

Proofs of Four Conjectures in Number Theory : Beal's Conjecture, Riemann Hypothesis, The $c < R^{1.63}$ and abc Conjectures - v4.- July 2024 -

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Abstract

This monograph presents the last version (4.) of the proofs of 4 important conjectures in the field of the number theory, namely:

- Beal's conjecture.
- The Riemann Hypothesis.
- The $c < R^{1.63}$ conjecture is true.
- The abc conjecture is true.

We give the details of the different proofs.

Résumé

Cette monographie présente la dernière version (4.) des preuves des 4 conjectures importantes dans le domaine de la théorie des nombres à savoir:

- La conjecture de Beal.
- L'Hypothèse de Riemann.
- La conjecture $c < R^{1.63}$ est vraie.
- La conjecture abc est vraie.

Nous donnons les détails des différentes démonstrations.

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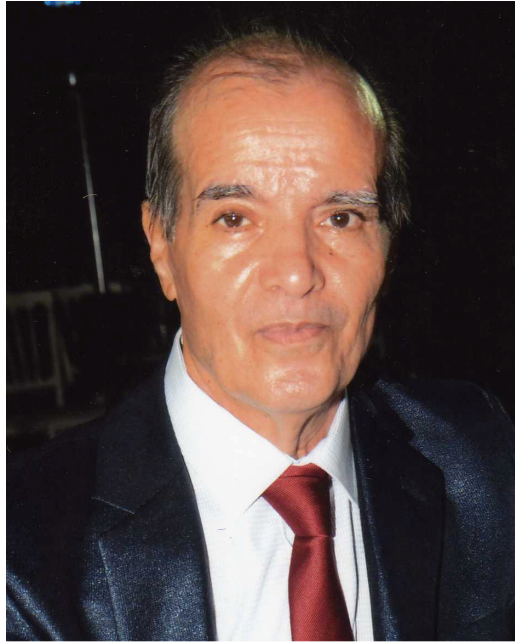


Figure 1: Photo of the Author

Dedication

**To the memory of my Parents, to my wife
Wahida, my daughter Sinda and my son
Mohamed Mazen.**

To my Teachers, to my Friends.

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Preface

This booklet is the fruit of ten years on working on four important conjectures in number theory:

- Beal's conjecture.
- The Riemann Hypothesis.
- The $c < R^{1.63}$ conjecture is true.
- The abc conjecture is true.

It is an update of the last edition (December 2022) with many modifications added. I had used elementary mathematics that can be understood by graduate and undergraduate students. All the conjectures are under review by mathematical journals.

Tunis,
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Chapter 1

A Complete Proof of Beal's Conjecture

Abstract

In 1997, Andrew Beal announced the following conjecture: *Let A, B, C, m, n , and l be positive integers with $m, n, l > 2$. If $A^m + B^n = C^l$ then A, B , and C have a common factor.* We begin to construct the polynomial $P(x) = (x - A^m)(x - B^n)(x + C^l) = x^3 - px + q$ with p, q integers depending of A^m, B^n and C^l . We resolve $x^3 - px + q = 0$ and we obtain the three roots x_1, x_2, x_3 as functions of p, q and a parameter θ . Since $A^m, B^n, -C^l$ are the only roots of $x^3 - px + q = 0$, we discuss the conditions that x_1, x_2, x_3 are integers and have or not a common factor. Three numerical examples are given.

Résumé

En 1997, Andrew Beal avait annoncé la conjecture suivante: *Soient A, B, C, m, n , et l des entiers positifs avec $m, n, l > 2$. Si $A^m + B^n = C^l$ alors A, B , et C ont un facteur commun.* Je commence par construire le polynôme $P(x) = (x - A^m)(x - B^n)(x + C^l) = x^3 - px + q$ avec p, q des entiers qui dépendent de A^m, B^n et C^l . Nous résolvons $x^3 - px + q = 0$ et nous obtenons les trois racines x_1, x_2, x_3 comme fonctions de p, q et d'un paramètre θ . Comme $A^m, B^n, -C^l$ sont les seules racines de $x^3 - px + q = 0$, nous discutons les conditions pour que x_1, x_2, x_3 soient des entiers. Trois exemples numériques sont présentés.

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1.1 Introduction

In 1997, Andrew Beal [1] announced the following conjecture :

Conjecture 1.1.1. *Let $A, B, C, m, n,$ and l be positive integers with $m, n, l > 2$. If:*

$$A^m + B^n = C^l \quad (1.1.1)$$

then $A, B,$ and C have a common factor.

The purpose of this paper is to give a complete proof of Beal's conjecture. Our idea is to construct a polynomial $P(x)$ of order three having as roots A^m, B^n and $-C^l$ with the condition (1.1.1). We obtain $P(x) = x^3 - px + q$ where p, q are depending of A^m, B^n and C^l . Then we express $A^m, B^n, -C^l$ the roots of $P(x) = 0$ in function of p and a parameter θ that depends of the A, B, C . The calculations give that $A^{2m} = \frac{4p}{3} \cos^2 \frac{\theta}{3}$. As A^{2m} is an integer, it follows that $\cos^2 \frac{\theta}{3}$ must be written as $\frac{a}{b}$ where a, b are two positive coprime integers. Beside the trivial cases, there are two main hypothesis to study:

- the first hypothesis is: $3 \mid a$ and $b \mid 4p$,
- the second hypothesis is: $3 \mid p$ and $b \mid 4p$.

We discuss the conditions of divisibility of p, a, b so that the expression of A^{2^m} is an integer. Depending of each individual case, we obtain that A, B, C have or do have not a common factor. Our proof of the conjecture contains many cases to study. there are many cases where we use elementary number theory and some cases need more research to obtain finally the solution. I think that my new idea detailed above overcomes the apparent limitations of the methods I am using.

The paper is organized as follows. In section 1, it is an introduction of the paper. The trivial case, where $A^m = B^n$, is studied in section 2. The preliminaries needed for the proof are given in section 3 where we consider the polynomial $P(x) = (x - A^m)(x - B^n)(x + C^l) = x^3 - px + q$. The section 4 is the preamble of the proof of the main theorem. Section 5 treats the cases of the first hypothesis $3 \mid a$ and $b \mid 4p$. We study the cases of the second hypothesis $3 \mid p$ and $b \mid 4p$ in section 6. Finally, we present three numerical examples and the conclusion in section 7.

In 1997, Andrew Beal [1] announced the following conjecture :

Conjecture 1.1.2. *Let A, B, C, m, n , and l be positive integers with $m, n, l > 2$. If:*

$$A^m + B^n = C^l \tag{1.1.2}$$

then A, B , and C have a common factor.

1.2 Trivial Case

We consider the trivial case when $A^m = B^n$. The equation (1.1.2) becomes:

$$2A^m = C^l \tag{1.2.1}$$

then $2 \mid C^l \implies 2 \mid C \implies C = 2^q \cdot C_1$ with $q \geq 1$, $2 \nmid C_1$ and $2A^m = 2^{ql} C_1^l \implies A^m = 2^{q(l-1)} C_1^l$. As $l > 2$, $q \geq 1$, then $2 \mid A^m \implies 2 \mid A \implies A = 2^r A_1$ with $r \geq 1$ and $2 \nmid A_1$. The equation (1.2.1), becomes:

$$2 \times 2^{rm} A_1^m = 2^{ql} C_1^l \tag{1.2.2}$$

As $2 \nmid A_1$ and $2 \nmid C_1$, we obtain the first condition :

$$\text{There exists two positive integers } r, q \text{ with } r, q \geq 1 \text{ so that } \boxed{ql = mr + 1} \tag{1.2.3}$$

Then from (1.2.2):

$$A_1^m = C_1^l \tag{1.2.4}$$

1.2.1 Case 1 $A_1 = 1 \implies C_1 = 1$

Using the condition (1.2.3) above, we obtain $2 \cdot (2^r)^m = (2^q)^l$ and the Beal conjecture is verified.

1.2.2 Case 2 $A_1 > 1 \implies C_1 > 1$

From the fundamental theorem of the arithmetic, we can write:

$$A_1 = a_1^{\alpha_1} \dots a_I^{\alpha_I}, \quad a_1 < a_2 < \dots < a_I \implies A_1^m = a_1^{m\alpha_1} \dots a_I^{m\alpha_I} \quad (1.2.5)$$

$$C_1 = c_1^{\beta_1} \dots c_J^{\beta_J}, \quad c_1 < c_2 < \dots < c_J \implies C_1^l = c_1^{l\beta_1} \dots c_J^{l\beta_J} \quad (1.2.6)$$

where a_i (respectively c_j) are distinct positive prime numbers and α_i (respectively β_j) are integers > 0 .

From (1.2.4) and using the uniqueness of the factorization of A_1^m and C_1^l , we obtain necessary:

$$\begin{cases} I = J \\ a_i = c_i, \quad i = 1, 2, \dots, I \\ m\alpha_i = l\beta_i \end{cases} \quad (1.2.7)$$

As one $a_i \mid A^m \implies a_i \mid B^m \implies a_i \mid B$ and $a_i = c_i \implies a_i \mid C^l \implies a_i \mid C$, in this case, the Beal conjecture is verified.

We suppose in the following that $A^m > B^n$.

1.3 Preliminaries

Let $m, n, l \in \mathbb{N}^* > 2$ and $A, B, C \in \mathbb{N}^*$ such:

$$A^m + B^n = C^l \quad (1.3.1)$$

We call:

$$\begin{aligned} P(x) &= (x - A^m)(x - B^n)(x + C^l) = x^3 - x^2(A^m + B^n - C^l) \\ &\quad + x[A^m B^n - C^l(A^m + B^n)] + C^l A^m B^n \end{aligned} \quad (1.3.2)$$

Using the equation (1.3.1), $P(x)$ can be written as:

$$\boxed{P(x) = x^3 + x[A^m B^n - (A^m + B^n)^2] + A^m B^n (A^m + B^n)} \quad (1.3.3)$$

We introduce the notations:

$$\begin{aligned} p &= (A^m + B^n)^2 - A^m B^n = A^{2m} + A^m B^n + B^{2n} \\ q &= A^m B^n (A^m + B^n) \end{aligned}$$

As $A^m \neq B^n$, we have $p > (A^m - B^n)^2 > 0$. Equation (1.3.3) becomes:

$$P(x) = x^3 - px + q$$

Using the equation (1.3.2), $P(x) = 0$ has three different real roots : A^m, B^n and $-C^l$.

Now, let us resolve the equation:

$$P(x) = x^3 - px + q = 0 \quad (1.3.4)$$

To resolve (1.3.4) let:

$$x = u + v$$

Then $P(x) = 0$ gives:

$$P(x) = P(u + v) = (u + v)^3 - p(u + v) + q = 0 \implies u^3 + v^3 + (u + v)(3uv - p) + q = 0 \quad (1.3.5)$$

To determine u and v , we obtain the conditions:

$$\begin{aligned} u^3 + v^3 &= -q \\ uv &= p/3 > 0 \end{aligned}$$

Then u^3 and v^3 are solutions of the second order equation:

$$X^2 + qX + p^3/27 = 0 \quad (1.3.6)$$

Its discriminant Δ is written as :

$$\Delta = q^2 - 4p^3/27 = \frac{27q^2 - 4p^3}{27} = \frac{\bar{\Delta}}{27}$$

Let:

$$\begin{aligned} \bar{\Delta} &= 27q^2 - 4p^3 = 27(A^m B^n (A^m + B^n))^2 - 4[(A^m + B^n)^2 - A^m B^n]^3 \\ &= 27A^{2m} B^{2n} (A^m + B^n)^2 - 4[(A^m + B^n)^2 - A^m B^n]^3 \end{aligned} \quad (1.3.7)$$

Denoting :

$$\begin{aligned} \alpha &= A^m B^n > 0 \\ \beta &= (A^m + B^n)^2 \end{aligned}$$

we can write (1.3.7) as:

$$\bar{\Delta} = 27\alpha^2\beta - 4(\beta - \alpha)^3 \quad (1.3.8)$$

As $\alpha \neq 0$, we can also rewrite (1.3.8) as :

$$\bar{\Delta} = \alpha^3 \left(27\frac{\beta}{\alpha} - 4\left(\frac{\beta}{\alpha} - 1\right)^3 \right)$$

We call t the parameter :

$$t = \frac{\beta}{\alpha}$$

$\bar{\Delta}$ becomes :

$$\bar{\Delta} = \alpha^3(27t - 4(t - 1)^3)$$

Let us calling :

$$y = y(t) = 27t - 4(t - 1)^3$$

Since $\alpha > 0$, the sign of $\bar{\Delta}$ is also the sign of $y(t)$. Let us study the sign of y . We obtain $y'(t)$:

$$y'(t) = y' = 3(1 + 2t)(5 - 2t)$$

$y' = 0 \implies t_1 = -1/2$ and $t_2 = 5/2$, then the table of variations of y is given below:

The table of the variations of the function y shows that $y < 0$ for $t > 4$. In our case, we are interested for $t > 0$. For $t = 4$ we obtain $y(4) = 0$ and for $t \in]0, 4[\implies y > 0$. As we have $t = \frac{\beta}{\alpha} > 4$ as $A^m \neq B^n$:

$$(A^m - B^n)^2 > 0 \implies \beta = (A^m + B^n)^2 > 4\alpha = 4A^m B^n$$

Then $y < 0 \implies \bar{\Delta} < 0 \implies \Delta < 0$. Then, the equation (1.3.6) does not have real solutions u^3 and v^3 . Let us find the solutions u and v with $x = u + v$ is a positive or a negative real and $u.v = p/3$.

t	$-\infty$		-1/2		5/2		4		$+\infty$
1+2t	-		0		+				+
5-2t	+				+		0		-
y'(t)	-		0		+		0		-
y(t)	$+\infty$						54		
			0						0
									$-\infty$

Figure 1.1: The table of variations

1.3.1 Expressions of the roots

Proof. The solutions of (1.3.6) are:

$$X_1 = \frac{-q + i\sqrt{-\Delta}}{2}$$

$$X_2 = \overline{X_1} = \frac{-q - i\sqrt{-\Delta}}{2}$$

We may resolve:

$$u^3 = \frac{-q + i\sqrt{-\Delta}}{2}$$

$$v^3 = \frac{-q - i\sqrt{-\Delta}}{2}$$

Writing X_1 in the form:

$$X_1 = \rho e^{i\theta}$$

with:

$$\rho = \frac{\sqrt{q^2 - \Delta}}{2} = \frac{p\sqrt{p}}{3\sqrt{3}}$$

and $\sin\theta = \frac{\sqrt{-\Delta}}{2\rho} > 0$

$$\cos\theta = -\frac{q}{2\rho} < 0$$

Then $\theta [2\pi] \in] + \frac{\pi}{2}, +\pi[$, let:

$$\boxed{\frac{\pi}{2} < \theta < +\pi \Rightarrow \frac{\pi}{6} < \frac{\theta}{3} < \frac{\pi}{3} \Rightarrow \frac{1}{2} < \cos \frac{\theta}{3} < \frac{\sqrt{3}}{2}} \tag{1.3.9}$$

and:

$$\boxed{\frac{1}{4} < \cos^2 \frac{\theta}{3} < \frac{3}{4}} \tag{1.3.10}$$

hence the expression of X_2 :

$$X_2 = \rho e^{-i\theta} \quad (1.3.11)$$

Let:

$$u = re^{i\psi} \quad (1.3.12)$$

$$\text{and } j = \frac{-1 + i\sqrt{3}}{2} = e^{i\frac{2\pi}{3}} \quad (1.3.13)$$

$$j^2 = e^{i\frac{4\pi}{3}} = -\frac{1 + i\sqrt{3}}{2} = \bar{j} \quad (1.3.14)$$

j is a complex cubic root of the unity $\iff j^3 = 1$. Then, the solutions u and v are:

$$u_1 = re^{i\psi_1} = \sqrt[3]{\rho} e^{i\frac{\theta}{3}} \quad (1.3.15)$$

$$u_2 = re^{i\psi_2} = \sqrt[3]{\rho} j e^{i\frac{\theta}{3}} = \sqrt[3]{\rho} e^{i\frac{\theta+2\pi}{3}} \quad (1.3.16)$$

$$u_3 = re^{i\psi_3} = \sqrt[3]{\rho} j^2 e^{i\frac{\theta}{3}} = \sqrt[3]{\rho} e^{i\frac{4\pi}{3}} e^{i\frac{\theta}{3}} = \sqrt[3]{\rho} e^{i\frac{\theta+4\pi}{3}} \quad (1.3.17)$$

and similarly:

$$v_1 = re^{-i\psi_1} = \sqrt[3]{\rho} e^{-i\frac{\theta}{3}} \quad (1.3.18)$$

$$v_2 = re^{-i\psi_2} = \sqrt[3]{\rho} j^2 e^{-i\frac{\theta}{3}} = \sqrt[3]{\rho} e^{i\frac{4\pi}{3}} e^{-i\frac{\theta}{3}} = \sqrt[3]{\rho} e^{i\frac{4\pi-\theta}{3}} \quad (1.3.19)$$

$$v_3 = re^{-i\psi_3} = \sqrt[3]{\rho} j e^{-i\frac{\theta}{3}} = \sqrt[3]{\rho} e^{i\frac{2\pi-\theta}{3}} \quad (1.3.20)$$

We may now choose u_k and v_h so that $u_k + v_h$ will be real. In this case, we have necessary :

$$v_1 = \bar{u}_1 \quad (1.3.21)$$

$$v_2 = \bar{u}_2 \quad (1.3.22)$$

$$v_3 = \bar{u}_3 \quad (1.3.23)$$

We obtain as real solutions of the equation (1.3.5):

$$x_1 = u_1 + v_1 = 2\sqrt[3]{\rho} \cos \frac{\theta}{3} > 0 \quad (1.3.24)$$

$$x_2 = u_2 + v_2 = 2\sqrt[3]{\rho} \cos \frac{\theta+2\pi}{3} = -\sqrt[3]{\rho} \left(\cos \frac{\theta}{3} + \sqrt{3} \sin \frac{\theta}{3} \right) < 0 \quad (1.3.25)$$

$$x_3 = u_3 + v_3 = 2\sqrt[3]{\rho} \cos \frac{\theta+4\pi}{3} = \sqrt[3]{\rho} \left(-\cos \frac{\theta}{3} + \sqrt{3} \sin \frac{\theta}{3} \right) > 0 \quad (1.3.26)$$

We compare the expressions of x_1 and x_3 , we obtain:

$$\begin{aligned} 2\sqrt[3]{\rho} \cos \frac{\theta}{3} &\stackrel{?}{>} \sqrt[3]{\rho} \left(-\cos \frac{\theta}{3} + \sqrt{3} \sin \frac{\theta}{3} \right) \\ 3\cos \frac{\theta}{3} &\stackrel{?}{>} \sqrt{3} \sin \frac{\theta}{3} \end{aligned} \quad (1.3.27)$$

As $\frac{\theta}{3} \in] + \frac{\pi}{6}, + \frac{\pi}{3}[$, then $\sin \frac{\theta}{3}$ and $\cos \frac{\theta}{3}$ are > 0 . Taking the square of the two members of the last equation, we get:

$$\frac{1}{4} < \cos^2 \frac{\theta}{3} \quad (1.3.28)$$

which is true since $\frac{\theta}{3} \in] + \frac{\pi}{6}, + \frac{\pi}{3}[$ then $x_1 > x_3$. As A^m, B^n and $-C^l$ are the only real solutions of (1.3.4), we consider, as A^m is supposed great than B^n , the expressions:

$$\left\{ \begin{array}{l} A^m = x_1 = u_1 + v_1 = 2\sqrt[3]{\rho}\cos\frac{\theta}{3} \\ B^n = x_3 = u_3 + v_3 = 2\sqrt[3]{\rho}\cos\frac{\theta+4\pi}{3} = \sqrt[3]{\rho}\left(-\cos\frac{\theta}{3} + \sqrt{3}\sin\frac{\theta}{3}\right) \\ -C^l = x_2 = u_2 + v_2 = 2\sqrt[3]{\rho}\cos\frac{\theta+2\pi}{3} = -\sqrt[3]{\rho}\left(\cos\frac{\theta}{3} + \sqrt{3}\sin\frac{\theta}{3}\right) \end{array} \right. \quad (1.3.29)$$

□

1.4 Preamble of the Proof of the Main Theorem

Theorem 1.4.1. *Let A, B, C, m, n , and l be positive integers with $m, n, l > 2$. If:*

$$A^m + B^n = C^l \quad (1.4.1)$$

then A, B , and C have a common factor.

Proof. $A^m = 2\sqrt[3]{\rho}\cos\frac{\theta}{3}$ is an integer $\Rightarrow A^{2m} = 4\sqrt[3]{\rho^2}\cos^2\frac{\theta}{3}$ is also an integer. But :

$$\sqrt[3]{\rho^2} = \frac{p}{3} \quad (1.4.2)$$

Then:

$$A^{2m} = 4\sqrt[3]{\rho^2}\cos^2\frac{\theta}{3} = 4\frac{p}{3}\cos^2\frac{\theta}{3} = p\cdot\frac{4}{3}\cos^2\frac{\theta}{3} \quad (1.4.3)$$

As A^{2m} is an integer and p is an integer, then $\cos^2\frac{\theta}{3}$ must be written under the form:

$$\boxed{\cos^2\frac{\theta}{3} = \frac{1}{b} \quad \text{or} \quad \cos^2\frac{\theta}{3} = \frac{a}{b}} \quad (1.4.4)$$

with $b \in \mathbb{N}^*$; for the last condition $a \in \mathbb{N}^*$ and a, b coprime.

Notations: In the following of the paper, the scalars $a, b, \dots, z, \alpha, \beta, \dots, A, B, C, \dots$ and Δ, Φ, \dots represent positive integers except the parameters θ, ρ , or others cited in the text, are reals.

1.4.1 Case $\cos^2\frac{\theta}{3} = \frac{1}{b}$

We obtain:

$$A^{2m} = p\cdot\frac{4}{3}\cos^2\frac{\theta}{3} = \frac{4\cdot p}{3\cdot b} \quad (1.4.5)$$

As $\frac{1}{4} < \cos^2\frac{\theta}{3} < \frac{3}{4} \Rightarrow \frac{1}{4} < \frac{1}{b} < \frac{3}{4} \Rightarrow b < 4 < 3b \Rightarrow b = 1, 2, 3$.

$$b = 1$$

$b = 1 \Rightarrow 4 < 3$ which is impossible.

$$b = 2$$

$b = 2 \Rightarrow A^{2m} = p \cdot \frac{4}{3} \cdot \frac{1}{2} = \frac{2 \cdot p}{3} \Rightarrow 3 \mid p \Rightarrow p = 3p'$ with $p' \neq 1$ because $3 \ll p$, we obtain:

$$\begin{aligned} A^{2m} &= (A^m)^2 = \frac{2p}{3} = 2 \cdot p' \implies 2 \mid p' \implies p' = 2^\alpha p_1^2 \\ &\text{with } 2 \nmid p_1, \quad \alpha + 1 = 2\beta \\ &A^m = 2^\beta p_1 \end{aligned} \quad (1.4.6)$$

$$B^n C^l = \sqrt[3]{\rho^2} \left(3 - 4 \cos^2 \frac{\theta}{3} \right) = p' = 2^\alpha p_1^2 \quad (1.4.7)$$

From the equation (1.4.6), it follows that $2 \mid A^m \implies A = 2^i A_1$, $i \geq 1$ and $2 \nmid A_1$. Then, we have $\beta = i \cdot m = im$. The equation (1.4.7) implies that $2 \mid (B^n C^l) \implies 2 \mid B^n$ or $2 \mid C^l$.

Case 2 | B^n : - If $2 \mid B^n \implies 2 \mid B \implies B = 2^j B_1$ with $2 \nmid B_1$. The expression of $B^n C^l$ becomes:

$$B_1^n C^l = 2^{2im-1-jn} p_1^2$$

- If $2im - 1 - jn \geq 1$, $2 \mid C^l \implies 2 \mid C$ according to $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$ and the conjecture (3.1.1) is verified.

- If $2im - 1 - jn \leq 0 \implies 2 \nmid C^l$, then the contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$.

Case 2 | C^l : If $2 \mid C^l$: with the same method used above, we obtain the identical results.

$$b = 3$$

$b = 3 \Rightarrow A^{2m} = p \cdot \frac{4}{3} \cdot \frac{1}{3} = \frac{4p}{9} \Rightarrow 9 \mid p \Rightarrow p = 9p'$ with $p' \neq 1$, as $9 \ll p$ then $A^{2m} = 4p'$. If p' is prime, it is impossible. We suppose that p' is not a prime, as $m \geq 3$, it follows that $2 \mid p'$, then $2 \mid A^m$. But $B^n C^l = 5p'$ and $2 \mid (B^n C^l)$. Using the same method for the case $b = 2$, we obtain the identical results.

1.4.2 Case $a > 1$, $\cos^2 \frac{\theta}{3} = \frac{a}{b}$

We have:

$$\cos^2 \frac{\theta}{3} = \frac{a}{b}; \quad A^{2m} = p \cdot \frac{4}{3} \cdot \cos^2 \frac{\theta}{3} = \frac{4 \cdot p \cdot a}{3 \cdot b} \quad (1.4.8)$$

where a, b verify one of the two conditions:

$$\boxed{\{3 \mid a \text{ and } b \mid 4p\}} \text{ or } \boxed{\{3 \mid p \text{ and } b \mid 4p\}} \quad (1.4.9)$$

and using the equation (1.3.10), we obtain a third condition:

$$\boxed{b < 4a < 3b} \quad (1.4.10)$$

For these conditions, $A^{2m} = 4 \sqrt[3]{\rho^2} \cos^2 \frac{\theta}{3} = 4 \frac{p}{3} \cdot \cos^2 \frac{\theta}{3}$ is an integer.

Let us study the conditions given by the equation (1.4.9) in the following two sections.

1.5 Hypothesis : $\{3 \mid a \text{ and } b \mid 4p\}$

We obtain :

$$3 \mid a \implies \exists a' \in \mathbb{N}^* / a = 3a' \quad (1.5.1)$$

1.5.1 Case $b = 2$ and $3 \mid a$

A^{2m} is written as:

$$A^{2m} = \frac{4p}{3} \cdot \cos^2 \frac{\theta}{3} = \frac{4p}{3} \cdot \frac{a}{b} = \frac{4p}{3} \cdot \frac{a}{2} = \frac{2 \cdot p \cdot a}{3} \quad (1.5.2)$$

Using the equation (1.5.1), A^{2m} becomes :

$$A^{2m} = \frac{2 \cdot p \cdot 3a'}{3} = 2 \cdot p \cdot a' \quad (1.5.3)$$

but $\cos^2 \frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{2} > 1$ which is impossible, then $b \neq 2$.

1.5.2 Case $b = 4$ and $3 \mid a$

A^{2m} is written :

$$A^{2m} = \frac{4 \cdot p}{3} \cos^2 \frac{\theta}{3} = \frac{4 \cdot p}{3} \cdot \frac{a}{b} = \frac{4 \cdot p}{3} \cdot \frac{a}{4} = \frac{p \cdot a}{3} = \frac{p \cdot 3a'}{3} = p \cdot a' \quad (1.5.4)$$

$$\text{and } \cos^2 \frac{\theta}{3} = \frac{a}{b} = \frac{3 \cdot a'}{4} < \left(\frac{\sqrt{3}}{2} \right)^2 = \frac{3}{4} \implies a' < 1 \quad (1.5.5)$$

which is impossible. Then the case $b = 4$ is impossible.

1.5.3 Case $b = p$ and $3 \mid a$

We have :

$$\cos^2 \frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{p} \quad (1.5.6)$$

and:

$$A^{2m} = \frac{4p}{3} \cdot \cos^2 \frac{\theta}{3} = \frac{4p}{3} \cdot \frac{3a'}{p} = 4a' = (A^m)^2 \quad (1.5.7)$$

$$\exists a'' / a' = a''^2 \quad (1.5.8)$$

$$\text{and } B^n C^l = p - A^{2m} = b - 4a' = b - 4a''^2 \quad (1.5.9)$$

The calculation of $A^m B^n$ gives :

$$\begin{aligned} A^m B^n &= p \cdot \frac{\sqrt{3}}{3} \sin \frac{2\theta}{3} - 2a' \\ \text{or } A^m B^n + 2a' &= p \cdot \frac{\sqrt{3}}{3} \sin \frac{2\theta}{3} \end{aligned} \quad (1.5.10)$$

The left member of (1.5.10) is an integer and p also, then $2 \frac{\sqrt{3}}{3} \sin \frac{2\theta}{3}$ is written under the form :

$$2 \frac{\sqrt{3}}{3} \sin \frac{2\theta}{3} = \frac{k_1}{k_2} \quad (1.5.11)$$

where k_1, k_2 are two coprime integers and $k_2 \mid p \implies p = b = k_2 \cdot k_3, k_3 \in \mathbb{N}^*$.

We suppose that $k_3 \neq 1$

We obtain :

$$A^m(A^m + 2B^n) = k_1.k_3 \quad (1.5.12)$$

Let μ be a prime integer with $\mu \mid k_3$, then $\mu \mid b$ and $\mu \mid A^m(A^m + 2B^n) \implies \mu \mid A^m$ or $\mu \mid (A^m + 2B^n)$.

** A-1-1- If $\mu \mid A^m \implies \mu \mid A$ and $\mu \mid A^{2m}$, but $A^{2m} = 4a' \implies \mu \mid 4a' \implies (\mu = 2, \text{ but } 2 \nmid a')$ or $(\mu \mid a')$. Then $\mu \mid a$ it follows the contradiction with a, b coprime.

** A-1-2- If $\mu \mid (A^m + 2B^n) \implies \mu \nmid A^m$ and $\mu \nmid 2B^n$ then $\mu \neq 2$ and $\mu \nmid B^n$. We write $\mu \mid (A^m + 2B^n)$ as:

$$A^m + 2B^n = \mu.t' \quad (1.5.13)$$

It follows :

$$A^m + B^n = \mu t' - B^n \implies A^{2m} + B^{2n} + 2A^m B^n = \mu^2 t'^2 - 2t' \mu B^n + B^{2n}$$

Using the expression of p :

$$p = t'^2 \mu^2 - 2t' B^n \mu + B^n (B^n - A^m) \quad (1.5.14)$$

As $p = b = k_2.k_3$ and $\mu \mid k_3$ then $\mu \mid b \implies \exists \mu'$ and $b = \mu \mu'$, so we can write:

$$\mu' \mu = \mu(\mu t'^2 - 2t' B^n) + B^n (B^n - A^m) \quad (1.5.15)$$

From the last equation, we obtain $\mu \mid B^n (B^n - A^m) \implies \mu \mid B^n$ or $\mu \mid (B^n - A^m)$.

