{0k})^2\right)$ from (207) we obtain the equalities

$$\varphi^{\#}_{\varkappa}(f)Ext\text{-}\exp(-iM^{0k}\beta)\psi = Ext\text{-}\exp(-iM^{0k}\beta)\varphi^{\#}_{\varkappa}\left(f_{\Lambda_\beta}\right)\psi + \left(Ext\text{-}\int_{\mathfrak{R}^4_\Delta}\delta(\beta,x,t)f(x,t)d^{\#3}xd^{\#}t\right)\psi. \tag{215}$$

Since $\left\|\varphi^{\#}_{\varkappa}\left(f_{\Lambda_\beta}\right)\psi\right\| \leq c_1\left\|(H+b)^{\frac{1}{2}}\psi\right\|$ and $D\left((M^{0k})^2\right)$ for any $k = 1,2,3$ is a #-core for H, by Theorem 11.31, the equalities (215) extends by #-closure to $D(H)$ and (215) holds for $\psi \in D(H)$. Since the domain $D(H)$ is a #-core for the operator $\varphi^{\#}_{\varkappa}\left(f_{\Lambda_\beta}\right)$, we conclude that (214) extends by #-closure to $D\left(\varphi^{\#}_{\varkappa}\left(f_{\Lambda_\beta}\right)\right)$ and therefore the equalities (215) hold for all $k = 1,2,3$ and for any ψ such that $\psi \in D\left(\varphi^{\#}_{\varkappa}\left(f_{\Lambda_\beta}\right)\right)$. Thus we have proved that

$$Ext\text{-}\exp(-iM^{0k}\beta)D\left(\varphi^{\#}_{\varkappa}\left(f_{\Lambda_\beta}\right)\right) \subset D\left(\varphi^{\#}_{\varkappa}(f)\right).$$

By similar consideration one obtains that

$$Ext\text{-}\exp(iM^{0k}\beta)D\left(\varphi^{\#}_{\varkappa}(f)\right) \subset D\left(\varphi^{\#}_{\varkappa}\left(f_{\Lambda_\beta}\right)\right).$$

This proves (214) as an equality between self- #-adjoint operators, completing the proof of the theorem.
Theorem 13.42 If $M^{0k}, k = 1,2,3$ satisfies only the conditions (6.2) and (6.3), the conclusions of Theorem 13.41 still hold.
Proof By (6.3) there is an $\varepsilon > 0$ so that for all $k = 1,2,3$

$$\alpha + x_k g_0^{(k)}(x_1, x_2, x_3) = x_k = x_k g_1(x_1, x_2, x_3)$$

for $(x_1, x_2, x_3) \in I_{2\varepsilon}^3 = [a - 2\varepsilon, b + 2\varepsilon]^3$. Let $\hat{g}_1$ be a $C^{*\infty}$ function so that

$$x_k \hat{g}_1 = \hat{h}_1^2$$

for $\hat{h}_1 \geq 0, \hat{h}_1 \in S_{\text{fin}}^{\#}({}^*\mathbb{R}_c^{\#3}), g_1(x_1, x_2, x_3) = 0$ for $(x_1, x_2, x_3) \notin I_{2\varepsilon}^3$ and $g_1(x_1, x_2, x_3) = 1$ for $(x_1, x_2, x_3) \in I_{\varepsilon}^3$.
Then conditions (6.2)-(6.4) hold for the pairs $\left\{g_0^{(k)}, \hat{g}_1\right\}$ and

$$\delta g_1 = g_1 - \hat{g}_1$$

is non-zero only in the complement of I_{ε}^3. Let for all $k = 1,2,3$

$$\widehat{M}^{0k} = H_{0.\varkappa} + T_0\left(x_k g_0^{(k)}\right) + T_I(x_k \hat{g}_1),$$

$$\delta M^{0k} = M^{0k} - \widehat{M}^{0k} = T_I(x_k \delta g_1).$$

By Theorem **5.3**, both M^{0k} and $\widehat{M}^{0k}$ are essentially self-#-adjoint on $D(H_{0.\varkappa}^2)$. The operators $\widehat{M}^{0k}$ are satisfies the conditions of Theorem 13.41. Note that M^{0k} is also essentially self-#-adjoint on this domain. By [**1, Theorem 3.**2], the spectral projections of δM^{0k} commute with $\varphi_{\varkappa}^{\#}(f)$ for $\text{supp}(f) \subset \mathfrak{R}_I^4 \subset M_4^{\#}$. Hence if $E_{\varkappa}$ is a spectral projection of $\varphi_{\varkappa}^{\#}(f)$

$$\{Ext\text{-}\exp(-iM^{0k}\beta)\}E_{\varkappa}\{Ext\text{-}\exp(-iM^{0k}\beta)\} = \lim_{n \to {}^*\infty} \left\{\left(Ext\text{-}\exp\left(\frac{i\widehat{M}^{0k}\beta}{n}\right) Ext\text{-}\exp\left(\frac{i\delta M^{0k}\beta}{n}\right)\right)^n\right\} E_{\varkappa} \times$$

$$\times \left\{\left(Ext\text{-}\exp\left(-i\widehat{M}^{0k}\beta/n\right) Ext\text{-}\exp(-i\delta M^{0k}\beta/n)\right)^n\right\},$$

where we use the fact that

$$\bigcup_{0 \leq \gamma \leq 1} \text{supp}\left(f_{\Lambda_\beta}\right) \subset \mathfrak{R}_I^4$$

if

$$\text{supp}(f) \cup \text{supp}\left(f_{\Lambda_\beta}\right) \subset \mathfrak{R}_I^4.$$

Thus M^{0k} and $\widehat{M}^{0k}$ generate the same transformations on the spectral projections of $\varphi_{\varkappa}^{\#}(f)$, if $\text{supp}(f) \cup \text{supp}\left(f_{\Lambda_\beta}\right) \subset \mathfrak{R}_I^4$. By Lemma **6.2**, Theorem **5.3**, and Theorem 2.4,chapt.2 and Theorem 4.3,chapt.2,

$$D(H^2) \subset D(M^{0k}) \cap D\left(\widehat{M}^{0k}\right)$$

$$D(M^{0k}) \cup D\left(\widehat{M}^{0k}\right) \subset D(H_{0.\varkappa}) \subset D\left((H + b)^{\frac{1}{2}}\right) \subset D\left(\varphi_{\varkappa}^{\#}(f)\right).$$

So $Ext\text{-}\exp(-iM^{0k}\beta): D(H^2) \to D\left(\varphi_{\varkappa}^{\#}(f)\right)$ and $Ext\text{-}\exp\left(-i\widehat{M}^{0k}\beta\right): D(H^2) \to D\left(\varphi_{\varkappa}^{\#}(f)\right)$

Since we can express $\varphi_{\varkappa}^{\#}(f)$ as a strong #-limit of an integral over its spectral projections on its domain $D(\varphi_{\varkappa}^{\#}(f))$, we obtain, on $D(H^2)$

$$\{Ext\text{-}\exp(-iM^{0k}\beta)\}\varphi_{\varkappa}^{\#}(f)\{Ext\text{-}\exp(-iM^{0k}\beta)\} =$$

$$= \{Ext\text{-}\exp(-i\widehat{M}^{0k}\beta)\}\varphi_{\varkappa}^{\#}(f)\{Ext\text{-}\exp(-i\widehat{M}^{0k}\beta)\} = \varphi_{\varkappa}^{\#}\left(f_{\Lambda_\beta}\right),$$

by Theorem **6.1**. Since $D(H^2)$ is a #-core for $\varphi_{\varkappa}^{\#}\left(f_{\Lambda_\beta}\right)$, this equality extends by #-closure to the domain $D\left(\varphi_{\varkappa}^{\#}\left(f_{\Lambda_\beta}\right)\right)$. Thus, part a) of Theorem **6.1** holds for M^{0k} satisfying (6.2)-(6.3). Part b) of Theorem 6.1 follows from this since the form $\varphi_{\varkappa}^{\#}(\boldsymbol{x},t)$ is #-continuous in $(\boldsymbol{x},t) \in M_4^{\#}$. on

§ 14. ESTIMATES ON THE INTERACTION HAMILTONIAN

Let $\mathcal{F}^{\#}$ be the Pock space for a massive, neutral scalar Geld in two-dimensional space-time. The elements of $\mathcal{F}^{\#}$ are sequences of functions on momentum space. Let the annihilation and creation operators be normalized by the relation

$$[a(\boldsymbol{k}), a^*(\boldsymbol{k}')] = \delta^{\#}(\boldsymbol{k}-\boldsymbol{k}'). \tag{14.1}$$

Thus the free-field Hamiltonian is

$$H_{0,\varkappa} = Ext\text{-}\int_{|\boldsymbol{k}|\leq\varkappa} a^*(\boldsymbol{k})a(\boldsymbol{k})\omega(\boldsymbol{k})d^{\#3}k. \tag{14.2}$$

The $t = 0$ field with hyperfinite ultraviolet cut-oft $\varkappa$ is

$$\varphi_{\varkappa}^{\#}(x) = Ext\text{-}\int_{|\boldsymbol{k}|\leq\varkappa} Ext\text{-}\exp(-i\langle\boldsymbol{k},\boldsymbol{x}\rangle)[a^*(\boldsymbol{k}) + a(\boldsymbol{k})]\, d^{\#3}k \tag{14.3}$$

The spatially cut-off interaction Hamiltonian reads

$$H_{I,\varkappa}(g) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} :\varphi_{\varkappa}^{\#4}(\boldsymbol{x}):g(\boldsymbol{x})\, d^{\#3}x = \tag{14.4}$$

$$\sum_{j=0}^{4}\binom{4}{j}\Big\{Ext\text{-}\int_{|\boldsymbol{k_1}|\leq\varkappa} d^{\#3}k_1 \cdots Ext\text{-}\int_{|\boldsymbol{k_m}|\leq\varkappa} d^{\#3}k_m\, a^*(\boldsymbol{k}_1)\cdots a^*(\boldsymbol{k}_j)a(-\boldsymbol{k_{j+1}})\cdots$$

$$\times\, a(-\boldsymbol{k_4})\hat{g}\left(\sum_{i=1}^{4} k_i^{(1)}, \sum_{i=1}^{4} k_i^{(2)}, \sum_{i=1}^{4} k_i^{(3)}\right)\prod_{i=1}^{4}[\omega(\boldsymbol{k}_i)]^{-1/2} d^{\#3}k_1 \ldots d^{\#3}k_4\Big\},$$

where we let $\boldsymbol{k}_i = \left(k_i^{(1)}, k_i^{(2)}, k_i^{(3)}\right), i = 1,2,3.$

The total Hamiltonian reads

$$H_{\varkappa}(g) = H_{0,\varkappa} + H_{I,\varkappa}(g) \tag{14.5}$$

We let

$$N_{\varkappa} = Ext\text{-}\int_{|\boldsymbol{k}|\leq\varkappa} a^*(\boldsymbol{k})a(\boldsymbol{k})d^{\#3}k, \tag{14.6}$$

and

$$D_{0,\varkappa}^{\#} = \bigcap_{n=0}^{{}^*\infty} D\left(H_{0,\varkappa}^n\right). \tag{14.7}$$

Theorem 14.1 For any $\varepsilon \in {}^*\mathbb{R}^{\#}_{\text{fin}+}$ and for fixed $g(x) \in S^{\#}_{\text{fin}}({}^*\mathbb{R}^{\#3}_c)$ there is a constant b such that as bilinear forms on $D^{\#}_{0,\varkappa} \times D^{\#}_{0,\varkappa}$

$$-\left[H_{0,\varkappa}^{\frac{1}{2}}, \left[H_{0,\varkappa}^{\frac{1}{2}}, H_{I,\varkappa}(g)\right]\right] \leq \varepsilon H_{0,\varkappa}^{2} + b, \tag{14.8}$$

$$-\left[N_{\varkappa}, \left[N_{\varkappa}, H_{I,\varkappa}(g)\right]\right] \leq \varepsilon N_{\varkappa}^{2} + b. \tag{14.9}$$

Theorem 14.2 Let $W: \mathcal{F}^{\#} \to \mathcal{F}^{\#}$ be an operator of the form

$$W = Ext\text{-}\int_{|\boldsymbol{k_1}|\leq\varkappa} d^{\#3}k_1 \cdots Ext\text{-}\int_{|\boldsymbol{k_m}|\leq\varkappa} d^{\#3}k_m\, w(\boldsymbol{k}_1, \dots, \boldsymbol{k}_m)\, a^*(\boldsymbol{k}_{\boldsymbol{1}}) \cdots a(-\boldsymbol{k_m}), \tag{14.10}$$

where $w(\boldsymbol{k}_1, \dots, \boldsymbol{k}_m) \in L_2^{\#}\left(({}^*\mathbb{R}^{\#\boldsymbol{3m}}_c)\right)$. Then

$$\left\|(N_{\varkappa} + I)^{-j/2} W (N_{\varkappa} + I)^{-(m-j)/2}\right\|_{\#} \leq \text{const}\|w(\boldsymbol{k}_1, \dots, \boldsymbol{k}_m)\|_{L_2^{\#}}, \tag{14.11}$$

$$\left\|\left(H_{0,\varkappa} + I\right)^{-1}\left[H_{0,\varkappa}^{\frac{1}{2}}, \left[H_{0,\varkappa}^{\frac{1}{2}}, W\right]\right]\left(H_{0,\varkappa} + I\right)^{-1}(N_{\varkappa} + I)^{-\frac{(m-4)}{2}}\right\|_{\#} \leq$$

$$\leq \text{const}\left\|\omega^{\frac{1}{2}}\left(\sum_{i=1}^{4} k_{\boldsymbol{i}}^{(1)}, \sum_{\boldsymbol{i=1}}^{\boldsymbol{4}} k_i^{(2)}, \sum_{\boldsymbol{i=1}}^{\boldsymbol{4}} k_i^{(3)}\right) w(\boldsymbol{k}_1, \dots, \boldsymbol{k}_m)\right\|_{L_2^{\#}}, \tag{14.12}$$

$$\left\|\left[H_{0,\varkappa}^{\frac{1}{2}}, \left[H_{0,\varkappa}^{\frac{1}{2}}, W\right]\right](N_{\varkappa} + I)^{-m/2}\right\|_{\#} \leq \text{const} \times \varkappa^4 \left\|\sum_{i=1}^{m} \omega(\boldsymbol{k}_i) w(\boldsymbol{k}_1, \dots, \boldsymbol{k}_m)\right\|_{L_2^{\#}}. \tag{14.13}$$

Theorem 14.3 Let the operator W be as above. Then

$$\left\|\left[H_{0,\varkappa}^{\frac{1}{2}}, \left[H_{0,\varkappa}^{\frac{1}{2}}, W\right]\right](N_{\varkappa} + I)^{-m/2}\right\|_{\#} \leq \text{const}\|w(\boldsymbol{k}_1, \dots, \boldsymbol{k}_m)\|_{L_2^{\#}}. \tag{14.14}$$

Proof of Theorem 14.1.Introduce the $t = 0$ field $\varphi_\mu^{\#}(x)$ with an hyperfinite ultraviolet cut-oft $\mu < \varkappa$:

$$\varphi_\mu^{\#}(x) = Ext\text{-}\int_{|\boldsymbol{k}|\leq\mu} Ext\text{-}\exp(-i\langle\boldsymbol{k}, \boldsymbol{x}\rangle)[a^*(\boldsymbol{k}) + a(\boldsymbol{k})]\, d^{\#3}k$$

The spatially cut-off interaction Hamiltonian $H_{I,\mu}(g)$ corresponding to the $t = 0$ field $\varphi_\mu^{\#}(x)$ reads

$$H_{I,\varkappa}(g) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} :\varphi_\varkappa^{\#4}(\boldsymbol{x}): g(\boldsymbol{x})\, d^{\#3}x. \tag{14.15}$$

Note that

$$H_{I,\varkappa}(g) = \text{st.}\#\text{-}\lim_{\mu\to_{\#}\varkappa} H_{I,\mu}(g). \tag{14.16}$$

If we write $H_{I,\varkappa}(g)$ as a sum of five operators of the form W in (14.10), then by Theorem 14.2 taken for the case $m = 4$ we get

$$\left\|\left(H_{0,\varkappa} + I\right)^{-1}\left[H_{0,\varkappa}^{\frac{1}{2}}, \left[H_{0,\varkappa}^{\frac{1}{2}}, W\right]\right]\left(H_{0,\varkappa} + I\right)^{-1}\right\|_{\#} \leq$$

$$\leq \text{const} \left\| \omega^{\frac{1}{2}} \left(\sum_{i=1}^{4} k_{\boldsymbol{i}}^{(1)}, \sum_{\boldsymbol{i=1}}^{\boldsymbol{4}} k_i^{(2)}, \ \sum_{\boldsymbol{i=1}}^{\boldsymbol{4}} k_i^{(3)} \ \right) w(\boldsymbol{k}_1, \dots, \boldsymbol{k}_4) \right\|_{L_2^{\#}}. \tag{14.17}$$

Since the kernel $w(\boldsymbol{k}_1, \dots, \boldsymbol{k}_4)$ has an over-all factor $\hat{g}\left(\sum_{i=1}^{4} k_{\boldsymbol{i}}^{(1)}, \sum_{\boldsymbol{i=1}}^{\boldsymbol{4}} k_i^{(2)}, \ \sum_{\boldsymbol{i=1}}^{\boldsymbol{4}} k_i^{(3)} \ \right)$, where $\hat{g}(\boldsymbol{k})$ is the Fourier transform of the spatial cut-off $g(\boldsymbol{x})$, the fast decrease of $\hat{g}(\boldsymbol{k})$) ensures that

$$\omega^{\frac{1}{2}} \left(\sum_{i=1}^{4} k_{\boldsymbol{i}}^{(1)}, \sum_{\boldsymbol{i=1}}^{\boldsymbol{4}} k_i^{(2)}, \ \sum_{\boldsymbol{i=1}}^{\boldsymbol{4}} k_i^{(3)} \ \right) w(\boldsymbol{k}_1, \dots, \boldsymbol{k}_4) \in$$

Thus the kernel for the corresponding cut-off interaction term w_μ approximates $w_\varkappa$ in the sense that

$$\left\| \omega^{\frac{1}{2}} \left(\sum_{i=1}^{4} k_{\boldsymbol{i}}^{(1)}, \sum_{\boldsymbol{i=1}}^{\boldsymbol{4}} k_i^{(2)}, \ \sum_{\boldsymbol{i=1}}^{\boldsymbol{4}} k_i^{(3)} \ \right) \left(w_\varkappa(\boldsymbol{k}_1, \dots, \boldsymbol{k}_4) - w_\mu(\boldsymbol{k}_1, \dots, \boldsymbol{k}_4) \right) \right\|_{L_2^{\#}} \to_{\#} 0 \tag{14.18}$$

as $\mu \to_{\#} \varkappa$. This is holds for each W making up $H_{I,\varkappa}(g)$, so we infer that there exists a μ_0 such that for any μ such that: $\mu_0 < \mu < \varkappa$

$$\left\| \left(H_{0,\varkappa} + I \right)^{-1} \left[H_{0,\varkappa}^{\frac{1}{2}}, \left[H_{0,\varkappa}^{\frac{1}{2}}, \left(H_{I,\varkappa}(g) - H_{I,\mu}(g) \right) \right] \right] \left(H_{0,\varkappa} + I \right)^{-1} \right\|_{\#} \leq \tfrac{1}{2} \varepsilon. \tag{14.19}$$

§ 15. SELF ADJOINTNESS OF THE INTERACTION HAMILTONIAN

For a real spatial cut-off $g(x)$ in the Schwartz space $S_{\text{fin}}^{\#}({}^*\mathbb{R}_c^{\#\boldsymbol{3}})$, the interaction part of the Hamiltonian $H_{I,\varkappa}(g)$ is self #-adjoint.

Theorem 15.1 If $g \in S_{\text{fin}}^{\#}({}^*\mathbb{R}_c^{\#\boldsymbol{3}})$ is real, then

$$H_{I,\varkappa}(g) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#\boldsymbol{3}}} : \varphi_\varkappa^{\#4}(\boldsymbol{x}) : g(\boldsymbol{x}) \, d^{\#3}x \tag{15.1}$$

is essentially self #-adjoint on $D_{0,\varkappa}^{\#} = \bigcap_{n=0}^{{}^*\infty} D\left(H_{0,\varkappa}^{n} \right)$.

