J-Integral

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Abstract

The J-integral is a path independent integral, used for characterizing the severity of the loading at a crack tip. In this paper an alternative derivation of J-integral and its path independence for plane stress using vector calculus has been presented.

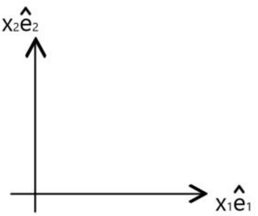
Keywords : J-Integral, Path independence.

1 DERIVATION

Consider a vector field

$\mathbf{F} = F_1\hat{\mathbf{e}}_1 + F_2\hat{\mathbf{e}}_2$

with components



$$F_1 = w - \left(\sigma_{11} \frac{\partial u_1}{\partial x_1} + \sigma_{12} \frac{\partial u_2}{\partial x_1} \right)$$

$$F_2 = -\left(\sigma_{21} \frac{\partial u_1}{\partial x_1} + \sigma_{22} \frac{\partial u_2}{\partial x_1} \right)$$

where

w = strain energy density

σ_{ij} = stress [i, j = 1, 2]

u_j = displacement

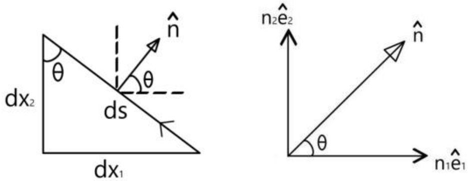
now let $\hat{\mathbf{n}}$ be the unit vector normal to a contour of integration then

$n_1 = \cos\theta$

$n_2 = \cos(90 - \theta) = \sin\theta$

$dx_1 = -ds \times \sin\theta = -ds \times n_2$

$dx_2 = ds \times \cos\theta = ds \times n_1$



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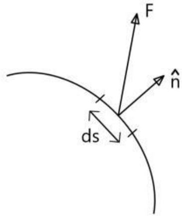
let



$$J = \int \mathbf{F}.\,\hat{\mathbf{n}}ds = \int \left(F_1\hat{\mathbf{e}}_1 + F_2\hat{\mathbf{e}}_2\right).\left(n_1\hat{\mathbf{e}}_1 + n_2\hat{\mathbf{e}}_2\right)ds$$

$$= \int \left(F_1 n_1 + F_2 n_2\right)ds$$

$$= \int \left(F_1 n_1 ds + F_2 n_2 ds\right)$$

$$= \int F_1 n_1 ds + \int F_2 n_2 ds$$

$$= \int \left[w - \left(\sigma_{11}\frac{\partial u_1}{\partial x_1} + \sigma_{12}\frac{\partial u_2}{\partial x_1} \right) \right] n_1 ds - \int \left(\sigma_{21}\frac{\partial u_1}{\partial x_1} + \sigma_{22}\frac{\partial u_2}{\partial x_1} \right) n_2 ds$$

$$= \int w n_1 ds - \int \left(\sigma_{11}\frac{\partial u_1}{\partial x_1} + \sigma_{12}\frac{\partial u_2}{\partial x_1} \right) n_1 ds - \int \left(\sigma_{21}\frac{\partial u_1}{\partial x_1} + \sigma_{22}\frac{\partial u_2}{\partial x_1} \right) n_2 ds$$

$$= \int w dx_2 - \int \left[\left(\sigma_{11} n_1 \frac{\partial u_1}{\partial x_1} + \sigma_{21} n_2 \frac{\partial u_1}{\partial x_1} \right) + \left(\sigma_{12} n_1 \frac{\partial u_2}{\partial x_1} + \sigma_{22} n_2 \frac{\partial u_2}{\partial x_1} \right) \right] ds$$

$$= \int w dx_2 - \int \left[T_1 \frac{\partial u_1}{\partial x_1} + T_2 \frac{\partial u_2}{\partial x_1} \right] ds$$

where

$$T_1 = \sigma_{11} n_1 + \sigma_{21} n_2 = \sigma_{11} n_1 + \sigma_{12} n_2 \qquad [\sigma_{12} = \sigma_{21}]$$

$$T_2 = \sigma_{12} n_1 + \sigma_{22} n_2 = \sigma_{21} n_1 + \sigma_{22} n_2$$

$$\Rightarrow J = \int \left[w dx_2 - T_i \frac{\partial u_i}{\partial x_1} ds \right] \qquad [\, i = 1, 2 \,]$$