** A-1-2-1- If $\mu \mid B^n$ which is in contradiction with $\mu \nmid B^n$.

** A-1-2-2- If $\mu \mid (B^n - A^m)$ and using that $\mu \mid (A^m + 2B^n)$, we arrive to :

$$\mu \mid 3B^n \begin{cases} \mu \mid B^n \\ \text{or} \\ \mu = 3 \end{cases} \quad (1.5.16)$$

** A-1-2-2-1- If $\mu \mid B^n \implies \mu \mid B$, it is the contradiction with $\mu \nmid B$ cited above.

** A-1-2-2-2- If $\mu = 3$, then $3 \mid b$, but $3 \nmid a$ then the contradiction with a, b coprime.

We assume now $k_3 = 1$

Then :

$$A^{2m} + 2A^m B^n = k_1 \quad (1.5.17)$$

$$b = k_2 \quad (1.5.18)$$

$$\frac{2\sqrt{3}}{3} \sin \frac{2\theta}{3} = \frac{k_1}{b} \quad (1.5.19)$$

Taking the square of the last equation, we obtain:

$$\frac{4}{3} \sin^2 \frac{2\theta}{3} = \frac{k_1^2}{b^2}$$

$$\frac{16}{3} \sin^2 \frac{\theta}{3} \cos^2 \frac{\theta}{3} = \frac{k_1^2}{b^2}$$

$$\frac{16}{3} \sin^2 \frac{\theta}{3} \cdot \frac{3a'}{b} = \frac{k_1^2}{b^2}$$

Finally:

$$4^2 a' (p - a) = k_1^2 \quad (1.5.20)$$

but $a' = a''^2$, then $p - a$ is a square. Let:

$$\lambda^2 = p - a = b - a = b - 3a''^2 \implies \lambda^2 + 3a''^2 = b \quad (1.5.21)$$

The equation (1.5.20) becomes:

$$4^2 a''^2 \lambda^2 = k_1^2 \implies k_1 = 4a'' \lambda \quad (1.5.22)$$

taking the positive root, but $k_1 = A^m (A^m + 2B^n) = 2a'' (A^m + 2B^n)$, then :

$$A^m + 2B^n = 2\lambda \implies \lambda = a'' + B^n \quad (1.5.23)$$

** A-2-1- As $A^m = 2a'' \implies 2 \mid A^m \implies 2 \mid A \implies A = 2^i A_1$, with $i \geq 1$ and $2 \nmid A_1$, then $A^m = 2a'' = 2^{im} A_1^m \implies a'' = 2^{im-1} A_1^m$, but $im \geq 3 \implies 4 \mid a''$. As $\lambda = a'' + B^n$, taking its square, we obtain $\lambda^2 = a''^2 + 2a'' \cdot B^n + B^{2n} \implies \lambda^2 \equiv B^{2n} \pmod{4} \implies \lambda^2 \equiv B^{2n} \equiv 0 \pmod{4}$ or $\lambda^2 \equiv B^{2n} \equiv 1 \pmod{4}$.

** A-2-1-1- We suppose that $\lambda^2 \equiv B^{2n} \equiv 0 \pmod{4} \implies 4 \mid \lambda^2 \implies 2 \mid (b - a)$. But $2 \mid a$ because $a = 3a' = 3a''^2 = 3 \times 2^{2(im-1)} A_1^{2m}$ and $im \geq 3$. Then $2 \mid b$, it follows the contradiction with a, b coprime.

** A-2-1-2- We suppose now that $\lambda^2 \equiv B^{2n} \equiv 1 \pmod{4}$. As $A^m = 2^{im-1} A_1^m$ and $im - 1 \geq 2$, then $A^m \equiv 0 \pmod{4}$. As $B^{2n} \equiv 1 \pmod{4}$, then B^n verifies $B^n \equiv 1 \pmod{4}$ or $B^n \equiv 3 \pmod{4}$ which gives for the two cases $B^n C^l \equiv 1 \pmod{4}$.

We have also $p = b = A^{2m} + A^m B^n + B^{2n} = 4a' + B^n \cdot C^l = 4a''^2 + B^n C^l \implies B^n C^l = \lambda^2 - a''^2 = B^n \cdot C^l$, then $\lambda, a'' \in \mathbb{N}^*$ are solutions of the Diophantine equation :

$$x^2 - y^2 = N \quad (1.5.24)$$

with $N = B^n C^l > 0$. Let $Q(N)$ be the number of the solutions of (1.5.24) and $\tau(N)$ is the number of suitable factorization of N , then we announce the following result concerning the solutions of the equation (1.5.24) (see theorem 27.3 in [2]):

- If $N \equiv 2 \pmod{4}$, then $Q(N) = 0$.
- If $N \equiv 1$ or $N \equiv 3 \pmod{4}$, then $Q(N) = [\tau(N)/2]$.
- If $N \equiv 0 \pmod{4}$, then $Q(N) = [\tau(N/4)/2]$.

$[x]$ is the integral part of x for which $[x] \leq x < [x] + 1$.

In our case, we have $N = B^n \cdot C^l \equiv 1 \pmod{4}$, then $Q(N) = [\tau(N)/2]$. As λ, a'' is a couple of solutions of the Diophantine equation (1.5.24), then $\exists d, d'$ positive integers with $d > d'$ and $N = d \cdot d'$ so that :

$$d + d' = 2\lambda \quad (1.5.25)$$

$$d - d' = 2a'' \quad (1.5.26)$$

** A-2-1-2-1- As $C^l > B^n$, we take $d = C^l$ and $d' = B^n$. It follows:

$$C^l + B^n = 2\lambda = A^m + 2B^n \quad (1.5.27)$$

$$C^l - B^n = 2a'' = A^m \quad (1.5.28)$$

Then the case $d = C^l$ and $d' = B^n$ gives *a priori* no contradictions.

** A-2-1-2-2- Now, we consider the case $d = B^n C^l$ and $d' = 1$. We rewrite the equations (1.5.25-1.5.26):

$$B^n C^l + 1 = 2\lambda \quad (1.5.29)$$

$$B^n C^l - 1 = 2a'' \quad (1.5.30)$$

We obtain $1 = \lambda - a''$, but from (1.5.23), we have $\lambda = a'' + B^n$, it follows $B^n = 1$ and $C^l - A^m = 1$, we know [?] that the only positive solution of the last equation is $C = 3, A = 2, m = 3$ and $l = 2 < 3$, then the contradiction.

** A-2-1-2-3- Now, we consider the case $d = c_1^{lr-1} C_1^l$ where c_1 is a prime integer with $c_1 \nmid C_1$ and $C = c_1^r C_1, r \geq 1$. It follows that $d' = c_1 \cdot B^n$. We rewrite the equations (1.5.25-1.5.26):

$$c_1^{lr-1} C_1^l + c_1 \cdot B^n = 2\lambda \quad (1.5.31)$$

$$c_1^{lr-1} C_1^l - c_1 \cdot B^n = 2a'' \quad (1.5.32)$$

As $l \geq 3$, from the last two equations above, it follows that $c_1 \mid (2\lambda)$ and $c_1 \mid (2a'')$. Then $c_1 = 2$, or $c_1 \mid \lambda$ and $c_1 \mid a''$.

** A-2-1-2-3-1- We suppose $c_1 = 2$. As $2 \mid A^m$ and $2 \mid C^l$ because $l \geq 3$, it follows $2 \mid B^n$, then $2 \mid (p = b)$. Then the contradiction with a, b coprime.

** A-2-1-2-3-2- We suppose $c_1 \neq 2$ and $c_1 \mid a''$ and $c_1 \mid \lambda$. $c_1 \mid a'' \implies c_1 \mid a$ and $c_1 \mid (A^m = 2a'')$. $B^n = C^l - A^m \implies c_1 \mid B^n$. It follows that $c_1 \mid (p = b)$. Then the contradiction with a, b coprime.

The other cases of the expressions of d and d' with d, d' not coprime so that $N = B^n C^l = d \cdot d'$ give also contradictions.

** A-2-1-2-4- Now, let $C = c_1^r C_1$ with c_1 a prime, $r \geq 1$ and $c_1 \nmid C_1$, we consider the case $d = C_1^l$ and $d' = c_1^r B^n$ so that $d > d'$. We rewrite the equations (1.5.25-1.5.26):

$$C_1^l + c_1^r B^n = 2\lambda \quad (1.5.33)$$

$$C_1^l - c_1^r B^n = 2a'' \quad (1.5.34)$$

We obtain $c_1^r B^n = \lambda - a'' = B^n \implies c_1^r = 1$, then the contradiction.

** A-2-1-2-5- Now, let $C = c_1^r C_1$ with c_1 a prime, $r \geq 1$ and $c_1 \nmid C_1$, we consider the case $d = C_1^l B^n$ and $d' = c_1^r$ so that $d > d'$. We rewrite the equations (1.5.25-1.5.26):

$$C_1^l B^n + c_1^r = 2\lambda \quad (1.5.35)$$

$$C_1^l B^n - c_1^r = 2a'' \quad (1.5.36)$$

We obtain $c_1^r = \lambda - a'' = B^n \implies c_1 \mid B^n$, then $c_1 \mid A^m = 2a''$. If $c_1 = 2$, the contradiction with $B^n C^l \equiv 1 \pmod{4}$. Then $c_1 \mid a'' \implies c_1 \mid a \implies c_1 \mid (p = b)$, it follows a, b are not

coprime, then the contradiction.

Cases like $d' < C^l$ a divisor of C^l or $d' < B^n$ a divisor of B^n with $d' < d$ and $d.d' = N = B^n C^l$ give contradictions.

** A-2-1-2-6- Now, we consider the case $d = b_1.C^l$ where b_1 is a prime integer with $b_1 \nmid B_1$ and $B = b_1^r B_1$, $r \geq 1$. It follows that $d' = b_1^{nr-1} B_1^n$. We rewrite the equations (1.5.25-1.5.26):

$$b_1 C^l + b_1^{nr-1} B_1^n = 2\lambda \quad (1.5.37)$$

$$b_1 C^l - b_1^{nr-1} B_1^n = 2a'' \quad (1.5.38)$$

As $n \geq 3$, from the last two equations above, it follows that $b_1 \mid 2\lambda$ and $b_1 \mid (2a'')$. Then $b_1 = 2$, or $b_1 \mid \lambda$ and $b_1 \mid a''$.

** A-2-1-2-6-1- We suppose $b_1 = 2 \implies 2 \mid B^n$. As $2 \mid (A^m = 2a'' \implies 2 \mid a'' \implies 2 \mid a$, but $2 \mid B^n$ and $2 \mid A^m$ then $2 \mid (p = b)$. It follows the contradiction with a, b coprime.

** A-2-1-2-6-2- We suppose $b_1 \neq 2$, then $b_1 \mid \lambda$ and $b_1 \mid a'' \implies b_1 \mid A^m$ and $b_1 \mid a'' \implies b_1 \mid a$, but $b_1 \mid B^n$ and $b_1 \mid A^m$ then $b_1 \mid (p = b)$. It follows the contradiction with a, b coprime.

The other cases of the expressions of d and d' with d, d' not coprime and $d > d'$ so that $N = C^l B^m = d.d'$ give also contradictions.

Finally, from the cases studied in the above paragraph A-2-1-2, we have found one suitable factorization of N that gives a priori no contradictions, it is the case $N = B^n.C^l = d.d'$ with $d = C^l, d' = B^n$ but $1 \ll \tau(N)$, it follows the contradiction with $Q(N) = [\tau(N)/2] \leq 1$. We conclude that the case A-2-1-2 is to reject.

Hence, the case $k_3 = 1$ is impossible.

Let us verify the condition (1.4.10) given by $b < 4a < 3b$. In our case, the condition becomes :

$$p < 3A^{2m} < 3p \quad \text{with} \quad p = A^{2m} + B^{2n} + A^m B^n \quad (1.5.39)$$

and $3A^{2m} < 3p \implies A^{2m} < p$ that is verified. If :

$$p < 3A^{2m} \implies 2A^{2m} - A^m B^n - B^{2n} \stackrel{?}{>} 0$$

Studying the sign of the polynomial $Q(Y) = 2Y^2 - B^n Y - B^{2n}$ and taking $Y = A^m > B^n$, the condition $2A^{2m} - A^m B^n - B^{2n} > 0$ is verified, then the condition $b < 4a < 3b$ is true.

In the following of the paper, we verify easily that the condition $b < 4a < 3b$ implies to verify that $A^m > B^n$ which is true.

1.5.4 Case $b \mid p \implies p = b.p', p' > 1, b \neq 2, b \neq 4$ and $3 \mid a$

$$A^{2m} = \frac{4.p}{3} \cdot \frac{a}{b} = \frac{4.b.p'.3.a'}{3.b} = 4.p'a' \quad (1.5.40)$$

We calculate $B^n C^l$:

$$B^n C^l = \sqrt[3]{\rho^2} \left(3 \sin^2 \frac{\theta}{3} - \cos^2 \frac{\theta}{3} \right) = \sqrt[3]{\rho^2} \left(3 - 4 \cos^2 \frac{\theta}{3} \right) \quad (1.5.41)$$

but $\sqrt[3]{\rho^2} = \frac{p}{3}$, using $\cos^2 \frac{\theta}{3} = \frac{3.a'}{b}$, we obtain:

$$B^n C^l = \sqrt[3]{\rho^2} \left(3 - 4 \cos^2 \frac{\theta}{3} \right) = \frac{p}{3} \left(3 - 4 \frac{3.a'}{b} \right) = p \cdot \left(1 - \frac{4.a'}{b} \right) = p'(b - 4a') \quad (1.5.42)$$

As $p = b.p'$, and $p' > 1$, so we have :

$$B^n C^l = p'(b - 4a') \quad (1.5.43)$$

$$\text{and } A^{2m} = 4.p'.a' \quad (1.5.44)$$

** B-1- We suppose that p' is prime, then $A^{2m} = 4a'p' = (A^m)^2 \implies p' \mid a'$. But $B^n C^l = p'(b - 4a') \implies p' \mid B^n$ or $p' \mid C^l$.

** B-1-1- If $p' \mid B^n \implies p' \mid B \implies B = p'B_1$ with $B_1 \in \mathbb{N}^*$. Hence : $p'^{n-1}B_1^n C^l = b - 4a'$. But $n > 2 \implies (n-1) > 1$ and $p' \mid a'$, then $p' \mid b \implies a$ and b are not coprime, then the contradiction.

** B-1-2- If $p' \mid C^l \implies p' \mid C$. The same method used above, we obtain the same results.

** B-2- We consider that p' is not a prime integer.

** B-2-1- p', a are supposed coprime: $A^{2m} = 4a'p' \implies A^m = 2a''.p_1$ with $a' = a''^2$ and $p' = p_1^2$, then a'', p_1 are also coprime. As $A^m = 2a''.p_1$ then $2 \mid a''$ or $2 \mid p_1$.

** B-2-1-1- $2 \mid a''$, then $2 \nmid p_1$. But $p' = p_1^2$.

** B-2-1-1-1- If p_1 is prime, it is impossible with $A^m = 2a''.p_1$.

** B-2-1-1-2- We suppose that p_1 is not prime, we can write it as $p_1 = \omega^m \implies p' = \omega^{2m}$, then: $B^n C^l = \omega^{2m}(b - 4a')$.

** B-2-1-1-2-1- If ω is prime, it is different of 2, then $\omega \mid (B^n C^l) \implies \omega \mid B^n$ or $\omega \mid C^l$.

** B-2-1-1-2-1-1- If $\omega \mid B^n \implies \omega \mid B \implies B = \omega^j B_1$ with $\omega \nmid B_1$, then $B_1^n C^l = \omega^{2m-nj}(b - 4a')$.

** B-2-1-1-2-1-1-1- If $2m - nj = 0$, we obtain $B_1^n C^l = b - 4a'$. As $C^l = A^m + B^n \implies \omega \mid C^l \implies \omega \mid C$, and $\omega \mid (b - 4a')$. But $\omega \neq 2$ and ω is coprime with a' then coprime with a , then $\omega \nmid b$. The conjecture (3.1.1) is verified.

** B-2-1-1-2-1-1-2- If $2m - nj \geq 1$, in this case with the same method, we obtain $\omega \mid C^l \implies \omega \mid C$ and $\omega \mid (b - 4a')$ and $\omega \nmid a$ and $\omega \nmid b$. The conjecture (3.1.1) is verified.

** B-2-1-1-2-1-1-3- If $2m - nj < 0 \implies \omega^{n.j-2m} B_1^n C^l = b - 4a'$. As $\omega \mid C$ using $C^l = A^m + B^n$ then $C = \omega^h C_1 \implies \omega^{n.j-2m+h.l} B_1^n C_1^l = b - 4a'$. If $n.j - 2m + h.l < 0 \implies \omega \mid B_1^n C_1^l$, it follows the contradiction that $\omega \nmid B_1$ or $\omega \nmid C_1$. Then if $n.j - 2m + h.l > 0$ and $\omega \mid (b - 4a')$ with ω, a, b coprime and the conjecture (3.1.1) is verified.

** B-2-1-1-2-1-2- We obtain the same results if $\omega \mid C^l$.

** B-2-1-1-2-2- Now, $p' = \omega^{2m}$ and ω not prime, we write $\omega = \omega_1^f \cdot \Omega$ with ω_1 prime $\nmid \Omega$ and $f \geq 1$ an integer, and $\omega_1 \mid A$. Then $B^n C^l = \omega_1^{2f \cdot m} \Omega^{2m} (b - 4a') \implies \omega_1 \mid (B^n C^l) \implies \omega_1 \mid B^n$ or $\omega_1 \mid C^l$.

** B-2-1-1-2-2-1- If $\omega_1 \mid B^n \implies \omega_1 \mid B \implies B = \omega_1^j B_1$ with $\omega_1 \nmid B_1$, then $B_1^n \cdot C^l = \omega_1^{2mf - nj} \Omega^{2m} (b - 4a')$:

** B-2-1-1-2-2-1-1- If $2f \cdot m - n \cdot j = 0$, we obtain $B_1^n \cdot C^l = \Omega^{2m} (b - 4a')$. As $C^l = A^m + B^n \implies \omega_1 \mid C^l \implies \omega_1 \mid C \implies \omega_1 \mid (b - 4a')$. But $\omega_1 \neq 2$ and ω_1 is coprime with a' , then coprime with a , we deduce $\omega_1 \nmid b$. Then the conjecture (3.1.1) is verified.

** B-2-1-1-2-2-1-2- If $2f \cdot m - n \cdot j \geq 1$, we have $\omega_1 \mid C^l \implies \omega_1 \mid C \implies \omega_1 \mid (b - 4a')$ and $\omega_1 \nmid a$ and $\omega_1 \nmid b$. The conjecture (3.1.1) is verified.

** B-2-1-1-2-2-1-3- If $2f \cdot m - n \cdot j < 0 \implies \omega_1^{n \cdot j - 2m \cdot f} B_1^n \cdot C^l = \Omega^{2m} (b - 4a')$. As $\omega_1 \mid C$ using $C^l = A^m + B^n$, then $C = \omega_1^h \cdot C_1 \implies \omega_1^{n \cdot j - 2m \cdot f + h \cdot l} B_1^n \cdot C_1^l = \Omega^{2m} (b - 4a')$. If $n \cdot j - 2m \cdot f + h \cdot l < 0 \implies \omega_1 \mid B_1^n C_1^l$, it follows the contradiction with $\omega_1 \nmid B_1$ and $\omega_1 \nmid C_1$. Then if $n \cdot j - 2m \cdot f + h \cdot l > 0$ and $\omega_1 \mid (b - 4a')$ with ω_1, a, b coprime and the conjecture (3.1.1) is verified.

** B-2-1-1-2-2-2- We obtain the same results if $\omega_1 \mid C^l$.

** B-2-1-2- If $2 \mid p_1$, then $2 \mid p_1 \implies 2 \nmid a' \implies 2 \nmid a$. But $p' = p_1^2$.

** B-2-1-2-1- If $p_1 = 2$, we obtain $A^m = 4a'' \implies 2 \mid a''$ as $m \geq 3$, then the contradiction with a, b coprime.

** B-2-1-2-2- We suppose that p_1 is not prime and $2 \mid p_1$, as $A^m = 2a'' p_1$, p_1 is written as $p_1 = 2^{m-1} \omega^m \implies p' = 2^{2m-2} \omega^{2m}$. It follows $B^n C^l = 2^{2m-2} \omega^{2m} (b - 4a') \implies 2 \mid B^n$ or $2 \mid C^l$.

** B-2-1-2-2-1- If $2 \mid B^n \implies 2 \mid B$, as $2 \mid A$, then $2 \mid C$. From $B^n C^l = 2^{2m-2} \omega^{2m} (b - 4a')$, it follows if $2 \mid (b - 4a') \implies 2 \mid b$ but as $2 \nmid a'$, there is no contradiction with a, b coprime and the conjecture (3.1.1) is verified.

** B-2-1-2-2-2- If $2 \mid C^l$, using the same method as above, we obtain the identical results.

** B-2-2- p', a' are supposed not coprime. Let ω be a prime integer so that $\omega \mid a'$ and $\omega \mid p'$.

** B-2-2-1- We suppose firstly $\omega = 3$. As $A^{2m} = 4a' p' \implies 3 \mid A$, but $3 \mid p' \implies 3 \mid p$, as $p = A^{2m} + B^{2n} + A^m B^n \implies 3 \mid B^{2n} \implies 3 \mid B$, then $3 \mid C^l \implies 3 \mid C$. We write $A = 3^i A_1$, $B = 3^j B_1$, $C = 3^h C_1$ and 3 coprime with A_1, B_1 and C_1 and $p = 3^{2im} A_1^{2m} + 3^{2nj} B_1^{2n} + 3^{im+jn} A_1^m B_1^n = 3^k \cdot g$ with $k = \min(2im, 2jn, im + jn)$ and $3 \nmid g$. We have also $(\omega = 3) \mid a$ and $(\omega = 3) \mid p'$ that gives $a = 3^\alpha a_1 = 3a' \implies a' = 3^{\alpha-1} a_1$, $3 \nmid a_1$ and $p' = 3^\mu p_1$, $3 \nmid p_1$ with $A^{2m} = 4a' p' = 3^{2im} A_1^{2m} = 4 \times 3^{\alpha-1+\mu} \cdot a_1 \cdot p_1 \implies \alpha + \mu - 1 = 2im$. As $p = b p' = b \cdot 3^\mu p_1 = 3^\mu \cdot b \cdot p_1$. The exponent of the term 3 of p is k , the exponent of the term 3 of the left member of the last equation is μ . If $3 \mid b$ it is a contradiction with a, b coprime. Then, we suppose that $3 \nmid b$, and the equality of the exponents: $\min(2im, 2jn, im + jn) = \mu$, recall that $\alpha + \mu - 1 = 2im$. But $B^n C^l = p' (b - 4a')$ that gives $3^{(nj+hl)} B_1^n C_1^l = 3^\mu p_1 (b - 4 \times 3^{\alpha-1} a_1)$. We have also $A^m + B^n = C^l$ gives $3^{im} A_1^m + 3^{jn} B_1^n = 3^{hl} C_1^l$. Let $\epsilon = \min(im, jn)$, we have

$\epsilon = hl = \min(im, jn)$. Then, we obtain the conditions:

$$k = \min(2im, 2jn, im + jn) = \mu \quad (1.5.45)$$

$$\alpha + \mu - 1 = 2im \quad (1.5.46)$$

$$\epsilon = hl = \min(im, jn) \quad (1.5.47)$$

$$3^{(nj+hl)} B_1^n C_1^l = 3^\mu p_1 (b - 4 \times 3^{(\alpha-1)} a_1) \quad (1.5.48)$$

** B-2-2-1-1- $\alpha = 1 \implies a = 3a_1 = 3a'$ and $3 \nmid a_1$, the equation (1.5.46) becomes:

$$\mu = 2im$$

and the first equation (1.5.45) is written as:

$$k = \min(2im, 2jn, im + jn) = 2im$$

- If $k = 2im$, then $2im \leq 2jn \implies im \leq jn \implies hl = im$, and (1.5.48) gives $\mu = 2im = nj + hl = im + nj \implies im = jn = hl$. Hence $3 \mid A, 3 \mid B$ and $3 \mid C$ and the conjecture (3.1.1) is verified.

- If $k = 2jn \implies 2jn = 2im \implies im = jn = hl$. Hence $3 \mid A, 3 \mid B$ and $3 \mid C$ and the conjecture (3.1.1) is verified.

- If $k = im + jn = 2im \implies im = jn \implies \epsilon = hl = im = jn$ case that is seen above and we deduce that $3 \mid A, 3 \mid B$ and $3 \mid C$, and the conjecture (3.1.1) is verified.

** B-2-2-1-2- $\alpha > 1 \implies \alpha \geq 2$ and $a' = 3^{\alpha-1} a_1$.

- If $k = 2im \implies 2im = \mu$, but $\mu = 2im + 1 - \alpha$ that is impossible.

- If $k = 2jn = \mu \implies 2jn = 2im + 1 - \alpha$. We obtain $2jn < 2im \implies jn < im \implies 2jn < im + jn$, $k = 2jn$ is just the minimum of $(2im, 2jn, im + jn)$. We obtain $jn = hl < im$ and the equation (1.5.48) becomes:

$$B_1^n C_1^l = p_1 (b - 4 \times 3^{(\alpha-1)} a_1)$$

The conjecture (3.1.1) is verified.

- If $k = im + jn \leq 2im \implies jn \leq im$ and $k = im + jn \leq 2jn \implies im \leq jn \implies im = jn \implies k = im + jn = 2im = \mu$ but $\mu = 2im + 1 - \alpha$ that is impossible.

- If $k = im + jn < 2im \implies jn < im$ and $2jn < im + jn = k$ that is a contradiction with $k = \min(2im, 2jn, im + jn)$.

** B-2-2-2- We suppose that $\omega \neq 3$. We write $a = \omega^\alpha a_1$ with $\omega \nmid a_1$ and $p' = \omega^\mu p_1$ with $\omega \nmid p_1$. As $A^{2m} = 4a'p' = 4\omega^{\alpha+\mu} a_1 p_1 \implies \omega \mid A \implies A = \omega^i A_1$, $\omega \nmid A_1$. But $B^n C^l = p'(b - 4a') = \omega^\mu p_1 (b - 4a') \implies \omega \mid B^n C^l \implies \omega \mid B^n$ or $\omega \mid C^l$.

** B-2-2-2-1- $\omega \mid B^n \implies \omega \mid B \implies B = \omega^j B_1$ and $\omega \nmid B_1$. From $A^m + B^n = C^l \implies \omega \mid C^l \implies \omega \mid C$. As $p = bp' = \omega^\mu bp_1 = \omega^k (\omega^{2im-k} A_1^{2m} + \omega^{2jn-k} B_1^{2n} + \omega^{im+jn-k} A_1^m B_1^n)$ with $k = \min(2im, 2jn, im + jn)$. Then :

- If $\mu = k$, then $\omega \nmid b$ and the conjecture (3.1.1) is verified.

- If $k > \mu$, then $\omega \mid b$, but $\omega \nmid a$ we deduce the contradiction with a, b coprime.

- If $k < \mu$, it follows from :

$$\omega^\mu bp_1 = \omega^k (\omega^{2im-k} A_1^{2m} + \omega^{2jn-k} B_1^{2n} + \omega^{im+jn-k} A_1^m B_1^n)$$

that $\omega \mid A_1$ or $\omega \mid B_1$ that is a contradiction with the hypothesis.

** B-2-2-2-2- If $\omega \mid C^l \implies \omega \mid C \implies C = \omega^h C_1$ with $\omega \nmid C_1$. From $A^m + B^n = C^l \implies \omega \mid (C^l - A^m) \implies \omega \mid B$. Then, we obtain the same results as B-2-2-2-1- above.

1.5.5 Case $b = 2p$ and $3 \mid a$

We have :

$$\cos^2 \frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{2p} \implies A^{2m} = \frac{4p \cdot a}{3b} = \frac{4p \cdot 3a'}{3 \cdot 2p} = 2a' = (A^m)^2 \implies 2 \mid a' \implies 2 \mid a$$

Then $2 \mid a$ and $2 \mid b$ that is a contradiction with a, b coprime.

1.5.6 Case $b = 4p$ and $3 \mid a$

We have :

$$\cos^2 \frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{4p} \implies A^{2m} = \frac{4p \cdot a}{3b} = \frac{4p \cdot 3a'}{3 \cdot 4p} = a' = (A^m)^2 = a''^2$$

with $A^m = a''$

Let us calculate $A^m B^n$, we obtain:

$$A^m B^n = \frac{p\sqrt{3}}{3} \cdot \sin \frac{2\theta}{3} - \frac{2p}{3} \cos^2 \frac{\theta}{3} = \frac{p\sqrt{3}}{3} \cdot \sin \frac{2\theta}{3} - \frac{a'}{2} \implies$$

$$A^m B^n + \frac{A^{2m}}{2} = \frac{p\sqrt{3}}{3} \cdot \sin \frac{2\theta}{3}$$

Let:

$$A^{2m} + 2A^m B^n = \frac{2p\sqrt{3}}{3} \sin \frac{2\theta}{3} \tag{1.5.49}$$

The left member of (1.5.49) is an integer and p is an integer, then $\frac{2\sqrt{3}}{3} \sin \frac{2\theta}{3}$ will be written as :

$$\frac{2\sqrt{3}}{3} \sin \frac{2\theta}{3} = \frac{k_1}{k_2}$$

where k_1, k_2 are two integers coprime and $k_2 \mid p \implies p = k_2 \cdot k_3$.

** C-1- Firstly, we suppose that $k_3 \neq 1$. Then :

$$A^{2m} + 2A^m B^n = k_3 \cdot k_1$$

Let μ be a prime integer and $\mu \mid k_3$, then $\mu \mid A^m(A^m + 2B^n) \implies \mu \mid A^m$ or $\mu \mid (A^m + 2B^n)$.

** C-1-1- If $\mu \mid (A^m = a'') \implies \mu \mid (a''^2 = a') \implies \mu \mid (3a' = a)$. As $\mu \mid k_3 \implies \mu \mid p \implies \mu \mid (4p = b)$, then the contradiction with a, b coprime.

** C-1-2- If $\mu \mid (A^m + 2B^n) \implies \mu \nmid A^m$ and $\mu \nmid 2B^n$, then:

$$\mu \neq 2 \quad \text{and} \quad \mu \nmid B^n \tag{1.5.50}$$

$\mu \mid (A^m + 2B^n)$, we write:

$$A^m + 2B^n = \mu \cdot t'$$

Then:

$$A^m + B^n = \mu t' - B^n \implies A^{2m} + B^{2n} + 2A^m B^n = \mu^2 t'^2 - 2t' \mu B^n + B^{2n}$$

$$\implies p = t'^2 \mu^2 - 2t' B^n \mu + B^n (B^n - A^m)$$

As $b = 4p = 4k_2.k_3$ and $\mu \mid k_3$ then $\mu \mid b \implies \exists \mu'$ so that $b = \mu.\mu'$, we obtain:

$$\mu'.\mu = \mu(4\mu t'^2 - 8t'B^n) + 4B^n(B^n - A^m)$$

The last equation implies $\mu \mid 4B^n(B^n - A^m)$, but $\mu \neq 2$ then $\mu \mid B^n$ or $\mu \mid (B^n - A^m)$.

** C-1-1-1- If $\mu \mid B^n \implies$ then the contradiction with (1.5.50).

** C-1-1-2- If $\mu \mid (B^n - A^m)$ and using $\mu \mid (A^m + 2B^n)$, we have :

$$\mu \mid 3B^n \implies \begin{cases} \mu \mid B^n \\ or \\ \mu = 3 \end{cases}$$

** C-1-1-2-1- If $\mu \mid B^n$ then the contradiction with (1.5.50).

** C-1-1-2-2- If $\mu = 3$, then $3 \mid b$, but $3 \mid a$ then the contradiction with a, b coprime.

** C-2- We assume now that $k_3 = 1$, then:

$$\begin{aligned} A^{2m} + 2A^m B^n &= k_1 & (1.5.51) \\ p &= k_2 \\ \frac{2\sqrt{3}}{3} \sin \frac{2\theta}{3} &= \frac{k_1}{p} \end{aligned}$$

We take the square of the last equation, we obtain :

$$\begin{aligned} \frac{4}{3} \sin^2 \frac{2\theta}{3} &= \frac{k_1^2}{p^2} \\ \frac{16}{3} \sin^2 \frac{\theta}{3} \cos^2 \frac{\theta}{3} &= \frac{k_1^2}{p^2} \\ \frac{16}{3} \sin^2 \frac{\theta}{3} \cdot \frac{3a'}{b} &= \frac{k_1^2}{p^2} \end{aligned}$$

Finally:

$$a'(4p - 3a') = k_1^2 \quad (1.5.52)$$

but $a' = a''^2$, then $4p - 3a'$ is a square. Let :

$$\lambda^2 = 4p - 3a' = 4p - a = b - a$$

The equation (1.5.52) becomes :

$$a''^2 \lambda^2 = k_1^2 \implies k_1 = a'' \lambda \quad (1.5.53)$$

taking the positive root. Using (1.5.51), we have:

$$k_1 = A^m(A^m + 2B^n) = a''(A^m + 2B^n)$$

Then :

$$A^m + 2B^n = \lambda$$

Now, we consider that $b - a = \lambda^2 \implies \lambda^2 + 3a''^2 = b$, then the couple (λ, a'') is a solution of the Diophantine equation:

$$X^2 + 3Y^2 = b \quad (1.5.54)$$

with $X = \lambda$ and $Y = a''$. But using one theorem on the solutions of the equation given by (1.5.54), b is written under the form (see theorem 37.4 in [3]):

$$b = 2^{2s} \times 3^t \cdot p_1^{t_1} \cdots p_g^{t_g} q_1^{2s_1} \cdots q_r^{2s_r}$$

where p_i are prime integers so that $p_i \equiv 1 \pmod{6}$, the q_j are also prime integers so that $q_j \equiv 5 \pmod{6}$. Then, as $b = 4p$:

- If $t \geq 1 \implies 3 \mid b$, but $3 \nmid a$, then the contradiction with a, b coprime.