Let us introduce a domain $D_{1,\varkappa}^{\#}$ obtained by applying any polynomial of the $t = 0$ fields $\varphi_\varkappa^{\#}(f_i)$, for real $f_i \in S_{\text{fin}}^{\#}({}^*\mathbb{R}_c^{\#\boldsymbol{3}})$ the no particle state Ω_0. Clearly $D_{1,\varkappa}^{\#} \subset D_{0,\varkappa}^{\#}$, and any vector Ω in $D_{1,\varkappa}^{\#}$ is an entire vector for $\varphi_\varkappa^{\#}(f)$, which means that the hyperinfinite power series

$$Ext\text{-}\sum_{n=0}^{{}^*\infty} \frac{\| \varphi_\varkappa^{\#n}(f)\Omega \|_{\#}}{n!} z^n \tag{15.2}$$

defines an entire function of s. Since $D_{1,\varkappa}^{\#}$ is #-dense in Fock space, Theorem 6.5 (Generalized Nelson's analytic vector theorem) shows that for real f, $\varphi_\varkappa^{\#}(f)$ is essentially self #-adjoint on $D_{1,\varkappa}^{\#}$. A similar argument can be made for the canonically conjugate $t = 0$ fields $\pi_\varkappa^{\#}(f)$. Let $\mathcal{M}_\varkappa^{\#}$ denote the von Neumann algebra of operators generated by the spectral projections of all the $t = 0$ field $\varphi_\varkappa^{\#}(f), f \in S_{\text{fin}}^{\#}({}^*\mathbb{R}_c^{\#\boldsymbol{3}})$. The algebra $\boldsymbol{\mathcal{M}}_\varkappa^{\#}$ is maximal Abelian. In other words, a bounded operator which commutes with all operators in $\mathcal{M}_\varkappa^{\#}$ is itself in $\mathcal{M}_\varkappa^{\#}$.

Let us consider $\varphi_\varkappa^{\#}(f)$ for $\operatorname{supp}(f) \subset \boldsymbol{O} \subset {}^*\mathbb{R}_c^{\#\boldsymbol{3}}$, where $\boldsymbol{O}$ is an #-open region of space. (The support of a function is the smallest #-closed set outside of which the function vanishes identically.) Define $\boldsymbol{\mathfrak{C}}_\varkappa^{\#}(\boldsymbol{O})$ as the von Neumann algebra of operators generated by the spectral projections of all the fields $\varphi_\varkappa^{\#}(f)$ and $\pi_\varkappa^{\#}(f)$ with $\operatorname{supp}(f) \subset \boldsymbol{O}$.

Since

$$\varphi_{\varkappa}^{\#}(\boldsymbol{x},t) = Ext\text{-}\exp(itH_{0,\varkappa})\,\varphi_{\varkappa}^{\#}(\boldsymbol{x})Ext\text{-}\exp(-itH_{0,\varkappa}) = \tag{15.3}$$

$$= Ext\text{-}\int_{^*\mathbb{R}_c^{\#3}} d^{\#3}y \left\{\Delta_{\#}(\boldsymbol{x}-\boldsymbol{y},t)\,\pi_{\varkappa}^{\#}(\boldsymbol{y}) - \left[\frac{\partial^{\#}}{\partial^{\#}t}\Delta_{\#}(\boldsymbol{x}-\boldsymbol{y},t)\right]\,\varphi_{\varkappa}^{\#}(\boldsymbol{y})\right\},$$

where $\Delta_{\#}(\boldsymbol{x},t)$ is the solution of the generalized Klein-Gordon equation (13.9) and $\Delta_{\#}(\boldsymbol{x},t)$ vanishes outside the light cone, we infer that

$$Ext\text{-}\exp(itH_{0,\varkappa})\mathfrak{C}_{\varkappa}^{\#}(\boldsymbol{O})Ext\text{-}\exp(-itH_{0,\varkappa}) \subset \mathfrak{C}_{\varkappa}^{\#}(\boldsymbol{O}_t), \tag{15.4}$$

where $\boldsymbol{O}_t$ is the region $\boldsymbol{O}$ expanded by t.

Theorem 15.2 If $g(x) \in S_{\mathrm{fin}}^{\#}(^*\mathbb{R}_c^{\#3})$ is real and has its support in an #-open rectangular parallelepiped $\boldsymbol{O} \subset {^*\mathbb{R}_c^{\#3}}$, then for the $H_{I,\varkappa}(g)$ of (15.1)

$$Ext\text{-}\exp\left(itH_{I,\varkappa}(g)\right) \in \mathfrak{C}_{\varkappa}^{\#} \cap \boldsymbol{\mathcal{M}}_{\varkappa}^{\#}.$$

Theorem 15.3 Let T be any operator with domain $D_{1,\varkappa}^{\#}$ such that

$$T\,D_{1,\varkappa}^{\#} \subset D\left(\varphi_{\varkappa}^{\#n}(f)\right), \tag{15.5}$$

$$T\,D_{1,\varkappa}^{\#} \subset D\left(\left(T \upharpoonright D_{1,\varkappa}^{\#}\right)^*\right), \tag{15.6}$$

$$[T,\ \varphi_{\varkappa}^{\#n}(f)]\,D_{1,\varkappa}^{\#} = 0. \tag{15.7}$$

Then

$$\boldsymbol{\mathcal{M}}_{\varkappa}^{\#}D_{1,\varkappa}^{\#} \subset D\left(T \upharpoonright D_{1,\varkappa}^{\#}\right), \tag{15.8}$$

$$[\#\text{-}\overline{T},\boldsymbol{\mathcal{M}}_{\varkappa}^{\#}]\,D_{1,\varkappa}^{\#} = 0. \tag{15.9}$$

Proof For $\Omega \in D_{1,\varkappa}^{\#}$, from (15.5) and (15.7) we get

$$T\,\varphi_{\varkappa}^{\#n}(f)\Omega = \varphi_{\varkappa}^{\#n}(f)T\Omega.$$

But by (15.6), for real f

$$\|T\,\varphi_{\varkappa}^{\#n}(f)\Omega\|_{\#}^2 = \langle T\Omega,\ \varphi_{\varkappa}^{\#2n}(f)T\Omega\rangle_{\#} = \langle T^*T\Omega,\ \varphi_{\varkappa}^{\#2n}(f)\Omega\rangle_{\#} \leq \|T^*T\Omega\|_{\#}\|\,\varphi_{\varkappa}^{\#2n}(f)\Omega\|_{\#}.$$

Thus the #-convergent power series (3.2) shows that for $\Omega \in D_{1,\varkappa}^{\#}$,

$$\#\text{-}\overline{T}\left(Ext\text{-}\exp\left(i\varphi_{\varkappa}^{\#}(f)\right)\right)\Omega = Ext\text{-}\exp\left(i\varphi_{\varkappa}^{\#}(f)\right)T\Omega. \tag{15.10}$$

It is clear that (15.10) is still valid with $Ext\text{-}\exp\left(i\varphi_{\varkappa}^{\#}(f)\right)$ replaced by strong #-limits of sums of such exponentials, and hence (15.8) and (15.9).

Theorem 15.4 Let $\mathcal{M}$ is a maximal Abelian algebra of bounded operators on a non-Archimedean Hilbert space $\mathcal{H}$ with a cyclic vector Ω_0. Let T be a symmetric operator with domain $\mathcal{M}\Omega_0$, and let T commute with $\mathcal{M}$.Then T is essentially self #-adjoint.

Proof Without loss of generality, $\mathcal{M} = L_{^*\infty}^{\#}(X)$ and $\mathcal{H} = L_2^{\#}(X)$ for some #-measure space (X,Σ,μ), and Ω_0 is the function 1. Let $f \in L_2^{\#}(X)$. Then $t \in L_2^{\#}(X)$ and T is multiplication by t, with domain $L_{^*\infty}^{\#}(X)$. Let $f \in L_2^{\#}(X)$ and suppose $tf \in L_2^{\#}(X)$ also and let $f_n(x) = f(x)$ if $|f(x)| \leq n, n \in {}^*\mathbb{N}$ and $f_n(x) \equiv 0$ otherwise. Then $f_n \in L^{\#}_{{}^*\infty} = D(T)$ and $f_n \to_{\#} f, tf_n \to_{\#} tf$ in $L_2^{\#}$ norm by the bounded #-convergence theorem. Thus $\{f, tf\}$ is in the graph of the #-closure of T. Thus the #-closure of T is self #-adjoint, and T is essentially self #-adjoint.

Remark 15.1 Let $T_n, n \in {}^*\mathbb{N}$ be a hyperinfinite sequence of operators with the property of T in the Theorem 15.4. Then $T_n \to_{\#} T$ strongly on the domain $\mathcal{M}\Omega_0$ if and only if $T_n\Omega_0 \to_{\#} T\Omega_0$.

Proof of the Theorems 15.1 and 15.2**.** We apply now the Theorems 15.3 and 15.4 with the case $T = H_{I,\varkappa}(g)$, $\mathcal{M}$ in Theorem 15.4 as in Theorem 15.3, the non-Archimedean Hilbert space Fock space $\mathcal{F}^{\#}$, and Ω_0 the Fock no-particle state. The hypotheses (15.5) and (15.6) can be verified by a direct computation. Thus $H_{I,\varkappa}(g)$ is essentially self #-adjoint on $D_{1,\varkappa}^{\#} \subset D_{0,\varkappa}^{\#}$, and hence $H_{I,\varkappa}(g)$ is essentially self #-adjoint on $D_{0,\varkappa}^{\#}$.

If we assume that $\sup(g) \subset \boldsymbol{O}$, then as $\boldsymbol{O}$ is an #-open region, $\sup(g) \subset \boldsymbol{O}_1$ where $\boldsymbol{O}_1$ is $\boldsymbol{O}$ contracted by some small amount $\varepsilon > 0, \varepsilon \approx 0$. Since $H_{I,\varkappa}(g)$ commutes with $\mathcal{M}$, and $\mathcal{M}$ is maximal Abelian, -exp $\left(itH_{I,\varkappa}(g)\right) \in \mathcal{M}$. Furthermore the argument in the proof of Theorem 15.3, can be repeated to show that $H_{I,\varkappa}(g)$ commutes with $\mathfrak{C}_{\varkappa}^{\#}(\boldsymbol{O}'_1)$, where $\boldsymbol{O}'_1$ is the complement of the #-closure of $\boldsymbol{O}_1$. Since $\mathfrak{C}_{\varkappa}^{\#}({}^*\mathbb{R}_c^{\#3})$ is irreducible and $H_{I,\varkappa}(g)$ commutes with $\mathfrak{C}_{\varkappa}^{\#}(\boldsymbol{O}'_1)$, Ext-exp $\left(itH_{I,\varkappa}(g)\right) \in \mathfrak{C}_{\varkappa}^{\#}(\boldsymbol{O}_2)$ where $\boldsymbol{O}_2$ is $\boldsymbol{O}_1$ expanded by any amount $\varepsilon' > 0$. Taking $\varepsilon' < \varepsilon$, we have Ext-exp $\left(itH_{I,\varkappa}(g)\right) \in \mathfrak{C}_{\varkappa}^{\#}(\boldsymbol{O})$, which completes the proof.

§ 16. SELF ADJOINTNESS OF THE TOTAL HAMILTONIAN

Theorem 16.1 (a) For real $g(x) \in S_{\text{fin}}^{\#}({}^*\mathbb{R}_c^{\#3})$, the total Hamiltonian $H_{\varkappa}(g) = H_{0,\varkappa}(g) + H_{I,\varkappa}(g)$ is self #-adjoint with the domain $D\left(H_{\varkappa}(g)\right) = D\left(H_{0,\varkappa}(g)\right) \cap D\left(H_{I,\varkappa}(g)\right)$.

(b) The total Hamiltonian $H_{\varkappa}(g)$ is essentially self #-adjoint on the domain

$$D_{0,\varkappa}^{\#} = \bigcap_{n=0}^{{}^*\infty} D\left(H_{0,\varkappa}^n\right).$$

Remark 16.1 In order to prove the self #-adjointness of $H_{\varkappa}$, we combine the estimates of Sec. 14, the self #-adjointness of $H_{I,\varkappa}(g)$ proved in Sec. 15, and a singular perturbation theory developed in [19]. We need the following result which is a special case of **Theorem 8** of Ref. [19].

Theorem 16.2 Under the hypotheses (i)-(iii) below, the operator $H_{\varkappa} = H_{0,\varkappa} + H_{I,\varkappa}$z is self #-adjoint.

(i) Both $H_{0,\varkappa}$ and $H_{I,\varkappa}$ are self #-adjoint. The domain $D_{0,\varkappa}^{\#}$ is contained in the domain of $H_{I,\varkappa}$, and $H_{I,\varkappa}$ is essentially self #-adjoint on $D_{0,\varkappa}^{\#}$.

(ii) Let $N_{\varkappa}$ be a positive self #-adjoint operator, commuting with $H_{0,\varkappa}$, and such that $N_{\varkappa} \leq \text{const}\, H_{0,\varkappa}$. Suppose that the operators $(N_{\varkappa} + I)^{-1}H_{I,\varkappa}(N_{\varkappa} + I)^{-1}$ and $(N_{\varkappa} + I)^{-1}H_{I,\varkappa}(N_{\varkappa} + I)^{-3}$ are bounded.

(iii) Suppose that for any $\varepsilon > 0$, there exists a number $b \in {}^*\mathbb{R}_c^{\#}$ such that as bilinear forms on $D_{0,\varkappa}^{\#} \times D_{0,\varkappa}^{\#}$,

$$-H_{I,\varkappa} \leq \varepsilon N_{\varkappa} + bI, \tag{16.1}$$

$$-\left[H_{0,\varkappa}^{\frac{1}{2}}\left[H_{0,\varkappa}^{\frac{1}{2}} H_{I,\varkappa}\right]\right] \leq \varepsilon H_{0,\varkappa}^2 + bI, \tag{16.2}$$

$$-\left[N_{\varkappa},\left[N_{\varkappa},H_{I,\varkappa}\right]\right]\leq\varepsilon N_{\varkappa}^{3}+bI. \quad (16.3)$$

Proof of Theorem 16.1 In order to prove that $H_{\varkappa}(g)$ is self #-adjoint, we apply Theorem 16.2 in the case that $H_{0,\varkappa}$ is the free Hamiltonian, $N_{\varkappa}$ is the number operator, and $H_{I,\varkappa}$ is the interaction Hamiltonian $H_{I,\varkappa}(g)$. Thus we need to verify (i)-(iii). Condition (i) was dealt with in Theorem 15.1, while condition (ii) is a consequence of (14.11).In **Refs. 2**, and **3,** it is shown that for any $\varepsilon>0$, there is a number $b\in {}^{*}\mathbb{R}_{c}^{\#}$ such that

$$-H_{I,\varkappa}(g)\leq\varepsilon H_{0,\varkappa}+bI.$$

By following that proof, but using the smoothing operator $Ext\text{-}\exp(-tN_{\varkappa})$, in place of $Ext\text{-}\exp\left(-tH_{0,\varkappa}\right)$, one arrives at the estimate (16.1) required in (iii). The remaining estimates (16.2) and (16.3) were established in Theorem 14.1.Thus we conclude from Theorem 16.2 that $H_{\varkappa}(g)$ is self #-adjoint on the domain $D\left(H_{0,\varkappa}\right)\cap D\left(H_{I,\varkappa}(g)\right)$. We now show that $H_{\varkappa}(g)$ is essentially self #-adjoint on $D\left(H_{0,\varkappa}\right)$. We first show that $H_{\varkappa}(g)$ is essentially self #-adjoint on $D_{2}=D\left(H_{0,\varkappa}\right)\cap D(N_{\varkappa}^{2})$. By (14.11) it is clear that the domain of $H_{\varkappa}(g)$ contains D_{2}. For $\psi\in D\left(H_{\varkappa}(g)\right)=D\left(H_{0,\varkappa}\right)\cap D\left(H_{I,\varkappa}(g)\right)$, consider hyperinfinite sequence $\psi_{n}\in D_{2}, n\in {}^{*}\mathbb{N}$ defined by

$$\psi_{n}=n(nI+N_{\varkappa})^{-1}\psi. \quad (16.4)$$

Thus $\|\psi_{n}-\psi\|_{\#}+\left\|H_{0,\varkappa}\psi_{n}-H_{0,\varkappa}\psi\right\|_{\#}\to_{\#}0$ as $n\to {}^{*}\infty$.

We need to study the following differences

$$H_{I,\varkappa}\psi_{n}-H_{I,\varkappa}\psi=-N_{\varkappa}(nI+N_{\varkappa})^{-1}\,H_{I,\varkappa}\psi+n\left[H_{I,\varkappa},(nI+N_{\varkappa})^{-1}\right]\psi, n\in {}^{*}\mathbb{N}. \quad (16.5)$$

Since $N_{\varkappa}(nI+N_{\varkappa})^{-1}, n\in {}^{*}\mathbb{N}$ is a uniformly bounded hyperinfinite sequence #-converging to zero on the #-dense set $D(N_{\varkappa})$, it #-converges to zero and $\left\|N_{\varkappa}(nI+N_{\varkappa})^{-1}\,H_{I,\varkappa}\psi\right\|_{\#}$ as $n\to {}^{*}\infty$. But for the second term in (16.5) we get

$$n\left[H_{I,\varkappa},(nI+N_{\varkappa})^{-1}\right]\psi=\left[H_{I,\varkappa},(nI+N_{\varkappa})^{-1}\right](nI+N_{\varkappa})n(nI+N_{\varkappa})^{-1}\psi= \quad (16.6)$$

$$=(nI+N_{\varkappa})^{-1}\left[N_{\varkappa},H_{I,\varkappa}\right]n(nI+N_{\varkappa})^{-1}\psi=$$

$$=(nI+N_{\varkappa})^{-1}(I+N_{\varkappa})(I+N_{\varkappa})^{-1}\left[N_{\varkappa},H_{I,\varkappa}\right]\times$$

$$\times(I+N_{\varkappa})^{-1}n(nI+N_{\varkappa})^{-1}(I+N_{\varkappa})\psi.$$

Note that as $n\to {}^{*}\infty$, hyperinfinite sequence $\delta_{n}=n(nI+N_{\varkappa})^{-1}(I+N_{\varkappa})\psi, n\in {}^{*}\mathbb{N}$ #-converges strongly to $(I+N_{\varkappa})\psi$, that by (14.11), $(I+N_{\varkappa})^{-1}\left[N_{\varkappa},H_{I,\varkappa}\right](I+N_{\varkappa})^{-1}$ is bounded, and hyperinfinite sequence $\gamma_{n}= =(nI+N_{\varkappa})^{-1}(I+N_{\varkappa})\psi, n\in {}^{*}\mathbb{N}$ #-converges strongly to zero. Thus we get $\left\|\left[H_{I,\varkappa},(nI+N_{\varkappa})^{-1}\psi\right]\right\|_{\#}\to_{\#}0$ as $n\to {}^{*}\infty$, and so $\left\|H_{I,\varkappa}\psi_{n}-H_{I,\varkappa}\psi\right\|_{\#}\to_{\#}0$ as $n\to {}^{*}\infty$. Thus we can to conclude that $H_{\varkappa}(g)$ is the #-closure of $H_{\varkappa}(g)$ restricted to D_{2}, so $H_{\varkappa}(g)$ is essentially self #-adjoint on D_{2}. Let D_{2} be a Hilbert space endowed with the #-norm $\|\cdot\|_{\#}'$ such that

$$(\|\psi\|_{\#}')^{2}=\|\psi\|_{\#}^{2}+\left\|H_{0,\varkappa}\psi\right\|_{\#}^{2}+\|N_{\varkappa}\psi\|_{\#}^{2}. \quad (16.7)$$

From (14.11) we infer that

$$\|H_{\varkappa}(g)\,\psi\|_{\#}\leq\mathrm{const}\|\psi\|_{\#}',$$

so that $H_{\varkappa}(g)$ is essentially self #-adjoint on any subset of D_2 which is #-dense in the Hilbert space D_2. For any $\psi \in D_2, \psi_\lambda = Ext\text{-}\exp(-\lambda H_{0,\varkappa})\psi \in D_{0,\varkappa}^{\#} = \bigcap_{n=0}^{*\infty} D(H_{0,\varkappa}^n)$, and $\|\psi - \psi_\lambda\|_{D_{1,\varkappa}^{\#}} \to_{\#} 0$ as $\lambda \to_{\#} 0$. Thus $H_{\varkappa}(g)$ is essentially self #-adjoint on D_0.