2 PATH INDEPENDENCE

By normal form of Green's theorem

$$\oint \mathbf{F}.\,\hat{\mathbf{n}}ds = \iint \left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} \right) dx_1 dx_2 \qquad (\,i\,)$$

now

$$\frac{\partial F_1}{\partial x_1} = \frac{\partial w}{\partial x_1} - \left(\frac{\partial \sigma_{11}}{\partial x_1} \times \frac{\partial u_1}{\partial x_1} + \sigma_{11} \times \frac{\partial^2 u_1}{\partial x_1^2} \right) - \left(\frac{\partial \sigma_{12}}{\partial x_1} \times \frac{\partial u_2}{\partial x_1} + \sigma_{12} \times \frac{\partial^2 u_2}{\partial x_1^2} \right)$$

$$\frac{\partial F_2}{\partial x_2} = -\left(\frac{\partial \sigma_{21}}{\partial x_2} \times \frac{\partial u_1}{\partial x_1} + \sigma_{21} \times \frac{\partial^2 u_1}{\partial x_1 \partial x_2} \right) - \left(\frac{\partial \sigma_{22}}{\partial x_2} \times \frac{\partial u_2}{\partial x_1} + \sigma_{22} \times \frac{\partial^2 u_2}{\partial x_1 \partial x_2} \right)$$

so

$$\frac{\partial F_1}{\partial x_1}+\frac{\partial F_2}{\partial x_2}=\frac{\partial w}{\partial x_1}-\left(\frac{\partial \sigma_{11}}{\partial x_1}+\frac{\partial \sigma_{21}}{\partial x_2}\right)\times\frac{\partial u_1}{\partial x_1}$$

$$-\left(\frac{\partial \sigma_{12}}{\partial x_1}+\frac{\partial \sigma_{22}}{\partial x_2}\right)\times\frac{\partial u_2}{\partial x_1}$$

$$-\left(\sigma_{11}\times\frac{\partial^2 u_1}{\partial x_1^2}+\sigma_{12}\times\frac{\partial^2 u_2}{\partial x_1^2}+\sigma_{21}\times\frac{\partial^2 u_1}{\partial x_1\partial x_2}+\sigma_{22}\times\frac{\partial^2 u_2}{\partial x_1\partial x_2}\right)$$

now second and third terms become zero by equilibrium equations and also $\sigma_{12}=\sigma_{21}$,
so

$$\frac{\partial F_1}{\partial x_1}+\frac{\partial F_2}{\partial x_2}=\frac{\partial w}{\partial x_1}-\left(\sigma_{11}\times\frac{\partial^2 u_1}{\partial x_1^2}+\sigma_{12}\times\frac{\partial^2 u_2}{\partial x_1^2}+\sigma_{21}\times\frac{\partial^2 u_1}{\partial x_1\partial x_2}+\sigma_{22}\times\frac{\partial^2 u_2}{\partial x_1\partial x_2}\right) \qquad (\text{ ii })$$

now

$$\varepsilon_{11}=\frac{\partial u_1}{\partial x_1}\Rightarrow\frac{\partial \varepsilon_{11}}{\partial x_1}=\frac{\partial^2 u_1}{\partial x_1^2}$$

$$\varepsilon_{22}=\frac{\partial u_2}{\partial x_2}\Rightarrow\frac{\partial \varepsilon_{22}}{\partial x_1}=\frac{\partial^2 u_2}{\partial x_1\partial x_2}$$

$$\varepsilon_{12}=\frac{1}{2}\left(\frac{\partial u_2}{\partial x_1}+\frac{\partial u_1}{\partial x_2}\right)\Rightarrow\frac{\partial \varepsilon_{12}}{\partial x_1}=\frac{1}{2}\left(\frac{\partial^2 u_2}{\partial x_1^2}+\frac{\partial^2 u_1}{\partial x_1\partial x_2}\right)$$

substituting in (ii)