** C-2-2-1- Hence, we suppose that p is written under the form:

$$p = p_1^{t_1} \cdots p_g^{t_g} q_1^{2s_1} \cdots q_r^{2s_r}$$

with $p_i \equiv 1 \pmod{6}$ and $q_j \equiv 5 \pmod{6}$. Finally, we obtain that :

$$p \equiv 1 \pmod{6} \quad (1.5.55)$$

We will verify if this condition does not give contradictions.

We will present the table of the value modulo 6 of $p = A^{2m} + A^m B^n + B^{2n}$ in function of the values of $A^m, B^n \pmod{6}$. We obtain the table below:

Table 1.1: Table of $p \pmod{6}$

A^m, B^n	0	1	2	3	4	5
0	0	1	4	3	4	1
1	1	3	1	1	3	1
2	4	1	0	1	4	3
3	3	1	1	3	1	1
4	4	3	4	1	0	1
5	1	1	3	1	1	3

** C-2-2-1-1- Case $A^m \equiv 0 \pmod{6} \implies 2 \mid (A^m = a'') \implies 2 \mid (a' = a''^2) \implies 2 \mid a$, but $2 \nmid b$, then the contradiction with a, b coprime. All the cases of the first line of the table 1.1 are to reject.

** C-2-2-1-2- Case $A^m \equiv 1 \pmod{6}$ and $B^n \equiv 0 \pmod{6}$, then $2 \mid B^n \implies B^n = 2B'$ and p is written as $p = (A^m + B')^2 + 3B'^2$ with $(p, 3) = 1$, if not $3 \mid p$, then $3 \mid b$, but $3 \nmid a$, then the contradiction with a, b coprime. Hence, the pair $(A^m + B', B')$ verifies the equation:

$$(A^m + B')^2 + 3B'^2 = p \quad (1.5.56)$$

that we can write it as:

$$(A^m + B')^2 - B'^2 = p - 4B'^2 = A^{2m} + B^{2n} + A^m B^n - B^{2n} = C^l A^m = N \quad (1.5.57)$$

Then $(A^m + B', B')$ is a solution of the Diophantine equation:

$$x^2 - y^2 = N \quad (1.5.58)$$

where $N = C^l A^m \equiv 1 \pmod{6}$. Let $Q(N)$ be the number of the solutions of (1.5.58) and $\tau(N)$ is the number of suitable factorization of N , then we recall the following result concerning the solutions of the equation (1.5.58) (see theorem 27.3 in [2]):

- If $N \equiv 2 \pmod{4}$, then $Q(N) = 0$.
- If $N \equiv 1$ or $N \equiv 3 \pmod{4}$, then $Q(N) = [\tau(N)/2]$.
- If $N \equiv 0 \pmod{4}$, then $Q(N) = [\tau(N/4)/2]$.

As $N = C^l A^m \equiv 1 \pmod{6} \implies N$ is odd, the cases $Q(N) = 0$ and $Q(N) = [\tau(N/4)/2]$ are rejected, then $N \equiv 1$ or $N \equiv 3 \pmod{4}$, it follows $Q(N) = [\tau(N)/2]$.

As $A^m + B', B'$ is a couple of solutions of the Diophantine equation (1.5.58), then $\exists d, d'$ positive integers with $d > d'$ and $N = d.d'$ so that :

$$d + d' = 2(A^m + B') \quad (1.5.59)$$

$$d - d' = 2B' = B^n \quad (1.5.60)$$

We will use the same method used for the paragraph above A-2-1-2-.

** C-2-2-1-2-1- As $C^l > A^m$, we take $d = C^l$ and $d' = A^m$. It follows:

$$\begin{aligned} C^l + A^m &= 2(A^m + B') = 2A^m + B^n \\ C^l - A^m &= B^n = 2B' \end{aligned}$$

Then the case $d = C^l$ and $d' = A^m$ gives *a priori* no contradictions.

** C-2-2-1-2-2- Now, we consider the case $d = C^l A^m$ and $d' = 1$. We rewrite the equations (1.5.59-1.5.60):

$$C^l A^m + 1 = 2(A^m + B') \quad (1.5.61)$$

$$C^l A^m - 1 = 2B' \quad (1.5.62)$$

We obtain $1 = A^m$, it follows $C^l - B^n = 1$, we know [?] that the only positive solution of the last equation is $C = 3, B = 2, n = 3$ and $l = 2 < 3$, then the contradiction.

** C-2-2-1-2-3- Now, we consider the case $d = c_1^{lr-1} C_1^l$ where c_1 is a prime integer with $c_1 \nmid C_1$ and $C = c_1^r C_1, r \geq 1$. It follows that $d' = c_1.A^m$. We rewrite the equations (1.5.59-1.5.60):

$$c_1^{lr-1} C_1^l + c_1.A^m = 2(A^m + B') \quad (1.5.63)$$

$$c_1^{lr-1} C_1^l - c_1.A^m = 2B' = B^n \quad (1.5.64)$$

As $l \geq 3$, from the last two equations above, it follows that $c_1 \mid 2(A^m + B')$ and $c_1 \mid (2B')$. Then $c_1 = 2$, or $c_1 \mid (A^m + B')$ and $c_1 \mid B'$.

** C-2-2-1-2-3-1- We suppose $c_1 = 2$. As $l \geq 3$, from the equation (1.5.64) it follows that $2 \mid B^n$, then $2 \mid (A^m = a'') \implies 2 \mid (a''^2 = a') \implies 2 \mid (a = 3a')$, but $b = 4p$ (see 1.5.6), then the contradiction with a, b coprime.

** C-2-2-1-2-3-2- We suppose $c_1 \neq 2$, then $c_1 \mid (A^m + B')$ and $c_1 \mid B'$. It follows $c_1 \mid A^m$ and $c_1 \mid (B^n = 2B') \implies c_1 \mid p \implies c_1 \mid b = 4p$. From $c_1 \mid (A^m = a'') \implies c_1 \mid (a''^2 = a') \implies c_1 \mid (a = 3a')$, then the contradiction with a, b coprime.

The other cases of the expressions of d and d' with d, d' not coprime and $d > d'$ so that $N = C^l A^m = d.d'$ give also contradictions.

** C-2-2-1-2-4- Now, we consider the case $d = a_1.C^l$ where a_1 is a prime integer with $a_1 \nmid A_1$ and $A = a_1^r A_1$, $r \geq 1$. It follows that $d' = a_1^{mr-1} A_1^m$. We rewrite the equations (1.5.59-1.5.60):

$$a_1 C^l + a_1^{mr-1} A_1^m = 2(A^m + B') \quad (1.5.65)$$

$$a_1 C^l - a_1^{mr-1} A_1^m = 2B' = B^n \quad (1.5.66)$$

As $m \geq 3$, from the last two equations above, it follows that $a_1 \mid 2(A^m + B')$ and $a_1 \mid (2B')$. Then $a_1 = 2$, or $a_1 \mid (A^m + B')$ and $a_1 \mid B'$.

** C-2-2-1-2-4-1- We suppose $a_1 = 2 \implies 2 \mid (A^m = a'') \implies a_1 \mid (a''^2 = a') \implies a_1 \mid (a = 3a')$. But $b = 4p$, then the contradiction with a, b coprime.

** C-2-2-1-2-4-2- We suppose $a_1 \neq 2$, then $a_1 \mid (A^m + B')$ and $a_1 \mid B'$. It follows $a_1 \mid A^m$ and $a_1 \mid (B^n = 2B') \implies a_1 \mid p \implies a_1 \mid b = 4p$. From $a_1 \mid (A^m = a'') \implies a_1 \mid (a''^2 = a') \implies a_1 \mid (a = 3a')$, then the contradiction with a, b coprime.

The other cases of the expressions of d and d' with d, d' not coprime and $d > d'$ so that $N = C^l A^m = d.d'$ give also contradictions.

** C-2-2-1-2-5- Now, let $C = c_1^r C_1$ with c_1 a prime, $r \geq 1$ and $c_1 \nmid C_1$, we consider the case $d = C_1^l$ and $d' = c_1^{rl} A^m$ so that $d > d'$. We rewrite the equations (1.5.59-1.5.60):

$$C_1^l + c_1^{rl} A^m = 2(A^m + B') \quad (1.5.67)$$

$$C_1^l - c_1^{rl} A^m = 2B' = B^n \quad (1.5.68)$$

We obtain $c_1^{rl} A^m = A^m \implies c_1^{rl} = 1$, then the contradiction.

** C-2-2-1-2-6- Now, let $C = c_1^r C_1$ with c_1 a prime, $r \geq 1$ and $c_1 \nmid C_1$, we consider the case $d = C_1^l A^m$ and $d' = c_1^{rl}$ so that $d > d'$. We rewrite the equations (1.5.59-1.5.60):

$$C_1^l A^m + c_1^{rl} = 2(A^m + B') \quad (1.5.69)$$

$$C_1^l A^m - c_1^{rl} = 2B' = B^n \quad (1.5.70)$$

We obtain $c_1^{rl} = A^m \implies c_1 \mid A^m$, then $c_1 \mid A^m = a'' \implies c_1 \mid (a''^2 = a') \implies c_1 \mid (a = 3a')$. As $c_1 \mid C$ and $c_1 \mid A^m \implies c_1 \mid B^n$, it follows $c_1 \mid (p = b)$, then the contradiction with a, b coprime.

The other cases of the expressions of d and d' with d, d' coprime and $d > d'$ so that $N = C^l A^m = d.d'$ give also contradictions.

Finally, from the cases studied in the above paragraph C-2-2-1-2, we have found one suitable factorization of N that gives *a priori* no contradictions, it is the case $N = C^l A^m$, but $1 \ll \tau(N)$, it follows the contradiction with $Q(N) = [\tau(N)/2] \leq 1$. We conclude that the case $A^m \equiv 1 \pmod{6}$ and $B^n \equiv 0 \pmod{6}$ of the paragraph C-2-2-1-2 is to reject.

** C-2-2-1-3- Case $A^m \equiv 1 \pmod{6}$ and $B^n \equiv 2 \pmod{6}$, then B^n is even, see C-2-2-1-2-.

** C-2-2-1-4- Case $A^m \equiv 1 \pmod{6}$ and $B^n \equiv 3 \pmod{6}$, then $3 \mid B^n \implies B^n = 3B'$. As $p = A^{2m} + A^m B^n + B^{2n} \implies p \equiv 5 \pmod{6} \not\equiv 1 \pmod{6}$ (see (1.5.55)), then the contradiction and the case C-2-2-1-4- is to reject.

** C-2-2-1-5- Case $A^m \equiv 1 \pmod{6}$ and $B^n \equiv 5 \pmod{6}$, then $C^l \equiv 0 \pmod{6} \implies 2 \mid C^l$, see C-2-2-1-2-.

** C-2-2-1-6- Case $A^m \equiv 2 \pmod{6} \implies 2 \mid a'' \implies 2 \mid a$, but $2 \mid b$, then the contradiction with a, b coprime.

** C-2-2-1-7- Case $A^m \equiv 3 \pmod{6}$ and $B^n \equiv 1 \pmod{6}$, then $C^l \equiv 4 \pmod{6} \implies 2 \mid C^l \implies C^l = 2C'$, and C is even, see C-2-2-1-2-.

** C-2-2-1-8- Case $A^m \equiv 3 \pmod{6}$ and $B^n \equiv 2 \pmod{6}$, then B^n is even, see C-2-2-1-2-.

** C-2-2-1-9- Case $A^m \equiv 3 \pmod{6}$ and $B^n \equiv 4 \pmod{6}$, then B^n is even, see C-2-2-1-2-.

** C-2-2-1-10- Case $A^m \equiv 3 \pmod{6}$ and $B^n \equiv 5 \pmod{6}$, then $C^l \equiv 2 \pmod{6} \implies 2 \mid C^l$, and C is even, see C-2-2-1-2-.

** C-2-2-1-11- Case $A^m \equiv 4 \pmod{6} \implies 2 \mid a'' \implies 2 \mid a$, but $2 \mid b$, then the contradiction with a, b coprime.

** C-2-2-1-12- Case $A^m \equiv 5 \pmod{6}$ and $B^n \equiv 0 \pmod{6}$, then B^n is even, see C-2-2-1-2-.

** C-2-2-1-13- Case $A^m \equiv 5 \pmod{6}$ and $B^n \equiv 1 \pmod{6}$, then $C^l \equiv 0 \pmod{6} \implies 2 \mid C^l$, C is even, see C-2-2-1-2-.

** C-2-2-1-14- Case $A^m \equiv 5 \pmod{6}$ and $B^n \equiv 3 \pmod{6}$, then $C^l \equiv 2 \pmod{6} \implies 2 \mid C^l \implies C^l = 2C'$, C is even, C-2-2-1-2-.

** C-2-2-1-15- Case $A^m \equiv 5 \pmod{6}$ and $B^n \equiv 4 \pmod{6}$, then B^n is even, see C-2-2-1-2-.

We have achieved the study all the cases of the table 1.1 giving contradictions.

Then the case $k_3 = 1$ is impossible.

1.5.7 Case $3 \mid a$ and $b = 2p'$, $b \neq 2$ with $p' \mid p$

$3 \mid a \implies a = 3a'$, $b = 2p'$ with $p = k.p'$, then:

$$A^{2m} = \frac{4.p}{3} \cdot \frac{a}{b} = \frac{4.k.p'.3.a'}{6p'} = 2.k.a'$$

We calculate $B^n C^l$:

$$B^n C^l = \sqrt[3]{\rho^2} \left(3 \sin^2 \frac{\theta}{3} - \cos^2 \frac{\theta}{3} \right) = \sqrt[3]{\rho^2} \left(3 - 4 \cos^2 \frac{\theta}{3} \right)$$

but $\sqrt[3]{\rho^2} = \frac{p}{3}$, then using $\cos^2 \frac{\theta}{3} = \frac{3.a'}{b}$:

$$B^n C^l = \sqrt[3]{\rho^2} \left(3 - 4 \cos^2 \frac{\theta}{3} \right) = \frac{p}{3} \left(3 - 4 \frac{3.a'}{b} \right) = p \cdot \left(1 - \frac{4.a'}{b} \right) = k(p' - 2a')$$

As $p = b.p'$, and $p' > 1$, then we have:

$$B^n C^l = k(p' - 2a') \quad (1.5.71)$$

$$\text{and } A^{2m} = 2k.a' \quad (1.5.72)$$

** D-1- We suppose that k is prime.

** D-1-1- If $k = 2$, then we have $p = 2p' = b \implies 2 \mid b$, but $A^{2m} = 4a' = (A^m)^2 \implies A^m = 2a''$ with $a' = a''^2$, then $2 \mid a'' \implies 2 \mid (a = 3a''^2)$, it follows the contradiction with a, b coprime.

** D-1-2- We suppose $k \neq 2$. From $A^{2m} = 2k.a' = (A^m)^2 \implies k \mid a'$ and $2 \mid a' \implies a' = 2.k.a''^2 \implies A^m = 2.k.a''$. Then $k \mid A^m \implies k \mid A \implies A = k^i.A_1$ with $i \geq 1$ and $k \nmid A_1$. $k^{im}A_1^m = 2ka'' \implies 2a'' = k^{im-1}A_1^m$. From $B^n C^l = k(p' - 2a') \implies k \mid (B^n C^l) \implies k \mid B^n$ or $k \mid C^l$.

** D-1-2-1- We suppose that $k \mid B^n \implies k \mid B \implies B = k^j.B_1$ with $j \geq 1$ and $k \nmid B_1$. It follows $k^{nj-1}B_1^n C^l = p' - 2a' = p' - 4ka''^2$. As $n \geq 3 \implies nj - 1 \geq 2$, then $k \mid p'$ but $k \neq 2 \implies k \mid (2p' = b)$, but $k \mid a' \implies k \mid (3a' = a)$. It follows the contradiction with a, b coprime.

** D-1-2-2- If $k \mid C^l$ we obtain the identical results.

** D-2- We suppose that k is not prime. Let ω be an integer prime so that $k = \omega^s.k_1$, with $s \geq 1$, $\omega \nmid k_1$. The equations (1.5.71-1.5.72) become:

$$B^n C^l = \omega^s.k_1(p' - 2a')$$

$$\text{and } A^{2m} = 2\omega^s.k_1.a'$$

** D-2-1- We suppose that $\omega = 2$, then we have the equations:

$$A^{2m} = 2^{s+1}.k_1.a' \quad (1.5.73)$$

$$B^n C^l = 2^s.k_1(p' - 2a') \quad (1.5.74)$$

** D-2-1-1- Case: $2 \mid a' \implies 2 \mid a$, but $2 \mid b$, then the contradiction with a, b coprime.

** D-2-1-2- Case: $2 \nmid a'$. As $2 \nmid k_1$, the equation (1.5.73) gives $2 \mid A^{2m} \implies A = 2^i.A_1$, with $i \geq 1$ and $2 \nmid A_1$. It follows that $2im = s + 1$.

** D-2-1-2-1- We suppose that $2 \nmid (p' - 2a') \implies 2 \nmid p'$. From the equation (1.5.74), we obtain that $2 \mid B^n C^l \implies 2 \mid B^n$ or $2 \mid C^l$.

** D-2-1-2-1-1- We suppose that $2 \mid B^n \implies 2 \mid B \implies B = 2^j.B_1$ with $2 \nmid B_1$ and $j \geq 1$, then $B_1^n C^l = 2^{s-jn}k_1(p' - 2a')$:

- If $s - jn \geq 1$, then $2 \mid C^l \implies 2 \mid C$, and no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$, and the conjecture (3.1.1) is verified.

- If $s - jn \leq 0$, from $B_1^n C^l = 2^{s-jn} k_1 (p' - 2a') \implies 2 \nmid C^l$, then the contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n \implies 2 \mid C^l$.

** D-2-1-2-1-2- Using the same method of the proof above, we obtain the identical results if $2 \mid C^l$.

** D-2-1-2-2- We suppose now that $2 \mid (p' - 2a') \implies p' - 2a' = 2^\mu \cdot \Omega$, with $\mu \geq 1$ and $2 \nmid \Omega$. We recall that $2 \nmid a'$. The equation (1.5.74) is written as:

$$B^n C^l = 2^{s+\mu} \cdot k_1 \cdot \Omega$$

This last equation implies that $2 \mid (B^n C^l) \implies 2 \mid B^n$ or $2 \mid C^l$.

** D-2-1-2-2-1- We suppose that $2 \mid B^n \implies 2 \mid B \implies B = 2^j B_1$ with $j \geq 1$ and $2 \nmid B_1$. Then $B_1^n C^l = 2^{s+\mu-jn} \cdot k_1 \cdot \Omega$:

- If $s + \mu - jn \geq 1$, then $2 \mid C^l \implies 2 \mid C$, no contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$, and the conjecture (3.1.1) is verified.

- If $s + \mu - jn \leq 0$, from $B_1^n C^l = 2^{s+\mu-jn} k_1 \cdot \Omega \implies 2 \nmid C^l$, then contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n \implies 2 \mid C^l$.

** D-2-1-2-2-2- We obtain the identical results if $2 \mid C^l$.

** D-2-2- We suppose that $\omega \neq 2$. We have then the equations:

$$A^{2m} = 2\omega^s \cdot k_1 \cdot a' \quad (1.5.75)$$

$$B^n C^l = \omega^s \cdot k_1 \cdot (p' - 2a') \quad (1.5.76)$$

As $\omega \neq 2$, from the equation (1.5.75), we have $2 \mid (k_1 \cdot a')$. If $2 \mid a' \implies 2 \mid a$, but $2 \nmid b$, then the contradiction with a, b coprime.

** D-2-2-1- Case: $2 \nmid a'$ and $2 \mid k_1 \implies k_1 = 2^\mu \cdot \Omega$ with $\mu \geq 1$ and $2 \nmid \Omega$. From the equation (1.5.75), we have $2 \mid A^{2m} \implies 2 \mid A \implies A = 2^i A_1$ with $i \geq 1$ and $2 \nmid A_1$, then $2im = 1 + \mu$. The equation (1.5.76) becomes:

$$B^n C^l = \omega^s \cdot 2^\mu \cdot \Omega \cdot (p' - 2a') \quad (1.5.77)$$

From the equation (1.5.77), we obtain $2 \mid (B^n C^l) \implies 2 \mid B^n$ or $2 \mid C^l$.

** D-2-2-1-1- We suppose that $2 \mid B^n \implies 2 \mid B \implies B = 2^j B_1$, with $j \in \mathbb{N}^*$ and $2 \nmid B_1$.

** D-2-2-1-1-1- We suppose that $2 \nmid (p' - 2a')$, then we have $B_1^n C^l = \omega^s 2^{\mu-jn} \Omega (p' - 2a')$:

- If $\mu - jn \geq 1 \implies 2 \mid C^l \implies 2 \mid C$, no contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$ and the conjecture (3.1.1) is verified.

- If $\mu - jn \leq 0 \implies 2 \nmid C^l$ then the contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$.

** D-2-2-1-1-2- We suppose that $2 \mid (p' - 2a') \implies p' - 2a' = 2^\alpha \cdot P$, with $\alpha \in \mathbb{N}^*$ and $2 \nmid P$. It follows that $B_1^n C^l = \omega^s 2^{\mu+\alpha-jn} \Omega \cdot P$:

- If $\mu + \alpha - jn \geq 1 \implies 2 \mid C^l \implies 2 \mid C$, no contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$ and the conjecture (3.1.1) is verified.

- If $\mu + \alpha - jn \leq 0 \implies 2 \nmid C^l$ then the contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$.

** D-2-2-1-2- We suppose now that $2 \mid C^n \implies 2 \mid C$. Using the same method described above, we obtain the identical results.

1.5.8 Case 3 | a and $b = 4p'$, $b \neq 4$ with $p' | p$

$3 | a \implies a = 3a'$, $b = 4p'$ with $p = k.p'$, $k \neq 1$ if not $b = 4p$ this case has been studied (see paragraph 1.5.6), then we have :

$$A^{2m} = \frac{4.p}{3} \cdot \frac{a}{b} = \frac{4.k.p'.3.a'}{12p'} = k.a'$$

We calculate $B^n C^l$:

$$B^n C^l = \sqrt[3]{\rho^2} \left(3 \sin^2 \frac{\theta}{3} - \cos^2 \frac{\theta}{3} \right) = \sqrt[3]{\rho^2} \left(3 - 4 \cos^2 \frac{\theta}{3} \right)$$

but $\sqrt[3]{\rho^2} = \frac{p}{3}$, then using $\cos^2 \frac{\theta}{3} = \frac{3.a'}{b}$:

$$B^n C^l = \sqrt[3]{\rho^2} \left(3 - 4 \cos^2 \frac{\theta}{3} \right) = \frac{p}{3} \left(3 - 4 \frac{3.a'}{b} \right) = p \cdot \left(1 - \frac{4.a'}{b} \right) = k(p' - a')$$

As $p = b.p'$, and $p' > 1$, we have :

$$B^n C^l = k(p' - a') \quad (1.5.78)$$

$$\text{and } A^{2m} = k.a' \quad (1.5.79)$$

** E-1- We suppose that k is prime. From $A^{2m} = k.a' = (A^m)^2 \implies k | a'$ and $a' = k.a'' \implies A^m = k.a''$. Then $k | A^m \implies k | A \implies A = k^i.A_1$ with $i \geq 1$ and $k \nmid A_1$. $k^{mi}A_1^m = ka'' \implies a'' = k^{mi-1}A_1^m$. From $B^n C^l = k(p' - a') \implies k | (B^n C^l) \implies k | B^n$ or $k | C^l$.

** E-1-1- We suppose that $k | B^n \implies k | B \implies B = k^j.B_1$ with $j \geq 1$ and $k \nmid B_1$. Then $k^{n.j-1}B_1^n C^l = p' - a'$. As $n.j - 1 \geq 2 \implies k | (p' - a')$. But $k | a' \implies k | a$, then $k | p' \implies k | (4p' = b)$ and we arrive to the contradiction that a, b are coprime.

** E-1-2- We suppose that $k | C^l$, using the same method with the above hypothesis $k | B^n$, we obtain the identical results.

** E-2- We suppose that k is not prime.

** E-2-1- We take $k = 4 \implies p = 4p' = b$, it is the case 1.5.3 studied above.

** E-2-2- We suppose that $k \geq 6$ not prime. Let ω be a prime so that $k = \omega^s.k_1$, with $s \geq 1$, $\omega \nmid k_1$. The equations (1.5.78-1.5.79) become:

$$B^n C^l = \omega^s.k_1(p' - a') \quad (1.5.80)$$

$$\text{and } A^{2m} = \omega^s.k_1.a' \quad (1.5.81)$$

** E-2-2-1- We suppose that $\omega = 2$.

** E-2-2-1-1- If $2 | a' \implies 2 | (3a' = a)$, but $2 | (4p' = b)$, then the contradiction with a, b coprime.

** E-2-2-1-2- We consider that $2 \nmid a'$. From the equation (1.5.81), it follows that $2 | A^{2m} \implies 2 | A \implies A = 2^i A_1$ with $2 \nmid A_1$ and:

$$B^n C^l = 2^s k_1 (p' - a')$$

** E-2-2-1-2-1- We suppose that $2 \nmid (p' - a')$, from the above expression, we have $2 \mid (B^n C^l) \implies 2 \mid B^n$ or $2 \mid C^l$.

** E-2-2-1-2-1-1- If $2 \mid B^n \implies 2 \mid B \implies B = 2^j B_1$ with $2 \nmid B_1$. Then $B_1^n C^l = 2^{2im-jn} k_1 (p' - a')$:
 - If $2im - jn \geq 1 \implies 2 \mid C^l \implies 2 \mid C$, no contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$ and the conjecture (3.1.1) is verified.

- If $2im - jn \leq 0 \implies 2 \nmid C^l$, then the contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n \implies 2 \mid C^l$.

** E-2-2-1-2-1-2- If $2 \mid C^l \implies 2 \mid C$, using the same method described above, we obtain the identical results.

** E-2-2-1-2-2- We suppose that $2 \mid (p' - a')$. As $2 \nmid a' \implies 2 \nmid p'$, $2 \mid (p' - a') \implies p' - a' = 2^\alpha .P$ with $\alpha \geq 1$ and $2 \nmid P$. The equation (1.5.80) is written as :

$$B^n C^l = 2^{s+\alpha} k_1 .P = 2^{2im+\alpha} k_1 .P \quad (1.5.82)$$

then $2 \mid (B^n C^l) \implies 2 \mid B^n$ or $2 \mid C^l$.

** E-2-2-1-2-2-1- We suppose that $2 \mid B^n \implies 2 \mid B \implies B = 2^j B_1$, with $2 \nmid B_1$. The equation (1.5.82) becomes $B_1^n C^l = 2^{2im+\alpha-jn} k_1 P$:

- If $2im + \alpha - jn \geq 1 \implies 2 \mid C^l \implies 2 \mid C$, no contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$ and the conjecture (3.1.1) is verified.

- If $2im + \alpha - jn \leq 0 \implies 2 \nmid C^l$, then the contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n \implies 2 \mid C^l$.

** E-2-2-1-2-2-2- We suppose that $2 \mid C^l \implies 2 \mid C$. Using the same method described above, we obtain the identical results.

** E-2-2-2- We suppose that $\omega \neq 2$. We recall the equations:

$$A^{2m} = \omega^s .k_1 .a' \quad (1.5.83)$$

$$B^n C^l = \omega^s .k_1 (p' - a') \quad (1.5.84)$$

** E-2-2-2-1- We suppose that ω, a' are coprime, then $\omega \nmid a'$. From the equation (1.5.83), we have $\omega \mid A^{2m} \implies \omega \mid A \implies A = \omega^i A_1$ with $\omega \nmid A_1$ and $s = 2im$.

** E-2-2-2-1-1- We suppose that $\omega \nmid (p' - a')$. From the equation (1.5.84) above, we have $\omega \mid (B^n C^l) \implies \omega \mid B^n$ or $\omega \mid C^l$.

** E-2-2-2-1-1-1- If $\omega \mid B^n \implies \omega \mid B \implies B = \omega^j B_1$ with $\omega \nmid B_1$. Then $B_1^n C^l = 2^{2im-jn} k_1 (p' - a')$:

- If $2im - jn \geq 1 \implies \omega \mid C^l \implies \omega \mid C$, no contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n$ and the conjecture (3.1.1) is verified.

- If $2im - jn \leq 0 \implies \omega \nmid C^l$, then the contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n \implies \omega \mid C^l$.

** E-2-2-2-1-1-2- If $\omega \mid C^l \implies \omega \mid C$, using the same method described above, we obtain the identical results.

** E-2-2-2-1-2- We suppose that $\omega \mid (p' - a') \implies \omega \nmid p'$ as ω and a' are coprime. $\omega \mid (p' - a') \implies p' - a' = \omega^\alpha \cdot P$ with $\alpha \geq 1$ and $\omega \nmid P$. The equation (1.5.84) becomes :

$$B^n C^l = \omega^{s+\alpha} k_1 \cdot P = \omega^{2im+\alpha} k_1 \cdot P \quad (1.5.85)$$

then $\omega \mid (B^n C^l) \implies \omega \mid B^n$ or $\omega \mid C^l$.

** E-2-2-2-1-2-1- We suppose that $\omega \mid B^n \implies \omega \mid B \implies B = \omega^j B_1$, with $\omega \nmid B_1$. The equation (1.5.85) is written as $B_1^n C^l = 2^{2im+\alpha-jn} k_1 P$:

- If $2im + \alpha - jn \geq 1 \implies \omega \mid C^l \implies \omega \mid C$, no contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n$ and the conjecture (3.1.1) is verified.

- If $2im + \alpha - jn \leq 0 \implies \omega \nmid C^l$, then the contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n \implies \omega \mid C^l$.

** E-2-2-2-1-2-2- We suppose that $\omega \mid C^l \implies \omega \mid C$, using the same method described above, we obtain the identical results.

** E-2-2-2-2- We suppose that ω, a' are not coprime, then $a' = \omega^\beta \cdot a''$ with $\omega \nmid a''$. The equation (1.5.83) becomes:

$$A^{2m} = \omega^s k_1 a' = \omega^{s+\beta} k_1 \cdot a''$$

We have $\omega \mid A^{2m} \implies \omega \mid A \implies A = \omega^i A_1$ with $\omega \nmid A_1$ and $s + \beta = 2im$.

** E-2-2-2-2-1- We suppose that $\omega \nmid (p' - a') \implies \omega \nmid p' \implies \omega \nmid (b = 4p')$. From the equation (1.5.84), we obtain $\omega \mid (B^n C^l) \implies \omega \mid B^n$ or $\omega \mid C^l$.

** E-2-2-2-2-1-1- If $\omega \mid B^n \implies \omega \mid B \implies B = \omega^j B_1$ with $\omega \nmid B_1$. Then $B_1^n C^l = 2^{s-jn} k_1 (p' - a')$:

- If $s - jn \geq 1 \implies \omega \mid C^l \implies \omega \mid C$, no contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n$ and the conjecture (3.1.1) is verified.