§ 17. REMOVING THE SPATIAL CUTOFF AND LOCALITY

For the reader's convenience, we sketch a proof of generalized Segal's theorem that the self #-adjointness of $H_{\varkappa}(g)$ allows the removal of the spatial cut-off. In fact, if A is a bounded function of the free fields localized in a bounded region of space at $t = 0$, then

$$\sigma_t(A) = Ext\text{-}\exp(itH_{\varkappa}(g))A Ext\text{-}\exp(-itH_{\varkappa}(g))$$

is independent of $g(x)$ provided that $g(x) = \lambda$, the desired coupling constant, on a sufficiently large region, depending on t. Furthermore, if A is localized in the region of space O, then $\sigma_t(A)$ is localized in the region O_t, where O_t is the region O expanded by t. (We have taken the velocity of light to be one.) In other words, the time translation σ_t gives rise to a local theory. If one chooses for the operator A a spectral projection of the $t = 0$ field $\varphi_{\varkappa}^{\#}(f)$, one can piece together the time translation operator for the fields themselves. In section 16, we showed that $H_{\varkappa} = H_{0,\varkappa} + H_{I,\varkappa}$, which is sum of two self #-adjoint operators, is itself self #-adjoint. As a consequence of this fact, the generalized Trotter product formula (7.1) (see section 7) says that for all $\psi \in \mathcal{F}^{\#}$

$$Ext\text{-}\exp(itH_{\varkappa}(g))\psi = \#\text{-}\lim_{n\to{}^*\infty}\left(\left[Ext\text{-}\exp\left(\frac{itH_{0,\varkappa}(g)}{n}\right)\right]\left[Ext\text{-}\exp\left(\frac{itH_{I,\varkappa}(g)}{n}\right)\right]\right)\psi.$$

And therefore we obtain

$$\sigma_t(A)\psi =$$

$$\#\text{-}\lim_{n\to{}^*\infty}\left(\left[Ext\text{-}\exp\left(\frac{itH_{0,\varkappa}(g)}{n}\right)\right]\left[Ext\text{-}\exp\left(\frac{itH_{I,\varkappa}(g)}{n}\right)\right]\right)^n A\left(\left[Ext\text{-}\exp\left(\frac{-itH_{0,\varkappa}(g)}{n}\right)\right]\left[Ext\text{-}\exp\left(\frac{-itH_{I,\varkappa}(g)}{n}\right)\right]\right)^n\psi.$$

I.et $\boldsymbol{O}$ be the region of space defined by $|\boldsymbol{x}| < M, t = 0$, and let $A \in \mathfrak{C}_{\varkappa}^{\#}(\boldsymbol{O})$, where $\mathfrak{C}_{\varkappa}^{\#}(\boldsymbol{O})$ is defined in Sec. 15. Given an arbitrary, positive ε, split $g(\boldsymbol{x})$ into two infinitely #-differentiable parts $g_1(\boldsymbol{x}), g_2(\boldsymbol{x})$ such that

$$g(\boldsymbol{x}) = g_1(\boldsymbol{x}) + g_2(\boldsymbol{x}),$$

where $\operatorname{supp}(g_1(\boldsymbol{x})) \subset \boldsymbol{O}_\varepsilon$ and $\operatorname{supp}(g_2(\boldsymbol{x})) \cap \boldsymbol{O}_{\frac{\varepsilon}{2}} = \emptyset$ is empty. Write now

$$H_{I,\varkappa}(g) = H_{I,\varkappa}(g_1) + H_{I,\varkappa}(g_2),$$

so that as a consequence of theorems 15.1 and 15.2, $H_{I,\varkappa}(g_1)$ and $H_{I,\varkappa}(g_2)$ commute, and

$$Ext\text{-}\exp\left(\frac{itH_{I,\varkappa}(g)}{n}\right) = \left[Ext\text{-}\exp\left(\frac{itH_{I,\varkappa}(g_1)}{n}\right)\right]\left[Ext\text{-}\exp\left(\frac{itH_{I,\varkappa}(g_2)}{n}\right)\right].$$

Furthermore,

$$Ext\text{-}\exp\left(\frac{itH_{I,\varkappa}(g_1)}{n}\right) \in \mathfrak{C}_{\varkappa}^{\#}(\boldsymbol{O}_\varepsilon),$$

and $Ext\text{-}\exp\left(\frac{itH_{I,\varkappa}(g_2)}{n}\right)$ commutes with $\mathfrak{C}_{\varkappa}^{\#}(\boldsymbol{O}_{\varepsilon/4})$. Therefore,

$$A_1(t) = \left[Ext\text{-}\exp\left(\frac{itH_{0,\varkappa}(g_1)}{n}\right)\right]\left[Ext\text{-}\exp\left(\frac{itH_{I,\varkappa}(g_2)}{n}\right)\right] A \left[Ext\text{-}\exp\left(-\frac{itH_{I,\varkappa}(g_1)}{n}\right)\right]\left[Ext\text{-}\exp\left(-\frac{itH_{0,\varkappa}(g_2)}{n}\right)\right]$$

depends on $g(\boldsymbol{x})$ only in the region $\boldsymbol{O}_\varepsilon$, and by the free propagation property (15.4),

$$A_1 \in \mathfrak{C}_\varkappa^\#(\boldsymbol{O}_{(t/n)+\varepsilon})$$

We continue step by step, and after $n \in {}^*\mathbb{N}\backslash\mathbb{N}$ steps by using hyperinfinite induction principle, see ref. [10], we conclude that

$$A_n(t) = \left(\left[Ext\text{-}\exp\left(\frac{itH_{0,\varkappa}(g_1)}{n}\right)\right]\left[Ext\text{-}\exp\left(\frac{itH_{I,\varkappa}(g_2)}{n}\right)\right]\right)^n A \times$$

$$\times\left(\left[Ext\text{-}\exp\left(-\frac{itH_{I,\varkappa}(g_1)}{n}\right)\right]\left[Ext\text{-}\exp\left(-\frac{itH_{0,\varkappa}(g_2)}{n}\right)\right]\right)^n$$

depends on $g(\boldsymbol{x})$ only in the region $\boldsymbol{O}_{t+n\varepsilon}$ and

$$A_1(t) \in \mathfrak{C}_\varkappa^\#(\boldsymbol{O}_{t+n\varepsilon})$$

Since ε can be chosen arbitrarily, $A_n(t)$ depends on $g(\boldsymbol{x})$ only in the region #- $\overline{\boldsymbol{O}}_t$, the #- closure of $\boldsymbol{O}_t$, and

$$A_n(t) \in \bigcap_{\varepsilon>0} \mathfrak{C}_\varkappa^\#(\boldsymbol{O}_{t+\varepsilon}).$$

Thus $A_n(t)$ commutes with any local observable B localized in #-open region of space $\boldsymbol{O}'$ such that $\boldsymbol{O}'$ and $\boldsymbol{O}_t$ are disjoint. As this is true for each $n \in {}^*\mathbb{N}\backslash\mathbb{N}$, it is true for

$$\sigma_t(A) = \text{strong } \#\text{-}\lim_{n\to{}^*\infty} A_n(t).$$

Hence $\sigma_t(A)$ is local and it depends on $g(\boldsymbol{x})$ only in the region #- $\overline{\boldsymbol{O}}_t$, where we choose $g(\boldsymbol{x}) = \lambda$. Thus we conclude that the spatial cut-off has been removed and the resulting theory is local.

§ 18. Semiboundedness of the total Hamiltonian

§ 18.1. Reduction to a Problem with Discrete Momentum We use the non-Archimedean Fock space representation for our field $\varphi_\varkappa^\#(\boldsymbol{x}), \boldsymbol{x} \in {}^*\mathbb{R}_c^{\#3}$. The Fock non-Archimedean Hubert space $\mathcal{F}^\#$ is a direct sum

$$\mathcal{F}^\# = Ext\text{-}\bigoplus_{n=0}^{^*\infty}\mathcal{F}_n^\#,$$

where $\mathcal{F}_n^\#$ is the space of n non-interacting particles, i.e. $\mathcal{F}_n^\#$ is the space of symmetric square #-integrable functions, i.e.$L_2^\#({}^*\mathbb{R}_c^{\#3})$ functions of n variables. Let $\boldsymbol{k} = (k_1, k_2, k_2) \in {}^*\mathbb{R}_c^{\#3}$

$$\mu(\boldsymbol{k}) = (\boldsymbol{k}^2 + \mu_0^2)^{1/2} = (k_1^2 + k_2^2 + k_3^2 + m_0^2)^{1/2},$$

$$\varphi_\varkappa^{\#-}(\boldsymbol{x}) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} Ext\text{-}\exp(i\langle\boldsymbol{k},\boldsymbol{x}\rangle)\, a(\boldsymbol{k})\theta(|\boldsymbol{k}|,\varkappa)[\mu(\boldsymbol{k})]^{-1/2}d^{\#3}k, \qquad (18.1.1)$$

$$\varphi_\varkappa^{\#+}(\boldsymbol{x}) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} Ext\text{-}\exp(i\langle\boldsymbol{k},\boldsymbol{x}\rangle)\, a^*(-\boldsymbol{k})\theta(|\boldsymbol{k}|,\varkappa)[\mu(\boldsymbol{k})]^{-1/2}d^{\#3}k, \qquad (18.1.2)$$

$$\theta(|\boldsymbol{k}|,\varkappa) = 1 \text{ if } |\boldsymbol{k}| \le \varkappa \text{ and } (|\boldsymbol{k}|,\varkappa) = 0 \text{ if } |\boldsymbol{k}| > \varkappa,$$

and $\varphi_\varkappa^\#(\boldsymbol{x}) = \varphi_\varkappa^{\#-}(\boldsymbol{x}) + \varphi_\varkappa^{\#+}(\boldsymbol{x})$, where $a(\boldsymbol{k})$ and $a^*(\boldsymbol{k})$ are the annihilation and creation operators,

$$[a(\boldsymbol{k}), \boldsymbol{a}^*(\boldsymbol{k}')] = \delta^\#(\boldsymbol{k} - \boldsymbol{k}'). \qquad (18.1.3)$$

By definition,

$$: \varphi_{\varkappa}^{\#p}(\boldsymbol{x}) := \sum_j \binom{p}{j} \ \varphi_{\varkappa}^{\#+}(\boldsymbol{x})^j \ \varphi_{\varkappa}^{\#-}(\boldsymbol{x})^{p-j}. \quad (18.1.4)$$

Remark 18.1.1 Remind that Wick product differs from the ordinary product in that all the annihilators are placed to the right and the creators are placed to the left. $: \varphi_{\varkappa}^{\#p}(\boldsymbol{x}):$ is not an operator, but it is a densely defined bilinear form. We take Fourier transforms to compute

$$Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} : \varphi_{\varkappa}^{\#p}(\boldsymbol{x}): d^{\#3}x = \sum_j \binom{p}{j} Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3p}} a^*(-\boldsymbol{k}_1) \cdots a^*(-\boldsymbol{k}_j) a(\boldsymbol{k}_j) \cdots a(\boldsymbol{k}_p) \times \quad (18.1.5)$$

$$\times \ Ext\text{-}\hat{h}(\boldsymbol{k}_1 + \cdots + \boldsymbol{k}_p) \prod_{i=1}^p \theta(\|\boldsymbol{k}_i\|, \varkappa)\left[\mu(\boldsymbol{k}_i)\right]^{-1/2} d^{\#3}k_i,$$

where $Ext\text{-}\hat{h}$ is the Fourier transform of h. We assume h is in $L_2^{\#}$ and so $Ext\text{-}\hat{h}$ is in $L_2^{\#}$ also. Since $\mu\,(\boldsymbol{k}) \sim |\boldsymbol{k}|$ for large $|\boldsymbol{k}|$, one can show that

$$Ext\text{-}\hat{h}(\boldsymbol{k}_1 + \cdots + \boldsymbol{k}_p) \prod_{i=1}^p \theta(\|\boldsymbol{k}_i\|, \varkappa)\left[\mu(\boldsymbol{k}_i)\right]^{-1/2} \in L_2^{\#}. \quad (18.1.6)$$

It is well known that (18.1.6) implies that each integral on the right side of (18.1.5) is an operator defined on the domain $D(N^{p/2})$ of $N^{p/2}$. This domain is the set of $\psi = \psi_0, \psi_1, \dots, \psi_j \in \mathcal{F}_j^{\#}$ with

$$Ext\text{-}\sum_n n^{p/2} \, \|Ext\text{-}\prod_{i=1}^n \theta(\|\boldsymbol{k}_i\|, \varkappa)\, \psi_n\|_{\#2}^2 < {}^*\infty. \quad (18.1.7)$$

Thus (18.1.5) is an operator defined on $D(N^{p/2})$. Similarly $H_{0.\varkappa} + Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} P(:\varphi_{\varkappa}^{\#}(\boldsymbol{x})): d^{\#3}x$ is an operator defined on the #-dense domain, $D(H_{0.\varkappa}) \cap D(N^{d/2})$, where d is the degree of the polynomial P. We approximate now (18.1.5) by a hyperfinite sum. Choose numbers $\delta \approx 0$ and $\varkappa \in {}^*\mathbb{R}_c^{\#} \backslash {}^*\mathbb{R}_{c,\mathrm{fin}}^{\#}$. We define now an hyperfinite approximation in configuration space. Under this approximation, the momentum space variable $\boldsymbol{k} = (k_1, k_2, k_3) \in {}^*\mathbb{R}_c^{\#\boldsymbol{3}}$ is replaced by a discrete variable $\boldsymbol{k} \in \Gamma_\delta^3$

$$\Gamma_\delta^3 = \{\boldsymbol{k} = (k_1, k_2, k_3) | k_i = \delta n_i, n_i \in {}^*\mathbb{Z}; i = 1,2,3\} \quad (18.1.8)$$

Thus we define $\mathcal{F}_V^{\#}$, the Fock space for hyperfinite volume $V^3 = \delta^{-3}$ as

$$\mathcal{F}_V^{\#} = \mathfrak{C}\left(l_2^{\#}(\Gamma_V^3)\right) = {}^*\mathbb{C}^{\#} \oplus l_2^{\#}(\Gamma_V^3) \oplus \{l_2^{\#}(\Gamma_V^3) \otimes_s l_2^{\#}(\Gamma_V^3)\} \cdots \quad (18.1.9)$$

We choose now one to one correspondence ${}^*\mathbb{Z} \leftrightarrow {}^*\mathbb{Z}\delta \times {}^*\mathbb{Z}\delta \times {}^*\mathbb{Z}\delta = \Gamma_\delta^3$ given by vector-function $\wp(m)$

$$\wp(m) = \{k_1(m), k_2(m), k_3(m)\} = \boldsymbol{k}(m) \quad (18.1.10)$$

and such that

$$\wp(-m) = -\wp(m). \quad (18.1.11)$$

And we define now

$$\Gamma_{\varkappa,\delta}^3 = \{\boldsymbol{k} \in \Gamma_\delta^3 | |\boldsymbol{k}| \le \varkappa\}. \quad (18.1.12)$$

We set now

$$a_\delta(\boldsymbol{k}(m)) = (\delta)^{-3/2}\left[Ext\text{-}\int_0^\delta d^{\#}l_1 \, Ext\text{-}\int_0^\delta d^{\#}l_2 \, Ext\text{-}\int_0^\delta d^{\#}l_3 \, a(\boldsymbol{k}(m) + \boldsymbol{l})\right], \quad (18.1.13)$$

$$a^{\,*}_{\delta}\big(\boldsymbol{k}(m)\big) = (\delta)^{-3/2}\left[Ext\text{-}\int_0^\delta d^{\#}l_1\, Ext\text{-}\int_0^\delta d^{\#}l_2\, Ext\text{-}\int_0^\delta d^{\#}l_3\, a^*(\boldsymbol{k}(m)+\boldsymbol{l})\right]. \tag{18.1.14}$$

Then one obtains

$$\left[a^{\,*}_{\delta}\big(\boldsymbol{k}(m_1)\big), a_\delta\big(\boldsymbol{k}(m_2)\big)\right] = \boldsymbol{\delta}_{m_1 m_2} = \begin{cases} 1 \text{ if } m_1 = m_2 \\ 0 \text{ if } m_1 \neq m_2 \end{cases} \tag{18.1.15}$$

Let

$$H_{0,\varkappa,\delta} = Ext\text{-}\textstyle\sum_{\boldsymbol{k}\in\Gamma^3_{\varkappa,\delta}} \mu(\boldsymbol{k}\,)\, a^{\,*}_{\delta}(\boldsymbol{k}) a_\delta(\boldsymbol{k}). \tag{18.1.16}$$

One can check that each ψ in $D(H_{0.\varkappa})$ is in $D\big(H_{0,\varkappa,\delta}\big)$ also and that

$$\#\text{-}\lim_{\delta\to_{\#}} H_{0,\varkappa,\delta}\psi = H_{0.\varkappa}\,\psi. \tag{18.1.17}$$

Next we approximate (18.1.5) by

$$:\varphi^{\#p}_{\varkappa,\delta}(\boldsymbol{x}) := \delta^{3p/2} \textstyle\sum_j \binom{p}{j} Ext\text{-}\sum_{\boldsymbol{k}\in\Gamma^3_{\varkappa,\delta}} a^{\,*}_{\delta}(-\boldsymbol{k}_1)\cdots a^{\,*}_{\delta}\big(-\boldsymbol{k}_j\big) a_\delta\big(\boldsymbol{k}_j\big)\cdots a_\delta\big(\boldsymbol{k}_p\big)\times \tag{18.1.18}$$

$$\times\, Ext\text{-}\hat{h}\big([\boldsymbol{k}_1]+\cdots+\big[\boldsymbol{k}_p\big]\big)\textstyle\prod_i [\mu([\boldsymbol{k}_i]\,)]^{-1/2},$$

where

$$\hat{h}_\delta(\boldsymbol{k}) = Ext\text{-}\int_{-\pi_{\#}/\delta}^{\pi_{\#}/\delta}\int_{-\pi_{\#}/\delta}^{\pi_{\#}/\delta}\int_{-\pi_{\#}/\delta}^{\pi_{\#}/\delta}\big(Ext\text{-}\exp(i\langle\boldsymbol{k},\boldsymbol{x}\rangle)\big) h(\boldsymbol{x})\, d^{\#3}x$$

and $[\boldsymbol{k}] = ([k_1],[k_2],[k_3])$, where

$$[k_1] = \sup\{l_1 | (l_1,l_2,l_3)\in\Gamma^3_\delta, l_1 \le k_1\},\ [k_2] = \sup\{l_2 | (l_1,l_2,l_3)\in\Gamma^3_\delta, l_2\le k_2\},$$

$$[k_3] = \sup\{l_3 | (l_1,l_2,l_3)\in\Gamma^3_\delta, l_3 \le k_3\}$$

is the integral part of $\boldsymbol{k}$ relative to the lattice Γ^3_δ. Since $h\in L^{\#}_1$, $\hat{h}_\delta$ is #-continuous and

$$Ext\text{-}\hat{h}\big([\boldsymbol{k}_1]+\cdots+\big[\boldsymbol{k}_p\big]\big)\textstyle\prod_i[\mu([\boldsymbol{k}_i]\,)]^{-1/2} \to_{\#} Ext\text{-}\hat{h}\big(\boldsymbol{k}_1+\cdots+\boldsymbol{k}_p\big)\prod_{i=1}^p \theta(\|\boldsymbol{k}_i\|,\varkappa)\big[\mu\big(\boldsymbol{k}_i\,\big)\big]^{-1/2}$$

uniformly. Let $D^{\#}_0$ be the set of states $\psi = \{\psi_0,\psi_1,\dots\}$ with $\psi_n(\boldsymbol{k}_1,\dots,\boldsymbol{k}_n) = 0$ for $n < {}^*\infty$ or $Ext\text{-}\sum_i |k_i| < {}^*\infty$ large. If $\phi,\psi\in D^{\#}_0$ then

$$\#\text{-}\lim_{\delta\to_{\#}0}\langle\phi, :\varphi^{\#p}_{\varkappa,\delta}(\boldsymbol{x}):\psi\rangle_{\#} = \langle\phi, Ext\text{-}\textstyle\int_{{}^*\mathbb{R}^{\#3}_c} :\varphi^{\#p}_{\varkappa}(\boldsymbol{x}): d^{\#3}x\psi\rangle. \tag{18.1.18}$$

Thus the bilinear form of

$$H_{\varkappa,\delta} = H_{0,\varkappa,\delta} + \textstyle\sum_p :\varphi^{\#p}_{\varkappa,\delta}(h): \tag{18.1.19}$$

#-converges to $H_\varkappa$ on $D^{\#}_0\times D^{\#}_0$ where $b_0,\dots,b_r$ are the coefficients of $y_0, y_1, \dots, y_r$ in the polynomial $P(y)$. Hence if the $H_{\varkappa,\delta}$ are semibounded with a lower bound independent of δ then $H_\varkappa$ is semibounded also. Let $\mathcal{F}^{\#}_\delta$ be the subspace of $\mathcal{F}^{\#}$ consisting of functions which are piece wise constant between lattice points. In other words,

$$\psi = \{\psi_0,\psi_1,\dots,\psi_n,\dots\}\in\mathcal{F}^{\#}_\delta \text{ if}$$

$$\psi_n(\boldsymbol{k}_1,\dots,\boldsymbol{k}_n)=\psi_n([\boldsymbol{k}_1],\dots,[\boldsymbol{k}_n]).$$

Let $\mathcal{F}^{\#}_{\varkappa,\delta}$ be the subspace of $\mathcal{F}^{\#}_{\delta}$ defined by the restriction

$$\psi_n(\boldsymbol{k}_1,\dots,\boldsymbol{k}_n)=0 \text{ if } [\boldsymbol{k}_i]\notin\Gamma^3_{\varkappa,\delta}$$

for some $i, 1\leq i\leq n$.
The operators $a^*_{\delta}(\boldsymbol{k})$ and $a_{\delta}(\boldsymbol{k}), \boldsymbol{k}\in\Gamma^3_{\varkappa,\delta}$, leave $\mathcal{F}^{\#}_{\varkappa,\delta}$ invariant and act irreducibly on $\mathcal{F}^{\#}_{\varkappa,\delta}$. We set now $\delta=2^{-\nu}$, $\varkappa=2^{\nu}$ and observe that $\mathcal{F}^{\#}_{2^{\nu},2^{-\nu}}$ increases monotonically with ν and that

$$D^{\#\prime}_0=D^{\#}_0\cap\bigcup_{\nu}\mathcal{F}^{\#}_{2^{\nu},2^{-\nu}}$$

is #-dense in $\mathcal{F}^{\#}$ and $H_{\varkappa}\subset\#\text{-}\overline{(H_{\varkappa}\restriction D^{\#\prime}_0)}$**.** Thus it is sufficient to prove the semiboundedness of

$$H_{\varkappa,\delta}\restriction\left(D\left(H_{0,\varkappa}\right)\cap D\left(N^{d/2}_{\varkappa}\right)\cap\mathcal{F}^{\#}_{\varkappa,\delta}\right)$$

with a lower bound independent of δ.