$$\frac{\partial F_1}{\partial x_1}+\frac{\partial F_2}{\partial x_2}=\frac{\partial w}{\partial x_1}-\left[\sigma_{11}\times\frac{\partial \varepsilon_{11}}{\partial x_1}+2\sigma_{12}\times\frac{\partial \varepsilon_{12}}{\partial x_1}+\sigma_{22}\times\frac{\partial \varepsilon_{22}}{\partial x_1}\right] \qquad (\text{ iii })$$

now for linear or non-linear elastic loading or monotonic plastic loading

$$w=\int_0^{\varepsilon_{11}}\sigma_{11}d\varepsilon_{11}+\int_0^{\varepsilon_{12}}\sigma_{12}d\varepsilon_{12}+\int_0^{\varepsilon_{21}}\sigma_{21}d\varepsilon_{21}+\int_0^{\varepsilon_{22}}\sigma_{22}d\varepsilon_{22}$$

$$=\int_0^{\varepsilon_{11}}\sigma_{11}d\varepsilon_{11}+2\int_0^{\varepsilon_{12}}\sigma_{12}d\varepsilon_{12}+\int_0^{\varepsilon_{22}}\sigma_{22}d\varepsilon_{22} \qquad (\because \sigma_{12}=\sigma_{21}\quad and\quad \varepsilon_{12}=\varepsilon_{21})$$

now by Leibniz's rule for differentiation of an integral

$$\frac{\partial w}{\partial x_1}=\sigma_{11}\times\frac{\partial \varepsilon_{11}}{\partial x_1}+2\sigma_{12}\times\frac{\partial \varepsilon_{12}}{\partial x_1}+\sigma_{22}\times\frac{\partial \varepsilon_{22}}{\partial x_1}$$

substituting in (iii)

$$\Rightarrow\frac{\partial F_1}{\partial x_1}+\frac{\partial F_2}{\partial x_2}=0 \qquad (\text{ iv })$$

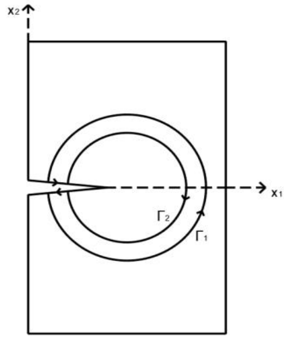
substituting (iv) in (i)

$$\Rightarrow \oint \mathbf{F}.\hat{\mathbf{n}}ds = 0$$

$$\Rightarrow \int_{\Gamma_1} \mathbf{F}.\hat{\mathbf{n}}ds + \int_{\substack{crack\\surface\,1}} \mathbf{F}.\hat{\mathbf{n}}ds + \int_{\Gamma_2} \mathbf{F}.\hat{\mathbf{n}}ds + \int_{\substack{crack\\surface\,2}} \mathbf{F}.\hat{\mathbf{n}}ds = 0 \qquad (\text{ v })$$

if the crack surfaces are free of traction (i.e., $\sigma_{12} = \sigma_{21} = \sigma_{22} = 0$ at the crack surfaces) and are approximately parallel to x_1 (i.e., $\Delta x_2 \approx 0$), then

$$\int_{\substack{crack\\surface\,1}} \mathbf{F}.\hat{\mathbf{n}}ds = \int_{\substack{crack\\surface\,2}} \mathbf{F}.\hat{\mathbf{n}}ds = 0$$



substituting in (v)

$$\Rightarrow \int_{\Gamma_1} \mathbf{F}.\hat{\mathbf{n}}ds + \int_{\Gamma_2} \mathbf{F}.\hat{\mathbf{n}}ds = 0$$

$$\Rightarrow \int_{\Gamma_1} \mathbf{F}.\hat{\mathbf{n}}ds = -\int_{\Gamma_2} \mathbf{F}.\hat{\mathbf{n}}ds$$

$$\Rightarrow \int_{\Gamma_1} \mathbf{F}.\hat{\mathbf{n}}ds = \int_{-\Gamma_2} \mathbf{F}.\hat{\mathbf{n}}ds = \text{constant} = J$$

3 CONCLUSION

Thus J is independent of path, provided that :

- ➢ Contour of integration starts from one crack surface and ends on the other crack surface enclosing the crack tip.
- ➢ Crack surfaces are free of traction.
- ➢ Crack surfaces are approximately parallel to x_1.

References

1. Prashant Kumar, *“Elements of Fracture Mechanics.”*