- If $s - jn \leq 0 \implies \omega \nmid C^l$, then the contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n \implies \omega \mid C^l$.

** E-2-2-2-2-1-2- If $\omega \mid C^l \implies \omega \mid C$, using the same method described above, we obtain the identical results.

** E-2-2-2-2-2- We suppose that $\omega \mid (p' - a' = p' - \omega^\beta \cdot a'') \implies \omega \mid p' \implies \omega \mid (4p' = b)$, but $\omega \mid a' \implies \omega \mid a$. Then the contradiction with a, b coprime.

The study of the cases of 1.5.8 is achieved.

1.5.9 Case 3 | a and b | $4p$

$a = 3a'$ and $4p = k_1 b$. As $A^{2m} = \frac{4p}{3} \cos^2 \frac{\theta}{3} = \frac{4p}{3} \frac{3a'}{b} = k_1 a'$ and $B^n C^l$:

$$B^n C^l = \sqrt[3]{\rho^2} \left(3 \sin^2 \frac{\theta}{3} - \cos^2 \frac{\theta}{3} \right) = \frac{p}{3} \left(3 - 4 \cos^2 \frac{\theta}{3} \right) = \frac{p}{3} \left(3 - 4 \frac{3a'}{b} \right) = \frac{k_1}{4} (b - 4a')$$

As $B^n C^l$ is an integer, we must obtain $4 \mid k_1$, or $4 \mid (b - 4a')$ or $(2 \mid k_1$ and $2 \mid (b - 4a'))$.

** F-1- If $k_1 = 1 \implies b = 4p$: it is the case 1.5.6.

** F-2- If $k_1 = 4 \Rightarrow p = b$: it is the case 1.5.3.

** F-3- If $k_1 = 2$ and $2 \mid (b - 4a')$: in this case, we have $A^{2m} = 2a' \Rightarrow 2 \mid a' \Rightarrow 2 \mid a$. $2 \mid (b - 4a') \Rightarrow 2 \mid b$ then the contradiction with a, b coprime.

** F-4- If $2 \mid k_1$ and $2 \mid (b - 4a')$: $2 \mid (b - 4a') \Rightarrow b - 4a' = 2^\alpha \lambda$, α and $\lambda \in \mathbb{N}^* \geq 1$ with $2 \nmid \lambda$; $2 \mid k_1 \Rightarrow k_1 = 2^t k'_1$ with $t \geq 1 \in \mathbb{N}^*$ with $2 \nmid k'_1$ and we have:

$$A^{2m} = 2^t k'_1 a' \quad (1.5.86)$$

$$B^n C^l = 2^{t+\alpha-2} k'_1 \lambda \quad (1.5.87)$$

From the equation (1.5.86), we have $2 \mid A^{2m} \Rightarrow 2 \mid A \Rightarrow A = 2^i A_1$, $i \geq 1$ and $2 \nmid A_1$.

** F-4-1- We suppose that $t = \alpha = 1$, then the equations (1.5.86-1.5.87) become :

$$A^{2m} = 2k'_1 a' \quad (1.5.88)$$

$$B^n C^l = k'_1 \lambda \quad (1.5.89)$$

From the equation (1.5.88) it follows that $2 \mid a' \Rightarrow 2 \mid (a = 3a')$. But $b = 4a' + 2\lambda \Rightarrow 2 \mid b$, then the contradiction with a, b coprime.

** F-4-2- We suppose that $t + \alpha - 2 \geq 1$ and we have the expressions:

$$A^{2m} = 2^t k'_1 a' \quad (1.5.90)$$

$$B^n C^l = 2^{t+\alpha-2} k'_1 \lambda \quad (1.5.91)$$

** F-4-2-1- We suppose that $2 \mid a' \Rightarrow 2 \mid a$, but $b = 2^\alpha \lambda + 4a' \Rightarrow 2 \mid b$, then the contradiction with a, b coprime.

** F-4-2-2- We suppose that $2 \nmid a'$. From (1.5.90), we have $2 \mid A^{2m} \Rightarrow 2 \mid A \Rightarrow A = 2^i A_1$ and $B^n C^l = 2^{t+\alpha-2} k'_1 \lambda \Rightarrow 2 \mid B^n C^l \Rightarrow 2 \mid B^n$ or $2 \mid C^l$.

** F-4-2-2-1- We suppose that $2 \mid B^n$. We have $2 \mid B \Rightarrow B = 2^j B_1$, $j \geq 1$ and $2 \nmid B_1$. The equation (1.5.91) becomes $B_1^n C^l = 2^{t+\alpha-2-jn} k'_1 \lambda$:

- If $t + \alpha - 2 - jn > 0 \Rightarrow 2 \mid C^l \Rightarrow 2 \mid C$, no contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$ and the conjecture (3.1.1) is verified.

- If $t + \alpha - 2 - jn < 0 \Rightarrow 2 \mid k'_1 \lambda$, but $2 \nmid k'_1$ and $2 \nmid \lambda$. Then this case is impossible.

- If $t + \alpha - 2 - jn = 0 \Rightarrow B_1^n C^l = k'_1 \lambda \Rightarrow 2 \nmid C^l$ then it is a contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$.

** F-4-2-2-2- We suppose that $2 \mid C^l$. We use the same method described above, we obtain the identical results.

** F-5- We suppose that $4 \mid k_1$ with $k_1 > 4 \Rightarrow k_1 = 4k'_2$, we have :

$$A^{2m} = 4k'_2 a' \quad (1.5.92)$$

$$B^n C^l = k'_2 (b - 4a') \quad (1.5.93)$$

** F-5-1- We suppose that k'_2 is prime, from (1.5.92), we have $k'_2 \mid a'$. From (1.5.93), $k'_2 \mid (B^n C^l) \Rightarrow k'_2 \mid B^n$ or $k'_2 \mid C^l$.

** F-5-1-1- We suppose that $k'_2 \mid B^n \implies k'_2 \mid B \implies B = k_2'^\beta \cdot B_1$ with $\beta \geq 1$ and $k'_2 \nmid B_1$. It follows that we have $k_2'^{n\beta-1} B_1^n C^l = b - 4a' \implies k'_2 \mid b$ then the contradiction with a, b coprime.

** F-5-1-2- We obtain identical results if we suppose that $k'_2 \mid C^l$.

** F-5-2- We suppose that k'_2 is not prime.

** F-5-2-1- We suppose that k'_2 and a' are coprime. From (1.5.92), k'_2 can be written under the form $k'_2 = q_1^{2j} \cdot q_2^2$ and $q_1 \nmid q_2$ and q_1 prime. We have $A^{2m} = 4q_1^{2j} \cdot q_2^2 a' \implies q_1 \mid A$ and $B^n C^l = q_1^{2j} \cdot q_2^2 (b - 4a') \implies q_1 \mid B^n$ or $q_1 \mid C^l$.

** F-5-2-1-1- We suppose that $q_1 \mid B^n \implies q_1 \mid B \implies B = q_1^f \cdot B_1$ with $q_1 \nmid B_1$. We obtain $B_1^n C^l = q_1^{2j-fn} q_2^2 (b - 4a')$:

- If $2j - f \cdot n \geq 1 \implies q_1 \mid C^l \implies q_1 \mid C$ but $C^l = A^m + B^n$ gives also $q_1 \mid C$ and the conjecture (3.1.1) is verified.

- If $2j - f \cdot n = 0$, we have $B_1^n C^l = q_2^2 (b - 4a')$, but $C^l = A^m + B^n$ gives $q_1 \mid C$, then $q_1 \mid (b - 4a')$. As q_1 and a' are coprime, then $q_1 \nmid b$, and the conjecture (3.1.1) is verified.

- If $2j - f \cdot n < 0 \implies q_1 \mid (b - 4a') \implies q_1 \nmid b$ because a' is coprime with q_1 , and $C^l = A^m + B^n$ gives $q_1 \mid C$, and the conjecture (3.1.1) is verified.

** F-5-2-1-2- We obtain identical results if we suppose that $q_1 \mid C^l$.

** F-5-2-2- We suppose that k'_2, a' are not coprime. Let q_1 be a prime so that $q_1 \mid k'_2$ and $q_1 \mid a'$. We write k'_2 under the form $q_1^j \cdot q_2$ with $j \geq 1$, $q_1 \nmid q_2$. From $A^{2m} = 4k'_2 a' \implies q_1 \mid A^{2m} \implies q_1 \mid A$. Then from $B^n C^l = q_1^j q_2 (b - 4a')$, it follows that $q_1 \mid (B^n C^l) \implies q_1 \mid B^n$ or $q_1 \mid C^l$.

** F-5-2-2-1- We suppose that $q_1 \mid B^n \implies q_1 \mid B \implies B = q_1^\beta \cdot B_1$ with $\beta \geq 1$ and $q_1 \nmid B_1$. Then, we have $q_1^{n\beta} B_1^n C^l = q_1^j q_2 (b - 4a') \implies B_1^n C^l = q_1^{j-n\beta} q_2 (b - 4a')$.

- If $j - n\beta \geq 1$, then $q_1 \mid C^l \implies q_1 \mid C$, but $C^l = A^m + B^n$ gives $q_1 \mid C$, then the conjecture (3.1.1) is verified.

- If $j - n\beta = 0$, we obtain $B_1^n C^l = q_2 (b - 4a')$, but $C^l = A^m + B^n$ gives $q_1 \mid C$, then $q_1 \mid (b - 4a') \implies q_1 \mid b$ because $q_1 \mid a' \implies q_1 \mid a$, then the contradiction with a, b coprime.

- If $j - n\beta < 0 \implies q_1 \mid (b - 4a') \implies q_1 \mid b$, because $q_1 \mid a' \implies q_1 \mid a$, then the contradiction with a, b coprime.

** F-5-2-2-2- We obtain identical results if we suppose that $q_1 \mid C^l$.

** F-6- If $4 \nmid (b - 4a')$ and $4 \nmid k_1$ it is impossible. We suppose that $4 \mid (b - 4a') \implies 4 \mid b$, and $b - 4a' = 4^t \cdot g$, $t \geq 1$ with $4 \nmid g$, then we have :

$$\begin{aligned} A^{2m} &= k_1 a' \\ B^n C^l &= k_1 \cdot 4^{t-1} \cdot g \end{aligned}$$

** F-6-1- We suppose that k_1 is prime. From $A^{2m} = k_1 a'$ we deduce easily that $k_1 \mid a'$. From $B^n C^l = k_1 \cdot 4^{t-1} \cdot g$ we obtain that $k_1 \mid (B^n C^l) \implies k_1 \mid B^n$ or $k_1 \mid C^l$.

** F-6-1-1- We suppose that $k_1 \mid B^n \implies k_1 \mid B \implies B = k_1^j \cdot B_1$ with $j > 0$ and $k_1 \nmid B_1$, then $k_1^{n \cdot j} B_1^n C^l = k_1 \cdot 4^{t-1} \cdot g \implies k_1^{n \cdot j - 1} B_1^n C^l = 4^{t-1} \cdot g$. But $n \geq 3$ and $j \geq 1$, then $n \cdot j - 1 \geq 2$. We deduce as $k_1 \neq 2$ that $k_1 \mid g \implies k_1 \mid (b - 4a')$, but $k_1 \mid a' \implies k_1 \mid b$, then the contradiction with a, b coprime.

** F-6-1-2- We obtain identical results if we suppose that $k_1 \mid C^l$.

** F-6-2- We suppose that k_1 is not prime $\neq 4$, ($k_1 = 4$ see case F-2, above) with $4 \nmid k_1$.

** F-6-2-1- If $k_1 = 2k'$ with k' odd > 1 . Then $A^{2m} = 2k'a' \implies 2 \mid a' \implies 2 \mid a$, as $4 \mid b$ it follows the contradiction with a, b coprime.

** F-6-2-2- We suppose that k_1 is odd with k_1 and a' coprime. We write k_1 under the form $k_1 = q_1^j \cdot q_2$ with $q_1 \nmid q_2$, q_1 prime and $j \geq 1$. $B^n C^l = q_1^j \cdot q_2 4^{t-1} g \implies q_1 \mid B^n$ or $q_1 \mid C^l$.

** F-6-2-2-1- We suppose that $q_1 \mid B^n \implies q_1 \mid B \implies B = q_1^f \cdot B_1$ with $q_1 \nmid B_1$. We obtain $B_1^n C^l = q_1^{j-f \cdot n} q_2 4^{t-1} g$.

- If $j - f \cdot n \geq 1 \implies q_1 \mid C^l \implies q_1 \mid C$, but $C^l = A^m + B^n$ gives also $q_1 \mid C$ and the conjecture (3.1.1) is verified.

- If $j - f \cdot n = 0$, we have $B_1^n C^l = q_2 4^{t-1} g$, but $C^l = A^m + B^n$ gives $q_1 \mid C$, then $q_1 \mid (b - 4a')$. As q_1 and a' are coprime then $q_1 \nmid b$ and the conjecture (3.1.1) is verified.

- If $j - f \cdot n < 0 \implies q_1 \mid (b - 4a') \implies q_1 \nmid b$ because q_1, a' are primes. $C^l = A^m + B^n$ gives $q_1 \mid C$ and the conjecture (3.1.1) is verified.

** F-6-2-2-2- We obtain identical results if we suppose that $q_1 \mid C^l$.

** F-6-2-3- We suppose that k_1 and a' are not coprime. Let q_1 be a prime so that $q_1 \mid k_1$ and $q_1 \mid a'$. We write k_1 under the form $q_1^j \cdot q_2$ with $q_1 \nmid q_2$. From $A^{2m} = k_1 a' \implies q_1 \mid A^{2m} \implies q_1 \mid A$. From $B^n C^l = q_1^j q_2 (b - 4a')$, it follows that $q_1 \mid (B^n C^l) \implies q_1 \mid B^n$ or $q_1 \mid C^l$.

** F-6-2-3-1- We suppose that $q_1 \mid B^n \implies q_1 \mid B \implies B = q_1^\beta \cdot B_1$ with $\beta \geq 1$ and $q_1 \nmid B_1$. Then we have $q_1^{n\beta} B_1^n C^l = q_1^j q_2 (b - 4a') \implies B_1^n C^l = q_1^{j-n\beta} q_2 (b - 4a')$:

- If $j - n\beta \geq 1$, then $q_1 \mid C^l \implies q_1 \mid C$, but $C^l = A^m + B^n$ gives $q_1 \mid C$, and the conjecture (3.1.1) is verified.

- If $j - n\beta = 0$, we obtain $B_1^n C^l = q_2 (b - 4a')$, but $q_1 \mid A$ and $q_1 \mid B$ then $q_1 \mid C$ and we obtain $q_1 \mid (b - 4a') \implies q_1 \mid b$ because $q_1 \mid a' \implies q_1 \mid a$, then the contradiction with a, b coprime.

- If $j - n\beta < 0 \implies q_1 \mid (b - 4a') \implies q_1 \mid b$, then the contradiction with a, b coprime.

** F-6-2-3-2- We obtain identical results as above if we suppose that $q_1 \mid C^l$.

1.6 Hypothesis: $\{3 \mid p \text{ and } b \mid 4p\}$

1.6.1 Case $b = 2$ and $3 \mid p$

$3 \mid p \implies p = 3p'$ with $p' \neq 1$ because $3 \ll p$, and $b = 2$, we obtain:

$$A^{2m} = \frac{4p \cdot a}{3b} = \frac{4 \cdot 3p' \cdot a}{3b} = \frac{4 \cdot p' \cdot a}{2} = 2 \cdot p' \cdot a$$

As:

$$\frac{1}{4} < \cos^2 \frac{\theta}{3} = \frac{a}{b} = \frac{a}{2} < \frac{3}{4} \Rightarrow 1 < 2a < 3 \Rightarrow a = 1 \Rightarrow \cos^2 \frac{\theta}{3} = \frac{1}{2}$$

but this case was studied (see case 1.4.1).

1.6.2 Case $b = 4$ and $3 \mid p$

we have $3 \mid p \Rightarrow p = 3p'$ with $p' \in \mathbb{N}^*$, it follows :

$$A^{2m} = \frac{4p.a}{3b} = \frac{4.3p'.a}{3 \times 4} = p'.a$$

and:

$$\frac{1}{4} < \cos^2 \frac{\theta}{3} = \frac{a}{b} = \frac{a}{4} < \frac{3}{4} \Rightarrow 1 < a < 3 \Rightarrow a = 2$$

as a, b are coprime, then the case $b = 4$ and $3 \mid p$ is impossible.

1.6.3 Case: $b \neq 2, b \neq 4, b \neq 3, b \mid p$ and $3 \mid p$

As $3 \mid p$, then $p = 3p'$ and :

$$A^{2m} = \frac{4p}{3} \cos^2 \frac{\theta}{3} = \frac{4p.a}{3b} = \frac{4 \times 3p'.a}{3b} = \frac{4p'.a}{b}$$

We consider the case: $b \mid p' \Rightarrow p' = bp''$ and $p'' \neq 1$ (If $p'' = 1$, then $p = 3b$, see paragraph 1.6.8 Case $k' = 1$). Finally, we obtain:

$$A^{2m} = \frac{4bp''a}{b} = 4ap''; \quad B^n C^l = p''.(3b - 4a)$$

** G-1- We suppose that p'' is prime, then $A^{2m} = 4ap'' = (A^m)^2 \Rightarrow p'' \mid a$. But $B^n C^l = p''(3b - 4a) \Rightarrow p'' \mid B^n$ or $p'' \mid C^l$.

** G-1-1- If $p'' \mid B^n \Rightarrow p'' \mid B \Rightarrow B = p''B_1$ with $B_1 \in \mathbb{N}^*$. Then $p''^{n-1}B_1^n C^l = 3b - 4a$. As $n > 2$, then $(n-1) > 1$ and $p'' \mid a$, then $p'' \mid 3b \Rightarrow p'' = 3$ or $p'' \mid b$.

** G-1-1-1- If $p'' = 3 \Rightarrow 3 \mid a$, with a that we write as $a = 3a'^2$, but $A^m = 6a' \Rightarrow 3 \mid A^m \Rightarrow 3 \mid A \Rightarrow A = 3A_1$, then $3^{m-1}A_1^m = 2a' \Rightarrow 3 \mid a' \Rightarrow a' = 3a''$. As $p''^{n-1}B_1^n C^l = 3^{n-1}B_1^n C^l = 3b - 4a \Rightarrow 3^{n-2}B_1^n C^l = b - 36a''^2$. As $n > 2 \Rightarrow n-2 \geq 1$, then $3 \mid b$ and the contradiction with a, b coprime.

** G-1-1-2- We suppose that $p'' \mid b$, as $p'' \mid a$, then the contradiction with a, b coprime.

** G-1-2- If we suppose $p'' \mid C^l$, we obtain identical results (contradictions).

** G-2- We consider now that p'' is not prime.

** G-2-1- p'', a coprime: $A^{2m} = 4ap'' \Rightarrow A^m = 2a'.p_1$ with $a = a'^2$ and $p'' = p_1^2$, then a', p_1 are also coprime. As $A^m = 2a'.p_1$, then $2 \mid a'$ or $2 \mid p_1$.

** G-2-1-1- We suppose that $2 \mid a'$, then $2 \mid a' \Rightarrow 2 \nmid p_1$, but $p'' = p_1^2$.

** G-2-1-1-1- If p_1 is prime, it is impossible with $A^m = 2a'.p_1$.

** G-2-1-1-2- We suppose that p_1 is not prime so we can write $p_1 = \omega^m \implies p'' = \omega^{2m}$. Then $B^n C^l = \omega^{2m}(3b - 4a)$.

** G-2-1-1-2-1- If ω is prime, $\omega \neq 2$, then $\omega \mid (B^n C^l) \implies \omega \mid B^n$ or $\omega \mid C^l$.

** G-2-1-1-2-1-1- If $\omega \mid B^n \implies \omega \mid B \implies B = \omega^j B_1$ with $\omega \nmid B_1$, then $B_1^n.C^l = \omega^{2m-nj}(3b - 4a)$.

** G-2-1-1-2-1-1-1- If $2m - n.j = 0$, we obtain $B_1^n.C^l = 3b - 4a$. As $C^l = A^m + B^n \implies \omega \mid C^l \implies \omega \mid C$, and $\omega \mid (3b - 4a)$. But $\omega \neq 2$ and ω, a' are coprime, then ω, a are coprime, it follows $\omega \nmid (3b)$, then $\omega \neq 3$ and $\omega \nmid b$, the conjecture (3.1.1) is verified.

** G-2-1-1-2-1-1-2- If $2m - n.j \geq 1$, using the method as above, we obtain $\omega \mid C^l \implies \omega \mid C$ and $\omega \mid (3b - 4a)$ and $\omega \nmid a$ and $\omega \neq 3$ and $\omega \nmid b$, then the conjecture (3.1.1) is verified.

** G-2-1-1-2-1-1-3- If $2m - n.j < 0 \implies \omega^{n.j-2m} B_1^n.C^l = 3b - 4a$. From $A^m + B^n = C^l \implies \omega \mid C^l \implies \omega \mid C$, then $C = \omega^h.C_1$, with $\omega \nmid C_1$, we obtain $\omega^{n.j-2m+h.l} B_1^n.C_1^l = 3b - 4a$. If $n.j - 2m + h.l < 0 \implies \omega \mid B_1^n.C_1^l$ then the contradiction with $\omega \nmid B_1$ or $\omega \nmid C_1$. It follows $n.j - 2m + h.l > 0$ and $\omega \mid (3b - 4a)$ with ω, a, b coprime and the conjecture is verified.

** G-2-1-1-2-1-2- Using the same method above, we obtain identical results if $\omega \mid C^l$.

** G-2-1-1-2-2- We suppose that $p'' = \omega^{2m}$ and ω is not prime. We write $\omega = \omega_1^f.\Omega$ with ω_1 prime $\nmid \Omega$, $f \geq 1$, and $\omega_1 \mid A$. Then $B^n C^l = \omega_1^{2f.m} \Omega^{2m}(3b - 4a) \implies \omega_1 \mid (B^n C^l) \implies \omega_1 \mid B^n$ or $\omega_1 \mid C^l$.

** G-2-1-1-2-2-1- If $\omega_1 \mid B^n \implies \omega_1 \mid B \implies B = \omega_1^j B_1$ with $\omega_1 \nmid B_1$, then $B_1^n.C^l = \omega_1^{2.m-nj} \Omega^{2m}(3b - 4a)$:

** G-2-1-1-2-2-1-1- If $2f.m - n.j = 0$, we obtain $B_1^n.C^l = \Omega^{2m}(3b - 4a)$. As $C^l = A^m + B^n \implies \omega_1 \mid C^l \implies \omega_1 \mid C$, and $\omega_1 \mid (3b - 4a)$. But $\omega_1 \neq 2$ and ω_1, a' are coprime, then ω, a are coprime, it follows $\omega_1 \nmid (3b)$, then $\omega_1 \neq 3$ and $\omega_1 \nmid b$, and the conjecture (3.1.1) is verified.

** G-2-1-1-2-2-1-2- If $2f.m - n.j \geq 1$, we have $\omega_1 \mid C^l \implies \omega_1 \mid C$ and $\omega_1 \mid (3b - 4a)$ and $\omega_1 \nmid a$ and $\omega_1 \neq 3$ and $\omega_1 \nmid b$, it follows that the conjecture (3.1.1) is verified.

** G-2-1-1-2-2-1-3- If $2f.m - n.j < 0 \implies \omega_1^{n.j-2m.f} B_1^n.C^l = \Omega^{2m}(3b - 4a)$. As $\omega_1 \mid C$ using $C^l = A^m + B^n$, then $C = \omega_1^h.C_1 \implies \omega_1^{n.j-2m.f+h.l} B_1^n.C_1^l = \Omega^{2m}(3b - 4a)$. If $n.j - 2m.f + h.l < 0 \implies \omega_1 \mid B_1^n.C_1^l$, then the contradiction with $\omega_1 \nmid B_1$ and $\omega_1 \nmid C_1$. Then if $n.j - 2m.f + h.l > 0$ and $\omega_1 \mid (3b - 4a)$ with ω_1, a, b coprime and the conjecture (3.1.1) is verified.

** G-2-1-1-2-2-2- Using the same method above, we obtain identical results if $\omega_1 \mid C^l$.

** G-2-1-2- We suppose that $2 \mid p_1$: then $2 \mid p_1 \implies 2 \nmid a' \implies 2 \nmid a$, but $p'' = p_1^2$.

** G-2-1-2-1- We suppose that $p_1 = 2$, we obtain $A^m = 4a' \implies 2 \mid a'$, then the contradiction with a, b coprime.

** G-2-1-2-2- We suppose that p_1 is not prime and $2 \mid p_1$. As $A^m = 2a'p_1$, p_1 can be written as $p_1 = 2^{m-1}\omega^m \implies p'' = 2^{2m-2}\omega^{2m}$. Then $B^n C^l = 2^{2m-2}\omega^{2m}(3b - 4a) \implies 2 \mid B^n$ or $2 \mid C^l$.

** G-2-1-2-2-1- We suppose that $2 \mid B^n \implies 2 \mid B$. As $2 \mid A$, then $2 \mid C$. From $B^n C^l = 2^{2m-2}\omega^{2m}(3b - 4a)$ it follows that if $2 \mid (3b - 4a) \implies 2 \mid b$ but as $2 \nmid a$ there is no contradiction with a, b coprime and the conjecture (3.1.1) is verified.

** G-2-1-2-2-2- We suppose that $2 \mid C^l$, using the same method above, we obtain identical results.

** G-2-2- We suppose that p'', a are not coprime: let ω be a prime integer so that $\omega \mid a$ and $\omega \mid p''$.

** G-2-2-1- We suppose that $\omega = 3$. As $A^{2m} = 4ap'' \implies 3 \mid A$, but $3 \nmid p$. As $p = A^{2m} + B^{2n} + A^m B^n \implies 3 \mid B^{2n} \implies 3 \mid B$, then $3 \mid C^l \implies 3 \mid C$. We write $A = 3^i A_1$, $B = 3^j B_1$, $C = 3^h C_1$ with 3 coprime with A_1, B_1 and C_1 and $p = 3^{2im} A_1^{2m} + 3^{2nj} B_1^{2n} + 3^{im+jn} A_1^m B_1^n = 3^k \cdot g$ with $k = \min(2im, 2jn, im + jn)$ and $3 \nmid g$. We have also $(\omega = 3) \mid a$ and $(\omega = 3) \mid p''$ that gives $a = 3^\alpha a_1$, $3 \nmid a_1$ and $p'' = 3^\mu p_1$, $3 \nmid p_1$ with $A^{2m} = 4ap'' = 3^{2im} A_1^{2m} = 4 \times 3^{\alpha+\mu} \cdot a_1 \cdot p_1 \implies \alpha + \mu = 2im$. As $p = 3p' = 3b \cdot p'' = 3b \cdot 3^\mu p_1 = 3^{\mu+1} \cdot b \cdot p_1$, the exponent of the factor 3 of p is k , the exponent of the factor 3 of the left member of the last equation is $\mu + 1$ added of the exponent β of 3 of the term b , with $\beta \geq 0$, let $\min(2im, 2jn, im + jn) = \mu + 1 + \beta$ and we recall that $\alpha + \mu = 2im$. But $B^n C^l = p''(3b - 4a)$, we obtain $3^{(nj+hl)} B_1^n C_1^l = 3^{\mu+1} p_1 (b - 4 \times 3^{(\alpha-1)} a_1) = 3^{\mu+1} p_1 (3^\beta b_1 - 4 \times 3^{(\alpha-1)} a_1)$, $3 \nmid b_1$. We have also $A^m + B^n = C^l \implies 3^{im} A_1^m + 3^{jn} B_1^n = 3^{hl} C_1^l$. We call $\epsilon = \min(im, jn)$, we have $\epsilon = hl = \min(im, jn)$. We obtain the conditions:

$$k = \min(2im, 2jn, im + jn) = \mu + 1 + \beta \quad (1.6.1)$$

$$\alpha + \mu = 2im \quad (1.6.2)$$

$$\epsilon = hl = \min(im, jn)$$

$$3^{(nj+hl)} B_1^n C_1^l = 3^{\mu+1} p_1 (3^\beta b_1 - 4 \times 3^{(\alpha-1)} a_1)$$

** G-2-2-1-1- $\alpha = 1 \implies a = 3a_1$ and $3 \nmid a_1$, the equation (1.6.2) becomes:

$$1 + \mu = 2im$$

and the first equation (1.6.1) is written as:

$$k = \min(2im, 2jn, im + jn) = 2im + \beta$$

- If $k = 2im \implies \beta = 0$ then $3 \nmid b$. We obtain $2im \leq 2jn \implies im \leq jn$, and $2im \leq im + jn \implies im \leq jn$. The third equation gives $hl = im$ and the last equation gives $nj + hl = \mu + 1 = 2im \implies im = nj$, then $im = nj = hl$ and $B_1^n C_1^l = p_1 (b - 4a_1)$. As a, b are coprime, the conjecture (3.1.1) is verified.

- If $k = 2jn$ or $k = im + jn$, we obtain $\beta = 0$, $im = jn = hl$ and $B_1^n C_1^l = p_1 (b - 4a_1)$. As a, b are coprime, the conjecture (3.1.1) is verified.

** G-2-2-1-2- $\alpha > 1 \implies \alpha \geq 2$.

- If $k = 2im \implies 2im = \mu + 1 + \beta$, but $\mu = 2im - \alpha$ that gives $\alpha = 1 + \beta \geq 2 \implies \beta \neq 0 \implies 3 \mid b$, but $3 \nmid a$ then the contradiction with a, b coprime.

- If $k = 2jn = \mu + 1 + \beta \leq 2im \implies \mu + 1 + \beta \leq \mu + \alpha \implies 1 + \beta \leq \alpha \implies \beta \geq 1$. If $\beta \geq 1 \implies 3 \mid b$ but $3 \nmid a$, then the contradiction with a, b coprime.

- If $k = im + jn \implies im + jn \leq 2im \implies jn \leq im$, and $im + jn \leq 2jn \implies im \leq jn$, then $im = jn$. As $k = im + jn = 2im = 1 + \mu + \beta$ and $\alpha + \mu = 2im$, we obtain $\alpha = 1 + \beta \geq 2 \implies \beta \geq 1 \implies 3 \mid b$, then the contradiction with a, b coprime.

** G-2-2-2- We suppose that $\omega \neq 3$. We write $a = \omega^\alpha a_1$ with $\omega \nmid a_1$ and $p'' = \omega^\mu p_1$ with $\omega \nmid p_1$. As $A^{2m} = 4ap'' = 4\omega^{\alpha+\mu} \cdot a_1 \cdot p_1 \implies \omega \mid A \implies A = \omega^i A_1$, $\omega \nmid A_1$. But $B^n C^l = p''(3b - 4a) = \omega^\mu p_1(3b - 4a) \implies \omega \mid B^n C^l \implies \omega \mid B^n$ or $\omega \mid C^l$.

** G-2-2-2-1- We suppose that $\omega \mid B^n \implies \omega \mid B \implies B = \omega^j B_1$ and $\omega \nmid B_1$. From $A^m + B^n = C^l \implies \omega \mid C^l \implies \omega \mid C$. As $p = bp' = 3bp'' = 3\omega^\mu bp_1 = \omega^k (\omega^{2im-k} A_1^{2m} + \omega^{2jn-k} B_1^{2n} + \omega^{im+jn-k} A_1^m B_1^n)$ with $k = \min(2im, 2jn, im + jn)$. Then:

- If $k = \mu$, then $\omega \nmid b$ and the conjecture (3.1.1) is verified.
- If $k > \mu$, then $\omega \mid b$, but $\omega \nmid a$ then the contradiction with a, b coprime.
- If $k < \mu$, it follows from:

$$3\omega^\mu bp_1 = \omega^k (\omega^{2im-k} A_1^{2m} + \omega^{2jn-k} B_1^{2n} + \omega^{im+jn-k} A_1^m B_1^n)$$

that $\omega \mid A_1$ or $\omega \mid B_1$ then the contradiction with $\omega \nmid A_1$ or $\omega \nmid B_1$.