§ 18.2. Diagonalizing the potential. In this subsection we give a new representation of $\mathcal{F}^{\#}_{\varkappa,\delta}$ in which the interaction term $::\varphi^{\#p}_{\varkappa}(h)$: is a multiplication operator. Let

$$q\left(\boldsymbol{k}(|m|)\right)=\left(2^{-2}\mu(\boldsymbol{k}(m))\right)^{1/2}\left[a_{\delta}\left(\boldsymbol{k}(m)\right)+a^*_{\delta}\left(\boldsymbol{k}(m)\right)+a_{\delta}\left(-\boldsymbol{k}(m)\right)+a^*_{\delta}\left(-\boldsymbol{k}(m)\right)\right],$$

$$q\left(\boldsymbol{k}(-|m|)\right)=i\left(2^{-2}\mu(\boldsymbol{k}(m))\right)^{1/2}\left[-a_{\delta}\left(\boldsymbol{k}(|m|)\right)+a^*_{\delta}\left(\boldsymbol{k}(|m|)\right)+a_{\delta}\left(-\boldsymbol{k}(|m|)\right)-a^*_{\delta}\left(-\boldsymbol{k}(|m|)\right)\right],$$

$$p\left(\boldsymbol{k}(|m|)\right)=i\left(2^{-2}\mu(\boldsymbol{k}(m))\right)^{1/2}\left[a_{\delta}\left(\boldsymbol{k}(m)\right)-a^*_{\delta}\left(\boldsymbol{k}(m)\right)+a_{\delta}\left(-\boldsymbol{k}(m)\right)-a^*_{\delta}\left(-\boldsymbol{k}(m)\right)\right],$$

$$p\left(\boldsymbol{k}(-|m|)\right)=\left(2^{-2}\mu(\boldsymbol{k}(m))\right)^{\mathbf{1/2}}\left[a_{\delta}\left(\boldsymbol{k}(|m|)\right)+a^*_{\delta}\left(\boldsymbol{k}(|m|)\right)-a_{\delta}\left(-\boldsymbol{k}(|m|)\right)-a^*_{\delta}\left(-\boldsymbol{k}(|m|)\right)\right],$$

$$p_m=p\left(\boldsymbol{k}(m)\right),\ q_m=q\left(\boldsymbol{k}(m)\right)$$

for $0\neq\boldsymbol{k}\in\Gamma^3_{\delta}$ and let

$$q_0=(\mu_0/2)^{1/2}\left[a_{\delta}\left(\mathbf{0}\right)+\boldsymbol{a}^*_{\boldsymbol{\delta}}\left(\mathbf{0}\right)\right],$$

$$p_0=i(\mu_0/2)^{1/2}\left[a_{\delta}\left(\mathbf{0}\right)-\boldsymbol{a}^*_{\boldsymbol{\delta}}\left(\mathbf{0}\right)\right].$$

Using the equations mentioned above one can compute that

$$H_{0,\varkappa,\delta}=Ext\text{-}\sum_{m\in{}^*\mathbb{Z},|\boldsymbol{k}(m)\leq\varkappa|}2^{-1}\left[p^2_m+\mu^2(\boldsymbol{k}(m))q^2_m-\mu(\boldsymbol{k}(m))\right]. \tag{18.2.1}$$

We replace now p_m and q_m by unitarily equivalent operators. Let

$$\mathcal{H}^{\#}_{\varkappa,\delta}=Ext\text{-}\bigotimes_{\boldsymbol{k}\in\Gamma^3_{\varkappa,\delta}}\mathcal{H}^{\#}_{\boldsymbol{k}},$$

where $\mathcal{H}^{\#}_{\boldsymbol{k}}$ is $L^{\#}_2({}^*\mathbb{R}^{\#}_c)$ with respect to the Gaussian #-measure

$$\phi^2_{\boldsymbol{k}}(q)d^{\#}q=(\mu(\boldsymbol{k})/\pi_{\#})^{1/2}\left(Ext\text{-}\exp(-\mu(\boldsymbol{k})q^2)\right)d^{\#}q. \tag{18.2.2}$$

There is a unitary equivalence between $\mathcal{H}^{\#}_{\varkappa,\delta}$ and $\mathcal{F}^{\#}_{\varkappa,\delta}$ which sends q_m into multiplication by q in the factor $\mathcal{H}^{\#}_{\boldsymbol{k}(m)}$ and p_m into the operator

$$\phi_{\boldsymbol{k}}^{-1}(q) i \left(\frac{d^{\#}}{d^{\#}q}\right) \phi_{\boldsymbol{k}}(q)$$

again acting in the factor $\mathcal{H}^{\#}_{\boldsymbol{k}}$. The proof of this statement is essentially generalized von Neumann's uniqueness theorem for irreducible representations of the commutation relations. We identify $\mathcal{H}^{\#}_{\varkappa,\delta}$ and $\mathcal{F}^{\#}_{\varkappa,\delta}$ and we identify q_m, etc. with its image, multiplication by q, etc. Let

$$H_{\mu(\boldsymbol{k})} = 2^{-1}\phi_{\boldsymbol{k}}^{-1}(q)\left[-\left(\frac{d^{\#}}{d^{\#}q}\right)^2 + \mu(\boldsymbol{k})q^2\right]\phi_{\boldsymbol{k}}(q) = \tag{18.2.3}$$

$$= -2^{-1}\left(\frac{d^{\#}}{d^{\#}q}\right)^2 + \mu(\boldsymbol{k})q\left(\frac{d^{\#}}{d^{\#}q}\right)$$

acting on $\mathcal{H}^{\#}_{\boldsymbol{k}}$. Now $-H_{\mu(\boldsymbol{k})}$ is the infinitesimal generator of a known Markoff process and furthermore the operator $Ext\text{-}\exp\left(-H_{\mu(\boldsymbol{k})}\right)$ is an integral operator and the kernel can be computed explicitly. In particular

$$\left(Ext\text{-}\exp\left(-H_{\mu(\boldsymbol{k})}\right)\psi\right)(q) = Ext\text{-}\int_{{}^{*}\mathbb{R}^{\#}_{c}} p^{t}(q,q')\psi(q')\phi_{\boldsymbol{k}}^{2}(q')d^{\#}q' \tag{18.2.4}$$

for $\psi \in \mathcal{H}^{\#}_{\boldsymbol{k}}$, where

$$p^{t}(q,q') = [1 - Ext\text{-}\exp(-\mu t)]\left\{Ext\text{-}\exp\left[-\frac{\mu\left(q' - (Ext\text{-}\exp(-\mu t))q\right)^2}{1 - Ext\text{-}\exp(-2\mu t)}\right] + \mu q'^2\right\}. \tag{18.2.5}$$

Let q now denote a variable in a Euclidean space $E_{\varkappa}$ and let q have coordinates $q_m = q\left(\boldsymbol{k}(m)\right)$. Then

$$\phi_{\varkappa}^{2}(q)d^{\#}q = Ext\text{-}\prod_{\boldsymbol{k}\in\Gamma^{3}_{\varkappa,\delta}} \phi_{\varkappa}^{2}\left(q(\boldsymbol{k})\right)d^{\#}q(\boldsymbol{k}) \tag{18.2.6}$$

is the product of the #-measures (18.2.2) and

§ 18.3. We will give an alternate derivation of the results of Nelson avoiding the use of functional integration, central in subsection 18.2. We consider a Hamiltonian of the form

$$H_{\varkappa} = H_{0.\varkappa} + V_{\varkappa}, \tag{18.3.1}$$

where $H_{0.\varkappa}$ is the free Hamiltonian of a particle of mass μ_0 expressed in terms of the neutral scalar field $\varphi^{\#}_{\varkappa}(\boldsymbol{x})$ and its momentum conjugate $\pi^{\#}_{\varkappa}(\boldsymbol{x})$:

$$H_{0.\varkappa} = Ext\text{-}\int_0^1 d^{\#}x_1\left(Ext\text{-}\int_0^1 d^{\#}x_2\left(Ext\text{-}\int_0^1 d^{\#}x_3 : [\nabla\varphi^{\#2}_{\varkappa}(\boldsymbol{x}) + \mu_0^2\varphi^{\#2}_{\varkappa}(\boldsymbol{x}) + \pi^{\#2}_{\varkappa}(\boldsymbol{x})]:\right)\right) \tag{18.3.2}$$

As is evident from (18.3.2) we are working in a periodic box $B = [0,1]^3$. $V_{\varkappa}$ is a polynomial function of the $\varphi^{\#}_{\varkappa}(\boldsymbol{x})$. We denote by ${}^{N}H_{0.\varkappa}$ and ${}^{N}V_{\varkappa}$, $N \in {}^{*}\mathbb{N}\backslash\mathbb{N}$ the parts of $H_{0.\varkappa}$ and $V_{\varkappa}$ depending only on the creation and annihilation operators of the N lowest-energy modes of the free Hamiltonian and such that $|\boldsymbol{k}| \leq \varkappa$. We always imagine we are working with ${}^{N}H_{0.\varkappa}$ and ${}^{N}V_{\varkappa}$, but derive inequalities independent of N.

Theorem 18.3.1 Assume for each finite $\alpha > 0$ that there is an M_α such that

$$\langle 0|Ext\text{-}\exp\big(-\alpha({}^N V_\varkappa)\big)|0\rangle \leq M_\alpha,$$

where $|0\rangle$ denotes the vacuum of the free field. Then there is a B such that

$${}^N H_{0.\varkappa} + {}^N V_\varkappa \geq B, \text{ for all } N.$$

Actually as will be seen it is not necessary to satisfy the condition above for all α, but only for some sufficiently large α that one can calculate. We refer to section 18.3 for the result that the conditions of the theorem are satisfied for a large class of self-interactions.
We apply the notation

$$\varphi^{\#}_{\varkappa,\delta}(\boldsymbol{x}) = Ext\text{-}\textstyle\sum_{\boldsymbol{k}\in\Gamma^3_{\varkappa,\delta}} Ext\text{-}\exp(i\langle\boldsymbol{k},\boldsymbol{x}\rangle)\Big(\boldsymbol{a}_{\boldsymbol{\delta}}(\boldsymbol{k}) + a^*_\delta(-\boldsymbol{k})\Big) \tag{18.3.3}$$

and define for $\boldsymbol{k} \in \Gamma^3_{\varkappa,\delta}$

$$q_0 = (\mu_0/2)^{1/2}\Big[a_\delta(\boldsymbol{0}) + \boldsymbol{a}^*_{\boldsymbol{\delta}}(\boldsymbol{0})\Big],\ p_0 = i(\mu_0/2)^{1/2}\Big[a_\delta(\boldsymbol{0}) - \boldsymbol{a}^*_{\boldsymbol{\delta}}(\boldsymbol{0})\Big], \tag{18.3.4}$$

$$q\big(\boldsymbol{k}(|m|)\big) = \big(2^{-2}\mu(\boldsymbol{k}(m))\big)^{1/2}\Big[a_\delta\big(\boldsymbol{k}(m)\big) + a^*_\delta\big(\boldsymbol{k}(m)\big) + a_\delta\big(-\boldsymbol{k}(m)\big) + a^*_\delta\big(-\boldsymbol{k}(m)\big)\Big],$$

$$q\big(\boldsymbol{k}(-|m|)\big) = i\big(2^{-2}\mu(\boldsymbol{k}(m))\big)^{1/2}\Big[-a_\delta\big(\boldsymbol{k}(|m|)\big) + a^*_\delta\big(\boldsymbol{k}(|m|)\big) + a_\delta\big(-\boldsymbol{k}(|m|)\big) - a^*_\delta\big(-\boldsymbol{k}(|m|)\big)\Big],$$

$$p\big(\boldsymbol{k}(|m|)\big) = i\big(2^{-2}\mu(\boldsymbol{k}(m))\big)^{1/2}\Big[a_\delta\big(\boldsymbol{k}(m)\big) - a^*_\delta\big(\boldsymbol{k}(m)\big) + a_\delta\big(-\boldsymbol{k}(m)\big) - a^*_\delta\big(-\boldsymbol{k}(m)\big)\Big],$$

$$p\big(\boldsymbol{k}(-|m|)\big) = \big(2^{-2}\mu(\boldsymbol{k}(m))\big)^{\boldsymbol{1/2}}\Big[a_\delta\big(\boldsymbol{k}(|m|)\big) + a^*_\delta\big(\boldsymbol{k}(|m|)\big) - a_\delta\big(-\boldsymbol{k}(|m|)\big) - a^*_\delta\big(-\boldsymbol{k}(|m|)\big)\Big],$$

$$p_m = p\big(\boldsymbol{k}(m)\big),\ q_m = q\big(\boldsymbol{k}(m)\big).$$

In terms of these variables,

$$H_{0,\varkappa,\delta} = Ext\text{-}\textstyle\sum_{m\in{}^*\mathbb{Z},|k(m)\leq\varkappa|} 2^{-1}\left[p_m^2 + \mu^2(k(m))q_m^2 - \mu(k(m))\right] = Ext\text{-}\sum_{m\in{}^*\mathbb{Z},|k(m)\leq\varkappa|} H_m. \tag{18.3.5}$$

We represent these operators on the $L_2^{\#}$ space of ${}^*\mathbb{R}_c^{\#N}$ with #-measure μ the product of the #-measures μ_m

$$d^{\#}\mu_m = (\omega_m/\pi_{\#})^{1/2}\big(Ext\text{-}\exp(-\omega_m q_m^2)\big)d^{\#}q_m \tag{18.3.6}$$

with q_m a multiplicative operator and

$$p_m = i(\partial^{\#}/\partial^{\#}q_m) - \omega_m q_m. \tag{18.3.7}$$

Where

$$\omega_m = (\boldsymbol{k}^2(m) + \mu_0^2)^{1/2} = (k_1^2(m) + k_2^2(m) + k_3^2(m) + \mu_0^2)^{1/2}.$$

A complete set of eigenfunctions for H_m is given by

$$\phi_{m,n}(q_m) = (2^n n!^{\#})^{-1/2} A_n\big(q_m(\omega_m)^{1/2}\big), n \in {}^*\mathbb{N}, \tag{18.3.8}$$

$$n!^{\#} = Ext\text{-}\prod_{0<p\leq n} p,\ 2^n = Ext\text{-}\prod_{0<p\leq n} 2,$$

$$A_n(z) = (-1)^n\big(Ext\text{-}\exp(z^2)\big)\frac{d^{\#n}}{d^{\#}z^n}\big(Ext\text{-}\exp(-z^2)\big).$$

The chief inequality we will exploit is the following numerical inequality for $x, y \in {}^*\mathbb{R}_c^{\#},\ y \geq 0$:

$$xy \leq Ext\text{-}\exp(x) + Ext\text{-}\ln(y). \tag{18.3.9}$$

The expectation value of the interaction $V_\varkappa$ in a state with ${}^*\mathbb{C}_c^{\#}$-function F is given by

$$\langle F|V_\varkappa|F\rangle = Ext\text{-}\int(|F|^2 V_\varkappa)d^{\#}\mu. \tag{18.3.11}$$

We apply (18.3.10) with $x = rV_\varkappa$ and $y = r^{-1}F^2$ to derive the inequality

$$-\langle F|V_\varkappa|F\rangle \leq Ext\text{-}\int\big(Ext\text{-}\exp(-rV_\varkappa)\big)d^{\#}\mu + \frac{1}{r}\Big[Ext\text{-}\int|F|^2\big(Ext\text{-}\ln(|F|^2)\big)d^{\#}\mu\Big] - \frac{1}{r}\big(Ext\text{-}\ln(r)\big). \tag{18.3.12}$$

Here r is a numerical factor to be fixed later. Note that

$$Ext\text{-}\int\big(Ext\text{-}\exp(-rV_\varkappa)\big)d^{\#}\mu = \langle 0|Ext\text{-}\exp(-r\,V_\varkappa)|0\rangle. \tag{18.3.13}$$

We intend now to bound the second term on the right side of (18.3.12) by the expectation value of $H_{0.\varkappa}$ in the state F. We consider the following equation:

$$\Big[Ext\text{-}\int|F|^2\big(Ext\text{-}\ln(|F|^2)\big)d^{\#}\mu\Big] = \tag{18.3.14}$$

$$= \frac{2}{\lambda}\Big(Ext\text{-}\int F^*\,H_{0.\varkappa}F\,d^{\#}\mu\Big) + \frac{1}{\lambda}\frac{d^{\#}}{d^{\#}t}\left(Ext\text{-}\int\Big[\big(Ext\text{-}\exp(-tH_{0.\varkappa})\big)^*\big(Ext\text{-}\exp(-tH_{0.\varkappa})\big)\Big]^{1+\lambda t}d^{\#}\mu\right)\Big|_{t=0},$$

which easily follows for functions F nice enough so that all the integrals exist and the differentiation may be moved inside the integral, a dense subspace in $L_2^{\#}$. We do not discuss domain questions. We rewrite (18.3.12) using (18.3.14):

$$-\langle F|V_\varkappa|F\rangle \leq Ext\text{-}\int\big(Ext\text{-}\exp(-rV_\varkappa)\big)d^{\#}\mu + \frac{2}{\lambda r}\langle F|H_{0.\varkappa}|F\rangle - \frac{1}{r}\big(Ext\text{-}\ln(r)\big) + \tag{18.3.15}$$

$$+\frac{1}{\lambda r}\frac{d^{\#}}{d^{\#}t}\left(Ext\text{-}\int\Big[\big(Ext\text{-}\exp(-tH_{0.\varkappa})\big)^*\big(Ext\text{-}\exp(-tH_{0.\varkappa})\big)\Big]^{1+\lambda t}d^{\#}\mu\right)\Big|_{t=0}.$$

The theorem we are after is established provided $\lambda r \geq 2$ and we can bound the last term in (18.3.15). The remainder of the paper is devoted to a study of

$$Ext\text{-}\int\Big[\big(Ext\text{-}\exp(-tH_{0.\varkappa})\big)^*\big(Ext\text{-}\exp(-tH_{0.\varkappa})\big)\Big]^{1+\lambda t}d^{\#}\mu = Ext\text{-}\int|Ext\text{-}\exp(-tH_{0.\varkappa})|^{2+2\lambda t}d^{\#}\mu. \tag{18.3.16}$$

We consider, corresponding to any g in $L_2^{\#}(\mu)$, its expression as a sum of products of the functions in (18.3.8):

$$g(q) = Ext\text{-}\sum_{i_1,..,i_N} C_{i_1,..,i_N}\Big\{Ext\text{-}\prod_s(2^{i_s}i_s!^{\#})^{-1}\Big(Ext\text{-}\exp(i_s)A_{i_s}\big(q_s(\omega_s)^{1/2}\big)\Big)\Big\}. \tag{18.3.17}$$

The q_s are merely the q_k in some order. The coefficient's $C_{i_1,..,i_N}$ are now considered as functions on the discrete space whose points are the indices of the C's. To the point $(i_{1,..,}i_N)$ is associated the point mass $Ext\text{-}\prod_s\big(Ext\text{-}\exp(2i_s)\big)$. With this measure, the transformation T that carries a set of C's into the corresponding function g as in (18.3.17) is norm preserving as a map from $l_2^{\#}$ to $L_2^{\#}$. We will later show that T is norm decreasing as a map from $l_1^\#$ to $L_4^\#$. Assuming this for a moment, we complete the proof of the theorem. We apply the generalized Riesz-Thorin convexity theorem to the transformation T obtaining

$$Ext\text{-}\int|Ext\text{-}\exp(-tH_{0.\varkappa})|^{2+2\lambda t}d^\#\mu \leq$$

$$\leq \left[Ext\text{-}\left(\sum_{i_1,\ldots,i_N} Ext\text{-}\prod_s\left(Ext\text{-}\exp(2i_s)\right)\times\left|\left(Ext\text{-}\exp\left(-\omega_{i_1,\ldots,i_N}t\right)\right)\times C_{i_1,\ldots,i_N}\right|^{\frac{2(1+\lambda t)}{1+3\lambda t}}\right)\right]^{\frac{1+3\lambda t}{2(1+\lambda t)}} \quad (18.3.18)$$

with

$$\omega_{i_1,\ldots,i_N} = Ext\text{-}\sum_s i_s\omega_s. \quad (18.3.19)$$

In the right-hand side of (18.3.18) we apply the generalized Holder inequality to obtain an expression involving the weighted sum of the squares of the absolute values of the C's which is equal to one:

$$Ext\text{-}\int|Ext\text{-}\exp(-tH_{0.\varkappa})|^{2+2\lambda t}d^\#\mu \leq$$

$$\leq \left[Ext\text{-}\left(\sum_{i_1,\ldots,i_N} Ext\text{-}\prod_s\left(Ext\text{-}\exp(2i_s)\right)\left(Ext\text{-}\exp\left(-\omega_{i_1,\ldots,i_N}\times\frac{2(1+\lambda t)}{2\lambda}\right)\right)\right)\right]^{2\lambda t}. \quad (18.3.20)$$

It follows that

$$\frac{d^\#}{d^\#t}\left(Ext\text{-}\int|Ext\text{-}\exp(-tH_{0.\varkappa})|^{2+2\lambda t}d^\#\mu\ \right)\Big|_{t=0} \leq$$

$$2\lambda\times\left\{Ext\text{-}\ln\left[Ext\text{-}\left(\sum_{i_1,\ldots,i_N} Ext\text{-}\prod_s\left(Ext\text{-}\exp(2i_s)\right)\left(Ext\text{-}\exp\left(-\omega_{i_1,\ldots,i_N}\times\frac{2(1+\lambda t)}{2\lambda}\right)\right)\right)\right]\right\}. \quad (18.3.21)$$

If $\mu_0/\lambda > 2$, this gives an inequality with finite right hand side in the #-limit $N\to{}^*\infty$. It is clear that the theorem is now reduced to establishing that T is #-norm decreasing from $l_1^\#$ to $L_4^\#$.