** G-2-2-2-2- If $\omega \mid C^l \implies \omega \mid C \implies C = \omega^h C_1$ with $\omega \nmid C_1$. From $A^m + B^n = C^l \implies \omega \mid (C^l - A^m) \implies \omega \mid B$. Then, using the same method as for the case G-2-2-2-1-, we obtain identical results.

1.6.4 Case $b = 3$ and $3 \mid p$

As $3 \mid p \implies p = 3p'$, We write :

$$A^{2m} = \frac{4p}{3} \cos^2 \frac{\theta}{3} = \frac{4p}{3} \frac{a}{b} = \frac{4 \times 3p'}{3} \frac{a}{3} = \frac{4p'a}{3}$$

As A^{2m} is an integer and a, b are coprime and $\cos^2 \frac{\theta}{3} < 1$ (see equation (1.3.9)), then we have necessary $3 \mid p' \implies p' = 3p''$ with $p'' \neq 1$, if not $p = 3p' = 3 \times 3p'' = 9$, but $9 \ll (p = A^{2m} + B^{2n} + A^m B^n)$, the hypothesis $p'' = 1$ is impossible, then $p'' > 1$, and we obtain:

$$A^{2m} = \frac{4p'a}{3} = \frac{4 \times 3p''a}{3} = 4p''a; \quad B^n C^l = p'' \cdot (9 - 4a)$$

As $\frac{1}{4} < \cos^2 \frac{\theta}{3} = \frac{a}{b} = \frac{a}{3} < \frac{3}{4} \implies 3 < 4a < 9 \implies$ as $a > 1$, $a = 2$ and we obtain:

$$A^{2m} = 4p''a = 8p''; \quad B^n C^l = \frac{3p''(9 - 4a)}{3} = p'' \tag{1.6.3}$$

The two last equations above imply that p'' is not a prime. We can write p'' as : $p'' = \prod_{i \in I} p_i^{\alpha_i}$ where p_i are distinct primes, α_i elements of \mathbb{N}^* and $i \in I$ a finite set of indexes. We can write also $p'' = p_1^{\alpha_1} \cdot q_1$ with $p_1 \nmid q_1$. From (1.6.3), we have $p_1 \mid A$ and $p_1 \mid B^n C^l \implies p_1 \mid B^n$ or $p_1 \mid C^l$.

** H-1- We suppose that $p_1 \mid B^n \implies B = p_1^{\beta_1} \cdot B_1$ with $p_1 \nmid B_1$ and $\beta_1 \geq 1$. Then, we obtain $B_1^n C^l = p_1^{\alpha_1 - n\beta_1} \cdot q_1$ with the following cases:

- If $\alpha_1 - n\beta_1 \geq 1 \implies p_1 \mid C^l \implies p_1 \mid C$, in accord with $p_1 \mid (C^l = A^m + B^n)$, it follows that the conjecture (3.1.1) is verified.

- If $\alpha_1 - n\beta_1 = 0 \implies B_1^n C^l = q_1 \implies p_1 \nmid C^l$, it is a contradiction with $p_1 \mid (A^m - B^n) \implies p_1 \mid C^l$. Then this case is impossible.

- If $\alpha_1 - n\beta_1 < 0$, we obtain $p_1^{n\beta_1 - \alpha_1} B_1^n C^l = q_1 \implies p_1 \mid q_1$, it is a contradiction with $p_1 \nmid q_1$. Then this case is impossible.

** H-2- We suppose that $p_1 \mid C^l$, using the same method as for the case $p_1 \mid B^n$, we obtain identical results.

1.6.5 Case 3 | p and b = p

We have $\cos^2 \frac{\theta}{3} = \frac{a}{b} = \frac{a}{p}$ and:

$$A^{2m} = \frac{4p}{3} \cos^2 \frac{\theta}{3} = \frac{4p}{3} \cdot \frac{a}{p} = \frac{4a}{3}$$

As A^{2m} is an integer, it implies that $3 \mid a$, but $3 \mid p \implies 3 \mid b$. As a and b are coprime, then the contradiction and the case $3 \mid p$ and $b = p$ is impossible.

1.6.6 Case 3 | p and b = 4p

$3 \mid p \implies p = 3p'$, $p' \neq 1$ because $3 \ll p$, then $b = 4p = 12p'$.

$$A^{2m} = \frac{4p}{3} \cos^2 \frac{\theta}{3} = \frac{4p}{3} \frac{a}{b} = \frac{a}{3} \implies 3 \mid a$$

as A^{2m} is an integer. But $3 \mid p \implies 3 \mid [(4p) = b]$, then the contradiction with a, b coprime and the case $b = 4p$ is impossible.

1.6.7 Case 3 | p and b = 2p

$3 \mid p \implies p = 3p'$, $p' \neq 1$ because $3 \ll p$, then $b = 2p = 6p'$.

$$A^{2m} = \frac{4p}{3} \cos^2 \frac{\theta}{3} = \frac{4p}{3} \frac{a}{b} = \frac{2a}{3} \implies 3 \mid a$$

as A^{2m} is an integer. But $3 \mid p \implies 3 \mid (2p) \implies 3 \mid b$, then the contradiction with a, b coprime and the case $b = 2p$ is impossible.

1.6.8 Case 3 | p and b ≠ 3 a divisor of p

We have $b = p' \neq 3$, and p is written as $p = kp'$ with $3 \mid k \implies k = 3k'$ and :

$$A^{2m} = \frac{4p}{3} \cos^2 \frac{\theta}{3} = \frac{4p}{3} \cdot \frac{a}{b} = 4ak'$$

$$B^n C^l = \frac{p}{3} \cdot \left(3 - 4 \cos^2 \frac{\theta}{3} \right) = k'(3p' - 4a) = k'(3b - 4a)$$

** I-1- $k' \neq 1$:

** I-1-1- We suppose that k' is prime, then $A^{2m} = 4ak' = (A^m)^2 \implies k' \mid a$. But $B^n C^l = k'(3b - 4a) \implies k' \mid B^n$ or $k' \mid C^l$.

** I-1-1-1- If $k' \mid B^n \implies k' \mid B \implies B = k'B_1$ with $B_1 \in \mathbb{N}^*$. Then $k'^{m-1}B_1^n C^l = 3b - 4a$. As $n > 2$, then $(n - 1) > 1$ and $k' \mid a$, then $k' \mid 3b \implies k' = 3$ or $k' \mid b$.

** I-1-1-1-1- If $k' = 3 \implies 3 \mid a$, with a that we can write it under the form $a = 3a'^2$. But $A^m = 6a' \implies 3 \mid A^m \implies 3 \mid A \implies A = 3A_1$ with $A_1 \in \mathbb{N}^*$. Then $3^{m-1}A_1^m = 2a' \implies 3 \mid a' \implies a' = 3a''$. But $k'^{m-1}B_1^n C^l = 3^{m-1}B_1^n C^l = 3b - 4a \implies 3^{n-2}B_1^n C^l = b - 36a''^2$. As $n \geq 3 \implies n - 2 \geq 1$, then $3 \mid b$. Hence the contradiction with a, b coprime.

** I-1-1-1-2- We suppose that $k' \mid b$, but $k' \nmid a$, then the contradiction with a, b coprime.

** I-1-1-2- We suppose that $k' \mid C^l$, using the same method as for the case $k' \mid B^n$, we obtain identical results.

** I-1-2- We consider that k' is not a prime.

** I-1-2-1- We suppose that k', a coprime: $A^{2m} = 4ak' \implies A^m = 2a'.p_1$ with $a = a'^2$ and $k' = p_1^2$, then a', p_1 are also coprime. As $A^m = 2a'.p_1$ then $2 \mid a'$ or $2 \mid p_1$.

** I-1-2-1-1- We suppose that $2 \mid a'$, then $2 \mid a' \implies 2 \nmid p_1$, but $k' = p_1^2$.

** I-1-2-1-1-1- If p_1 is prime, it is impossible with $A^m = 2a'.p_1$.

** I-1-2-1-1-2- We suppose that p_1 is not prime and it can be written as $p_1 = \omega^m \implies k' = \omega^{2m}$. Then $B^n C^l = \omega^{2m}(3b - 4a)$.

** I-1-2-1-1-2-1- If ω is prime $\neq 2$, then $\omega \mid (B^n C^l) \implies \omega \mid B^n$ or $\omega \mid C^l$.

** I-1-2-1-1-2-1-1- If $\omega \mid B^n \implies \omega \mid B \implies B = \omega^j B_1$ with $\omega \nmid B_1$, then $B_1^n C^l = \omega^{2m-nj}(3b - 4a)$.

- If $2m - n.j = 0$, we obtain $B_1^n C^l = 3b - 4a$, as $C^l = A^m + B^n \implies \omega \mid C^l \implies \omega \mid C$, and $\omega \mid (3b - 4a)$. But $\omega \neq 2$ and ω, a' are coprime, then $\omega \nmid (3b) \implies \omega \neq 3$ and $\omega \nmid b$. Hence, the conjecture (3.1.1) is verified.

- If $2m - nj \geq 1$, using the same method, we have $\omega \mid C^l \implies \omega \mid C$ and $\omega \mid (3b - 4a)$ and $\omega \nmid a$ and $\omega \neq 3$ and $\omega \nmid b$. Then the conjecture (3.1.1) is verified.

- If $2m - nj < 0 \implies \omega^{n.j-2m} B_1^n C^l = 3b - 4a$. As $C^l = A^m + B^n \implies \omega \mid C$ then $C = \omega^h C_1 \implies \omega^{n.j-2m+h.l} B_1^n C_1^l = 3b - 4a$. If $n.j - 2m + h.l < 0 \implies \omega \mid B_1^n C_1^l$, then the contradiction with $\omega \nmid B_1$ or $\omega \nmid C_1$. If $n.j - 2m + h.l > 0 \implies \omega \mid (3b - 4a)$ with ω, a, b coprime, it implies that the conjecture (3.1.1) is verified.

** I-1-2-1-1-2-1-2- We suppose that $\omega \mid C^l$, using the same method as for the case $\omega \mid B^n$, we obtain identical results.

** I-1-2-1-1-2-2- Now $k' = \omega^{2m}$ and ω not a prime, we write $\omega = \omega_1^f \Omega$ with ω_1 a prime $\nmid \Omega$ and $f \geq 1$ an integer, and $\omega_1 \mid A$, then $B^n C^l = \omega_1^{2f.m} \Omega^{2m}(3b - 4a) \implies \omega_1 \mid (B^n C^l) \implies \omega_1 \mid B^n$ or $\omega_1 \mid C^l$.

** I-1-2-1-1-2-2-1- If $\omega_1 \mid B^n \implies \omega_1 \mid B \implies B = \omega_1^j B_1$ with $\omega_1 \nmid B_1$, then $B_1^n.C^l = \omega_1^{2.fm-nj} \Omega^{2m}(3b-4a)$.

- If $2f.m - n.j = 0$, we obtain $B_1^n.C^l = \Omega^{2m}(3b-4a)$. As $C^l = A^m + B^n \implies \omega_1 \mid C^l \implies \omega_1 \mid C$, and $\omega_1 \mid (3b-4a)$. But $\omega_1 \neq 2$ and ω_1, a' are coprime, then ω, a are coprime, then $\omega_1 \nmid (3b) \implies \omega_1 \neq 3$ and $\omega_1 \nmid b$. Hence, the conjecture (3.1.1) is verified.

- If $2f.m - n.j \geq 1$, we have $\omega_1 \mid C^l \implies \omega_1 \mid C$ and $\omega_1 \mid (3b-4a)$ and $\omega_1 \nmid a$ and $\omega_1 \neq 3$ and $\omega_1 \nmid b$, then the conjecture (3.1.1) is verified.

- If $2f.m - n.j < 0 \implies \omega_1^{n.j-2m.f} B_1^n.C^l = \Omega^{2m}(3b-4a)$. As $C^l = A^m + B^n \implies \omega_1 \mid C$, then $C = \omega_1^h.C_1 \implies \omega_1^{n.j-2m.f+h.l} B_1^n.C_1^l = \Omega^{2m}(3b-4a)$. If $n.j - 2m.f + h.l < 0 \implies \omega_1 \mid B_1^n.C_1^l$, then the contradiction with $\omega_1 \nmid B_1$ and $\omega_1 \nmid C_1$. Then if $n.j - 2m.f + h.l > 0$ and $\omega_1 \mid (3b-4a)$ with ω_1, a, b coprime, then the conjecture (3.1.1) is verified.

** I-1-2-1-1-2-2-2- As in the case $\omega_1 \mid B^n$, we obtain identical results if $\omega_1 \mid C^l$.

** I-1-2-1-2- If $2 \mid p_1$: then $2 \mid p_1 \implies 2 \nmid a' \implies 2 \nmid a$, but $k' = p_1^2$.

** I-1-2-1-2-1- If $p_1 = 2$, we obtain $A^m = 4a' \implies 2 \mid a'$, then the contradiction with $2 \nmid a'$. Case to reject.

** I-1-2-1-2-2- We suppose that p_1 is not prime and $2 \mid p_1$. As $A^m = 2a'p_1$, p_1 is written under the form $p_1 = 2^{m-1}\omega^m \implies p_1^2 = 2^{2m-2}\omega^{2m}$. Then $B^n.C^l = k'(3b-4a) = 2^{2m-2}\omega^{2m}(3b-4a) \implies 2 \mid B^n$ or $2 \mid C^l$.

** I-1-2-1-2-2-1- If $2 \mid B^n \implies 2 \mid B$, as $2 \mid A \implies 2 \mid C$. From $B^n.C^l = 2^{2m-2}\omega^{2m}(3b-4a)$ it follows that if $2 \mid (3b-4a) \implies 2 \mid b$ but as $2 \nmid a$, there is no contradiction with a, b coprime and the conjecture (3.1.1) is verified.

** I-1-2-1-2-2-2- We obtain identical results as above if $2 \mid C^l$.

** I-1-2-2- We suppose that k', a are not coprime: let ω be a prime integer so that $\omega \mid a$ and $\omega \mid p_1^2$.

** I-1-2-2-1- We suppose that $\omega = 3$. As $A^{2m} = 4ak' \implies 3 \mid A$, but $3 \mid p$. As $p = A^{2m} + B^{2n} + A^m B^n \implies 3 \mid B^{2n} \implies 3 \mid B$, then $3 \mid C^l \implies 3 \mid C$. We write $A = 3^i A_1$, $B = 3^j B_1$, $C = 3^h C_1$ with 3 coprime with A_1, B_1 and C_1 and $p = 3^{2im} A_1^{2m} + 3^{2nj} B_1^{2n} + 3^{im+jn} A_1^m B_1^n = 3^s.g$ with $s = \min(2im, 2jn, im+jn)$ and $3 \nmid g$. We have also $(\omega = 3) \mid a$ and $(\omega = 3) \mid k'$ that give $a = 3^\alpha a_1$, $3 \nmid a_1$ and $k' = 3^\mu p_2$, $3 \nmid p_2$ with $A^{2m} = 4ak' = 3^{2im} A_1^{2m} = 4 \times 3^{\alpha+\mu}.a_1.p_2 \implies \alpha + \mu = 2im$. As $p = 3p' = 3b.k' = 3b.3^\mu p_2 = 3^{\mu+1}.b.p_2$. The exponent of the factor 3 of p is s , the exponent of the factor 3 of the left member of the last equation is $\mu + 1$ added of the exponent β of 3 of the factor b , with $\beta \geq 0$, let $\min(2im, 2jn, im+jn) = \mu + 1 + \beta$, we recall that $\alpha + \mu = 2im$. But $B^n.C^l = k'(4b-3a)$ that gives $3^{(nj+hl)} B_1^n.C_1^l = 3^{\mu+1} p_2 (b - 4 \times 3^{(\alpha-1)} a_1) = 3^{\mu+1} p_2 (3^\beta b_1 - 4 \times 3^{(\alpha-1)} a_1)$, $3 \nmid b_1$. We have also $A^m + B^n = C^l$ that gives $3^{im} A_1^m + 3^{jn} B_1^n = 3^{hl} C_1^l$. We call $\epsilon = \min(im, jn)$, we obtain $\epsilon = hl = \min(im, jn)$. We have then the conditions:

$$s = \min(2im, 2jn, im+jn) = \mu + 1 + \beta \quad (1.6.4)$$

$$\alpha + \mu = 2im \quad (1.6.5)$$

$$\epsilon = hl = \min(im, jn) \quad (1.6.6)$$

$$3^{(nj+hl)} B_1^n.C_1^l = 3^{\mu+1} p_2 (3^\beta b_1 - 4 \times 3^{(\alpha-1)} a_1) \quad (1.6.7)$$

** I-1-2-2-1-1- $\alpha = 1 \implies a = 3a_1$ and $3 \nmid a_1$, the equation (1.6.5) becomes:

$$1 + \mu = 2im$$

and the first equation (1.6.4) is written as :

$$s = \min(2im, 2jn, im + jn) = 2im + \beta$$

- If $s = 2im \implies \beta = 0 \implies 3 \nmid b$. We obtain $2im \leq 2jn \implies im \leq jn$, and $2im \leq im + jn \implies im \leq jn$. The third equation (1.6.6) gives $hl = im$. The last equation (1.6.7) gives $nj + hl = \mu + 1 = 2im \implies im = jn$, then $im = jn = hl$ and $B_1^n C_1^l = p_2(b - 4a_1)$. As a, b are coprime, the conjecture (3.1.1) is verified.

- If $s = 2jn$ or $s = im + jn$, we obtain $\beta = 0$, $im = jn = hl$ and $B_1^n C_1^l = p_2(b - 4a_1)$. Then as a, b are coprime, the conjecture (3.1.1) is verified.

** I-1-2-2-1-2- $\alpha > 1 \implies \alpha \geq 2$.

- If $s = 2im \implies 2im = \mu + 1 + \beta$, but $\mu = 2im - \alpha$ it gives $\alpha = 1 + \beta \geq 2 \implies \beta \neq 0 \implies 3 \mid b$, but $3 \mid a$ then the contradiction with a, b coprime and the conjecture (3.1.1) is not verified.

- If $s = 2jn = \mu + 1 + \beta \leq 2im \implies \mu + 1 + \beta \leq \mu + \alpha \implies 1 + \beta \leq \alpha \implies \beta = 1$. If $\beta = 1 \implies 3 \mid b$ but $3 \mid a$, then the contradiction with a, b coprime and the conjecture (3.1.1) is not verified.

- If $s = im + jn \implies im + jn \leq 2im \implies jn \leq im$, and $im + jn \leq 2jn \implies im \leq jn$, then $im = jn$. As $s = im + jn = 2im = 1 + \mu + \beta$ and $\alpha + \mu = 2im$ it gives $\alpha = 1 + \beta \geq 2 \implies \beta \geq 1 \implies 3 \mid b$, then the contradiction with a, b coprime and the conjecture (3.1.1) is not verified.

** I-1-2-2-2- We suppose that $\omega \neq 3$. We write $a = \omega^\alpha a_1$ with $\omega \nmid a_1$ and $k' = \omega^\mu p_2$ with $\omega \nmid p_2$. As $A^{2m} = 4ak' = 4\omega^{\alpha+\mu} a_1 p_2 \implies \omega \mid A \implies A = \omega^i A_1$, $\omega \nmid A_1$. But $B^n C^l = k'(3b - 4a) = \omega^\mu p_2(3b - 4a) \implies \omega \mid B^n C^l \implies \omega \mid B^n$ or $\omega \mid C^l$.

** I-1-2-2-2-1- $\omega \mid B^n \implies \omega \mid B \implies B^n = \omega^j B_1^n$ and $\omega \nmid B_1$. From $A^m + B^n = C^l \implies \omega \mid C^l \implies \omega \mid C$. As $p = bp' = 3bk' = 3\omega^\mu bp_2 = \omega^s (\omega^{2im-s} A_1^{2m} + \omega^{2jn-s} B_1^{2n} + \omega^{im+jn-s} A_1^m B_1^n)$ with $s = \min(2im, 2jn, im + jn)$. Then:

- If $s = \mu$, then $\omega \nmid b$ and the conjecture (3.1.1) is verified.

- If $s > \mu$, then $\omega \mid b$, but $\omega \mid a$ then the contradiction with a, b coprime and the conjecture (3.1.1) is not verified.

- If $s < \mu$, it follows from:

$$3\omega^\mu bp_1 = \omega^s (\omega^{2im-s} A_1^{2m} + \omega^{2jn-s} B_1^{2n} + \omega^{im+jn-s} A_1^m B_1^n)$$

that $\omega \mid A_1$ or $\omega \mid B_1$ that is the contradiction with the hypothesis and the conjecture (3.1.1) is not verified.

** I-1-2-2-2-2- If $\omega \mid C^l \implies \omega \mid C \implies C = \omega^h C_1$ with $\omega \nmid C_1$. From $A^m + B^n = C^l \implies \omega \mid (C^l - A^m) \implies \omega \mid B$. Then we obtain identical results as the case above I-1-2-2-2-1-.

** I-2- We suppose $k' = 1$: then $k' = 1 \implies p = 3b$, then we have $A^{2m} = 4a = (2a')^2 \implies A^m = 2a'$, then $a = a'^2$ is even and :

$$A^m B^n = 2\sqrt[3]{\rho} \cos \frac{\theta}{3} \cdot \sqrt[3]{\rho} \left(\sqrt{3} \sin \frac{\theta}{3} - \cos \frac{\theta}{3} \right) = \frac{p\sqrt{3}}{3} \sin \frac{2\theta}{3} - 2a$$

and we have also:

$$A^{2m} + 2A^m B^n = \frac{2p\sqrt{3}}{3} \sin \frac{2\theta}{3} = 2b\sqrt{3} \sin \frac{2\theta}{3} \quad (1.6.8)$$

The left member of the equation (1.6.8) is a natural number and also b , then $2\sqrt{3} \sin \frac{2\theta}{3}$ can be written under the form :

$$2\sqrt{3} \sin \frac{2\theta}{3} = \frac{k_1}{k_2}$$

where k_1, k_2 are two natural numbers coprime and $k_2 \mid b \implies b = k_2.k_3$.

** I-2-1- $k' = 1$ and $k_3 \neq 1$: then $A^{2m} + 2A^m B^n = k_3.k_1$. Let μ be a prime integer so that $\mu \mid k_3$. If $\mu = 2 \implies 2 \mid b$, but $2 \nmid a$, it is a contradiction with a, b coprime. We suppose that $\mu \neq 2$ and $\mu \mid k_3$, then $\mu \mid A^m(A^m + 2B^n) \implies \mu \mid A^m$ or $\mu \mid (A^m + 2B^n)$.

** I-2-1-1- $\mu \mid A^m$: If $\mu \mid A^m \implies \mu \mid A^{2m} \implies \mu \mid 4a \implies \mu \mid a$. As $\mu \mid k_3 \implies \mu \mid b$, the contradiction with a, b coprime.

** I-2-1-2- $\mu \mid (A^m + 2B^n)$: If $\mu \mid (A^m + 2B^n) \implies \mu \nmid A^m$ and $\mu \nmid 2B^n$, then $\mu \neq 2$ and $\mu \nmid B^n$. $\mu \mid (A^m + 2B^n)$, we can write $A^m + 2B^n = \mu.t'$. It follows:

$$A^m + B^n = \mu t' - B^n \implies A^{2m} + B^{2n} + 2A^m B^n = \mu^2 t'^2 - 2t' \mu B^n + B^{2n}$$

Using the expression of p , we obtain:

$$p = t'^2 \mu^2 - 2t' B^n \mu + B^n (B^n - A^m)$$

As $p = 3b = 3k_2.k_3$ and $\mu \mid k_3$ then $\mu \mid p \implies p = \mu.\mu'$, then we obtain:

$$\mu'.\mu = \mu(\mu t'^2 - 2t' B^n) + B^n (B^n - A^m)$$

and $\mu \mid B^n (B^n - A^m) \implies \mu \mid B^n$ or $\mu \mid (B^n - A^m)$.

** I-2-1-2-1- $\mu \mid B^n$: If $\mu \mid B^n \implies \mu \mid B$, that is the contradiction with I-2-1-2- above.

** I-2-1-2-2- $\mu \mid (B^n - A^m)$: If $\mu \mid (B^n - A^m)$ and using that $\mu \mid (A^m + 2B^n)$, we obtain :

$$\mu \mid 3B^n \implies \begin{cases} \mu \mid B^n \implies \mu \mid B \\ or \\ \mu = 3 \end{cases}$$

** I-2-1-2-2-1- $\mu \mid B^n$: If $\mu \mid B^n \implies \mu \mid B$, that is the contradiction with I-2-1-2- above.

** I-2-1-2-2-2- $\mu = 3$: If $\mu = 3 \implies 3 \mid k_3 \implies k_3 = 3k'_3$, and we have $b = k_2 k_3 = 3k_2 k'_3$, it follows $p = 3b = 9k_2 k'_3$, then $9 \mid p$, but $p = (A^m - B^n)^2 + 3A^m B^n$ then:

$$9k_2 k'_3 - 3A^m B^n = (A^m - B^n)^2$$

that we write as:

$$3(3k_2 k'_3 - A^m B^n) = (A^m - B^n)^2 \quad (1.6.9)$$

then:

$$3 \mid (3k_2 k'_3 - A^m B^n) \implies 3 \mid A^m B^n \implies 3 \mid A^m \text{ or } 3 \mid B^n$$

** I-2-1-2-2-2-1- 3 | A^m : If $3 | A^m \implies 3 | A$ and we have also $3 | A^{2m}$, but $A^{2m} = 4a \implies 3 | 4a \implies 3 | a$. As $b = 3k_2k'_3$ then $3 | b$, but a, b are coprime, then the contradiction and $3 \nmid A$.

** I-2-1-2-2-2-2- 3 | B^m : If $3 | B^m \implies 3 | B$, but the equation (1.6.9) implies $3 | (A^m - B^n)^2 \implies 3 | (A^m - B^n) \implies 3 | A^m \implies 3 | A$. The last case above has given that $3 \nmid A$. Then the case $3 | B^m$ is to reject.

Finally the hypothesis $k_3 \neq 1$ is impossible.

** I-2-2- Now, we suppose that $k_3 = 1 \implies b = k_2$ and $p = 3b = 3k_2$, then we have:

$$2\sqrt{3}\sin\frac{\theta}{3} = \frac{k_1}{b} \quad (1.6.10)$$

with k_1, b coprime. We write (1.6.10) as :

$$4\sqrt{3}\sin\frac{\theta}{3}\cos\frac{\theta}{3} = \frac{k_1}{b}$$

Taking the square of the two members and replacing $\cos^2\frac{\theta}{3}$ by $\frac{a}{b}$, we obtain:

$$3 \times 4^2 \cdot a(b-a) = k_1^2 \implies k_1^2 = 3 \times 4^2 \cdot a^2(b-a)$$

it implies that :

$$b-a = 3\alpha^2, \alpha \in \mathbb{N}^* \implies b = a'^2 + 3\alpha^2 \implies k_1 = 12a'\alpha$$

As:

$$k_1 = 12a'\alpha = A^m(A^m + 2B^n) \implies 3\alpha = a' + B^n$$

We consider now that $3 | (b-a)$ with $b = a'^2 + 3\alpha^2$. The case $\alpha = 1$ gives $a' + B^n = 3$ that is impossible. We suppose $\alpha > 1$, the pair (a', α) is a solution of the Diophantine equation:

$$X^2 + 3Y^2 = b \quad (1.6.11)$$

with $X = a'$ and $Y = \alpha$. But using a theorem on the solutions of the equation given by (1.6.11), b is written as (see theorem in [2]):

$$b = 2^{2s} \times 3^t \cdot p_1^{t_1} \cdots p_g^{t_g} q_1^{2s_1} \cdots q_r^{2s_r}$$

where p_i are prime numbers verifying $p_i \equiv 1 \pmod{6}$, the q_j are also prime numbers so that $q_j \equiv 5 \pmod{6}$, then :

- If $s \geq 1 \implies 2 | b$, as $2 | a$, then the contradiction with a, b coprime.
- If $t \geq 1 \implies 3 | b$, but $3 | (b-a) \implies 3 | a$, then the contradiction with a, b coprime.

** I-2-2-1- We suppose that b is written as :

$$b = p_1^{t_1} \cdots p_g^{t_g} q_1^{2s_1} \cdots q_r^{2s_r}$$

with $p_i \equiv 1 \pmod{6}$ and $q_j \equiv 5 \pmod{6}$. Finally, we obtain that $b \equiv 1 \pmod{6}$. We will verify then this condition.

Table 1.2: Table of $C^l \pmod{6}$

A^m, B^n	1	2	4	5
1	2	3	5	0
2	3	4	0	1
4	5	0	2	3
5	0	1	3	4

** I-2-2-1-1- We present the table below giving the value of $A^m + B^n = C^l$ modulo 6 in function of the value of $A^m, B^n \pmod{6}$. We obtain the table below after retiring the lines (respectively the colones) of $A^m \equiv 0 \pmod{6}$ and $A^m \equiv 3 \pmod{6}$ (respectively of $B^n \equiv 0 \pmod{6}$ and $B^n \equiv 3 \pmod{6}$), they present cases with contradictions:

** I-2-2-1-1-1- For the case $C^l \equiv 0 \pmod{6}$ and $C^l \equiv 3 \pmod{6}$, we deduce that $3 \mid C^l \implies 3 \mid C \implies C = 3^h C_1$, with $h \geq 1$ and $3 \nmid C_1$. It follows that $p - B^n C^l = 3b - 3^{lh} C_1^l B^n = A^{2m} \implies 3 \mid (A^{2m} = 4a) \implies 3 \mid a \implies 3 \mid b$, then the contradiction with a, b coprime.

** I-2-2-1-1-2- For the case $C^l \equiv 0 \pmod{6}$, $C^l \equiv 2 \pmod{6}$ and $C^l \equiv 4 \pmod{6}$, we deduce that $2 \mid C^l \implies 2 \mid C \implies C = 2^h C_1$, with $h \geq 1$ and $2 \nmid C_1$. It follows that $p = 3b = A^{2m} + B^n C^l = 4a + 2^{lh} C_1^l B^n \implies 2 \mid 3b \implies 2 \mid b$, then the contradiction with a, b coprime.

** I-2-2-1-1-3- We consider the cases $A^m \equiv 1 \pmod{6}$ and $B^n \equiv 4 \pmod{6}$ (respectively $B^n \equiv 2 \pmod{6}$): then $2 \mid B^n \implies 2 \mid B \implies B = 2^j B_1$ with $j \geq 1$ and $2 \nmid B_1$. It follows from $3b = A^{2m} + B^n C^l = 4a + 2^{jn} B_1^n C^l$ that $2 \mid b$, then the contradiction with a, b coprime.

** I-2-2-1-1-4- We consider the case $A^m \equiv 5 \pmod{6}$ and $B^n \equiv 2 \pmod{6}$: then $2 \mid B^n \implies 2 \mid B \implies B = 2^j B_1$ with $j \geq 1$ and $2 \nmid B_1$. It follows that $3b = A^{2m} + B^n C^l = 4a + 2^{jn} B_1^n C^l$, then $2 \mid b$ and we obtain the contradiction with a, b coprime.