Lemma18.3.1. Let S be the space of sequences $\{C_\gamma\}$, $\gamma = 0,1,\ldots,N$ with #-measure at γ, $Ext\text{-}\exp(2\gamma)$; and Y the space of functions on ${}^*\mathbb{R}_c^\#$ with #-measure

$$(1/\pi_\#)^{1/2}\left(Ext\text{-}\exp(-x^2)\right)d^\#x, \quad (18.3.22)$$

and T the operator from S to Y given by

$$T\{C_\gamma\} = Ext\text{-}\sum_\gamma C_\gamma\frac{(Ext\text{-}\exp(\gamma))A_\gamma(x)}{[2^\gamma\gamma!^\#]} \quad (18.3.23)$$

with $A_\gamma(x)$ the γ-th Hermite polynomial $A_\gamma(x) = (-1)^n\left(Ext\text{-}\exp(x^2)\right)\frac{d^{\#n}}{d^\#x^n}\left(Ext\text{-}\exp(-x^2)\right)$; then, T is #-norm decreasing from $l_1^\#$ to $L_4^\#$.

It is easy to see that this lemma would follow from establishing the inequality

$$\left|\left(\frac{1}{\pi_\#}\right)^{\frac{1}{2}}\left(Ext\text{-}\exp(-a-b-c-d)\right)\times\right.$$

$$\left.\times\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^\#}[2^{a+b+c+d}(a!^\#)(b!^\#)(c!^\#)(d!^\#)]^{-1/2}A_a(x)A_b(x)A_c(x)A_d(x)\left(Ext\text{-}\exp(-x^2)\right)d^\#x\right)\right| \leq 1 \quad (18.3.24)$$

for all integers $a, b, c \in {}^*\mathbb{N}$ and $d \geq 0$; actually, it is sufficient to let $a = b = c = d$. We use the generating function [21]

$$Ext\text{-}\exp(-t^2 + 2tZ) = Ext\text{-}\sum_{N\in{}^*\mathbb{N}} \frac{t^N}{N!^{\#}} A_N(Z) \tag{18.3.25}$$

to obtain

$$\left(\frac{1}{\pi_{\#}}\right)^{\frac{1}{2}} \left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#}} \left(Ext\text{-}\exp(-x^2)\right) A_a(x) A_b(x) A_c(x) A_d(x) d^{\#}x\right) = \tag{18.3.26}$$

$$= \frac{(a!^{\#})(b!^{\#})(c!^{\#})(d!^{\#})}{\frac{1}{2}(a+b+c+d)!^{\#}} \times 2^{\frac{1}{2}(a+b+c+d)} \times (rs + rt + ru + st + su + tu)^{\frac{1}{2}(a+b+c+d)}_{\text{pick-a-power}}$$

where pick-a-power means to find the coefficient of the monomial $r^a s^b t^c u^q$ in the expansion of the expression. Note that $a + b + c + d$ is even or the integral vanishes. We make the crude estimate

$$(rs + rt + ru + st + su + tu)^{\frac{1}{2}(a+b+c+d)}_{\text{pick-a-power}} \leq \tag{18.3.27}$$

$$\leq 2^{-\frac{1}{2}(a+b+c+d)} \times (r + s + t + u)^{\frac{1}{2}(a+b+c+d)}_{\text{pick-a-power}}$$

Now,

$$(r + s + t + u)^{\frac{1}{2}(a+b+c+d)}_{\text{pick-a-power}} = \frac{(a+b+c+d)!^{\#}}{(a!^{\#})(b!^{\#})(c!^{\#})(d!^{\#})}. \tag{18.3.28}$$

Denoting the left-hand side of (18.3.24) by $\mathfrak{I}$ and using (18.3.27) we obtain

$$\mathfrak{I} \leq \left(Ext\text{-}\exp(-a - b - c - d)\right) \times \frac{\left[(a+b+c+d)!^{\#}\right] \times 2^{-\frac{1}{2}(a+b+c+d)}}{\left[(a!^{\#})(b!^{\#})(c!^{\#})(d!^{\#})\right]^{1/2} \left[\frac{1}{2}(a+b+c+d)\right]!^{\#}}. \tag{18.3.29}$$

It is easily verify that

$$\mathfrak{I} \leq 1. \tag{18.3.30}$$

The inequality (18.3.30) finalized the proof of theorem.

CHAPTER II

§ 1. INTRODUCTION

§ 1. 1 We can consider a somewhat different cut-off theory, namely the $\lambda\varphi_4^4$ theory in a periodic box. This gives a cut-off interaction which is translation invariant, and therefore it is useful for the study of the vacuum state. In a finite interval we prove that the total Hamiltonian is self #-adjoint and has a complete set of normalizable eigenstates.

§ 1. 2 Definitions and notation The Fock space $\mathcal{F}^{\#}$ is the Hilbert space completion of the symmetric tensor algebra over $L_2^{\#}({}^*\mathbb{R}_c^{\#\mathbf{3}})$

$$\mathcal{F}^{\#} = \mathfrak{C}\left(L_2^{\#}({}^*\mathbb{R}_c^{\#\mathbf{3}})\right) = Ext\text{-}\bigoplus_{n=0}^{{}^*\infty}\mathcal{F}_n^{\#}, \tag{1.2.1}$$

where $\mathcal{F}_n^{\#}$ is the space of n non-interacting particles,

$$\mathcal{F}_n^{\#} = L_2^{\#}({}^*\mathbb{R}_c^{\#\mathbf{3}})\otimes_s L_2^{\#}({}^*\mathbb{R}_c^{\#\mathbf{3}})\otimes_s \cdots \otimes_s L_2^{\#}({}^*\mathbb{R}_c^{\#\mathbf{3}}). \tag{1.2.2}$$

The variable $\boldsymbol{k} = (k_1, k_2, k_3) \in {}^*\mathbb{R}_c^{\#\mathbf{3}}$ denotes momentum vector. For $\psi = \{\psi_0, \psi_1, \dots\} \in \mathcal{F}^{\#} = \mathcal{F}_0^{\#}\oplus\mathcal{F}_1^{\#}\oplus\cdots$

We define on Fock space $\mathcal{F}^{\#}$ the ${}^*\mathbb{R}_c^{\#}$- valued #-norm $\|\cdot\|_{\#}$ by $\|\psi\|_{\#}^2 = Ext\text{-}\sum_{n=0}^{{}^*\infty}\|\psi_2\|_{\#2}^2$, where $\|\cdot\|_{\#2}$is a #-norm in $L_2^{\#}({}^*\mathbb{R}_c^{\#\mathbf{3}})$ The no particle space $\mathcal{F}_0^{\#} = {}^*\mathbb{C}^{\#}$ is the complex numbers, and

$$\Omega_0 = \{1,0,0,\dots\} \in \mathcal{F}^{\#} \tag{1.2.3}$$

is the (bare) vacuum or (bare) no-particle state vector. We define operators N and $H_{0,\varkappa}$ by

$$(N\psi)_n = n(Ext\text{-}\textstyle\prod_{i=1}^{n}\theta(\|\boldsymbol{k}_i\|,\varkappa)\,\psi_n), \tag{1.2.4}$$

$$\left(H_{0,\varkappa}\psi\right)_n(\boldsymbol{k}_1,\dots,\boldsymbol{k}_n) = Ext\text{-}\textstyle\sum_{j=1}^{n}\theta\left(\|\boldsymbol{k}_j\|,\varkappa\right)\mu\left(\boldsymbol{k}_j\right)\psi_n(\boldsymbol{k}_1,\dots,\boldsymbol{k}_n), \tag{1.2.5}$$

where $\varkappa \in {}^*\mathbb{R}_{c+}^{\#}\backslash{}^*\mathbb{R}_{\text{fin}+}^{\#}$ and

$$\theta\left(\|\boldsymbol{k}_j\|,\varkappa\right) = 1 \text{ if } \|\boldsymbol{k}_j\| \leq \varkappa \text{ and } \left(\|\boldsymbol{k}_j\|,\varkappa\right) = 0 \text{ if } \|\boldsymbol{k}_j\| > \varkappa,\ \mu\left(\boldsymbol{k}_j\right) = \sqrt{\langle\boldsymbol{k}_j,\boldsymbol{k}_j\rangle + m_0^2} \tag{1.2.6}$$

Here N is the number of particles operator, and $H_{0,\varkappa}$ is the free energy operator (the free Hamiltonian). The rest mass of the non-interacting particles is m_0, and $\mu(\boldsymbol{k})$ is the energy of a free particle with momentum vector $\boldsymbol{k}$. We use the standard annihilation and creation operators $a(\boldsymbol{k})$ and $a^*(\boldsymbol{k})$,

$$(a(\boldsymbol{k})\psi)_{n-1}(\boldsymbol{k}_1,\dots,\boldsymbol{k}_{n-1}) = \sqrt{n}\psi_n(\boldsymbol{k},\boldsymbol{k}_1,\dots,\boldsymbol{k}_{n-1}).$$

As a convenient minimal domain for $a(\boldsymbol{k})$, we use the set $\mathcal{E}^{\#}$ of vectors $\psi \in \mathcal{F}^{\#}$ with $\psi_n = 0$ for large $n \in {}^*\mathbb{N}$ and $\psi_n \in S_{\text{fin}}^{\#}({}^*\mathbb{R}_c^{\#\mathbf{3n}})$ for all $n \in {}^*\mathbb{N}$.

$$(a^*(\boldsymbol{k})\psi)_{n+1}(\boldsymbol{k}_1,\dots,\boldsymbol{k}_n,\boldsymbol{k}_{n+1}) = {}^{-1/2}\!\!\sqrt{n+1}Ext\text{-}\textstyle\sum_{j=1}^{n+1}\delta^{\#}\left(\boldsymbol{k}-\boldsymbol{k}_j\right)\psi_n\left(\boldsymbol{k}_1,\dots,\widehat{\boldsymbol{k}}_j,\dots\boldsymbol{k}_n\right). \tag{1.2.7}$$

Here the variable $\boldsymbol{k}_j$ is omitted. While a*((k) is not an operator, it is a densely defined bilinear form on $\mathcal{E}^{\#} \times \mathcal{E}^{\#}$.
Remark 1.2,1 Note for a ${}^*\mathbb{C}_c^{\#}$- valued function or ${}^*\mathbb{C}_c^{\#}$- valued distribution b we can define ${}^*\mathbb{C}_c^{\#}$- valued bilinear form

$$B = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}\times\cdot\times\cdot{}^*\mathbb{R}_c^{\#3}\cdot} b(\boldsymbol{k}_1,\dots,\boldsymbol{k}_m;\boldsymbol{k}_1',\dots,\boldsymbol{k}_n')a^*(\boldsymbol{k}_1)\cdots a^*(\boldsymbol{k}_m)a(-\boldsymbol{k}_1')\cdots a(-\boldsymbol{k}_n')d^{\#3}\boldsymbol{k}_1'\dots d^{\#3}\boldsymbol{k}_n'. \tag{1.2.8}$$

The integration helps in (1.2.8) and B is not only a bilinear form, but often an operator. This is the case if, for example, b is the kernel of a bounded operator B_0 from $\mathcal{F}_n^{\#}$ to $\mathcal{F}_m^{\#}$. In this case

$$\left\|(N_{\varkappa}+I)^{-\alpha/2}B(N_{\varkappa}+I)^{-\beta/2}\right\|_{\#} \leq \text{const}\cdot\|B_0\|_{\#}, \tag{1.2.9}$$

provided that $m+n\leq\alpha+\beta$. The constant depends only on α,β,m and n. Intuitively we think of B as being dominated by $N_{\varkappa}^{(m+n)/2}$; in particular B is an operator on $D\left(N_{\varkappa}^{(m+n)/2}\right)$ the domain of $N_{\varkappa}^{(m+n)/2}$. The inequality (1.2.9) is one of our basic estimates and in using it we will often dominate $\|B_0\|_{\#}$ by the Hilbert Schmidt #-norm $\|B_0\|_{\#HS}\leq\|b\|_{\#2}$, $\|B_0\|_{\#HS}=\sqrt{Ext\text{-}\sum_{i\in{}^*\infty}\|Ae_i\|_{\#}}$, and where $\{e_i|i\in{}^*\infty\}$ is an orthonormal basis in $\mathcal{F}^{\#}$.
By definition the field with hyperfinite momentum cut-off $\varphi_{\varkappa}^{\#}(\boldsymbol{x}),\boldsymbol{x}=(x_1,x_2,x_3)\in{}^*\mathbb{R}_c^{\#3},\varkappa\in{}^*\mathbb{R}_{c+}^{\#}\backslash{}^*\mathbb{R}_{\text{fin}+}^{\#}$ is

$$\varphi_{\varkappa}^{\#}(\boldsymbol{x})=Ext\text{-}\int_{|\boldsymbol{k}|\leq\varkappa}\left(Ext\text{-}\exp(-i\langle\boldsymbol{k},\boldsymbol{x}\rangle)\right)\{a^*(\boldsymbol{k})+a(\boldsymbol{k})\}[\mu(\boldsymbol{k})]^{-1/2}d^{\#3}k= \tag{1.2.10}$$

$$=Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}}\theta(\|\boldsymbol{k}\|,\varkappa)\left(Ext\text{-}\exp(-i\langle\boldsymbol{k},\boldsymbol{x}\rangle)\right)\{a^*(\boldsymbol{k})+a(\boldsymbol{k})\}[\mu(\boldsymbol{k})]^{-1/2}d^{\#3}k.$$

We also define the bilinear form

$$\pi_{\varkappa}^{\#}(x)=Ext\text{-}\int_{|\boldsymbol{k}|\leq\varkappa}i\left(Ext\text{-}\exp(-i\langle\boldsymbol{k},\boldsymbol{x}\rangle)\right)\{a^*(\boldsymbol{k})+a(\boldsymbol{k})\}[\mu(\boldsymbol{k})]^{\frac{1}{2}}d^{\#3}k= \tag{1.2.11}$$

$$Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}}i\theta(\|\boldsymbol{k}\|,\varkappa)\left(Ext\text{-}\exp(-i\langle\boldsymbol{k},\boldsymbol{x}\rangle)\right)\{a^*(\boldsymbol{k})+a(\boldsymbol{k})\}[\mu(\boldsymbol{k})]^{\frac{1}{2}}d^{\#3}k,$$

the conjugate momentum to $\varphi_{\varkappa}^{\#}(x)$. Since the kernels $b(\boldsymbol{k})=\theta(\|\boldsymbol{k}\|,\varkappa)\left(Ext\text{-}\exp(-i\langle\boldsymbol{k},\boldsymbol{x}\rangle)\right)[\mu(\boldsymbol{k})]^{-1/2}$ in $L_2^{\#}$ the bilinear forms (1.2.10)-(1.2.11) define operator-valued functions $\varphi_{\varkappa}^{\#}(\boldsymbol{x})\colon{}^*\mathbb{R}_c^{\#3}\to L(\mathcal{F}^{\#})$ and $\pi_{\varkappa}^{\#}(\boldsymbol{x})\colon{}^*\mathbb{R}_c^{\#3}\to L(\mathcal{F}^{\#})$. For real $f(x),g(x)$ such that $\theta(\|\boldsymbol{k}\|,\varkappa)[\mu(\boldsymbol{k})]^{-1/2}\hat{f}(x)\in L_2^{\#}$ and $\theta(\|\boldsymbol{k}\|,\varkappa)[\mu(\boldsymbol{k})]^{\frac{1}{2}}\hat{g}(x)\in L_2^{\#}$, , the bilinear forms $\varphi_{\varkappa}^{\#}(f)$ and $\pi_{\varkappa}^{\#}(g)$ define operators whose #-closures on $D\left(N_{\varkappa}^{1/2}\right)$ are self-#-adjoint. They satisfy the canonical commutation relations

$$Ext\text{-}\exp\left(i\pi_{\varkappa}^{\#}(g)\right)Ext\text{-}\exp\left(i\varphi_{\varkappa}^{\#}(g)\right)=Ext\text{-}\exp(i\langle f,g\rangle_{\#})\left\{Ext\text{-}\exp\left(i\varphi_{\varkappa}^{\#}(g)\right)Ext\text{-}\exp\left(i\pi_{\varkappa}^{\#}(g)\right)\right\}. \tag{1.2.12}$$

It is furthermore possible to define polynomial functions of the field $\varphi_{\varkappa}^{\#}(x)$, the Wick polynomials $:\varphi_{\varkappa}^{\#}(x):$ (see chapter I for a definition of the Wick dots : :). Explicitly, as a bilinear form on $D\left(N_{\varkappa}^{n/2}\right)\times D\left(N_{\varkappa}^{n/2}\right)$,

$$:\varphi_{\varkappa}^{\#n}(x)\coloneqq\sum_{j=0}^{n}\binom{n}{j}b_x(\boldsymbol{k}_1,\dots,\boldsymbol{k}_n)\,a^*(\boldsymbol{k}_1)\cdots a^*(\boldsymbol{k}_j)a(-\boldsymbol{k}_{j+1})\cdots a(-\boldsymbol{k}_n), \tag{1.2.13}$$

where

$$b_x(\boldsymbol{k}_1,\dots,\boldsymbol{k}_n)=\prod_{j=1}^{n}\theta\left(\|\boldsymbol{k}_j\|,\varkappa\right)\left[\mu(\boldsymbol{k}_j)\right]^{-1/2}Ext\text{-}\exp\left(-i\langle\textstyle\sum_{j=1}^{n}\boldsymbol{k}_j,\boldsymbol{x}\rangle\right).$$

Thus for real $f(x)\in S^{\#}({}^*\mathbb{R}_c^{\#3})$, the bilinear form

$$:\varphi_{\varkappa}^{\#n}(f)\coloneqq Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}}:\varphi_{\varkappa}^{\#n}(x)d^{\#3}x$$

has a kernel proportional to $\prod_{j=1}^{n}\theta\left(\|\boldsymbol{k}_j\|,\varkappa\right)\left[\mu(\boldsymbol{k}_j)\right]^{-1/2}\hat{f}\left(\sum_{j=1}^{n}\boldsymbol{k}_j\right)$. Thus from (1.2.9) we conclude that $:\varphi_{\varkappa}^{\#n}(f):$ defines a symmetric operator on the domain $D\left(N_{\varkappa}^{n/2}\right)$. It was shown in chapter I sect. 15 that $:\varphi_{\varkappa}^{\#n}(f):$ is essentially self-#-adjoint on this domain.

§ 2. THE PEREODIC HYPERFINITE APPROXIMATION IN

CONFIGURATION SPACE

§ 2.1 The cut-off Hamiltonian $\boldsymbol{H_{\varkappa}(g)}$. The cut-off Hamiltonian $H_{\varkappa}(g)$ acts on $\mathcal{F}_V^{\#}$ and can be written in terms of the field operator $\varphi_{\varkappa}^{\#}(\boldsymbol{x}), \boldsymbol{x} = (x_1, x_2, x_3)$ as

$$H_{\varkappa}(g) = H_{0,\varkappa} + Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \varphi_{\varkappa}^{\#4}(x)\, d^{\#3}x = \tag{2.1.1}$$

$$= H_{0,\varkappa} + H_{I,\varkappa,g},$$

where $H_{0,\varkappa} = H_{\varkappa}(0)$ is the free hamiltonian, and $0 \leq g$. Let

$$\mathrm{C}^{*\infty}(H_{0,\varkappa}) = \bigcap_{n=0}^{*\infty} D(H_{0,\varkappa}^n)$$

be the set of $\mathrm{C}^{*\infty}$ vectors for $H_{0,\varkappa}$. It was shown in sect.15 chapt.1 that $H_{\varkappa}(g)$ and $H_{I,\varkappa,g}$ are essentially self-#-adjoint on $\mathrm{C}^{*\infty}(H_{0,\varkappa})$, that

$$D(H_{\varkappa}(g)) = D(H_{0,\varkappa}) \cap D(H_{I,\varkappa,g}) \tag{2.1.2}$$

and that there are finite or hyperfinite $a, b = b(g)$ such that

$$\| H_{0,\varkappa}\psi\|_{\#} + \| H_{I,\varkappa,g}\psi\|_{\#} \leq \|(H_{\varkappa}(g) + b)\psi\|_{\#} \tag{2.1.3}$$

for all $\psi \in D(H_{\varkappa}(g))$.

Note that it is convenient to introduce a periodic hyperfinite approximation in configuration space. Under this approximation, the momentum space variable $\boldsymbol{k} = (k_1, k_2, k_3) \in {}^*\mathbb{R}_c^{\#\boldsymbol{3}}$ is replaced by a discrete variable $\boldsymbol{k} \in \Gamma_V^3$

$$\Gamma_V^3 = \left\{\boldsymbol{k} = (k_1, k_2, k_3) | k_i = \frac{2\pi n_i}{V}, n_i \in {}^*\mathbb{Z}; i = 1,2,3\right\}$$

with $V \in {}^*\mathbb{R}_{c+}^{\#} \backslash {}^*\mathbb{R}_{\text{fin}+}^{\#}$. Thus we define $\mathcal{F}_V^{\#}$, the Fock space for volume V^3 as

$$\mathcal{F}_V^{\#} = \mathfrak{C}\left(l_2^{\#}(\Gamma_V^3)\right) = {}^*\mathbb{C}^{\#} \oplus l_2^{\#}(\Gamma_V^3) \oplus \{l_2^{\#}(\Gamma_V^3) \otimes_s l_2^{\#}(\Gamma_V^3)\} \cdots$$

We identify $\mathcal{F}_V^{\#}$ with the subspace of $\mathcal{F}^{\#}$ consisting of piecewise constant functions which are constant on each cube of volume $(2\pi/V)^{3j}$ cantered about a lattice point

$$\{\boldsymbol{k}_1, \ldots \boldsymbol{k}_j\} \in \Gamma_V^3 \times \Gamma_V^3 \times \cdots \times \Gamma_V^3 = \Gamma_V^{3j}.$$

The periodic annihilation and creation operators $a(\boldsymbol{k})$ and $a^*(\boldsymbol{k})$ can be extended from $\mathcal{F}_V^{\#}$ to $\mathcal{F}^{\#}$ by the formulas

$$a_V(\boldsymbol{k}) = \left(\frac{V}{2\pi}\right)^{3/2} \left[Ext\text{-}\int_{-\pi/V}^{\pi/V} d^{\#}l_1\, Ext\text{-}\int_{-\pi/V}^{\pi/V} d^{\#}l_2\, Ext\text{-}\int_{-\pi/V}^{\pi/V} d^{\#}l_3\, a(\boldsymbol{k} + \boldsymbol{l})\right], \tag{2.1.4}$$

$$a_V^*(\boldsymbol{k}) = \left(\frac{V}{2\pi}\right)^{3/2} \left[Ext\text{-}\int_{-\pi/V}^{\pi/V} d^{\#}l_1\, Ext\text{-}\int_{-\pi/V}^{\pi/V} d^{\#}l_2\, Ext\text{-}\int_{-\pi/V}^{\pi/V} d^{\#}l_3\, a^*(\boldsymbol{k} + \boldsymbol{l})\right]. \tag{2.1.5}$$

Therefore the periodic field $\varphi_{\varkappa,V}^{\#}(\boldsymbol{x})$ and the periodic hamiltonian $H_{\varkappa,V}(g)$ can be extended to act on $\mathcal{F}^{\#}$ by the formulas

$$\varphi_{\varkappa,V}^{\#}(\boldsymbol{x}) = (2V)^{-3/2} Ext\text{-}\sum_{\boldsymbol{k}\in\Gamma_V^3, |\boldsymbol{k}|\leq\varkappa} Ext\text{-}\exp(-i\langle\boldsymbol{k}, \boldsymbol{x}\rangle)[a^*(\boldsymbol{k}) + a(-\boldsymbol{k})]\big(\mu(\boldsymbol{k})\big)^{-1/2}, \tag{2.1.6}$$

$$H_{\varkappa,V} = H_{0,\varkappa,V} + H_{I,\varkappa,V}, \tag{2.1.7}$$

$$H_{I,\varkappa,V} = Ext\text{-}\int_{-V/2}^{V/2} Ext\text{-}\int_{-V/2}^{V/2} Ext\text{-}\int_{-V/2}^{V/2} \varphi_{\varkappa,V}^{\#4}(\boldsymbol{x})\, d^{\#3}x, \tag{2.1.8}$$

$$H_{0,\varkappa,V} = Ext\text{-}\int_{|\boldsymbol{k}|\leq\varkappa} a^*(\boldsymbol{k})\, a(\boldsymbol{k})\mu(\boldsymbol{k}_V) d^{\#3}k \tag{2.1.9}$$

with $\boldsymbol{k_V}$, a lattice point infinite close to $\boldsymbol{k}$,

$$\boldsymbol{k_V} \in \Gamma_V^3,\ \|\boldsymbol{k} - \boldsymbol{k_V}\| \leq \frac{\pi}{V} \approx 0. \tag{2.1.10}$$

Remark 2.1.1 Note the absence of a V in the $a(\boldsymbol{k})$ and $a^*(\boldsymbol{k})$ in (2.1.9). On $\mathcal{F}_V^{\#}$, this definition of $H_{0,\varkappa,V}$ agrees with the standard definition

$$Ext\text{-}\sum_{\boldsymbol{k}\in\Gamma_{V,|\boldsymbol{k}|\leq\varkappa}^3} a_V^*(\boldsymbol{k}) a_V(\boldsymbol{k})\mu(\boldsymbol{k}).$$

The operators $H_{I,\varkappa,V}$ and $H_{\varkappa,V}$ are essentially self adjoint on ${}^*C^{\infty}(H_{0,\varkappa,V})$, and

$$D(H_{\varkappa,V}) = D(H_{0,\varkappa,V}) \cap D(H_{I,\varkappa,V}). \tag{2.1.11}$$

For all $\psi \in D(H_{\varkappa,V})$,

$$\|H_{0,\varkappa,V}\psi\|_{\#} + \|H_{I,\varkappa,V}\psi\|_{\#} \leq a\|(H_{\varkappa,V} + b)\psi\|_{\#}, \tag{2.1.12}$$

where b depend on V. On $\mathcal{F}_V^{\#}$, the operator $H_{I,\varkappa,V}$ has a #-compact resolvent. We want to approximate $H_{I,\varkappa}(g)$ by operators with #-compact resolvents on $\mathcal{F}_V^{\#}$, so we define

$$H_{\varkappa}(g,V) = H_{0,\varkappa,V} + Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} :\varphi_{\varkappa,V}^{\#4}(x) g(\boldsymbol{x}) d^{\#3}x = \tag{2.1.13}$$

$$= H_{0,\varkappa,V} + H_{I,\varkappa}(g,V).$$

As in chapter I sect. we can show that $H_{\varkappa}(g,V)$, and $H_{I,\varkappa}(g,V)$ are essentially self-#-adjoint on ${}^*C^{\infty}(H_{0,\varkappa,V})$, and that

$$D(H_{\varkappa}(g,V)) = D(H_{0,\varkappa,V}) \cap D\left(H_{I,\varkappa}(g,V)\right). \tag{2.1.14}$$

Furthermore, for all $\psi \in D(H_{\varkappa}(g,V))$,

$$\|H_{0,\varkappa,V}\psi\|_{\#} + \|H_{I,\varkappa}(g,V)\psi\|_{\#} \leq a\|(H_{\varkappa}(g,V) + b)\psi\|_{\#}. \tag{2.1.15}$$

In this case both g and V serve as volume cutoffs, and the constant $b = b(g,V)$ can be chosen independently of V for fixed g. On the space $\mathcal{F}_V^{\#}$, the operator $H_{\varkappa}(g,V)$ has a #-compact resolvent. Our hamiltonians are semi-bounded and for each $\varepsilon > 0$, there is a constant b such that

$$0 \leq \varepsilon H_{0,\varkappa} + H_{I,\varkappa}(g) + b, \tag{2.1.16}$$

$$0 \leq \varepsilon H_{0,\varkappa,V} + H_{I,\varkappa,V} + b, \tag{2.1.17}$$

$$0 \leq \varepsilon H_{0,\varkappa,V} + H_{I,\varkappa}(g,V) + b, \tag{2.1.18}$$

see chapter I sect.18. In (2.1.18), the b can be chosen to be independent of V. Taking $\varepsilon = 1/2$, we have

$$\frac{1}{2}H_{0,\varkappa} \leq H_{I,\varkappa}(g) + b,$$

which implies that for all $\psi \in D\left(\left(H_{\varkappa}(g)\right)^{\frac{1}{2}}\right)$,

$$\left\|H_{0,\varkappa}^{1/2}\psi\right\|_{\#} \leq \sqrt{2}\left\|(H_{\varkappa}(g)+b)^{1/2}\psi\right\|_{\#}. \tag{2.1.19}$$

Here we must choose b at least$|E_{\varkappa}(2g)|$, where $E_{\varkappa}(2g)$ is the vacuum energy for the cut-off $2g$.

§ 3. THE EXISTENCE OF A VACUUM VECTOR $\Omega_{\varkappa,g}$ FOR $H_{\varkappa}(g)$

In this section we prove the existence of a vacuum vector $\Omega_{\varkappa,g}$ for $H_{\varkappa}(g)$, and we prove that the vacuum is unique.

§ 3.1 The existence of a vacuum vector In this subsection we prove the existence of a vacuum vector $\Omega_{\varkappa,g}$ for $H_{\varkappa}(g)$. Since the Hamiltonian $H_{\varkappa}(g)$ is bounded from below, we can define the vacuum energy $E_{\varkappa,g} \triangleq E(\varkappa,g)$ to be the infimum of the spectrum of $H_{\varkappa}(g)$ and we also refer to $E_{\varkappa,g}$ as the *lower bound* of $H_{\varkappa}(g)$. We show that $E_{\varkappa,g}$ is an isolated point in the spectrum. In a relativistic theory, the gap between the ground state and the first excited state is the mass of the interacting particle. For this reason we say that $H_{\varkappa}(g)$ has a mass gap. A vacuum vector $\Omega_{\varkappa,g}$ is defined as a normalized eigenvector of $H_{\varkappa}(g)$ corresponding to the eigenvalue $E_{\varkappa,g}$.

$$H_{\varkappa}(g)\Omega_{\varkappa,g} = E_{\varkappa,g}\Omega_{\varkappa,g}, \left\|\Omega_{\varkappa,g}\right\|_{\#} = 1. \tag{3.1.1}$$

Theorem 3.1.1 There is exists a vacuum vector $\boldsymbol{\Omega_{\varkappa,g}}$ for Hamiltonian $\boldsymbol{H_{\varkappa}(g)}$. For any $\boldsymbol{\varepsilon > 0, \varepsilon \approx 0}$ the operator $\boldsymbol{H_{\varkappa}(g)}$, restricted to the spectral interval $\boldsymbol{[E_{\varkappa,g}, E_{\varkappa,g} + m_0 - \varepsilon]}$ is #-compact.
Theorem 3.1.2 The approximate Hamiltonian $\boldsymbol{H_{\varkappa,V}(g)}$, has a vacuum vector $\boldsymbol{\Omega_{\varkappa,g,V}}$. Any hyperinfinite sequence of volumes $\boldsymbol{V_j}$ tending to hyperinfinity ${}^*\infty$ has a hyperinfinite subsequence $\boldsymbol{V_l}, \boldsymbol{l} \in {}^*\mathbb{N}$ such that #-limit

.
$$\Omega_{\varkappa,g} = \#\text{-}\lim_{l\to{}^*\infty}\Omega_{\varkappa,g,V_l} \tag{3.1.2}$$

exists and satisfies (3.1.1).
Remark 3.1.1 Let $E_{\varkappa,g,V}$ be the lower bound of $H_{\varkappa,V}(g)$ on $\mathcal{F}_V^{\#}$. Since $H_{\varkappa,V}(g)$ has a #-compact resolvent on $\mathcal{F}_V^{\#}$, there is a vacuum vector $\Omega_{\varkappa,g,V}$ for $H_{\varkappa,V}(g) \upharpoonright \mathcal{F}_V^{\#}$. We now see that $E_{\varkappa,g,V}$ is the lower bound for $H_{\varkappa,V}(g)$ on $\mathcal{F}_V^{\#}$, so that $\Omega_{\varkappa,g,V}$ is a vacuum vector for $H_{\varkappa,V}(g)$.
Remark 3.1.2 Let $\mathcal{F}_V^{\#\perp}$ be the orthogonal complement of $\mathcal{F}_V^{\#}$. Since $H_{\varkappa,V}(g)$ leaves $\mathcal{F}_V^{\#}$ invariant and is self-#-adjoint, $H_{\varkappa,V}(g)$ also leaves $\mathcal{F}_V^{\#\perp}$ invariant.
Theorem 3.1.3 The lower bound of $H_{\varkappa,V}(g)$ on $\mathcal{F}_V^{\#}$ is $E_{\varkappa,g,V} + m_0$, where m_0, is the rest mass of the Fock space bosons.
Remark 3.1.3 Theorem 3.1.3 shows that $\Omega_{\varkappa,g,V}$ is a vacuum for $H_{\varkappa,V}(g)$.
Proof We have an orthogonal decomposition in the single particle space

$$\mathcal{F}_1^{\#} = L_2^{\#}({}^*\mathbb{R}_c^{\#3}) = \mathcal{F}_{1V}^{\#} \oplus \mathcal{F}_{1V}^{\#\perp}. \tag{3.1.3}$$

Here $\mathcal{F}_{1V}^{\#} = \mathcal{F}_1^{\#} \cap \mathcal{F}_V^{\#}$ consists of functions piecewise constant on intervals cantered at lattice points. Thus we may write

$$\mathcal{F}^{\#} = Ext\text{-}\bigoplus_{j=0}^{{}^*\infty}\mathcal{F}^{\#(j)},\ \mathcal{F}_V^{\#\perp} = Ext\text{-}\bigoplus_{j=1}^{{}^*\infty}\mathcal{F}^{\#(j)}, \tag{3.1.4}$$

where $\mathcal{F}^{\#(j)}$ consists of vectors with exactly j particles from $\mathcal{F}_{1V}^{\#\perp}$ and

$$\mathcal{F}^{\#(j)} = (Ext\text{-}\,\mathcal{F}_{1V}^{\#\perp} \otimes_s \cdots \otimes_s \mathcal{F}_{1V}^{\#\perp}) \otimes_s \mathscr{F}_V^{\#} \tag{3.1.5}$$

In this tensor product decomposition there are j factors $\mathcal{F}_{1V}^{\#\perp}$. The Hamiltonian $H_{\varkappa,V}(g)$ leaves each subspace $\mathcal{F}^{\#(j)}$ invariant, and on $\mathcal{F}^{\#(j)}$ we have $H_{\varkappa,V}(g) = I \otimes A + B \otimes I$, where $A = H_{\varkappa,V}(g) \upharpoonright \mathcal{F}_V^{\#}$ and B is a sum of j copies of $H_{0,\varkappa,V}$ each acting on a single factor $\mathcal{F}_{1V}^{\#\perp}$. Since

$$jm_0 \leq B, \tag{3.1.6}$$

the Theorem follows from this decomposition.

Theorem 3.1.4 For $V \leq {}^*\infty$, and for b sufficiently large we have

$$D(H_{0,\varkappa}) \subset D\left(H_{0,\varkappa}^{\frac{1}{2}}\right) \cap D(N_\varkappa) \subset D\big(H_{\varkappa,V}(g) + b\big), \tag{3.1.7}$$

$$D(H_{0,\varkappa}) \subset D\left(\left[(N_\varkappa + I)^{-1}\big(H_{\varkappa,V}(g) + b\big)\right]^{\#-}\right). \tag{3.1.8}$$

Here we denote by $A^{\#-}$ #-closure of the operator A.

Proof We take b large enough so that $H_{\varkappa,V}(g) + b$ is positive, see (2.1.18). By (1.2.9) and (2.1.14) we get

$$D(H_{0,\varkappa}) \cap D(N_\varkappa^2) \subset D(H_{0,\varkappa}) \cap D\left(H_{I,\varkappa,V}(g)\right) = D\left(H_{\varkappa,V}(g)\right) \subset D\left(\big(H_{\varkappa,V}(g) + b\big)^{\frac{1}{2}}\right).$$

Thus for all $\psi \in D(H_{0,\varkappa}) \cap D(N_\varkappa^2)$,

$$\left\|\big(H_{\varkappa,V}(g) + b\big)^{\frac{1}{2}}\psi\right\|_{\#}^{2} = \langle\psi, \big(H_{\varkappa,V}(g) + b\big)\psi\rangle_{\#} \leq \langle\psi, \big(H_{\varkappa,V} + b\big)\psi\rangle_{\#} +$$

$$+\left\|(N_\varkappa + I)^{-1} H_{I,\varkappa,V}(N_\varkappa + I)^{-1}\right\|_{\#} \|(N_\varkappa + I)\psi\|_{\#}^{2}.$$

Since $\big(H_{\varkappa,V}(g) + b\big)^{\frac{1}{2}}$ is a #-closed operator, we can extend this inequality by #- continuity. As $N_\varkappa$ and $H_{0,\varkappa,V}$ commute, the inequality extends by #-continuity to all $\psi \in D\left(H_{0,\varkappa}^{1/2}\right) \cap D(N_\varkappa) \supset D(H_{0,\varkappa})$. The proof of (3.1.8) is similar.