** I-2-2-1-1-5- We consider the case $A^m \equiv 2 \pmod{6}$ and $B^n \equiv 5 \pmod{6}$: as $A^m \equiv 2 \pmod{6} \implies A^m \equiv 2 \pmod{3}$, then A^m is not a square and also for B^n . Hence, we can write A^m and B^n as:

$$\begin{aligned} A^m &= a_0 \cdot \mu A^2 \\ B^n &= b_0 \mu B^2 \end{aligned}$$

where a_0 (respectively b_0) regroups the product of the prime numbers of A^m with exponent 1 (respectively of B^n) with not necessary $(a_0, \mu A) = 1$ and $(b_0, \mu B) = 1$. We have also $p = 3b = A^{2m} + A^m B^n + B^{2n} = (A^m - B^n)^2 + 3A^m B^n \implies 3 \mid (b - A^m B^n) \implies A^m B^n \equiv b \pmod{3}$ but $b = a + 3a^2 \implies b \equiv a \equiv a'^2 \pmod{3}$, then $A^m B^n \equiv a'^2 \pmod{3}$. But $A^m \equiv 2 \pmod{6} \implies 2a' \equiv 2 \pmod{6} \implies 4a'^2 \equiv 4 \pmod{6} \implies a'^2 \equiv 1 \pmod{3}$. It follows that $A^m B^n$ is a square, let $A^m B^n = \mu N^2 = \mu A^2 \cdot \mu B^2 \cdot a_0 \cdot b_0$. We call $\mu N_1^2 = a_0 \cdot b_0$. Let p_1 be a prime number so that $p_1 \mid a_0 \implies a_0 = p_1 \cdot a_1$ with $p_1 \nmid a_1$. $p_1 \mid \mu N_1^2 \implies p_1 \mid \mu N_1 \implies \mu N_1 = p_1^t \mu N_1'$ with $t \geq 1$ and $p_1 \nmid \mu N_1'$, then $p_1^{2t-1} \mu N_1'^2 = a_1 \cdot b_0$. As $2t \geq 2 \implies 2t - 1 \geq 1 \implies p_1 \mid a_1 \cdot b_0$ but $(p_1, a_1) = 1$, then $p_1 \mid b_0 \implies p_1 \mid B^n \implies p_1 \mid B$. But $p_1 \mid (A^m = 2a')$, and $p_1 \neq 2$ because $p_1 \mid B^n$ and B^n is odd, then the contradiction. Hence, $p_1 \mid a' \implies p_1 \mid a$. If $p_1 = 3$, from $3 \mid (b - a) \implies 3 \mid b$ then the contradiction with a, b coprime. Then $p_1 > 3$ a prime

that divides A^m and B^n , then $p_1 \mid (p = 3b) \implies p_1 \mid b$, it follows the contradiction with a, b coprime, knowing that $p = 3b \equiv 3 \pmod{6}$ and we choose the case $b \equiv 1 \pmod{6}$ of our interest.

** I-2-2-1-1-6- We consider the last case of the table above $A^m \equiv 4 \pmod{6}$ and $B^n \equiv 1 \pmod{6}$. We return to the equation (1.6.11) that b verifies :

$$b = X^2 + 3Y^2 \quad (1.6.12)$$

with $X = a'$; $Y = \alpha$
and $3\alpha = a' + B^n$

But $p = A^{2m} + A^m B^n + B^{2n} = 3b = 3(3\alpha^2 + a'^2) \implies A^{2m} + C^l B^n = 3a'^2 + 9\alpha^2$. As $A^{2m} = (2a')^2 = 4a'^2$, we obtain:

$$9\alpha^2 - a'^2 = C^l \cdot B^n$$

Then the pair $(3\alpha, a') \in \mathbb{N}^* \times \mathbb{N}^*$ is a solution of the Diophantine equation:

$$x^2 - y^2 = N \quad (1.6.13)$$

where $N = C^l \cdot B^n > 0$.

Let $Q(N)$ be the number of the solutions of (1.6.13) and $\tau(N)$ the number of ways to write the factors of N , then we announce the following result concerning the number of the solutions of (1.6.13) (see theorem 27.3 in [2]):

Theorem 1.6.1. *Let $Q(N)$ be the number of the solutions of (1.6.13) and $\tau(N)$ the number of ways to write the factors of N , then the number of the solutions of (1.6.13) :*

- If $N \equiv 2 \pmod{4}$, then $Q(N) = 0$.
- If $N \equiv 1$ or $N \equiv 3 \pmod{4}$, then $Q(N) = \lceil \tau(N)/2 \rceil$.
- If $N \equiv 0 \pmod{4}$, then $Q(N) = \lceil \tau(N/4)/2 \rceil$.

As $A^m = 2a', m \geq 3 \implies A^m \equiv 0 \pmod{4}$. Concerning B^n , for $B^n \equiv 0 \pmod{4}$ or $B^n \equiv 2 \pmod{4}$, we find that $2 \mid B^n \implies 2 \mid \alpha \implies 2 \mid b$, then the contradiction with a, b coprime.

For the last case $B^n \equiv 3 \pmod{4} \implies C^l \equiv 3 \pmod{4} \implies N = B^n C^l \equiv 1 \pmod{4} \implies Q(N) = \lceil \tau(N)/2 \rceil$.

As $(3\alpha, a')$ is a couple of solutions of the Diophantine equation (1.6.13) and $3\alpha > a'$, then $\exists d, d'$ positive integers with $d > d'$ and $N = d \cdot d'$ so that :

$$d + d' = 6\alpha \quad (1.6.14)$$

$$d - d' = 2a' \quad (1.6.15)$$

We will use the same method used in the above paragraph A-2-1-2-

** I-2-2-1-1-6-1- As $C^l > B^n$, we take $d = C^l$ and $d' = B^n$. It follows:

$$C^l + B^n = 6\alpha = 2a' + 2B^n = A^m + 2B^n \quad (1.6.16)$$

$$C^l - B^n = 2a' = A^m \quad (1.6.17)$$

Then the case $d = C^l$ and $d' = B^n$ gives *a priori* no contradictions.

** I-2-2-1-1-6-2- Now, we consider the case $d = B^n C^l$ and $d' = 1$. We rewrite the equations (3.2.24-3.2.25):

$$B^n C^l + 1 = 6\alpha \quad (1.6.18)$$

$$B^n C^l - 1 = 2a' \quad (1.6.19)$$

We obtain $1 = B^n$, it follows $C^l - A^m = 1$, we know [?] that the only positive solution of the last equation is $C = 3, A = 2, m = 3$ and $l = 2 < 3$, then the contradiction.

** I-2-2-1-1-6-3- Now, we consider the case $d = c_1^{lr-1} C_1^l$ where c_1 is a prime integer with $c_1 \nmid C_1$ and $C = c_1^r C_1, r \geq 1$. It follows that $d' = c_1 \cdot B^n$. We rewrite the equations (3.2.24-3.2.25):

$$c_1^{lr-1} C_1^l + c_1 \cdot B^n = 6\alpha \quad (1.6.20)$$

$$c_1^{lr-1} C_1^l - c_1 \cdot B^n = 2a' \quad (1.6.21)$$

As $l \geq 3$, from the last two equations above, it follows that $c_1 \mid (6\alpha)$ and $c_1 \mid (2a')$. Then $c_1 = 2$, or $c_1 = 3$ and $3 \mid a'$ or $c_1 \neq 3 \mid \alpha$ and $c_1 \mid a'$.

** I-2-2-1-1-6-3-1- We suppose $c_1 = 2$. As $2 \mid (A^m = 2a') \Rightarrow 2 \mid (a = a'^2)$ and $2 \mid C^l$ because $l \geq 3$, it follows $2 \mid B^n$, then $2 \mid (p = 3b)$. Then the contradiction with a, b coprime.

** I-2-2-1-1-6-3-2- We suppose $c_1 = 3 \Rightarrow c_1 \mid 2a' \Rightarrow c_1 \mid a' \Rightarrow c_1 \mid (a = a'^2)$. It follows that $(c_1 = 3) \mid (b = a'^2 + 3\alpha^2)$, then the contradiction with a, b coprime.

** I-2-2-1-1-6-3-3- We suppose $c_1 \neq 3$ and $c_1 \mid 3\alpha$ and $c_1 \mid a'$. It follows that $c_1 \mid a$ and $c_1 \mid (b = a'^2 + 3\alpha^2)$, then the contradiction with a, b coprime.

The other cases of the expressions of d and d' not coprime so that $N = B^n C^l = d \cdot d'$ give also contradictions.

** I-2-2-1-1-6-4- Now, let $C = c_1^r C_1$ with c_1 a prime, $r \geq 1$ and $c_1 \nmid C_1$, we consider the case $d = C_1^l$ and $d' = c_1^r B^n$ so that $d > d'$. We rewrite the equations (3.2.24-3.2.25):

$$C_1^l + c_1^r B^n = 6\alpha \quad (1.6.22)$$

$$C_1^l - c_1^r B^n = 2a' \quad (1.6.23)$$

We obtain $c_1^r B^n = B^n \Rightarrow c_1^r = 1$, then the contradiction.

** I-2-2-1-1-6-5- Now, let $C = c_1^r C_1$ with c_1 a prime, $r \geq 1$ and $c_1 \nmid C_1$, we consider the case $d = C_1^l B^n$ and $d' = c_1^r$ so that $d > d'$. We rewrite the equations (3.2.24-3.2.25):

$$C_1^l B^n + c_1^r = 6\alpha \quad (1.6.24)$$

$$C_1^l B^n - c_1^r = 2a' \quad (1.6.25)$$

We obtain $c_1^r = B^n \Rightarrow c_1 \mid B^n$, as $c_1 \mid C$ then $c_1 \mid A^m = 2a'$. If $c_1 = 2$, the contradiction with $B^n C^l \equiv 1 \pmod{4}$. Then $c_1 \mid a' \Rightarrow c_1 \mid (a = a'^2) \Rightarrow c_1 \mid (p = b)$, it follows a, b are not coprime, then the contradiction.

Cases like $d' < C^l$ a divisor of C^l or $d' < B^l$ a divisor of B^n with $d' < d$ and $d \cdot d' = N = B^n C^l$ give contradictions.

** I-2-2-1-1-6-6- Now, we consider the case $d = b_1.C^l$ where b_1 is a prime integer with $b_1 \nmid B_1$ and $B = b_1^r B_1$, $r \geq 1$. It follows that $d' = b_1^{nr-1} B_1^n$. We rewrite the equations (3.2.24-3.2.25):

$$b_1 C^l + b_1^{nr-1} B_1^n = 6\alpha \quad (1.6.26)$$

$$b_1 C^l - b_1^{nr-1} B_1^n = 2a' \quad (1.6.27)$$

As $n \geq 3$, from the last two equations above, it follows that $b_1 \mid 6\alpha$ and $b_1 \mid (2a')$. Then $b_1 = 2$, or $b_1 \mid \alpha$ and $b_1 \mid a'$ or $b_1 = 3$ and $3 \mid a'$.

** I-2-2-1-1-6-6-1- We suppose $b_1 = 2 \implies 2 \mid B^n$. As $2 \mid (A^m = 2a' \implies 2 \mid a' \implies 2 \mid a$, but $2 \mid B^n$ and $2 \mid A^m$ then $2 \mid (p = 3b)$. It follows the contradiction with a, b coprime.

** I-2-2-1-1-6-6-2- We suppose $b_1 \neq 2, 3$, then $b_1 \mid \alpha$ and $b_1 \mid a' \implies b_1 \mid (a = a'^2)$, then $b_1 \mid (b = 3\alpha^2 + a'^2)$, it follows the contradiction with a, b coprime.

** I-2-2-1-1-6-6-3- We suppose $b_1 = 3 \implies 3 \mid 6\alpha$, and $3 \mid (A^m = 2a') \implies 3 \mid (a = a'^2)$, then $3 \mid (b = 3\alpha^2 + a'^2)$, it follows the contradiction with a, b coprime.

The other cases of the expressions of d and d' with d, d' not coprime and $d > d'$ so that $N = C^l B^m = d.d'$ give also contradictions.

Finally, from the cases studied in the above paragraph I-2-2-1-1-6-, we have found one suitable factorization of N that gives a priori no contradictions, it is the case $N = B^n.C^l = d.d'$ with $d = C^l, d' = B^n$ but $1 \ll \tau(N)$, it follows the contradiction with $Q(N) = [\tau(N)/2] \leq 1$. The last case $A^m \equiv 4 \pmod{6}$ and $B^n \equiv 1 \pmod{6}$ gives contradictions.

It follows that the condition $3 \mid (b - a)$ is a contradiction.

The study of the case 1.6.8 is achieved.

1.6.9 Case $3 \mid p$ and $b \mid 4p$

The following cases have been soon studied:

- * $3 \mid p, b = 2 \implies b \mid 4p$: case 1.6.1,
- * $3 \mid p, b = 4 \implies b \mid 4p$: case 1.6.2,
- * $3 \mid p \implies p = 3p', b \mid p' \implies p' = bp'', p'' \neq 1$: case 1.6.3,
- * $3 \mid p, b = 3 \implies b \mid 4p$: case 1.6.4,
- * $3 \mid p \implies p = 3p', b = p' \implies b \mid 4p$: case 1.6.8.

** J-1- Particular case: $b = 12$. In fact $3 \mid p \implies p = 3p'$ and $4p = 12p'$. Taking $b = 12$, we have $b \mid 4p$. But $b < 4a < 3b$, that gives $12 < 4a < 36 \implies 3 < a < 9$. As $2 \mid b$ and $3 \mid b$, the possible values of a are 5 and 7.

** J-1-1- $a = 5$ and $b = 12 \implies 4p = 12p' = bp'$. But $A^{2m} = \frac{4p}{3} \cdot \frac{a}{b} = \frac{5bp'}{3b} = \frac{5p'}{3} \implies 3 \mid p' \implies p' = 3p''$ with $p'' \in \mathbb{N}^*$, then $p = 9p''$, we obtain the expressions:

$$A^{2m} = 5p'' \quad (1.6.28)$$

$$B^n C^l = \frac{p}{3} \left(3 - 4\cos^2 \frac{\theta}{3} \right) = 4p'' \quad (1.6.29)$$

As $n, l \geq 3$, we deduce from the equation (1.6.29) that $2 \mid p'' \implies p'' = 2^\alpha p_1$ with $\alpha \geq 1$ and $2 \nmid p_1$. Then (1.6.28) becomes: $A^{2m} = 5p'' = 5 \times 2^\alpha p_1 \implies 2 \mid A \implies A = 2^i A_1$, $i \geq 1$ and $2 \nmid A_1$. We have also $B^n C^l = 2^{\alpha+2} p_1 \implies 2 \mid B^n$ or $2 \mid C^l$.

** J-1-1-1- We suppose that $2 \mid B^n \implies B = 2^j B_1$, $j \geq 1$ and $2 \nmid B_1$. We obtain $B_1^n C^l = 2^{\alpha+2-jn} p_1$:

- If $\alpha + 2 - jn > 0 \implies 2 \mid C^l$, there is no contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n \implies 2 \mid C^l$ and the conjecture (3.1.1) is verified.

- If $\alpha + 2 - jn = 0 \implies B_1^n C^l = p_1$. From $C = 2^{im} A_1^m + 2^{jn} B_1^n \implies 2 \mid C^l$ that implies that $2 \mid p_1$, then the contradiction with $2 \nmid p_1$.

- If $\alpha + 2 - jn < 0 \implies 2^{jn-\alpha-2} B_1^n C^l = p_1$, it implies that $2 \mid p_1$, then the contradiction as above.

** J-1-1-2- We suppose that $2 \mid C^l$, using the same method above, we obtain the identical results.

** J-1-2- We suppose that $a = 7$ and $b = 12 \implies 4p = 12p' = bp'$. But $A^{2m} = \frac{4p}{3} \cdot \frac{a}{b} = \frac{12p'}{3} \cdot \frac{7}{12} = \frac{7p'}{3} \implies 3 \mid p' \implies p = 9p''$, we obtain:

$$A^{2m} = 7p''$$

$$B^n C^l = \frac{p}{3} \left(3 - 4\cos^2 \frac{\theta}{3} \right) = 2p''$$

The last equation implies that $2 \mid B^n C^l$. Using the same method as for the case J-1-1- above, we obtain the identical results.

We study now the general case. As $3 \mid p \implies p = 3p'$ and $b \mid 4p \implies \exists k_1 \in \mathbb{N}^*$ and $4p = 12p' = k_1 b$.

** J-2- $k_1 = 1$: If $k_1 = 1$ then $b = 12p'$, ($p' \neq 1$, if not $p = 3 \ll A^{2m} + B^{2n} + A^m B^n$). But $A^{2m} = \frac{4p}{3} \cdot \cos^2 \frac{\theta}{3} = \frac{12p' a}{3 b} = \frac{4p' \cdot a}{12p'} = \frac{a}{3} \implies 3 \mid a$ because A^{2m} is a natural number, then the contradiction with a, b coprime.

** J-3- $k_1 = 3$: If $k_1 = 3$, then $b = 4p'$ and $A^{2m} = \frac{4p}{3} \cdot \cos^2 \frac{\theta}{3} = \frac{k_1 \cdot a}{3} = a = (A^m)^2 = a'^2 \implies A^m = a'$. The term $A^m B^n$ gives $A^m B^n = \frac{p\sqrt{3}}{3} \sin \frac{2\theta}{3} - \frac{a}{2}$, then:

$$A^{2m} + 2A^m B^n = \frac{2p\sqrt{3}}{3} \sin \frac{2\theta}{3} = 2p' \sqrt{3} \sin \frac{2\theta}{3} \quad (1.6.30)$$

The left member of (1.6.30) is an integer number and also p' , then $2\sqrt{3} \sin \frac{2\theta}{3}$ can be written under the form:

$$2\sqrt{3} \sin \frac{2\theta}{3} = \frac{k_2}{k_3}$$

where k_2, k_3 are two integer numbers and are coprime and $k_3 \mid p' \implies p' = k_3 \cdot k_4$.

** J-3-1- $k_4 \neq 1$: We suppose that $k_4 \neq 1$, then:

$$A^{2m} + 2A^m B^n = k_2 \cdot k_4 \quad (1.6.31)$$

Let μ be a prime number so that $\mu \mid k_4$, then $\mu \mid A^m(A^m + 2B^n) \implies \mu \mid A^m$ or $\mu \mid (A^m + 2B^n)$.

** J-3-1-1- $\mu \mid A^m$: If $\mu \mid A^m \implies \mu \mid A^{2m} \implies \mu \mid a$. As $\mu \mid k_4 \implies \mu \mid p' \implies \mu \mid (4p' = b)$. But a, b are coprime, then the contradiction.

** J-3-1-2- $\mu \mid (A^m + 2B^n)$: If $\mu \mid (A^m + 2B^n) \implies \mu \nmid A^m$ and $\mu \nmid 2B^n$, then $\mu \neq 2$ and $\mu \nmid B^n$. $\mu \mid (A^m + 2B^n)$, we can write $A^m + 2B^n = \mu.t'$. It follows:

$$A^m + B^n = \mu t' - B^n \implies A^{2m} + B^{2n} + 2A^m B^n = \mu^2 t'^2 - 2t' \mu B^n + B^{2n}$$

Using the expression of p , we obtain $p = t'^2 \mu^2 - 2t' B^n \mu + B^n (B^n - A^m)$. As $p = 3p'$ and $\mu \mid p' \implies \mu \mid (3p') \implies \mu \mid p$, we can write : $\exists \mu'$ and $p = \mu \mu'$, then we arrive to:

$$\mu' \cdot \mu = \mu(\mu t'^2 - 2t' B^n) + B^n (B^n - A^m)$$

and $\mu \mid B^n (B^n - A^m) \implies \mu \mid B^n$ or $\mu \mid (B^n - A^m)$.

** J-3-1-2-1- $\mu \mid B^n$: If $\mu \mid B^n \implies \mu \mid B$, it is in contradiction with J-3-1-2-.

** J-3-1-2-2- $\mu \mid (B^n - A^m)$: If $\mu \mid (B^n - A^m)$ and using $\mu \mid (A^m + 2B^n)$, we obtain :

$$\mu \mid 3B^n \implies \begin{cases} \mu \mid B^n \\ \text{or} \\ \mu = 3 \end{cases}$$

** J-3-1-2-2-1- $\mu \mid B^n$: If $\mu \mid B^n \implies \mu \mid B$, it is in contradiction with J-3-1-2-.

** J-3-1-2-2-2- $\mu = 3$: If $\mu = 3 \implies 3 \mid k_4 \implies k_4 = 3k'_4$, and we have $p' = k_3 k_4 = 3k_3 k'_4$, it follows that $p = 3p' = 9k_3 k'_4$, then $9 \mid p$, but $p = (A^m - B^n)^2 + 3A^m B^n$, then we obtain:

$$9k_3 k'_4 - 3A^m B^n = (A^m - B^n)^2$$

that we write : $3(3k_3 k'_4 - A^m B^n) = (A^m - B^n)^2$, then : $3 \mid (3k_3 k'_4 - A^m B^n) \implies 3 \mid A^m B^n \implies 3 \mid A^m$ or $3 \mid B^n$.

** J-3-1-2-2-2-1- $3 \mid A^m$: If $3 \mid A^m \implies 3 \mid A^{2m} \implies 3 \mid a$, but $3 \mid p' \implies 3 \mid (4p') \implies 3 \mid b$, then the contradiction with a, b coprime and $3 \nmid A$.

** J-3-1-2-2-2-2- $3 \mid B^n$: If $3 \mid B^n$ but $A^m = \mu t' - 2B^n = 3t' - 2B^n \implies 3 \mid A^m$, it is in contradiction with $3 \nmid A$.

Then the hypothesis $k_4 \neq 1$ is impossible.

** J-3-2- $k_4 = 1$: We suppose now that $k_4 = 1 \implies p' = k_3 k_4 = k_3$. Then we have:

$$2\sqrt{3} \sin \frac{2\theta}{3} = \frac{k_2}{p'} \quad (1.6.32)$$

with k_2, p' coprime, we write (1.6.32) as :

$$4\sqrt{3} \sin \frac{\theta}{3} \cos \frac{\theta}{3} = \frac{k_2}{p'}$$

Taking the square of the two members and replacing $\cos^2 \frac{\theta}{3}$ by $\frac{a}{b}$ and $b = 4p'$, we obtain:

$$3.a(b - a) = k_2^2$$

As $A^{2m} = a = a'^2$, it implies that :

$$3 \mid (b - a), \quad \text{and} \quad b - a = b - a'^2 = 3\alpha^2$$

As $k_2 = A^m(A^m + 2B^n)$ following the equation (1.6.31) and that $3 \mid k_2 \implies 3 \mid A^m(A^m + 2B^n) \implies 3 \mid A^m$ or $3 \mid (A^m + 2B^n)$.

** J-3-2-1- $3 \mid A^m$: If $3 \mid A^m \implies 3 \mid A^{2m} \implies 3 \mid a$, but $3 \mid (b - a) \implies 3 \mid b$, then the contradiction with a, b coprime.

** J-3-2-2- $3 \mid (A^m + 2B^n) \implies 3 \nmid A^m$ and $3 \nmid B^n$. As $k_2^2 = 9a\alpha^2 = 9a'^2\alpha^2 \implies k_2 = 3a'\alpha = A^m(A^m + 2B^n)$, then :

$$3\alpha = A^m + 2B^n \tag{1.6.33}$$

As b can be written under the form $b = a'^2 + 3\alpha^2$, then the pair (a', α) is a solution of the Diophantine equation:

$$x^2 + 3y^2 = b \tag{1.6.34}$$

As $b = 4p'$, then :

** J-3-2-2-1- If x, y are even, then $2 \mid a' \implies 2 \mid a$, it is a contradiction with a, b coprime.

** J-3-2-2-2- If x, y are odd, then a', α are odd, it implies $A^m = a' \equiv 1 \pmod{4}$ or $A^m \equiv 3 \pmod{4}$. If u, v verify (1.6.34), then $b = u^2 + 3v^2$, with $u \neq a'$ and $v \neq \alpha$, then u, v do not verify (1.6.33): $3v \neq u + 2B^n$, if not, $u = 3v - 2B^n \implies b = (3v - 2B^n)^2 + 3v^2 = a'^2 + 3\alpha^2$, the resolution of the obtained equation of second degree in v gives the positive root $v_1 = \alpha$, then $u = 3v - 2B^n = 3\alpha - 2B^n = a'$, then the uniqueness of the representation of b by the equation (1.6.34).

** J-3-2-2-2-1- We suppose that $A^m \equiv 1 \pmod{4}$ and $B^n \equiv 0 \pmod{4}$, then B^n is even and $B^n = 2B'$. The expression of p becomes:

$$\begin{aligned} p &= a'^2 + 2a'B' + 4B'^2 = (a' + B')^2 + 3B'^2 = 3p' \implies 3 \mid (a' + B') \implies a' + B' = 3B'' \\ p' &= B'^2 + 3B''^2 \implies b = 4p' = (2B')^2 + 3(2B'')^2 = a'^2 + 3\alpha^2 \end{aligned}$$

as b has a unique representation, it follows $2B' = B^n = a' = A^m$, then the contradiction with $A^m > B^n$.

** J-3-2-2-2-2- We suppose that $A^m \equiv 1 \pmod{4}$ and $B^n \equiv 1 \pmod{4}$, then C^l is even and $C^l = 2C'$. The expression of p becomes:

$$\begin{aligned} p &= C^{2l} - C^l B^n + B^{2n} = 4C'^2 - 2C'B^n + B^{2n} = (C' - B^n)^2 + 3C'^2 = 3p' \\ &\implies 3 \mid (C' - B^n) \implies C' - B^n = 3C'' \\ p' &= C'^2 + 3C''^2 \implies b = 4p' = (2C')^2 + 3(2C'')^2 = a'^2 + 3\alpha^2 \end{aligned}$$

as b has a unique representation, it follows $2C' = C^l = a' = A^m$, then the contradiction.

** J-3-2-2-2-3- We suppose that $A^m \equiv 1 \pmod{4}$ and $B^n \equiv 2 \pmod{4}$, then B^n is even, see J-3-2-2-2-1-.

** J-3-2-2-2-4- We suppose that $A^m \equiv 1 \pmod{4}$ and $B^n \equiv 3 \pmod{4}$, then C^l is even, see J-3-2-2-2-2-.

** J-3-2-2-2-5- We suppose that $A^m \equiv 3 \pmod{4}$ and $B^n \equiv 0 \pmod{4}$, then B^n is even, see J-3-2-2-2-1-.

** J-3-2-2-2-6- We suppose that $A^m \equiv 3 \pmod{4}$ and $B^n \equiv 1 \pmod{4}$, then C^l is even, see J-3-2-2-2-2-.

** J-3-2-2-2-7- We suppose that $A^m \equiv 3 \pmod{4}$ and $B^n \equiv 2 \pmod{4}$, then B^n is even, see J-3-2-2-2-1-.

** J-3-2-2-2-8- We suppose that $A^m \equiv 3 \pmod{4}$ and $B^n \equiv 3 \pmod{4}$, then C^l is even, see J-3-2-2-2-2-.

We have achieved the study of the case J-3-2-2- . It gives contradictions.

** J-4- We suppose that $k_1 \neq 3$ and $3 \mid k_1 \implies k_1 = 3k'_1$ with $k'_1 \neq 1$, then $4p = 12p' = k_1b = 3k'_1b \implies 4p' = k'_1b$. A^{2m} can be written as $A^{2m} = \frac{4p}{3} \cos^2 \frac{\theta}{3} = \frac{3k'_1b}{3} \frac{a}{b} = k'_1a$ and $B^n C^l = \frac{p}{3} \left(3 - 4\cos^2 \frac{\theta}{3} \right) = \frac{k'_1}{4} (3b - 4a)$. As $B^n C^l$ is an integer number, we must have $4 \mid (3b - 4a)$ or $4 \mid k'_1$ or $[2 \mid k'_1$ and $2 \mid (3b - 4a)]$.

** J-4-1- We suppose that $4 \mid (3b - 4a)$.

** J-4-1-1- We suppose that $3b - 4a = 4 \implies 4 \mid b \implies 2 \mid b$. Then, we have:

$$\begin{aligned} A^{2m} &= k'_1 a \\ B^n C^l &= k'_1 \end{aligned}$$

** J-4-1-1-1- If k'_1 is prime, from $B^n C^l = k'_1$, it is impossible.

** J-4-1-1-2- We suppose that $k'_1 > 1$ is not prime. Let ω be a prime number so that $\omega \mid k'_1$.

** J-4-1-1-2-1- We suppose that $k'_1 = \omega^s$, with $s \geq 6$. Then we have :

$$A^{2m} = \omega^s . a \tag{1.6.35}$$

$$B^n C^l = \omega^s \tag{1.6.36}$$

** J-4-1-1-2-1-1- We suppose that $\omega = 2$. If a, k'_1 are not coprime , then $2 \mid a$, as $2 \mid b$, it is the contradiction with a, b coprime.

** J-4-1-1-2-1-2- We suppose $\omega = 2$ and a, k'_1 are coprime, then $2 \nmid a$. From (1.6.36), we deduce that $B = C = 2$ and $n + l = s$, and $A^{2m} = 2^s . a$, but $A^m = 2^l - 2^n \implies A^{2m} = (2^l - 2^n)^2 = 2^{2l} + 2^{2n} - 2(2^{l+n}) = 2^{2l} + 2^{2n} - 2 \times 2^s = 2^s . a \implies 2^{2l} + 2^{2n} = 2^s (a + 2)$. If $l = n$, we obtain $a = 0$ then the contradiction. If $l \neq n$, as $A^m = 2^l - 2^n > 0 \implies n < l \implies 2n < s$, then $2^{2n} (1 + 2^{2l-2n} - 2^{s+1-2n}) = 2^n 2^l . a$. We call $l = n + n_1 \implies 1 + 2^{2l-2n} - 2^{s+1-2n} = 2^{n_1} . a$,

but the left member is odd and the right member is even, then the contradiction. Then the case $\omega = 2$ is impossible.

** J-4-1-1-2-1-3- We suppose that $k'_1 = \omega^s$ with $\omega \neq 2$:

** J-4-1-1-2-1-3-1- Suppose that a, k'_1 are not coprime, then $\omega \mid a \implies a = \omega^t .a_1$ and $t \nmid a_1$. Then, we have:

$$A^{2m} = \omega^{s+t} .a_1 \quad (1.6.37)$$

$$B^n C^l = \omega^s \quad (1.6.38)$$

From (1.6.38), we deduce that $B^n = \omega^n$, $C^l = \omega^l$, $s = n + l$ and $A^m = \omega^l - \omega^n > 0 \implies l > n$. We have also $A^{2m} = \omega^{s+t} .a_1 = (\omega^l - \omega^n)^2 = \omega^{2l} + \omega^{2n} - 2 \times \omega^s$. As $\omega \neq 2 \implies \omega$ is odd, then $A^{2m} = \omega^{s+t} .a_1 = (\omega^l - \omega^n)^2$ is even, then $2 \mid a_1 \implies 2 \mid a$, it is in contradiction with a, b coprime, then this case is impossible.

** J-4-1-1-2-1-3-2- Suppose that a, k'_1 are coprime, with :

$$A^{2m} = \omega^s .a \quad (1.6.39)$$

$$B^n C^l = \omega^s \quad (1.6.40)$$

From (1.6.40), we deduce that $B^n = \omega^n$, $C^l = \omega^l$ and $s = n + l$. As $\omega \neq 2 \implies \omega$ is odd and $A^{2m} = \omega^s .a = (\omega^l - \omega^n)^2$ is even, then $2 \mid a$. It follows the contradiction with a, b coprime and this case is impossible.

** J-4-1-1-2-2- We suppose that $k'_1 = \omega^s .k_2$, with $s \geq 6$, $\omega \nmid k_2$. We have :

$$A^{2m} = \omega^s .k_2 .a$$

$$B^n C^l = \omega^s .k_2$$

** J-4-1-1-2-2-1- If k_2 is prime, from the last equation above, $\omega = k_2$, it is in contradiction with $\omega \nmid k_2$. Then this case is impossible.