Theorem 3.1.5 Let z be non-real or real and sufficiently negative. Then as V tends to hyper infinity ${}^*\infty$,

$$\left\|\big(H_{\varkappa,V}(g) - zI\big)^{-1} - \big(H_\varkappa(g) - zI\big)^{-1}\right\|_{\#} = O(V^{-1}). \tag{3.1.9}$$

Proof Let us fix g and z and suppress g when possible. In chapter I sect 16 we have shown that $H_\varkappa(g)$ is essentially self-#-adjoint on $C^{*\infty}(H_{0,\varkappa})$. Thus vectors of the form $\chi = (H_\varkappa - zI)\psi, \psi \in C^{*\infty}(H_{0,\varkappa})$, are #-dense in $\mathcal{F}^{\#}$. On these vectors

$$\left\{\big(H_{\varkappa,V} - zI\big)^{-1} - (H_\varkappa - zI)^{-1}\right\}\chi = \big(H_{\varkappa,V} - zI\big)^{-1}\left\{(H_\varkappa - zI)\psi - \big(H_{\varkappa,V} - zI\big)\psi\right\} =$$

$$= \big(H_{\varkappa,V} - zI\big)^{-1}\big(H_\varkappa - H_{\varkappa,V}\big)(H_\varkappa - zI)^{-1}\chi =$$

$$= \big(H_{\varkappa,V} - zI\big)^{-1}(N_\varkappa + I)(N_\varkappa + I)^{-1}\big(H_\varkappa - H_{\varkappa,V}\big)(N_\varkappa + I)^{-1}(N_\varkappa + I)(H_\varkappa - zI)^{-1}\chi.$$

For $\theta \in \mathcal{F}^{\#}$,

$$\left|\langle\theta,\left\{\left(H_{\varkappa,V}-zI\right)^{-1}-\left(H_{\varkappa}-zI\right)^{-1}\right\}\chi\rangle_{\#}\right|\leq \tag{3.1.10}$$

$$\leq\left\|\left(N_{\varkappa}+I\right)\left(H_{\varkappa,V}-\bar{z}I\right)^{-1}\right\|_{\#}\cdot\|\theta\|_{\#}\cdot\left\|\left(N_{\varkappa}+I\right)^{-1}\left(H_{\varkappa}-H_{\varkappa,V}\right)\left(N_{\varkappa}+I\right)^{-1}\right\|_{\#}\times$$

$$\times\left\|\left(N_{\varkappa}+I\right)\left(H_{\varkappa}-zI\right)^{-1}\right\|_{\#}\|\chi\|_{\#}\cdot$$

Using (2.1.15), we find that $\left\|\left(N_{\varkappa}+I\right)\left(H_{\varkappa,V}-\bar{z}I\right)^{-1}\right\|_{\#}$ is bounded uniformly in V, since

$$\left\|\left(N_{\varkappa}+I\right)\left(H_{\varkappa,V}-\bar{z}I\right)^{-1}\psi\right\|_{\#}\leq\text{const}\cdot\left\|\left(H_{0,\varkappa,V}+I\right)\left(H_{\varkappa,V}-\bar{z}I\right)^{-1}\psi\right\|_{\#}\leq$$

$$\leq\text{const}\cdot\left\|H_{\varkappa,V}\left(H_{\varkappa,V}-\bar{z}I\right)^{-1}\psi\right\|_{\#}+\text{const}\cdot\left\|\left(H_{\varkappa,V}-\bar{z}I\right)^{-1}\psi\right\|_{\#},$$

where the constants can be chosen independently of V. By a similar consideration, the orthogonal decomposition (3.1.3) shows that $\left(N_{\varkappa}+I\right)\left(H_{\varkappa,V}-zI\right)^{-1}$ is a bounded operator. Thus from (3.1.10), and the fact that the χ are #-dense, we infer

$$\left\|\left(H_{\varkappa,V}-zI\right)^{-1}-\left(H_{\varkappa}-zI\right)^{-1}\right\|_{\#}\leq\text{const}\cdot\left\|\left(N_{\varkappa}+I\right)^{-1}\left(H_{\varkappa}-H_{\varkappa,V}\right)\left(N_{\varkappa}+I\right)^{-1}\right\|_{\#} \tag{3.1.11}$$

with a constant independent of V. The difference $H_{\varkappa}-H_{\varkappa,V}=\left(H_{0,\varkappa}-H_{0,\varkappa,V}\right)+\left(H_{I,\varkappa}(g)-H_{I,\varkappa,V}(g)\right)$ and for infinite large V,

$$\left\|\left(N_{\varkappa}+I\right)^{-1/2}\left(H_{0,\varkappa}-H_{0,\varkappa,V}\right)\left(N_{\varkappa}+I\right)^{-1}\right\|_{\#}=O\left(V^{-1}\right). \tag{3.1.12}$$

This is a simple direct computation, using $|\mu(\boldsymbol{k}_V)-\mu(\boldsymbol{k})|=O(V^{-1})$. For the interaction terms, we use (1.2.10) to estimate

$$\left\|\left(N_{\varkappa}+I\right)^{-1/2}\left(H_{I,\varkappa}(g)-H_{I,\varkappa,V}(g)\right)\left(N_{\varkappa}+I\right)^{-1}\right\|_{\#}=O\left(V^{-1}\right). \tag{3.1.13}$$

The kernel $b\left(\boldsymbol{k}_1,\dots,\boldsymbol{k}_4\right)$ corresponding to a monomial in $H_{I,\varkappa}(g)$ is

$$b\left(\boldsymbol{k}_1,\dots,\boldsymbol{k}_4\right)=\binom{4}{j}\prod_{j=1}^{4}\theta\left(\left\|\boldsymbol{k}_j\right\|,\varkappa\right)\left[\mu\left(\boldsymbol{k}_j\right)\right]^{-1/2}\hat{g}\left(k_1^{(1)}+k_2^{(1)}+k_3^{(1)}+k_4^{(1)},\dots,k_1^{(3)}+k_2^{(3)}+k_3^{(3)}+k_4^{(3)}\right),$$

$0\leq j\leq 4.$ The kernel $b_V\left(\boldsymbol{k}_1,\dots,\boldsymbol{k}_4\right)$) for the corresponding monomial in $H_{I,\varkappa,V}(g)$ is obtained by replacing the factor $\prod_{j=1}^{4}\theta\left(\left\|\boldsymbol{k}_j\right\|,\varkappa\right)\left[\mu\left(\boldsymbol{k}_j\right)\right]^{-1/2}$ by the factor $\prod_{j=1}^{4}\theta\left(\left\|\boldsymbol{k}_{jV}\right\|,\varkappa\right)\left[\mu\left(\boldsymbol{k}_{jV}\right)\right]^{-1/2}$. Inspection of the difference $b\left(\boldsymbol{k}_1,\dots,\boldsymbol{k}_4\right)-b_V\left(\boldsymbol{k}_1,\dots,\boldsymbol{k}_4\right)$ shows that $\left\|b\left(\boldsymbol{k}_1,\dots,\boldsymbol{k}_4\right)-b_V\left(\boldsymbol{k}_1,\dots,\boldsymbol{k}_4\right)\right\|_{L_2^{\#}}=O\left(V^{-1}\right)$.as $V\to{}^{*}\infty$, from which we conclude that (3.13) is $O\left(V^{-1}\right)$. The #-convergence of the resolvents follows from (3.1.11)-(3.1.13). The #-limit

$$E_{\varkappa,g,V}\to_{\#}E_{\varkappa,g}$$

follows from the #-convergence of the resolvents, since for large positive b,

$$\left(E_{\varkappa,g,V}+b\right)^{-1}=\left\|\left(H_{\varkappa,V}(g)+b\right)^{-1}\right\|_{\#}.$$

Proof of the theorems 3.1.1 and 3.1.2 Let $f(x)$ be a #-smooth positive function with support in the interval $[-\varepsilon,m_0-\varepsilon]$. Then $f\left(H_{\varkappa,V}(g)-E_{\varkappa,g,V}\right)\restriction\mathcal{F}_V^{\#}$ is #-compact, since the resolvent of $H_{\varkappa,V}(g)\restriction\mathcal{F}_V^{\#}$is #-compact on

$\mathcal{F}_V^\#$. By Theorem 3.1.3, $f\big(H_{\varkappa,V}(g) - E_{\varkappa,g,V}\big) \upharpoonright \mathcal{F}_V^{\#\perp} = 0$ and therefore #-compact on the full Fock space $\mathcal{F}_V^\#$. By Theorem 3.5, the resolvent $\big(H_{\varkappa,V}(g) - E_{\varkappa,g,V} - z\big)^{-1}$ #-converge in #-norm as $V \to {}^*\infty$, and therefore

$$\left\|f\big(H_{\varkappa,V}(g) - E_{\varkappa,g,V}\big) - f\big(H_\varkappa(g) - E_{\varkappa,g}\big)\right\|_\# \to_\# 0,$$

since $f\big(H_\varkappa(g) - E_{\varkappa,g}\big)$ is a bounded function of $\big(H_\varkappa(g) - E_{\varkappa,g} - z\big)^{-1}$ which vanishes at hyperinfinity. Since the uniform #-limit of #-compact operators is #-compact, $H_\varkappa(g)$ restricted to the spectral interval $[-\varepsilon, m_0 - \varepsilon]$ is #-compact. This means furthermore that only a finite or hyperfinite number of eigenvalues of $H_{\varkappa,V}(g)$ #-converge to $E_{\varkappa,g}$. Theorem 3.1.6 shows that the projection onto the corresponding set of eigenvectors of $H_{\varkappa,V}(g)$ #-converge as $V \to {}^*\infty$. Since $\Omega_{\varkappa,g,V}$ is an eigenvector of $f\big(H_{\varkappa,V}(g) - E_{\varkappa,g,V}\big)$ a hyperinfinite subsequence of the $\Omega_{\varkappa,g,V}$ #-converge to a #-limit as $V \to {}^*\infty$. For this #-limit

Notation 3.1.1 Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be a non-Archimedean Banach spaces. The set of all #-closed operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$ will be denoted by $\mathcal{E}^\#(\boldsymbol{X},\boldsymbol{Y})$. Also we write $\mathcal{E}^\#(\boldsymbol{X},\boldsymbol{X}) = \mathcal{E}^\#(\boldsymbol{X})$. The set of all linear operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$ will be denoted by $\mathcal{B}(\boldsymbol{X},\boldsymbol{Y})$. Also we write $\mathcal{B}(\boldsymbol{X},\boldsymbol{X}) = \mathcal{B}(\boldsymbol{X})$.

Theorem 3.1.6

[18, p. 21 $\mathscr{E}\mathcal{B}$
$\mathfrak{C}(B)$

§ 3.2 Uniqueness of the vacuum. In this subsection we prove the uniqueness of a vacuum vector $\Omega_{\varkappa,g}$ for $H_\varkappa(g)$.

Theorem 3.2.1 The vacuum vector $\Omega_{\varkappa,g,V}$ for $H_\varkappa(g)$ is unique.
Remark 3.2.1 In other words $E_{\varkappa,g}$, the lower bound of $H_\varkappa(g)$ is a simple eigenvalue.
Definition 3.2.1 Let $\mathcal{H}^\# = L_2^\#(Q, d^\#\mu)$ be a non-Archimedean Hilbert space. We say that a bounded operator $A\colon \mathcal{H}^\# \to \mathcal{H}^\#$ has a strictly positive kernel provided that

$$\langle \psi, A\chi\rangle_\# > 0 \tag{3.2.1}$$

whenever ψ and χ are non-negative $L_2^\#$ functions with non-zero #-norms. Such an operator transforms a function $\chi \geq 0, \|\chi\|_\# \neq 0$ into a function $A\chi$ which is strictly positive #-almost everywhere.
Definition 3.2.2 Let $\mathcal{H}^\# = L_2^\#(Q, d^\#\mu)$ be a non-Archimedean Hilbert space. We say that a bounded operator $A\colon \mathcal{H}^\# \to \mathcal{H}^\#$ has a positive, ergodic kernel if for each ψ, χ as above $\langle \psi. A\chi\rangle \geq 0$ and

$$\langle \psi, A^j\chi\rangle_\# > 0 \tag{3.2.2}$$

for some j, depending on ψ and χ. Clearly every A with a strictly positive kernel has a positive, ergodic kernel.
Theorem 3.2.2 Let A have a positive ergodic kernel, and suppose that $\|A\|_\#$ is an eigenvalue of A. Then $\|A\|_\#$ is a simple eigenvalue and the corresponding eigenvector can be chosen to be a strictly positive function.
Proof Since A maps positive functions into positive functions it also maps real functions into real functions. If

$\psi \in \mathcal{H}^{\#}$ satisfies $A\psi = \|A\|_{\#} \cdot \psi$, then so do Re$\psi$ and Imψ. Therefore without loss of generality we may assume that ψ is real. Since $\|A^j\|_{\#} = \|A\|_{\#}^j$, and $A^j\psi = \|A\|_{\#}^j \cdot \psi$, we infer that

$$\|A^j\|_{\#} \cdot \|\psi\|_{\#}^2 = \langle \psi . A^j\psi \rangle_{\#} \leq \langle |\psi|, A^j|\psi| \rangle_{\#} \leq \|\boldsymbol{A}\|_{\#}^{\boldsymbol{j}} \cdot \|\psi\|_{\#}^2,$$

$$\langle \psi . A^j\psi \rangle_{\#} = \langle |\psi|, A^j|\psi| \rangle_{\#}.$$

Writing now $\psi = \psi^+ - \psi^-$, where ψ^+ and ψ^- are the positive and negative parts of ψ,

$$\langle \psi^+, A^j\psi^+ \rangle_{\#} - \langle \psi^+, A^j\psi^- \rangle_{\#} - \langle \psi^-, A^j\psi^+ \rangle_{\#} + \langle \psi^-, A^j\psi^- \rangle_{\#} =$$

$$= \langle \psi^+, A^j\psi^+ \rangle_{\#} + \langle \psi^+, A^j\psi^- \rangle_{\#} + \langle \psi^-, A^j\psi^+ \rangle_{\#} + \langle \psi^-, A^j\psi^- \rangle_{\#}$$

or

$$\langle \psi^+, A^j\psi^- \rangle_{\#} + \langle \psi^-, A^j\psi^+ \rangle_{\#} = 0. \tag{3.2.3}$$

Unless $\psi^+ = 0$ or $\psi^- = 0$, each term of (3.2.3) could be made strictly positive by choosing an appropriate j. Thus either ψ^+ or ψ^- must vanish, and we may choose the eigenvector ψ to be non-negative. If $\chi \geq 0, \|\chi\|_{\#} \neq 0$, then for some integer j , $0 < \langle \chi, A^j\psi \rangle_{\#} = \|A\|_{\#}^j \cdot \langle \chi, \psi \rangle_{\#}$. This proves that $\chi\psi$ is not zero almost everywhere, and that ψ is strictly positive #-almost everywhere. Finally, if ψ and χ were linearly independent eigenvectors of A with the eigenvalue $\|A\|_{\#}$, then we could repeat the above argument with the component of χ orthogonal to ψ. This would yield two positive, orthogonal eigenvectors, which would be impossible, and the proof is complete.

Remark 3.2.2 Let $\varphi_{\varkappa}^{\#}(h) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#4}} \varphi_{\varkappa}^{\#}(x)h(x)\, d^{\#3}x$ denote the smeared, time zero free field operators. The spectral projections of the $\varphi_{\varkappa}^{\#}(h)$, or the functions Ext-exp $(i\varphi_{\varkappa}^{\#}(h))$ generate a maximal abelian algebra $\mathcal{M}^{\#}$ of bounded operators on $\mathcal{F}^{\#}$. Let Q be the spectrum of the algebra $\mathcal{M}^{\#}$. The no particle vector $\Omega_0 \in \mathcal{F}^{\#}$ is a cyclic vector for 'DR, namely $\mathcal{F}^{\#} = \#\text{-}\overline{(\mathscr{M}^{\#}\Omega_0)}$. Therefore we may introduce a #-measure $d^{\#}\mu$ on Q so that $\mathcal{F}^{\#}$ is unitarily equivalent to $L_2^{\#}(Q, d^{\#}\mu)$ and so that the equivalence carries $\mathcal{M}^{\#}$ into $L_{^*\infty}^{\#}$ and takes Ω_0 into the function 1.

Theorem 3.2.3 With $\mathcal{F}^{\#}$ represented as $L_2^{\#}(Q, d^{\#}\mu)$, Ext-exp$(-H_{0,\varkappa})$ has a positive, ergodic kernel.

Proof Let ψ and χ be non-negative. Write $\psi = \psi_1 + \psi_2$, where ψ_1 is the component of ψ along Ω_0. Thus the $L_1^{\#}$ #-norm of ψ is given by $\|\psi\|_{\#1} = \langle \psi, \Omega_0 \rangle_{\#} = \langle \psi_1, \Omega_0 \rangle_{\#}$. Note $\|\psi\|_{\#1} \neq 0$ whenever ψ is non-zero, and $\|Ext\text{-exp}(-tH_{0,\varkappa})\psi_2\|_{\#1} \leq (Ext\text{-exp}(-tm_0))\|\psi_2\|_{\#1}$, where m_0 is the boson mass. Thus

$$\langle \psi, Ext\text{-exp}(-tH_{0,\varkappa})\chi \rangle_{\#} \geq \|\psi\|_{\#1} \cdot \|\chi\|_{\#1} - \|\psi_2\|_{\#1} \cdot \|\chi_2\|_{\#1}(Ext\text{-exp}(-tm_0)). \tag{3.2.4}$$

By choosing t sufficiently large, (3.2.4) is positive, which proves (3.2.2). If the following inequality holds

$$Ext\text{-exp}(-tm_0) < \frac{1}{2}\frac{\|\psi\|_{\#1} \cdot \|\chi\|_{\#1}}{\|\psi_2\|_{\#1} \cdot \|\chi_2\|_{\#1}} = \frac{1}{2}\frac{\|\psi\|_{\#1} \cdot \|\chi\|_{\#1}}{(\|\psi\|_{\#}^2 - \|\psi\|_{\#1}^2)^{1/2}(\|\chi\|_{\#}^2 - \|\chi\|_{\#1}^2)^{1/2}} \tag{3.2.5}$$

then

$$\langle \psi, Ext\text{-exp}(-tH_{0,\varkappa})\chi \rangle_{\#} \geq \frac{1}{2}\|\psi\|_{\#1} \cdot \|\chi\|_{\#1}. \tag{3.2.6}$$

We need to show that $\langle \psi, Ext\text{-exp}(-tH_{0,\varkappa})\chi \rangle_{\#} > 0$ for all finite t. In fact, it is sufficient to prove this for a #-dense set of non-negative ψ and χ. Let us consider an approximate free energy operator

$$H_{0,\varkappa,V} = Ext\text{-}\int_{|\boldsymbol{k}| \leq \varkappa} a^*(\boldsymbol{k})a(\boldsymbol{k})\mu(\boldsymbol{k}_V)d^{\#3}k. \tag{3.2.7}$$

For vectors $\psi \in C^{*\infty}(H_{0,\varkappa})$, as $V \to {}^*\infty$. $\| H_{0,\varkappa,V}\psi - H_{0,\varkappa}\psi\|_{\#} \to_{\#} 0$. Since $H_{0,\varkappa}$ is essentially self-#-adjoint on $C^{*\infty}(H_{0,\varkappa})$, the resolvents of $H_{0,\varkappa,V}$ converge strongly **[18, p. 429]**. Thus the generalized semigroup #-convergence theorem **[18, p. 502]** ensures that for all $\psi \in \mathcal{F}^{\#}$

$$\left\|Ext\text{-}\exp\left(-tH_{0,\varkappa,V}\right)\psi - Ext\text{-}\exp\left(-tH_{0,\varkappa}\right)\psi\right\|_{\#} \to_{\#} 0$$

as $V \to {}^*\infty$, and the #-convergence is uniform on #-compact sets of t. Therefore we need only show that for a #-dense set of non-negative ψ and χ $\langle\psi, Ext\text{-}\exp\left(-tH_{0,\varkappa,V}\right)\chi\rangle_{\#} \geq 0$. Let $F(x_1, \dots, x_n)$ be a non- negative, hyper infinitely #-differentiable function with #-compact support, and let

$$\psi = F\left(\varphi^{\#}_{\varkappa}(f_1), \dots, \varphi^{\#}_{\varkappa}(f_n)\right)\Omega_0, \qquad (3.2.8)$$

where $f_1, \dots, f_n$ are real. The set of all such vectors are #-dense in $\mathcal{F}^{\#+}$, the non- negative vectors in $\mathcal{F}^{\#}$. Furthermore, we define

$$\psi_{\varkappa,V} = F\left(\varphi^{\#}_{\varkappa,V}(f_1), \dots, \varphi^{\#}_{\varkappa,V}(f_n)\right)\Omega_0, \qquad (3.2.9)$$

where $\varphi^{\#}_{\varkappa,V}(f_1)$ is defined by restricting the sum in (2.1.6) to those

$$\boldsymbol{k} \in \Gamma^3_{\varkappa,V} = \Gamma^3_V \cap \{\boldsymbol{k} | |\boldsymbol{k}| \leq \varkappa\}.$$

Then $\psi_{\varkappa,V} \in \mathcal{F}^{\#+}_{\varkappa,V} \subset \mathcal{F}^{\#+}$ where $\mathcal{F}^{\#+}_{\varkappa,V}$ is the Fock space corresponding to the modes $\boldsymbol{k} \in \Gamma^3_{\varkappa,V}$. For any vector $\chi \in C^{*\infty}(H_{0,\varkappa})$

$$\left\|\varphi^{\#}_{\varkappa,V}(f)\chi - \varphi^{\#}_{\varkappa}(f)\chi\right\|_{\#} \to_{\#} 0, \text{ as } V \to {}^*\infty,$$

and as $C^{*\infty}(H_{0,\varkappa})$ is a #-core for $\varphi^{\#}_{\varkappa}(f)$, the resolvents of $\varphi^{\#}_{\varkappa,V}(f)$ #-converge strongly to the resolvent of $\varphi^{\#}_{\varkappa}(f)$. **[18, p. 429]**. Thus the generalized semigroup #-convergence theorem **[18, p. 502]** ensures that for each $\chi \in \mathcal{F}^{\#}$, s real

$$\left\|Ext\text{-}\exp\left(is\varphi^{\#}_{\varkappa,V}(f)\right)\psi - Ext\text{-}\exp(is\,\varphi^{\#}_{\varkappa}(f))\psi\right\|_{\#} \to_{\#} 0, \text{ as } V \to {}^*\infty,$$

and the #-convergence is uniform for #-compact sets of s. By (3.2.9)

$$\psi_{\varkappa,V} = Ext\text{-}\int \hat{F}(s_1, \dots, s_n)\left[i \textstyle\sum_{j=1}^{n} Ext\text{-}\exp\left(is\varphi^{\#}_{\varkappa,V}\left(f_j\right)\right)\right] d^{\#}s_1 \cdots d^{\#}s_n,$$

and $\hat{F}(s_1, \dots, s_n)$ vanishes rapidly at hyperinfinity, so we conclude that

$$\left\|\psi_{\varkappa,V} - \psi\right\|_{\#} \to_{\#} 0, \text{ as } V \to {}^*\infty.$$

Thus for such vectors ψ, χ,

$$\langle\psi, Ext\text{-}\exp\left(-tH_{0,\varkappa}\right)\chi\rangle_{\#} = \#\text{-}\lim_{V \to {}^*\infty}\langle\psi_{\varkappa,V}, Ext\text{-}\exp\left(-tH_{0,\varkappa,V}\right)\chi_{\varkappa,V}\rangle_{\#}$$

and we need only show that

$$\langle\psi_{\varkappa,V}, Ext\text{-}\exp\left(-tH_{0,\varkappa,V}\right)\chi_{\varkappa,V}\rangle_{\#} \geq 0. \qquad (3.2.10)$$

However on $\mathcal{F}^{\#}_{\varkappa,V}$

$$H_{0,\varkappa,V} = Ext\text{-}\sum_{\boldsymbol{k}\in\Gamma^3_{\varkappa,V}} a^*_V(\boldsymbol{k})a_V(\boldsymbol{k})\mu(\boldsymbol{k}) = Ext\text{-}\sum_{\boldsymbol{k}\in\Gamma^3_{\varkappa,V}} H_{0,\varkappa,V},$$

so $Ext\text{-}\exp(-tH_{0,\varkappa,V}) = Ext\text{-}\prod_{\boldsymbol{k}\in\Gamma^3_{\varkappa,V}} \exp(-tH_{0,\varkappa,V})$. It easily verify by explicit computation that each operator $\exp(-tH_{0,\varkappa,V})$ have a strictly positive kernel, so (3.2.10) holds and the proof is complete.

Theorem 3.2.4 With $\mathcal{F}^\#$ represented as $L^\#_2(Q, d^\#\mu)$, the operator $Ext\text{-}\exp(-H_\varkappa(g))$ has a positive, ergodic kernel.
Remark 3.2.3 We expect that $Ext\text{-}\exp(-H_{0,\varkappa})$ and $Ext\text{-}\exp(-H_\varkappa(g))$ have strictly positive kernels.
Proof As in Theorem 3.2.3, formula (3.2.7), we consider $H_{\varkappa,V}(g) = H_{0,\varkappa} + H_{I,\varkappa,V}(g)$. The approximate interaction $H_{I,\varkappa,V}(g)$ is constructed with $\varphi^\#_{\varkappa,V}$ in place of $\varphi^\#_\varkappa$. Since $C^{*\infty}(H_{0,\varkappa})$ is a #-core for $H_\varkappa(g)$, we can argue as in the previous theorem that for all $\psi \in \mathcal{F}^\#$

$$Ext\text{-}\exp\left(-tH_{\varkappa,V}(g)\right)\psi \to_\# Ext\text{-}\exp(-tH_\varkappa(g))\psi, \text{ as } V \to {}^*\infty.$$

Thus we need only prove that for ψ, χ as in Theorem 3.2.3

$$0 < \varepsilon < \langle\psi_{\varkappa,V}, Ext\text{-}\exp\left(-tH_{\varkappa,V}(g)\right)\chi_{\varkappa,V}\rangle_\#. \quad (3.2.11)$$

and that for sufficiently large t, the constant $\varepsilon = \varepsilon(\psi, \chi, \varkappa, V)$ can be chosen independently of $\varkappa$ and V. On $\mathcal{F}^\#_{\varkappa,V}$ we have an explicit representation of $Ext\text{-}\exp\left(-tH_{\varkappa,V}(g)\right)$ given by generalized Feynman-Kac integral formula

$$\langle\psi_{\varkappa,V}, Ext\text{-}\exp\left(-tH_{\varkappa,V}(g)\right)\chi_{\varkappa,V}\rangle_\# = \quad (3.2.12)$$

$$Ext\text{-}\int_{C_{\varkappa,V}} Ext\text{-}\exp\left(-\left[Ext\text{-}\int_0^t H_{I,g,\varkappa,V}(q(s))\,d^\#t\right]\right)\psi_{\varkappa,V}(q(0))\,\chi_{\varkappa,V}(q(t))D^\#q(\cdot).$$

Here $q(s)$ denotes a points in the spectrum of the modes

$$q_V(\boldsymbol{k}) = a_V(\boldsymbol{k}) + a_V(\boldsymbol{k}) + a^*_V(\boldsymbol{k}) + a^*_V(-\boldsymbol{k})$$

$$q'_V(\boldsymbol{k}) = a_V(\boldsymbol{k}) - a_V(-\boldsymbol{k}) + a^*_V(\boldsymbol{k}) - a^*_V(-\boldsymbol{k})$$

for $\boldsymbol{k} \in \Gamma^3_{\varkappa,V} = \{\boldsymbol{k} | \boldsymbol{k} \in \Gamma^3_V \wedge |\boldsymbol{k}| \leq \varkappa\}$, and $C_{\varkappa,V}$ is the path space for these modes. Since $Ext\text{-}\exp\left(-tH_{0,\varkappa}(g)\right)$ has a strictly positive kernel, (3.2.12) exhibits $Ext\text{-}\exp\left(-tH_{\varkappa,V}(g)\right)$ explicitly as an operator with a strictly positive kernel. Thus (3.2.11) is valid, and taking the #-limit as $V \to {}^*\infty$ shows that

$$\langle\psi, Ext\text{-}\exp(-tH_\varkappa(g))\chi\rangle_\# \geq 0. \quad (3.2.13)$$

We now establish a uniform lower bound on ε in (3.2.11) to prove that for t sufficiently large (3.2.13) is strictly positive. Given any positive M we can split the integral (3.2.13) into two parts. Let $C^{(1)}_{\varkappa,V}$ be those paths such that the exponent in the Feynman-Kac formula satisfies $-\left[Ext\text{-}\int_0^t H_{I,g,\varkappa,V}(q(s))\,d^\#t\right] \geq -M$, and let $C^{(2)}_{\varkappa,V}$ be the complementary set of paths. Thus

$$\langle\psi_{\varkappa,V}, Ext\text{-}\exp\left(-tH_{\varkappa,V}(g)\right)\chi_{\varkappa,V}\rangle_\# \geq \left(Ext\text{-}\exp(-M)\right)Ext\text{-}\int_{C^{(1)}_{\varkappa,V}} \psi_{\varkappa,V}(q(0))\,\chi_{\varkappa,V}(q(t))D^\#q(\cdot) =$$

$$= \left(Ext\text{-}\exp(-M)\right)\left\{\langle\psi_{\varkappa,V}, Ext\text{-}\exp(-tH_{0,\varkappa,V})\chi_{\varkappa,V}\rangle_\# - Ext\text{-}\int_{C^{(2)}_{\varkappa,V}} \psi_{\varkappa,V}(q(0))\,\chi_{\varkappa,V}(q(t))D^\#q(\cdot)\right\}. \quad (3.2.14)$$

First we choose t by (3.2.5) so that (3.2.6) holds. Then for sufficiently infinitely large V (depending on t),

$$\langle \psi_{\varkappa,V}, Ext\text{-}\exp\left(-tH_{0,\varkappa,V}\right)\chi_{\varkappa,V}\rangle_{\#} \geq \frac{1}{2}\langle \psi\,, Ext\text{-}\exp\left(-tH_{0,\varkappa}\right)\chi\,\rangle_{\#} \geq \frac{1}{4}\|\psi\,\|_{\#1}\cdot\|\chi\,\|_{\#1}.$$

Thus (3.2.14) becomes

Let $\Pr\{\cdot\}$ denote the #-measure on path space, so that by the generalized Holder inequality

where $1 < r < 2$. By the smoothing property of Ext-$\exp\left(-tH_{0,\varkappa,V}\right)$ for sufficiently large t

and for V sufficiently infinitely large, this is dominated by $2\|\psi\,\|_{\#2}\cdot\|\chi\,\|_{\#2}$. Thus with the choices so far made for V, t, M,

provided in addition that

$$\Pr\left\{\mathrm{C}^{(2)}_{\varkappa,V}\right\} \leq \left(\frac{\|\psi\,\|_{\#1}\cdot\|\chi\,\|_{\#1}}{16\|\psi\,\|_{\#2}\cdot\|\chi\,\|_{\#2}}\right)^{\frac{r}{(r-1)}}. \qquad (3.2.15)$$

We now show that for M sufficiently large, (3.2.15) is satisfied and therefore theorem is proved.

§ 4. THE HEISENBERG PICTURE FIELD OPERATORS

§ 4.1 In the Heisenberg picture operators have the time dependence

$$A(t) = Ext\text{-}\exp\left(it\,H_{\varkappa}(g)\right)A(0)Ext\text{-}\exp\left(-i\,tH_{\varkappa}(g)\right) \qquad (4.1.1)$$

This definition of the dynamics contains the cutoff function $g(\boldsymbol{x})$ explicitly. For an important class of operators $A(0)$, however, $A(t)$ is independent of $g(\boldsymbol{x})$ provided that $g(\boldsymbol{x}) = \lambda$, the coupling constant, on a suitably large set. For example, we take $A(0)$ to be an observable representing a measurement performed in some 3-dimensional region $B \subset {}^*\mathbb{R}_c^{\#3}$ of space (at time $t = 0$). Then $A(t)$ represents the same measurement performed at time t. A hamiltonian with a hyperfinite ultraviolet cut-off $\varkappa \in {}^*\mathbb{R}_{c+}^{\#} \backslash {}^*\mathbb{R}_{\text{fin}+}^{\#}$, such as $H_\varkappa(g)$, propagates information with at most the speed of light. Therefore if $g(\boldsymbol{x}) = \lambda$ on a region containing B, and t is sufficiently small, the fact that $g(\boldsymbol{x})$ does not equal λ everywhere will never be recorded by a measurement $A(t)$. For each localized observable $A(0)$ and each t, we make an appropriate choice for $g(\boldsymbol{x})$. Therefore (4.1) provides the correct dynamics for the $(\varphi^4)_4$ quantum field theory with the cut quantum field theory with the cut-off removed. In this section we discuss the field operators $\varphi_\varkappa^\#(\boldsymbol{x}, t)$ or ${}^*\mathbb{R}_c^{\#4}$

$$\varphi_\varkappa^\#(f) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#4}} \varphi_\varkappa^\#(\boldsymbol{x}, t)\, d^{\#3}x d^{\#}t. \tag{4.1.2}$$

We see that integration helps in (4.2) because $\varphi_\varkappa^\#(f)$ is an operator while $\varphi_\varkappa^\#(\boldsymbol{x}, t)$ is a bilinear form. Actually the time integration is not required and for real f,

$$A(t) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \varphi_\varkappa^\#(\boldsymbol{x}, t)\, d^{\#3}x \tag{4.1.3}$$

is also a self-#-adjoint operator depending #-continuously on t. We expect that this is a special feature of the two dimensional model we are considering and that sharp time fields will not be operators in four dimensions. For this reason, basic physical concepts have been formulated in terms of the time averaged fields (4.2) rather than the sharp time fields (4.3). For example, Wightman's axioms for a quantum field theory are expressed in terms of the operators (4.2), and we will show that many of his axioms are satisfied for our model.

§ 4.2 An invariant domain for localized fields. In this section we study the Heisenberg picture field localized in a 4-dimensional region of space time $\mathcal{B}$. We find that $\varphi_\varkappa^\#(\boldsymbol{x}, t)$ is a bilinear form and that for real f, $\varphi_\varkappa^\#(f)$ is a #-densely defined symmetric operator. We start with the region B, a bounded open subset of space time. We require that $H_\varkappa(g)$ be a hamiltonian for $\mathcal{B}$. This means that the spatial cut-off $g(\boldsymbol{x})$ equals the coupling constant λ on a sufficiently large interval to contain the domain of dependence of $\mathcal{B}$. In other words, assuming that the velocity of light is one, for every point $(\boldsymbol{y}, t) \in \mathcal{B}$,

$$g(\boldsymbol{x}) = \lambda, \text{ if } \|\boldsymbol{x} - \boldsymbol{y}\| < t. \tag{4.2.1}$$

It is convenient to deal with the field

$$\varphi_{\varkappa,g}^\#(\boldsymbol{x}, t) = Ext\text{-}\exp\big(it\, H_\varkappa(g)\big)\varphi_\varkappa^\#(\boldsymbol{x}) Ext\text{-}\exp\big(-i\, tH_\varkappa(g)\big)$$

and its time #-derivative

$$\pi_{\varkappa,g}^\#(\boldsymbol{x}, t) = Ext\text{-}\exp\big(it\, H_\varkappa(g)\big)\pi_\varkappa^\#(\boldsymbol{x}) Ext\text{-}\exp\big(-i\, tH_\varkappa(g)\big) = \partial^\# \varphi_{\varkappa,g}^\#(\boldsymbol{x}, t)/\partial^\# t.$$

The time zero fields $\varphi_\varkappa^\#(\boldsymbol{x})$ and its conjugate momentum $\pi_\varkappa^\#(\boldsymbol{x})$ were defined in chapter I. We shall see that for $(\boldsymbol{x}, t) \in \mathcal{B}$, $\varphi_{\varkappa,g}^\#(\boldsymbol{x}, t)$ is independent of g, and equals the field $\varphi_\varkappa^\#(\boldsymbol{x}, t)$. Thus all the cut-offs have been removed in the definition of $\varphi_\varkappa^\#(\boldsymbol{x}, t)$. For each $C^{*\infty}$-function $f(\boldsymbol{x}, t)$ with support in $\mathcal{B}$, we show that

$$\varphi_\varkappa^\#(f) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#4}} \varphi_\varkappa^\#(\boldsymbol{x}, t) f(\boldsymbol{x}, t)\, d^{\#3}x d^{\#}t \tag{4.2.2}$$

is an operator whose domain contains

$$D_{\varkappa,g}^\# = C^{*\infty}\big(H_\varkappa(g)\big) = \bigcap_{n=0}^{*\infty} D\big(H_\varkappa^n(g)\big), \tag{4.2.3}$$

In fact $D_{\varkappa,g}^\#$ is an invariant domain, i.e.

$$\varphi_{\varkappa}^{\#}(f)D_{\varkappa,g}^{\#} \subset D_{\varkappa,g}^{\#}, \tag{4.2.4}$$

so that $D_{\varkappa,g}^{\#} \subset C^{{}^*\infty}(\varphi_{\varkappa}^{\#}(f))$. We note that this invariant domain may depend on the region $\mathcal{B}$ in which the field $\varphi_{\varkappa}^{\#}(f)$ is localized. For $\psi \in D_{\varkappa,g}^{\#}$ the expectation values

$$\langle \psi, \varphi_{\varkappa}^{\#}(\boldsymbol{x}_1, t_1) \cdots \varphi_{\varkappa}^{\#}(\boldsymbol{x}_n, t_n)\psi\rangle_{\#} \tag{4.2.5}$$

is ${}^*\mathbb{C}_c^{\#4}$- valued Schwartz distribution in $\mathcal{D}^{\#\prime}(\mathcal{B} \times \cdots \times \mathcal{B})$. If $f(\boldsymbol{x}, t)$ is a function in $S^{\#}({}^*\mathbb{R}_c^{\#4})$, then $\varphi_{\varkappa,g}^{\#}(f)$ still is defined on $D_{\varkappa,g}^{\#}$and leaves it invariant. The expectation values (4.2.5) of $\varphi_{\varkappa,g}^{\#}(\boldsymbol{x}, t)$ are tempered distributions in $S^{\#\prime}({}^*\mathbb{R}_c^{\#4})$. However, the fields $\varphi_{\varkappa,g}^{\#}(f)$ may depend on g.

Lemma 3.2.1 The field $\varphi_{\varkappa,g}^{\#}(\boldsymbol{x}, t)$ is a bilinear form on $D((H_{\varkappa}(g) + b)^{1/2}) \times D((H_{\varkappa}(g) + b)^{1/2})$ #-continuous in $\boldsymbol{x}$ and t. Namely for $\psi \in D\left((H_{\varkappa}(g) + b)^{\frac{1}{2}}\right)$, $\langle \psi, \varphi_{\varkappa}^{\#}(\boldsymbol{x}, t)\psi\rangle_{\#}$ is a #-continuous function. Furthermore

$$\left| Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \langle \psi, \varphi_{\varkappa}^{\#}, g(\boldsymbol{x}, t)\psi\rangle_{\#} \frac{\partial^{\#}}{\partial^{\#} x_i} f(\boldsymbol{x}) \right| \leq \text{const} \cdot \|f\|_{\#2} \langle \psi, (H_{\varkappa}(g) + b)\psi\rangle_{\#}, i = 1,2,3. \tag{4.2.6}$$

Proof The free field $\varphi_{\varkappa}^{\#}(\boldsymbol{x}, 0)$ is the sum of two expressions of the form (1.8). The kernels $\theta(\boldsymbol{k}, \varkappa)b(\boldsymbol{k})$ are in $L_2^{\#}$. Furthermore we have $\theta(\boldsymbol{k}, \varkappa)b(\boldsymbol{k})[\mu(\boldsymbol{k})]^{-1/2} \in L_2^{\#}$. The estimate (1.2. 9) has been generalized to cover such kernels, giving us

$$\left\| (H_{0,\varkappa} + I)^{-1/2} \varphi_{\varkappa,g}^{\#}(\boldsymbol{x}, 0)(H_{0,\varkappa} + I)^{-1/2} \right\|_{\#} \leq \text{const} \cdot \left\| \theta(\boldsymbol{k}, \varkappa)b(\boldsymbol{k})[\mu(\boldsymbol{k})]^{-1/2} \right\|_{\#} < {}^*\infty\,. \tag{4.2.7}$$

Thus for $\psi \in D\left((H_{\varkappa}(g) + b)^{\frac{1}{2}}\right)$, $Ext\text{-}\exp(-it\,H_{\varkappa}(g))\psi \in D\left((H_{\varkappa}(g) + b)^{\frac{1}{2}}\right) \subset D(H_{0,\varkappa}^{1/2})$, by (2.1.19) and therefore $\langle \psi, \varphi_{\varkappa,g}^{\#}(\boldsymbol{x}, t)\psi\rangle_{\#} = \langle Ext\text{-}\exp(-it\,H_{\varkappa}(g))\psi, \varphi_{\varkappa,g}^{\#}(\boldsymbol{x}, 0)Ext\text{-}\exp(-it\,H_{\varkappa}(g))\psi\rangle_{\#}$ is defined and

$$\left|\langle \psi, \varphi_{\varkappa,g}^{\#}(\boldsymbol{x}, t)\psi\rangle_{\#}\right| \leq \text{const} \cdot \left\|\theta(\boldsymbol{k}, \varkappa)b(\boldsymbol{k})[\mu(\boldsymbol{k})]^{-1/2}\right\|_{\#} \cdot \langle \psi, (H_{\varkappa}(g) + b)\psi\rangle_{\#}$$

Since $\left\|\theta(\boldsymbol{k}, \varkappa)b(\boldsymbol{k})[\mu(\boldsymbol{k})]^{-1/2}|\boldsymbol{k}|^2 \hat{f}(\boldsymbol{k})\right\|_{\#} \leq \left\|\theta(\boldsymbol{k}, \varkappa)b(\boldsymbol{k})[\mu(\boldsymbol{k})]^{-1/2}|\boldsymbol{k}|^2\right\|_{{}^*\infty} \cdot \|f\|_{\#2} \leq \text{const} \cdot \|f\|_{\#2}$ the inequality (4.2.6) holds. Let us write $b_{x_i}, i = 1,2,3$ for b to denote the dependence of b on x_i. Then $\left\|(b_{x_i} - b_{y_i})[\mu(\boldsymbol{k})]^{-1/2}\right\|_{\#2}$ is a function of $(x_i - y_i)$ only and it #-tends to zero as $|x - y| \to_{\#} 0$. Since

§ 5. THE ALGEBRA OF LOCAL OBSERVABLES

CONCLUSION

A new non-Archimedean approach to interacted quantum fields is presented. In proposed approach, a field operator $\varphi(x,t)$ no longer a standard tempered operator-valued distribution, but a non-classical operator-valued function. We prove using this novel approach that the quantum field theory with Hamiltonian $P(\varphi)_4$ exists and that the canonical C^*- algebra of bounded observables corresponding to this model satisfies all the Haag-Kastler axioms except Lorentz covariance. We prove that the $\lambda(\varphi^4)_4$ quantum field theory model is Lorentz covariant. For each Poincare transformation $a\,,\Lambda$ and each bounded region O of Minkowski space we obtain a unitary operator U which correctly transforms the field bilinear forms $\varphi(x,t)$ for $(x,t) \in O$. The von Neumann algebra $\mathfrak{C}(O)$ of local observables is obtained as standard part of external nonstandard algebra $\mathcal{B}_{\#}(O)$.

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