** J-4-1-1-2-2-2- We suppose that $k'_1 = \omega^s .k_2$, with $s \geq 6$, $\omega \nmid k_2$ and k_2 not a prime. Then, we have:

$$A^{2m} = \omega^s .k_2 .a$$

$$B^n C^l = \omega^s .k_2 \quad (1.6.41)$$

** J-4-1-1-2-2-2-1- We suppose that ω, a are coprime, then $\omega \nmid a$. As $A^{2m} = \omega^s .k_2 .a \implies \omega \mid A \implies A = \omega^i .A_1$ with $i \geq 1$ and $\omega \nmid A_1$, then $s = 2i.m$. From (1.6.41), we have $\omega \mid (B^n C^l) \implies \omega \mid B^n$ or $\omega \mid C^l$.

** J-4-1-1-2-2-2-1-1- We suppose that $\omega \mid B^n \implies \omega \mid B \implies B = \omega^j .B_1$ with $j \geq 1$ and $\omega \nmid B_1$. then :

$$B_1^n C^l = \omega^{2im-jn} k_2$$

- If $2im - jn > 0$, $\omega \mid C^l \implies \omega \mid C$, no contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n$ and the conjecture (3.1.1) is verified.

- If $2im - jn = 0 \implies B_1^n C^l = k_2$, as $\omega \nmid k_2 \implies \omega \nmid C^l$, then the contradiction with $\omega \mid (C^l = A^m + B^n)$.

- If $2im - jn < 0 \implies \omega^{jn-2im}B_1^n C^l = k_2 \implies \omega \mid k_2$, then the contradiction with $\omega \nmid k_2$.

** J-4-1-1-2-2-2-1-2- We suppose that $\omega \mid C^l$. Using the same method used above, we obtain identical results.

** J-4-1-1-2-2-2-2- We suppose that a, ω are not coprime, then $\omega \mid a \implies a = \omega^t.a_1$ and $\omega \nmid a_1$. So we have :

$$A^{2m} = \omega^{s+t}.k_2.a_1 \quad (1.6.42)$$

$$B^n C^l = \omega^s.k_2 \quad (1.6.43)$$

As $A^{2m} = \omega^{s+t}.k_2.a_1 \implies \omega \mid A \implies A = \omega^i A_1$ with $i \geq 1$ and $\omega \nmid A_1$, then $s + t = 2im$. From (1.6.43), we have $\omega \mid (B^n C^l) \implies \omega \mid B^n$ or $\omega \mid C^l$.

** J-4-1-1-2-2-2-2-1- We suppose that $\omega \mid B^n \implies \omega \mid B \implies B = \omega^j B_1$ with $j \geq 1$ and $\omega \nmid B_1$. then:

$$B_1^n C^l = \omega^{2im-t-jn} k_2$$

- If $2im - t - jn > 0$, $\omega \mid C^l \implies \omega \mid C$, no contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n$ and the conjecture (3.1.1) is verified.

- If $2im - t - jn = 0 \implies B_1^n C^l = k_2$, As $\omega \nmid k_2 \implies \omega \nmid C^l$, then the contradiction with $\omega \mid (C^l = A^m + B^n)$.

- If $2im - t - jn < 0 \implies \omega^{jn+t-2im} B_1^n C^l = k_2 \implies \omega \mid k_2$, then the contradiction with $\omega \nmid k_2$.

** J-4-1-1-2-2-2-2-2- We suppose that $\omega \mid C^l$. Using the same method used above, we obtain identical results.

** J-4-1-2- 3b - 4a $\neq 4$ and $4 \mid (3b - 4a) \implies 3b - 4a = 4^s \Omega$ with $s \geq 1$ and $4 \nmid \Omega$. We obtain:

$$A^{2m} = k'_1 a \quad (1.6.44)$$

$$B^n C^l = 4^{s-1} k'_1 \Omega \quad (1.6.45)$$

** J-4-1-2-1- We suppose that $k'_1 = 2$. From (1.6.44), we deduce that $2 \mid a$. As $4 \mid (3b - 4a) \implies 2 \mid b$, then the contradiction with a, b coprime and this case is impossible.

** J-4-1-2-2- We suppose that $k'_1 = 3$. From (1.6.44) we deduce that $3^3 \mid A^{2m}$. From (1.6.45), it follows that $3^3 \mid B^n$ or $3^3 \mid C^l$. In the last two cases, we obtain $3^3 \mid p$. But $4p = 3k'_1 b = 9b \implies 3 \mid b$, then the contradiction with a, b coprime. Then this case is impossible.

** J-4-1-2-3- We suppose that k'_1 is prime ≥ 5 :

** J-4-1-2-3-1- Suppose that k'_1 and a are coprime. The equation (1.6.44) gives $(A^m)^2 = k'_1.a$, that is impossible with $k'_1 \nmid a$. Then this case is impossible.

** J-4-1-2-3-2- Suppose that k'_1 and a are not coprime. Let $k'_1 \mid a \implies a = k_1'^{\alpha} a_1$ with $\alpha \geq 1$ and $k'_1 \nmid a_1$. The equation (1.6.44) is written as :

$$A^{2m} = k'_1 a = k_1'^{\alpha+1} a_1$$

The last equation gives $k'_1 \mid A^{2m} \implies k'_1 \mid A \implies A = k_1^i . A_1$, with $k'_1 \nmid A_1$. If $2i.m \neq (\alpha + 1)$, it is impossible. We suppose that $2i.m = \alpha + 1$, then $k'_1 \mid A^m$. We return to the equation (1.6.45). If k'_1 and Ω are coprime, it is impossible. We suppose that k'_1 and Ω are not coprime, then $k'_1 \mid \Omega$ and the exponent of k'_1 in Ω is so the equation (1.6.45) is satisfying. We deduce easily that $k'_1 \mid B^n$. Then $k_1^{l2} \mid (p = A^{2m} + B^{2n} + A^m B^n)$, but $4p = 3k_1^l b \implies k_1^l \mid b$, then the contradiction with a, b coprime.

** J-4-1-2-4- We suppose that $k'_1 \geq 4$ is not a prime.

** J-4-1-2-4-1- We suppose that $k'_1 = 4$, we obtain then $A^{2m} = 4a$ and $B^n C^l = 3b - 4a = 3p' - 4a$. This case was studied in the paragraph 1.6.8, case ** I-2-.

** J-4-1-2-4-2- We suppose that $k'_1 > 4$ is not a prime.

** J-4-1-2-4-2-1- We suppose that a, k'_1 are coprime. From the expression $A^{2m} = k_1^l . a$, we deduce that $a = a_1^2$ and $k'_1 = k''_1$. It gives :

$$\begin{aligned} A^m &= a_1 . k''_1 \\ B^n C^l &= 4^{s-1} k''_1^2 . \Omega \end{aligned}$$

Let ω be a prime so that $\omega \mid k''_1$ and $k''_1 = \omega^t . k''_2$ with $\omega \nmid k''_2$. The last two equations become :

$$A^m = a_1 . \omega^t . k''_2 \quad (1.6.46)$$

$$B^n C^l = 4^{s-1} \omega^{2t} . k''_2^2 . \Omega \quad (1.6.47)$$

From (1.6.46), $\omega \mid A^m \implies \omega \mid A \implies A = \omega^i . A_1$ with $\omega \nmid A_1$ and $im = t$. From (1.6.47), we obtain $\omega \mid B^n C^l \implies \omega \mid B^n$ or $\omega \mid C^l$.

** J-4-1-2-4-2-1-1- If $\omega \mid B^n \implies \omega \mid B \implies B = \omega^j . B_1$ with $\omega \nmid B_1$. From (1.6.46), we have $B_1^n C^l = \omega^{2t-j.n} 4^{s-1} . k''_2^2 . \Omega$.

** J-4-1-2-4-2-1-1-1- If $\omega = 2$ and $2 \nmid \Omega$, we have $B_1^n C^l = 2^{2t+2s-j.n-2} k''_2^2 . \Omega$:

- If $2t + 2s - jn - 2 \leq 0$ then $2 \nmid C^l$, then the contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n$.
- If $2t + 2s - jn - 2 \geq 1 \implies 2 \mid C^l \implies 2 \mid C$ and the conjecture (3.1.1) is verified.

** J-4-1-2-4-2-1-1-2- If $\omega = 2$ and if $2 \mid \Omega \implies \Omega = 2 . \Omega_1$ because $4 \nmid \Omega$, we have $B_1^n C^l = 2^{2t+2s+1-j.n-2} k''_2^2 \Omega_1$:

- If $2t + 2s - jn - 3 \leq 0$ then $2 \nmid C^l$, then the contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n$.
- If $2t + 2s - jn - 3 \geq 1 \implies 2 \mid C^l \implies 2 \mid C$ and the conjecture (3.1.1) is verified.

** J-4-1-2-4-2-1-1-3- If $\omega \neq 2$, we have $B_1^n C^l = \omega^{2t-j.n} 4^{s-1} . k''_2^2 . \Omega$:

- If $2t - jn \leq 0 \implies \omega \nmid C^l$ it is in contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n$.
- If $2t - jn \geq 1 \implies \omega \mid C^l \implies \omega \mid C$ and the conjecture (3.1.1) is verified.

** J-4-1-2-4-2-1-2- If $\omega \mid C^l \implies \omega \mid C \implies C = \omega^h . C_1$, with $\omega \nmid C_1$. Using the same method as in the case J-4-1-2-4-2-1-1 above, we obtain identical results.

** J-4-1-2-4-2-2- We suppose that a, k'_1 are not coprime. Let ω be a prime so that $\omega \mid a$ and $\omega \mid k'_1$. We write:

$$\begin{aligned} a &= \omega^\alpha . a_1 \\ k'_1 &= \omega^\mu . k''_1 \end{aligned}$$

with a_1, k''_1 coprime. The expression of A^{2m} becomes $A^{2m} = \omega^{\alpha+\mu} . a_1 . k''_1$. The term $B^n C^l$ becomes:

$$B^n C^l = 4^{s-1} . \omega^\mu . k''_1 . \Omega \quad (1.6.48)$$

** J-4-1-2-4-2-2-1- If $\omega = 2 \implies 2 \mid a$, but $2 \nmid b$, then the contradiction with a, b coprime, this case is impossible.

** J-4-1-2-4-2-2-2- If $\omega \geq 3$, we have $\omega \mid a$. If $\omega \mid b$ then the contradiction with a, b coprime. We suppose that $\omega \nmid b$. From the expression of A^{2m} , we obtain $\omega \mid A^{2m} \implies \omega \mid A \implies A = \omega^i . A_1$ with $\omega \nmid A_1$, $i \geq 1$ and $2i.m = \alpha + \mu$. From (1.6.48), we deduce that $\omega \mid B^n$ or $\omega \mid C^l$.

** J-4-1-2-4-2-2-2-1- We suppose that $\omega \mid B^n \implies \omega \mid B \implies B = \omega^j B_1$ with $\omega \nmid B_1$ and $j \geq 1$. Then, $B_1^n C^l = 4^{s-1} \omega^{\mu-jn} . k''_1 . \Omega$:

* $\omega \nmid \Omega$:

- If $\mu - jn \geq 1$, we have $\omega \mid C^l \implies \omega \mid C$, there is no contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n$ and the conjecture (3.1.1) is verified.

- If $\mu - jn \leq 0$, then $\omega \nmid C^l$ and it is a contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n$. Then this case is impossible.

* $\omega \mid \Omega$: we write $\Omega = \omega^\beta . \Omega_1$ with $\beta \geq 1$ and $\omega \nmid \Omega_1$. As $3b - 4a = 4^s . \Omega = 4^s . \omega^\beta . \Omega_1 \implies 3b = 4a + 4^s . \omega^\beta . \Omega_1 = 4\omega^\alpha . a_1 + 4^s . \omega^\beta . \Omega_1 \implies 3b = 4\omega(\omega^{\alpha-1} . a_1 + 4^{s-1} . \omega^{\beta-1} . \Omega_1)$. If $\omega = 3$ and $\beta = 1$, we obtain $b = 4(3^{\alpha-1} a_1 + 4^{s-1} \Omega_1)$ and $B_1^n C^l = 4^{s-1} 3^{\mu+1-jn} . k''_1 \Omega_1$.

- If $\mu - jn + 1 \geq 1$, then $3 \mid C^l$ and the conjecture (3.1.1) is verified.

- If $\mu - jn + 1 \leq 0$, then $3 \nmid C^l$ and it is the contradiction with $C^l = 3^{im} A_1^m + 3^{jn} B_1^n$.

Now, if $\beta \geq 2$ and $\alpha = im \geq 3$, we obtain $3b = 4\omega^2(\omega^{\alpha-2} a_1 + 4^{s-1} \omega^{\beta-2} \Omega_1)$. If $\omega = 3$ or not, then $\omega \mid b$, but $\omega \nmid a$, then the contradiction with a, b coprime.

** J-4-1-2-4-2-2-2-2- We suppose that $\omega \mid C^l \implies \omega \mid C \implies C = \omega^h C_1$ with $\omega \nmid C_1$ and $h \geq 1$. Then, $B^n C_1^l = 4^{s-1} \omega^{\mu-hl} . k''_1 . \Omega$. Using the same method as above, we obtain identical results.

** J-4-2- We suppose that $4 \mid k'_1$.

** J-4-2-1- $k'_1 = 4 \implies 4p = 3k'_1 b = 12b \implies p = 3b = 3p'$, this case has been studied (see case I-2- paragraph 1.6.8).

** J-4-2-2- $k'_1 > 4$ with $4 \mid k'_1 \implies k'_1 = 4^s k''_1$ and $s \geq 1$, $4 \nmid k''_1$. Then, we obtain:

$$\begin{aligned} A^{2m} &= 4^s k''_1 a = 2^{2s} k''_1 a \\ B^n C^l &= 4^{s-1} k''_1 (3b - 4a) = 2^{2s-2} k''_1 (3b - 4a) \end{aligned}$$

** J-4-2-2-1- We suppose that $s = 1$ and $k'_1 = 4k''_1$ with $k''_1 > 1$, so $p = 3p'$ and $p' = k''_1 b$, this is the case 1.6.3 already studied.

** J-4-2-2-2- We suppose that $s > 1$, then $k'_1 = 4^s k''_1 \implies 4p = 3 \times 4^s k''_1 b$ and we obtain:

$$A^{2m} = 4^s k''_1 a \quad (1.6.49)$$

$$B^n C^l = 4^{s-1} k''_1 (3b - 4a) \quad (1.6.50)$$

** J-4-2-2-2-1- We suppose that $2 \nmid (k''_1 \cdot a) \implies 2 \nmid k''_1$ and $2 \nmid a$. As $(A^m)^2 = (2^s)^2 \cdot (k''_1 \cdot a)$, we call $d^2 = k''_1 \cdot a$, then $A^m = 2^s \cdot d \implies 2 \mid A^m \implies 2 \mid A \implies A = 2^i A_1$ with $2 \nmid A_1$ and $i \geq 1$, then: $2^{im} A_1^m = 2^s \cdot d \implies s = im$. From the equation (1.6.50), we have $2 \mid (B^n C^l) \implies 2 \mid B^n$ or $2 \mid C^l$.

** J-4-2-2-2-1-1- We suppose that $2 \mid B^n \implies 2 \mid B \implies B = 2^j \cdot B_1$, with $j \geq 1$ and $2 \nmid B_1$. The equation (1.6.50) becomes:

$$B_1^n C^l = 2^{2s-jn-2} k''_1 (3b - 4a) = 2^{2im-jn-2} k''_1 (3b - 4a)$$

* We suppose that $2 \nmid (3b - 4a)$:

- If $2im - jn - 2 \geq 1$, then $2 \mid C^l$, there is no contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$ and the conjecture (3.1.1) is verified.

- If $2im - jn - 2 \leq 0$, then $2 \nmid C^l$, then the contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$.

* We suppose that $2^\mu \mid (3b - 4a)$, $\mu \geq 1$:

- If $2im + \mu - jn - 2 \geq 1$, then $2 \mid C^l$, no contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$ and the conjecture (3.1.1) is verified.

- If $2im + \mu - jn - 2 \leq 0$, then $2 \nmid C^l$, then the contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$.

** J-4-2-2-2-1-2- We suppose that $2 \mid C^l \implies 2 \mid C \implies C = 2^h \cdot C_1$, with $h \geq 1$ and $2 \nmid C_1$. With the same method used above, we obtain identical results.

** J-4-2-2-2-2- We suppose that $2 \mid (k''_1 \cdot a)$:

** J-4-2-2-2-2-1- We suppose that k''_1 and a are coprime:

** J-4-2-2-2-2-1-1- We suppose that $2 \nmid a$ and $2 \mid k''_1 \implies k''_1 = 2^{2\mu} \cdot k''_2$ and $a = a_1^2$, then the equations (1.6.49-1.6.50) become:

$$A^{2m} = 4^s \cdot 2^{2\mu} k''_2 a_1^2 \implies A^m = 2^{s+\mu} \cdot k''_2 \cdot a_1 \quad (1.6.51)$$

$$B^n C^l = 4^{s-1} 2^{2\mu} k''_2 (3b - 4a) = 2^{2s+2\mu-2} k''_2 (3b - 4a) \quad (1.6.52)$$

The equation (1.6.51) gives $2 \mid A^m \implies 2 \mid A \implies A = 2^i \cdot A_1$ with $2 \nmid A_1$, $i \geq 1$ and $im = s + \mu$. From the equation (1.6.52), we have $2 \mid (B^n C^l) \implies 2 \mid B^n$ or $2 \mid C^l$.

** J-4-2-2-2-2-1-1-1- We suppose that $2 \mid B^n \implies 2 \mid B \implies B = 2^j \cdot B_1$, $2 \nmid B_1$ and $j \geq 1$, then $B_1^n C^l = 2^{2s+2\mu-jn-2} k''_2 (3b - 4a)$:

* We suppose that $2 \nmid (3b - 4a)$:

- If $2im + 2\mu - jn - 2 \geq 1 \implies 2 \mid C^l$, then there is no contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$ and the conjecture (3.1.1) is verified.

- If $2im + 2\mu - jn - 2 \leq 0 \implies 2 \nmid C^l$, then the contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$.

* We suppose that $2^\alpha \mid (3b - 4a)$, $\alpha \geq 1$ so that a, b remain coprime:

- If $2im + 2\mu + \alpha - jn - 2 \geq 1 \Rightarrow 2 \mid C^l$, then no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (3.1.1) is verified.

- If $2im + 2\mu + \alpha - jn - 2 \leq 0 \Rightarrow 2 \nmid C^l$, then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$.

** J-4-2-2-2-2-1-1-2- We suppose that $2 \mid C^l \Rightarrow 2 \mid C \Rightarrow C = 2^h.C_1$, with $h \geq 1$ and $2 \nmid C_1$. With the same method used above, we obtain identical results.

** J-4-2-2-2-2-1-2- We suppose that $2 \nmid k''_1$ and $2 \mid a \Rightarrow a = 2^{2\mu}.a_1^2$ and $k''_1 = k''_2^2$, then the equations (1.6.49-1.6.50) become:

$$A^{2m} = 4^s.2^{2\mu}.a_1^2.k''_2^2 \Rightarrow A^m = 2^{s+\mu}.a_1.k''_2. \quad (1.6.53)$$

$$B^n.C^l = 4^{s-1}.k''_2^2.(3b-4a) = 2^{2s-2}.k''_2^2.(3b-4a) \quad (1.6.54)$$

The equation (1.6.53) gives $2 \mid A^m \Rightarrow 2 \mid A \Rightarrow A = 2^i.A_1$ with $2 \nmid A_1$, $i \geq 1$ and $im = s + \mu$. From the equation (1.6.54), we have $2 \mid (B^n.C^l) \Rightarrow 2 \mid B^n$ or $2 \mid C^l$.

** J-4-2-2-2-2-1-2-1- We suppose that $2 \mid B^n \Rightarrow 2 \mid B \Rightarrow B = 2^j.B_1$, $2 \nmid B_1$ and $j \geq 1$. Then we obtain $B_1^n.C^l = 2^{2s-jn-2}.k''_2^2.(3b-4a)$:

* We suppose that $2 \nmid (3b-4a) \Rightarrow 2 \nmid b$:

- If $2im - jn - 2 \geq 1 \Rightarrow 2 \mid C^l$, then no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (3.1.1) is verified.

- If $2im - jn - 2 \leq 0 \Rightarrow 2 \nmid C^l$, then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$.

* We suppose that $2^\alpha \mid (3b-4a)$, $\alpha \geq 1$, in this case a, b are not coprime, then the contradiction.

** J-4-2-2-2-2-1-2-2- We suppose that $2 \mid C^l \Rightarrow 2 \mid C \Rightarrow C = 2^h.C_1$, with $h \geq 1$ and $2 \nmid C_1$. With the same method used above, we obtain identical results.

** J-4-2-2-2-2-2- We suppose that k''_1 and a are not coprime $2 \mid a$ and $2 \mid k''_1$. Let $a = 2^t.a_1$ and $k''_1 = 2^\mu.k''_2$ and $2 \nmid a_1$ and $2 \nmid k''_2$. From (1.6.49), we have $\mu + t = 2\lambda$ and $a_1.k''_2 = \omega^2$. The equations (1.6.49-1.6.50) become:

$$A^{2m} = 4^s.k''_1.a = 2^{2s}.2^\mu.k''_2.2^t.a_1 = 2^{2s+2\lambda}.\omega^2 \Rightarrow A^m = 2^{s+\lambda}.\omega \quad (1.6.55)$$

$$B^n.C^l = 4^{s-1}.2^\mu.k''_2^2.(3b-4a) = 2^{2s+\mu-2}.k''_2^2.(3b-4a) \quad (1.6.56)$$

From (1.6.55) we have $2 \mid A^m \Rightarrow 2 \mid A \Rightarrow A = 2^i.A_1$, $i \geq 1$ and $2 \nmid A_1$. From (1.6.56), $2s + \mu - 2 \geq 1$, we deduce that $2 \mid (B^n.C^l) \Rightarrow 2 \mid B^n$ or $2 \mid C^l$.

** J-4-2-2-2-2-2-1- We suppose that $2 \mid B^n \Rightarrow 2 \mid B \Rightarrow B = 2^j.B_1$, $2 \nmid B_1$ and $j \geq 1$. Then we obtain $B_1^n.C^l = 2^{2s+\mu-jn-2}.k''_2^2.(3b-4a)$:

* We suppose that $2 \nmid (3b-4a)$:

- If $2s + \mu - jn - 2 \geq 1 \Rightarrow 2 \mid C^l$, then no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (3.1.1) is verified.

- If $2s + \mu - jn - 2 \leq 0 \Rightarrow 2 \nmid C^l$, then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$.

* We suppose that $2^\alpha \mid (3b-4a)$, for one value $\alpha \geq 1$. As $2 \mid a$, then $2^\alpha \mid (3b-4a) \Rightarrow 2 \mid (3b-4a) \Rightarrow 2 \mid (3b) \Rightarrow 2 \mid b$, then the contradiction with a, b coprime.

** J-4-2-2-2-2-2- We suppose that $2 \mid C^l \implies 2 \mid C \implies C = 2^h.C_1$, with $h \geq 1$ and $2 \nmid C_1$. With the same method used above, we obtain identical results.

** J-4-3- $2 \mid k'_1$ and $2 \mid (3b - 4a)$: then we obtain $2 \mid k'_1 \implies k'_1 = 2^t.k''_1$ with $t \geq 1$ and $2 \nmid k''_1$, $2 \mid (3b - 4a) \implies 3b - 4a = 2^\mu.d$ with $\mu \geq 1$ and $2 \nmid d$. We have also $2 \mid b$. If $2 \mid a$, it is a contradiction with a, b coprime.

We suppose, in the following, that $2 \nmid a$. The equations (1.6.49-1.6.50) become:

$$A^{2m} = 2^t.k''_1.a = (A^m)^2 \quad (1.6.57)$$

$$B^n C^l = 2^{t-1}k''_1.2^{\mu-1}d = 2^{t+\mu-2}k''_1.d \quad (1.6.58)$$

From (1.6.57), we deduce that the exponent t is even, let $t = 2\lambda$. Then we call $\omega^2 = k''_1.a$, it gives $A^m = 2^\lambda.\omega \implies 2 \mid A^m \implies 2 \mid A \implies A = 2^i.A_1$ with $i \geq 1$ and $2 \nmid A_1$. From (1.6.58), we have $2\lambda + \mu - 2 \geq 1$, then $2 \mid (B^n C^l) \implies 2 \mid B^n$ or $2 \mid C^l$:

** J-4-3-1- We suppose that $2 \mid B^n \implies 2 \mid B \implies B = 2^j B_1$, with $j \geq 1$ and $2 \nmid B_1$. Then we obtain $B_1^n C^l = 2^{2\lambda+\mu-jn-2}.k''_1.d$.

- If $2\lambda + \mu - jn - 2 \geq 1 \implies 2 \mid C^l \implies 2 \mid C$, there is no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (3.1.1) is verified.

- If $2s + t + \mu - jn - 2 \leq 0 \implies 2 \nmid C$, then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$.

** J-4-3-2- We suppose that $2 \mid C^l \implies 2 \mid C$. With the same method used above, we obtain identical results. □

The Main Theorem is proved.

1.7 Examples and Conclusion

1.7.1 Numerical Examples

Example 1:

We consider the example : $6^3 + 3^3 = 3^5$ with $A^m = 6^3$, $B^n = 3^3$ and $C^l = 3^5$. With the notations used in the paper, we obtain:

$$\begin{aligned} p &= 3^6 \times 73, & q &= 8 \times 3^{11}, & \bar{\Delta} &= 4 \times 3^{18}(3^7 \times 4^2 - 73^3) < 0 \\ \rho &= \frac{3^8 \times 73\sqrt{73}}{\sqrt{3}}, & \cos\theta &= -\frac{4 \times 3^3 \times \sqrt{3}}{73\sqrt{73}} \end{aligned} \quad (1.7.1)$$

As $A^{2m} = \frac{4p}{3}.\cos^2\frac{\theta}{3} \implies \cos^2\frac{\theta}{3} = \frac{3A^{2m}}{4p} = \frac{3 \times 2^4}{73} = \frac{a}{b} \implies a = 3 \times 2^4, b = 73$; then we obtain:

$$\cos\frac{\theta}{3} = \frac{4\sqrt{3}}{\sqrt{73}}, \quad p = 3^6.b \quad (1.7.2)$$

We verify easily the equation (1.7.1) to calculate $\cos\theta$ using (1.7.2). For this example, we can use the two conditions from (1.4.9) as $3 \mid a, b \mid 4p$ and $3 \mid p$. The cases 1.5.4 and 1.6.3 are respectively used. For the case 1.5.4, it is the case B-2-2-1- that was used and the conjecture (3.1.1) is verified. Concerning the case 1.6.3, it is the case G-2-2-1- that was used and the conjecture (3.1.1) is verified.

Example 2:

The second example is: $7^4 + 7^3 = 14^3$. We take $A^m = 7^4, B^n = 7^3$ and $C^l = 14^3$. We obtain $p = 57 \times 7^6 = 3 \times 19 \times 7^6$, $q = 8 \times 7^{10}$, $\bar{\Delta} = 27q^2 - 4p^3 = 27 \times 4 \times 7^{18}(16 \times 49 - 19^3) = -27 \times 4 \times 7^{18} \times 6075 < 0$, $\rho = 19 \times 7^9 \times \sqrt{19}$, $\cos\theta = -\frac{4 \times 7}{19\sqrt{19}}$. As $A^{2m} = \frac{4p}{3} \cdot \cos^2\frac{\theta}{3} \implies \cos^2\frac{\theta}{3} = \frac{3A^{2m}}{4p} = \frac{7^2}{4 \times 19} = \frac{a}{b} \implies a = 7^2, b = 4 \times 19$, then $\cos\frac{\theta}{3} = \frac{7}{2\sqrt{19}}$ and we have the two principal conditions $3 \mid p$ and $b \mid (4p)$. The calculation of $\cos\theta$ from the expression of $\cos\frac{\theta}{3}$ is confirmed by the value below:

$$\cos\theta = \cos 3(\theta/3) = 4\cos^3\frac{\theta}{3} - 3\cos\frac{\theta}{3} = 4\left(\frac{7}{2\sqrt{19}}\right)^3 - 3\frac{7}{2\sqrt{19}} = -\frac{4 \times 7}{19\sqrt{19}}$$

Then, we obtain $3 \mid p \implies p = 3p', b \mid (4p)$ with $b \neq 2, 4$ then $12p' = k_1b = 3 \times 7^6b$. It concerns the paragraph 1.6.9 of the second hypothesis. As $k_1 = 3 \times 7^6 = 3k'_1$ with $k'_1 = 7^6 \neq 1$. It is the case J-4-1-2-4-2-2- with the condition $4 \mid (3b - 4a)$. So we verify :

$$3b - 4a = 3 \times 4 \times 19 - 4 \times 7^2 = 32 \implies 4 \mid (3b - 4a)$$

with $A^{2m} = 7^8 = 7^6 \times 7^2 = k'_1 \cdot a$ and k'_1 not a prime, with a and k'_1 not coprime with $\omega = 7 \nmid \Omega (= 2)$. We find that the conjecture (3.1.1) is verified with a common factor equal to 7 (prime and divisor of $k'_1 = 7^6$).

Example 3:

The third example is: $19^4 + 38^3 = 57^3$ with $A^m = 19^4, B^n = 38^3$ and $C^l = 57^3$. We obtain $p = 19^6 \times 577$, $q = 8 \times 27 \times 19^{10}$, $\bar{\Delta} = 27q^2 - 4p^3 = 4 \times 19^{18}(27^3 \times 16 \times 19^2 - 577^3) < 0$, $\rho = \frac{19^9 \times 577\sqrt{577}}{3\sqrt{3}}$, $\cos\theta = -\frac{4 \times 3^4 \times 19\sqrt{3}}{577\sqrt{577}}$. As $A^{2m} = \frac{4p}{3} \cdot \cos^2\frac{\theta}{3} \implies \cos^2\frac{\theta}{3} = \frac{3A^{2m}}{4p} = \frac{3 \times 19^2}{4 \times 577} = \frac{a}{b} \implies a = 3 \times 19^2, b = 4 \times 577$, then $\cos\frac{\theta}{3} = \frac{19\sqrt{3}}{2\sqrt{577}}$ and we have the first hypothesis $3 \mid a$ and $b \mid (4p)$. Here again, the calculation of $\cos\theta$ from the expression of $\cos\frac{\theta}{3}$ is confirmed by the value below:

$$\cos\theta = \cos 3(\theta/3) = 4\cos^3\frac{\theta}{3} - 3\cos\frac{\theta}{3} = 4\left(\frac{19\sqrt{3}}{2\sqrt{577}}\right)^3 - 3\frac{19\sqrt{3}}{2\sqrt{577}} = -\frac{4 \times 3^4 \times 19\sqrt{3}}{577\sqrt{577}}$$

Then, we obtain $3 \mid a \implies a = 3a' = 3 \times 19^2, b \mid (4p)$ with $b \neq 2, 4$ and $b = 4p'$ with $p = kp'$ soit $p' = 577$ and $k = 19^6$. This concerns the paragraph 1.5.8 of the first hypothesis. It is the case E-2-2-2-2-1- with $\omega = 19, a', \omega$ not coprime and $\omega = 19 \nmid (p' - a') = (577 - 19^2)$ with $s - jn = 6 - 1 \times 3 = 3 \geq 1$, and the conjecture (3.1.1) is verified.

1.7.2 Conclusion

The method used to give the proof of the conjecture of Beal has discussed many possibles cases, using elementary number theory and the results of some theorems about Diophantine equations. We have confirmed the method by three numerical examples. In conclusion, we can announce the theorem:

Theorem 1.7.1. *Let $A, B, C, m, n,$ and l be positive natural numbers with $m, n, l > 2$. If:*

$$A^m + B^n = C^l \tag{1.7.3}$$

then $A, B,$ and C have a common factor.

Bibliography

- [1] Maildin D.R. 1977. A Generalization of Fermat's Last Theorem: The Beal Conjecture and Prize Problem. *Notice of AMS*, Vol. 44, n°11, (pp 1436-1437), <https://www.ams.org/notices/199711/beal.pdf>.
- [2] Stewart B.M. 1964. Theory of Numbers, Second edition. *The Macmillan Compagny*, New-York, (pp 196-197).
- [3] Bolker E.D. 1970. Elementary Number Theory: An Algebraic Approach. *W.A. Benjamin, Inc.*, New-York, (pp 121-122).

Chapter 2

Is The Riemann Hypothesis True? Yes It Is

Abstract

In 1859, Georg Friedrich Bernhard Riemann had announced the following conjecture, called Riemann Hypothesis : *The nontrivial roots (zeros) $s = \sigma + it$ of the zeta function, defined by:*

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) > 1$$

have real part $\sigma = \frac{1}{2}$.

We give a proof that $\sigma = \frac{1}{2}$ using an equivalent statement of the Riemann Hypothesis concerning the Dirichlet η function.

Résumé

En 1859, Georg Friedrich Bernhard Riemann avait annoncé la conjecture suivante, dite Hypothèse de Riemann: *Les zéros non triviaux $s = \sigma + it$ de la fonction zeta définie par:*

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ pour } \Re(s) > 1$$

ont comme parties réelles $\sigma = \frac{1}{2}$.

On donne une démonstration que $\sigma = \frac{1}{2}$ en utilisant une proposition équivalente de l'Hypothèse de Riemann.

'I feel that these aren't the right techniques to solve the Riemann hypothesis itself, it's going to need some big idea from somewhere else.'

James Maynard (07/15/2024) [1]

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2.1 Introduction

In 1859, G.F.B. Riemann had announced the following conjecture [2]:

Conjecture 2.1.1. Let $\zeta(s)$ be the complex function of the complex variable $s = \sigma + it$ defined by the analytic continuation of the function:

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) = \sigma > 1$$

over the whole complex plane, with the exception of $s = 1$. Then the nontrivial zeros of $\zeta(s) = 0$ are written as :

$$s = \frac{1}{2} + it$$

In this paper, our idea is to start from an equivalent statement of the Riemann Hypothesis, namely the one concerning the Dirichlet η function. The latter is related to Riemann's ζ function where we do not need to manipulate any expression of $\zeta(s)$ in the critical band $0 < \Re(s) < 1$. In our calculations, we will use the definition of the limit of real sequences.

We arrive to give the proof that $\sigma = \frac{1}{2}$.

2.1.1 The function ζ .

We denote $s = \sigma + it$ the complex variable of \mathbb{C} . For $\Re(s) = \sigma > 1$, let ζ_1 be the function defined by :

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) = \sigma > 1$$

We know that with the previous definition, the function ζ_1 is an analytical function of s . Denote by $\zeta(s)$ the function obtained by the analytic continuation of $\zeta_1(s)$ to the whole complex plane, minus the point $s = 1$, then we recall the following theorem [3]:

Theorem 2.1.2. The function $\zeta(s)$ satisfies the following :

1. $\zeta(s)$ has no zero for $\Re(s) > 1$;
2. the only pole of $\zeta(s)$ is at $s = 1$; it has residue 1 and is simple;
3. $\zeta(s)$ has trivial zeros at $s = -2, -4, \dots$;
4. the nontrivial zeros lie inside the region $0 \leq \Re(s) \leq 1$ (called the critical strip) and are symmetric about both the vertical line $\Re(s) = \frac{1}{2}$ and the real axis $\Im(s) = 0$.

The vertical line $\Re(s) = \frac{1}{2}$ is called the critical line.

The Riemann Hypothesis is formulated as:

Conjecture 2.1.3. (The Riemann Hypothesis,[3]) All nontrivial zeros of $\zeta(s)$ lie on the critical line $\Re(s) = \frac{1}{2}$.

In addition to the properties cited by the theorem 2.1.2 above, the function $\zeta(s)$ satisfies the functional relation [3] called also the reflection functional equation for $s \in \mathbb{C} \setminus \{0, 1\}$:

$$\zeta(1-s) = 2^{1-s} \pi^{-s} \cos \frac{s\pi}{2} \Gamma(s) \zeta(s) \quad (2.1.1)$$

where $\Gamma(s)$ is the *gamma function* defined only for $\Re(s) > 0$, given by the formula :

$$\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} dt, \quad \Re(s) > 0$$

So, instead of using the functional given by (2.1.1), we will use the one presented by G.H. Hardy [4] namely Dirichlet's eta function [3]:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s}) \zeta(s)$$

The function eta is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$ [3].

We have also the theorem (see page 16, [4]):

Theorem 2.1.4. For all $t \in \mathbb{R}$, $\zeta(1+it) \neq 0$.

So, we take the critical strip as the region defined as $0 < \Re(s) < 1$.

2.1.2 A Equivalent statement to the Riemann Hypothesis.

Among the equivalent statements to the Riemann Hypothesis is that of the Dirichlet function eta which is stated as follows [3]:

Equivalence 2.1.5. The Riemann Hypothesis is equivalent to the statement that all zeros of the Dirichlet eta function :

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad \sigma > 1 \quad (2.1.2)$$

that fall in the critical strip $0 < \Re(s) < 1$ lie on the critical line $\Re(s) = \frac{1}{2}$.

The series (2.1.2) is convergent, and represents $(1 - 2^{1-s})\zeta(s)$ for $\Re(s) = \sigma > 0$ ([4], pages 20-21). We can rewrite:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad \Re(s) = \sigma > 0 \quad (2.1.3)$$

$\eta(s)$ is a complex number, it can be written as :

$$\eta(s) = \rho.e^{i\alpha} \implies \rho^2 = \eta(s).\overline{\eta(s)} \quad (2.1.4)$$

and $\eta(s) = 0 \iff \rho = 0$.

2.2 Preliminaries of the proof

Proof. . We denote $s = \sigma + it$ with $0 < \sigma < 1$. We consider one zero of $\eta(s)$ that falls in critical strip and we write it as $s = \sigma + it$, then we obtain $0 < \sigma < 1$ and $\eta(s) = 0 \iff (1 - 2^{1-s})\zeta(s) = 0$. We verifies easily the two propositions:

$$\boxed{s, \text{ is one zero of } \eta(s) \text{ that falls in the critical strip, is also one zero of } \zeta(s)} \quad (2.2.1)$$

Conversely, if s is a zero of $\zeta(s)$ in the critical strip, let $\zeta(s) = 0 \implies \eta(s) = (1 - 2^{1-s})\zeta(s) = 0$, then s is also one zero of $\eta(s)$ in the critical strip. We can write:

$$\boxed{s, \text{ is one zero of } \zeta(s) \text{ that falls in the critical strip, is also one zero of } \eta(s)} \quad (2.2.2)$$

Let us write the function η :

$$\begin{aligned} \eta(s) &= \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-s \text{Log} n} = \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-(\sigma+it) \text{Log} n} = \\ &= \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-\sigma \text{Log} n} \cdot e^{-it \text{Log} n} \\ &= \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-\sigma \text{Log} n} (\cos(t \text{Log} n) - i \sin(t \text{Log} n)) \end{aligned}$$

The function η is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$, but not absolutely convergent. Let s be one zero of the function eta, then :

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = 0$$

or:

$$\forall \epsilon' > 0 \quad \exists n_0, \forall N > n_0, \left| \sum_{n=1}^N \frac{(-1)^{n-1}}{n^s} \right| < \epsilon'$$

We define the sequence of functions $((\eta_n)_{n \in \mathbb{N}^*}(s))$ as:

$$\eta_n(s) = \sum_{k=1}^n \frac{(-1)^{k-1}}{k^s} = \sum_{k=1}^n (-1)^{k-1} \frac{\cos(t \operatorname{Log} k)}{k^\sigma} - i \sum_{k=1}^n (-1)^{k-1} \frac{\sin(t \operatorname{Log} k)}{k^\sigma}$$

with $s = \sigma + it$ and $t \neq 0$.

Let s be one zero of η that lies in the critical strip, then $\eta(s) = 0$, with $0 < \sigma < 1$. It follows that we can write $\lim_{n \rightarrow +\infty} \eta_n(s) = 0 = \eta(s)$. We obtain:

$$\lim_{n \rightarrow +\infty} \sum_{k=1}^n (-1)^{k-1} \frac{\cos(t \operatorname{Log} k)}{k^\sigma} = 0$$

$$\lim_{n \rightarrow +\infty} \sum_{k=1}^n (-1)^{k-1} \frac{\sin(t \operatorname{Log} k)}{k^\sigma} = 0$$

Using the definition of the limit of a sequence, we can write:

$$\forall \epsilon_1 > 0 \exists n_r, \forall N > n_r, |\Re(\eta(s)_N)| < \epsilon_1 \implies \Re(\eta(s)_N)^2 < \epsilon_1^2 \quad (2.2.3)$$

$$\forall \epsilon_2 > 0 \exists n_i, \forall N > n_i, |\Im(\eta(s)_N)| < \epsilon_2 \implies \Im(\eta(s)_N)^2 < \epsilon_2^2 \quad (2.2.4)$$

Then:

$$0 < \sum_{k=1}^N \frac{\cos^2(t \operatorname{Log} k)}{k^{2\sigma}} + 2 \sum_{k,k'=1; k < k'}^N \frac{(-1)^{k+k'} \cos(t \operatorname{Log} k) \cdot \cos(t \operatorname{Log} k')}{k^\sigma k'^\sigma} < \epsilon_1^2$$

$$0 < \sum_{k=1}^N \frac{\sin^2(t \operatorname{Log} k)}{k^{2\sigma}} + 2 \sum_{k,k'=1; k < k'}^N \frac{(-1)^{k+k'} \sin(t \operatorname{Log} k) \cdot \sin(t \operatorname{Log} k')}{k^\sigma k'^\sigma} < \epsilon_2^2$$

Taking $\epsilon = \epsilon_1 = \epsilon_2$ and $N > \max(n_r, n_i)$, we get by making the sum member to member of the last two inequalities:

$$0 < \sum_{k=1}^N \frac{1}{k^{2\sigma}} + 2 \sum_{k,k'=1; k < k'}^N (-1)^{k+k'} \frac{\cos(t \operatorname{Log}(k/k'))}{k^\sigma k'^\sigma} < 2\epsilon^2 \quad (2.2.5)$$

We can write the above equation as :

$$0 < \rho_N^2 < 2\epsilon^2 \quad (2.2.6)$$

or $\rho(s) = 0$.

2.3 Case $\sigma = \frac{1}{2}$.

We suppose that $\sigma = \frac{1}{2}$. Let's start by recalling Hardy's theorem (1914) ([3], page 24):

Theorem 2.3.1. *There are infinitely many zeros of $\zeta(s)$ on the critical line.*

From the propositions (2.2.1-2.2.2), it follows the proposition :

Proposition 2.3.2. There are infinitely many zeros of $\eta(s)$ on the critical line.

Let $s_j = \frac{1}{2} + it_j$ one of the zeros of the function $\eta(s)$ on the critical line, so $\eta(s_j) = 0$. The equation (2.2.5) is written for s_j :

$$0 < \sum_{k=1}^N \frac{1}{k} + 2 \sum_{k,k'=1;k < k'}^N (-1)^{k+k'} \frac{\cos(t_j \text{Log}(k/k'))}{\sqrt{k}\sqrt{k'}} < 2\epsilon^2$$

or:

$$\sum_{k=1}^N \frac{1}{k} < 2\epsilon^2 - 2 \sum_{k,k'=1;k < k'}^N (-1)^{k+k'} \frac{\cos(t_j \text{Log}(k/k'))}{\sqrt{k}\sqrt{k'}}$$

If $N \rightarrow +\infty$, the series $\sum_{k=1}^N \frac{1}{k}$ is divergent and becomes infinite. then:

$$\sum_{k=1}^{+\infty} \frac{1}{k} \leq 2\epsilon^2 - 2 \sum_{k,k'=1;k < k'}^{+\infty} (-1)^{k+k'} \frac{\cos(t_j \text{Log}(k/k'))}{\sqrt{k}\sqrt{k'}}$$

Hence, we obtain the following result:

$$\lim_{N \rightarrow +\infty} \sum_{k,k'=1;k < k'}^N (-1)^{k+k'} \frac{\cos(t_j \text{Log}(k/k'))}{\sqrt{k}\sqrt{k'}} = -\infty \quad (2.3.1)$$

if not, we will have a contradiction with the fact that :

$$\lim_{N \rightarrow +\infty} \sum_{k=1}^N (-1)^{k-1} \frac{1}{k^{s_j}} = 0 \iff \eta(s) \text{ is convergent for } s_j = \frac{1}{2} + it_j$$

2.4 Case $0 < \Re(s) < \frac{1}{2}$.

2.4.1 Case where there are zeros of $\eta(s)$ with $s = \sigma + it$ and $0 < \sigma < \frac{1}{2}$.

Suppose that there exists $s = \sigma + it$ one zero of $\eta(s)$ or $\eta(s) = 0 \implies \rho^2(s) = 0$ with $0 < \sigma < \frac{1}{2} \implies s$ lies inside the critical band. We write the equation (2.2.5):

$$0 < \sum_{k=1}^N \frac{1}{k^{2\sigma}} + 2 \sum_{k,k'=1;k < k'}^N (-1)^{k+k'} \frac{\cos(t \text{Log}(k/k'))}{k^\sigma k'^\sigma} < 2\epsilon^2$$

or:

$$\sum_{k=1}^N \frac{1}{k^{2\sigma}} < 2\epsilon^2 - 2 \sum_{k,k'=1;k < k'}^N (-1)^{k+k'} \frac{\cos(t \text{Log}(k/k'))}{k^\sigma k'^\sigma}$$

But $2\sigma < 1$, it follows that $\lim_{N \rightarrow +\infty} \sum_{k=1}^N \frac{1}{k^{2\sigma}} \rightarrow +\infty$ and then, we obtain :

$$\sum_{k,k'=1;k < k'}^{+\infty} (-1)^{k+k'} \frac{\cos(t \text{Log}(k/k'))}{k^\sigma k'^\sigma} = -\infty \quad (2.4.1)$$

2.5 Case $\frac{1}{2} < \Re(s) < 1$.

Let $s = \sigma + it$ be the zero of $\eta(s)$ in $0 < \Re(s) < \frac{1}{2}$, object of the previous paragraph. From the proposition (2.2.1), $\zeta(s) = 0$. According to point 4 of theorem 2.1.2, the complex number $s' = 1 - \sigma + it = \sigma' + it'$ with $\sigma' = 1 - \sigma$, $t' = t$ and $\frac{1}{2} < \sigma' < 1$ verifies $\zeta(s') = 0$, so s' is also a zero of the function $\zeta(s)$ in the band $\frac{1}{2} < \Re(s) < 1$, it follows from the proposition (2.2.2) that $\eta(s') = 0 \implies \rho(s') = 0$. By applying (2.2.5), we get:

$$0 < \sum_{k=1}^N \frac{1}{k^{2\sigma'}} + 2 \sum_{k,k'=1;k < k'}^N \frac{(-1)^{k+k'} \cos(t' \text{Log}(k/k'))}{k^{\sigma'} k'^{\sigma'}} < 2\epsilon^2 \implies$$

$$-\frac{1}{2} \cdot \sum_{k=1}^N \frac{1}{k^{2\sigma'}} < \sum_{k,k'=1;k < k'}^N \frac{(-1)^{k+k'} \cos(t' \text{Log}(k/k'))}{k^{\sigma'} k'^{\sigma'}} < -\frac{1}{2} \cdot \sum_{k=1}^N \frac{1}{k^{2\sigma'}} + \epsilon^2 \quad (2.5.1)$$

As $0 < \sigma < \frac{1}{2} \implies 2 > 2\sigma' = 2(1 - \sigma) > 1$, then the series $\sum_{k=1}^N \frac{1}{k^{2\sigma'}}$ is convergent to a positive constant not null $C(\sigma')$. As $1/k^2 < 1/k^{2\sigma'}$ for all $k > 0$, then :

$$0 < \zeta(2) = \frac{\pi^2}{6} = \sum_{k=1}^{+\infty} \frac{1}{k^2} < \sum_{k=1}^{+\infty} \frac{1}{k^{2\sigma'}} = C(\sigma') = \zeta_1(2\sigma') = \zeta(2\sigma')$$

From the equation (2.5.1), it follows that :

$$\sum_{k,k'=1;k < k'}^{+\infty} \frac{(-1)^{k+k'} \cos(t' \text{Log}(k/k'))}{k^{\sigma'} k'^{\sigma'}} = -\frac{C(\sigma')}{2} = -\frac{\zeta(2\sigma')}{2} > -\infty \quad (2.5.2)$$

Case $t = 0$

We suppose that $t = 0 \implies t' = 0$. We known the following proposition:

Proposition 2.5.1. For all $s = \sigma$ real with $0 < \sigma < 1$, $\eta(s) > 0$ and $\zeta(s) < 0$.

We deduce the contradiction with the hypothesis $s' = \sigma'$ is a zero of $\eta(s)$ and:

$$\boxed{\text{The equation (2.5.2) is false for the case } t' = t = 0.} \quad (2.5.3)$$

Case $t' = t \neq 0$

We suppose that $t' \neq 0$. Let $s' = \sigma' + it' = 1 - \sigma + it$ a zero of $\eta(s)$, we have:

$$\sum_{k,k'=1;k < k'}^{+\infty} \frac{(-1)^{k+k'} \cos(t' \text{Log}(k/k'))}{k^{\sigma'} k'^{\sigma'}} = -\frac{C(\sigma')}{2} = -\frac{\zeta(2\sigma')}{2} > -\infty \quad (2.5.4)$$

the left member of the equation (2.5.4) above is finite and depends of σ' and t' , but the right member is a function only of σ' equal to $-\zeta(2\sigma')/2$.

We recall the following theorem (see page 140, [4]):

Theorem 2.5.2.

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \int_1^T |\zeta(\sigma'' + i\tau)|^2 d\tau = \zeta(2\sigma'') \quad (\sigma'' > \frac{1}{2}) \quad (2.5.5)$$

Let t_0 so that $t_0 \geq 1$. As the integral of the left member of the above equation is convergent, the equation (2.5.5) can be written as:

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \int_{t_0}^T |\zeta(\sigma'' + i\tau)|^2 d\tau = \zeta(2\sigma'')$$

and $\zeta(2\sigma'')$ is independent of any t_0 then in particular for $t_0 = t'$. As σ'' is any $\sigma'' > 1/2$, I choose $\sigma'' = \sigma'$ and $t_0 = t'$, it follows that $\zeta(2\sigma')$ does not depend of t' so that $s' = \sigma' + it'$ is a root of η . Hence, the contradiction with equation (2.5.2). Then the equation (2.5.4) is false.

$$\boxed{\text{It follows that the equation (2.5.4) is false for the case } t' \neq 0.} \quad (2.5.6)$$

It follows that the equation (2.5.2) is false and $\eta(s')$ does not vanish for $\sigma' \in]1/2, 1[$.

From (2.5.3-2.5.6), we conclude that the function $\eta(s)$ has no zeros for all $s' = \sigma' + it'$ with $\sigma' \in]1/2, 1[$, it follows that the case of the section (2.4) above concerning the case $0 < \Re(s) < \frac{1}{2}$ is false too. Then, the function $\eta(s)$ has all its zeros on the critical line $\sigma = \frac{1}{2}$. From the equivalent statement (2.1.5), it follows that **the Riemann hypothesis is verified.** \square

We therefore announce the important theorem as follows:

Theorem 2.5.3. *The Riemann Hypothesis is true:*

All nontrivial zeros of the function $\zeta(s)$ with $s = \sigma + it$ lie on the vertical line $\Re(s) = \frac{1}{2}$.

Bibliography

- [1] Quanta Magazine, '*Sensational*' Proof Delivers New Insights Into Prime Numbers. July 15 (2024).
- [2] E. BOMBIERI, The Riemann Hypothesis. In *The millennium prize problems*, edited by J. Carlson, A. Jaffe and A. Wiles, pp. 107–124, Clay Math. Institute, Amer. Math. Soc., Providence, RI, 2006.
- [3] P. BORWEIN, S. CHOI, B. ROONEY and A. WEIRATHMUELLER, *The Riemann hypothesis - a resource for the aficionado and virtuoso alike*. 1st Ed. CMS Books in Mathematics, Springer-Verlag, New-York, 2008. <https://doi.org/10.1007/978-0-387-72126-2>
- [4] E.C. TITCHMARSH, D.R. HEATH-BROWN: *The theory of the Riemann zeta-function*. 2sd Ed. revised by D.R. Heath-Brown. Oxford University Press, New-York, 1986.

Chapter 3

Is The Conjecture $c < rad^{1.63}(abc)$ True?

Abstract

In this paper, we consider the abc conjecture, we will give the proof that the conjecture $c < rad^{1.63}(abc)$ is true. It constitutes the key to resolve the abc conjecture.

Résumé:

Dans cet article, nous considérons la conjecture abc . Nous donnons la preuve de la conjecture $c < rad^{1.63}(abc)$ qui constitue la clé pour résoudre la conjecture abc .

To Prof. A. Nitaj for his work on the abc conjecture

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3.1 Introduction and Notations

Let a be a positive integer, $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as:

$$a = \prod_i a_i^{\alpha_i} = rad(a) \cdot \prod_i a_i^{\alpha_i - 1} \tag{3.1.1}$$

We denote:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \implies a = \mu_a \cdot rad(a) \tag{3.1.2}$$

The abc conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Oesterlé of Pierre et Marie Curie University (Paris 6) [1]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the abc conjecture is given below:

Conjecture 3.1.1. (abc Conjecture): For each $\epsilon > 0$, there exists $K(\epsilon)$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \quad (3.1.3)$$

where K is a constant depending only of ϵ .

We know that numerically, $\frac{\text{Log } c}{\text{Log}(rad(abc))} \leq 1.629912$ [2]. It concerned the best example given by E. Reyssat [2]:

$$2 + 3^{10}.109 = 23^5 \implies c < rad^{1.629912}(abc) \quad (3.1.4)$$

A conjecture was proposed that $c < rad^2(abc)$ [3]. In 2012, A. Nitaj [4] proposed the following conjecture:

Conjecture 3.1.2. Let a, b, c be positive integers relatively prime with $c = a + b$, then:

$$c < rad^{1.63}(abc) \quad (3.1.5)$$

$$abc < rad^{4.42}(abc) \quad (3.1.6)$$

In this paper, we will give the proof of the conjecture given by (3.1.5) that constitutes the key to obtain the proof of the abc conjecture using classical methods with the help of some theorems from the field of the number theory.

3.2 The Proof of The $c < rad^{1.63}(abc)$ Conjecture

Let a, b, c be positive integers, relatively prime, with $c = a + b$, $b < a$ and $R = rad(abc)$,

$$c = \prod_{j'=1}^{j'=J'} c_{j'}^{\beta_{j'}}, \beta_{j'} \geq 1, c_{j'} \geq 2 \text{ prime integers.}$$

In the following, we will give the proof of the conjecture $c < rad^{1.63}(abc)$.

Proof. :

3.2.1 Trivial cases:

- We suppose that $c < rad(abc)$, then we obtain:

$$c < rad(abc) < rad^{1.63}(abc) \implies \boxed{c < R^{1.63}}$$

and the condition (3.1.5) is satisfied.

- We suppose that $c = rad(abc)$, then a, b, c are not coprime, case to reject.

In the following, we suppose that $c > rad(abc)$ and a, b and c are not all prime numbers.

- We suppose $\mu_a \leq rad^{0.63}(a)$. We obtain :

$$c = a + b < 2a \leq 2rad^{1.63}(a) < rad^{1.63}(abc) \implies c < rad^{1.63}(abc) \implies \boxed{c < R^{1.63}}$$

Then (3.1.5) is satisfied.

- We suppose $\mu_c \leq rad^{0.63}(c)$. We obtain :

$$c = \mu_c rad(c) \leq rad^{1.63}(c) < rad^{1.63}(abc) \implies \boxed{c < R^{1.63}}$$

and the condition (3.1.5) is satisfied.

3.2.2 We suppose $\mu_c > rad^{0.63}(c)$ and $\mu_a > rad^{0.63}(a)$

Case : $rad^{0.63}(c) < \mu_c \leq rad^{1.63}(c)$ and $rad^{0.63}(a) < \mu_a \leq rad^{1.63}(a)$

We can write:

$$\left. \begin{array}{l} \mu_c \leq rad^{1.63}(c) \implies c \leq rad^{2.63}(c) \\ \mu_a \leq rad^{1.63}(a) \implies a \leq rad^{2.63}(a) \end{array} \right\} \implies ac \leq rad^{2.63}(ac) \implies a^2 < ac \leq rad^{2.63}(ac) \\ \implies a < rad^{1.315}(ac) \implies c < 2a < 2rad^{1.315}(ac) < rad^{1.63}(abc) \\ \implies \boxed{c = a + b < R^{1.63}}$$

Case : $rad^{1.63}(c) < \mu_c$ or $rad^{1.63}(a) < \mu_a$

I - We suppose that $rad^{1.63}(c) < \mu_c$ and $rad^{1.63}(a) < \mu_a \leq rad^2(a)$:

I-1- Case $rad(a) < rad(c)$:

In this case $a = \mu_a \cdot rad(a) \leq rad^3(a) \leq rad^{1.63}(a) \cdot rad^{1.37}(a) < rad^{1.63}(a) \cdot rad^{1.37}(c) \implies c < 2a < 2rad^{1.63}(a) \cdot rad^{1.37}(c) < rad^{1.63}(abc) \implies \boxed{c < R^{1.63}}$.

I-2- Case $rad(c) < rad(a) < rad^{\frac{1.63}{1.37}}(c)$: As $a \leq rad^{1.63}(a) \cdot rad^{1.37}(a) < rad^{1.63}(a) \cdot rad^{1.63}(c) \implies c < 2a < 2rad^{1.63}(a) \cdot rad^{1.63}(c) < R^{1.63} \implies \boxed{c < R^{1.63}}$.

I-3- Case $rad^{\frac{1.63}{1.37}}(c) < rad(a)$:

I-3-1- We suppose $rad^{1.63}(c) < \mu_c \leq rad^{2.26}(c)$, we obtain:

$$c \leq rad^{3.26}(c) \implies c \leq rad^{1.63}(c) \cdot rad^{1.63}(c) \implies \\ c < rad^{1.63}(c) \cdot rad^{1.37}(a) < rad^{1.63}(c) \cdot rad^{1.63}(a) \cdot rad^{1.63}(b) = R^{1.63} \implies \boxed{c < R^{1.63}}$$

I-3-2- We suppose $\mu_c > rad^{2.26}(c) \implies c > rad^{3.26}(c)$.

I-3-2-1- We consider the case $\mu_a = rad^2(a) \implies a = rad^3(a)$ and $c = a + 1$. Then, we obtain that $X = rad(a)$ is a solution in positive integers of the equation:

$$X^3 + 1 = c \tag{3.2.1}$$

I-3-2-1-1- We suppose that $c = rad^n(c)$ with $n \geq 4$, we obtain the equation:

$$rad^n(c) - rad^3(a) = 1 \quad (3.2.2)$$

But the solutions of the equation (3.2.2) are [5] : ($rad(c) = 3, n = 2, rad(a) = +2$), it follows the contradiction with $n \geq 4$ and the case $c = rad^n(c), n \geq 4$ is to reject.

I-3-2-1-2- In the following, we will study the cases $\mu_c = A.rad^n(c)$ with $rad(c) \nmid A, n \geq 0$. The above equation (3.2.1) can be written as :

$$(X + 1)(X^2 - X + 1) = c \quad (3.2.3)$$

Let δ one divisor of c so that :

$$X + 1 = \delta \quad (3.2.4)$$

$$X^2 - X + 1 = \frac{c}{\delta} = m = \delta^2 - 3X \quad (3.2.5)$$

We recall that $rad(a) > rad^{1.63}(c)$.

I-3-2-1-2-1- We suppose $\delta = l.rad(c)$. We have $\delta = l.rad(c) < c = \mu_c.rad(c) \implies l < \mu_c$. As $\frac{c}{\delta} = \frac{\mu_c.rad(c)}{l.rad(c)} = \frac{\mu_c}{l} = m = \delta^2 - 3X \implies \mu_c = l.m = l(\delta^2 - 3X)$. From $m = \delta^2 - 3X$ and $X = rad(a)$, we obtain:

$$m = l^2 rad^2(c) - 3rad(a) \implies 3rad(a) = l^2 rad^2(c) - m$$

A- Case $3|m \implies m = 3m', m' > 1$: As $\mu_c = ml = 3m'l \implies 3|rad(c)$ and $(rad(c), m')$ not coprime. We obtain:

$$rad(a) = l^2 rad(c) \cdot \frac{rad(c)}{3} - m'$$

It follows that a, c are not coprime, then the contradiction.

B - Case $m = 3 \implies \mu_c = 3l \implies c = 3lrad(c) = 3\delta = \delta(\delta^2 - 3X) \implies \delta^2 = 3(1 + X) = 3\delta \implies \delta = lrad(c) = 3 \implies c = 3\delta = 9 = a + 1 \implies a = 8 \implies c = 9 < (2 \times 3)^{1.63} \approx 18.55$, it is a trivial case and the conjecture is true.

I-3-2-1-2-2- We suppose $\delta = l.rad^2(c), l \geq 2$. If $n = 0$ then $\mu_c = A$ and from the equation above (3.2.5):

$$m = \frac{c}{\delta} = \frac{\mu_c.rad(c)}{lrad^2(c)} = \frac{A.rad(c)}{lrad^2(c)} = \frac{A}{lrad(c)} \Rightarrow rad(c)|A$$

It follows the contradiction with the hypothesis above $rad(c) \nmid A$.

I-3-2-1-2-3- We suppose $\delta = lrad^2(c), l \geq 2$ and in the following $n > 0$. As $m = \frac{c}{\delta} = \frac{\mu_c.rad(c)}{lrad^2(c)} = \frac{\mu_c}{lrad(c)}$, if $lrad(c) \nmid \mu_c$ then the case is to reject. We suppose $lrad(c)|\mu_c \implies \mu_c = m.lrad(c)$, with $m, rad(c)$ not coprime, then $\frac{c}{\delta} = m = \delta^2 - 3rad(a)$.

C - Case $m = 1 = c/\delta \implies \delta^2 - 3rad(a) = 1 \implies (\delta - 1)(\delta + 1) = 3rad(a) = rad(a)(\delta + 1) \implies \delta = 2 = l.rad^2(c)$, then the contradiction.

D - Case $m = 3$, we obtain $3(1 + rad(a)) = \delta^2 = 3\delta \implies \delta = 3 = lrad^2(c)$. Then the contradiction.

E - Case $m \neq 1, 3$, we obtain: $3rad(a) = l^2rad^4(c) - m \implies rad(a)$ and $rad(c)$ are not coprime. Then the contradiction.

I-3-2-1-2-4- We suppose $\delta = lrad^n(c), l \geq 2$ with $n \geq 3$. $c = \mu_c \cdot rad(c) = lrad^n(c)(\delta^2 - 3rad(a))$ and $m = \delta^2 - 3rad(a) = \delta^2 - 3X$.

F - As seen above (paragraphs C,D), the cases $m = 1$ and $m = 3$ give contradictions, it follows the reject of these cases.

G - Case $m \neq 1, 3$. Let q be a prime that divides m (q can be equal to m), it follows $q | (\mu_c = l.m) \implies q = c_{j_0} \implies c_{j_0} | \delta^2 \implies c_{j_0} | 3rad(a)$. Then $rad(a)$ and $rad(c)$ are not coprime. It follows the contradiction.

I-3-2-1-2-5- We suppose $\delta = \prod_{j \in J_1} c_j^{\beta_j}, \beta_j \geq 1$ with at least one $j_0 \in J_1$ with:

$$\beta_{j_0} \geq 2, \quad rad(c) \nmid \delta \quad (3.2.6)$$

We can write:

$$\delta = \mu_\delta \cdot rad(\delta), \quad rad(c) = r \cdot rad(\delta), \quad r > 1, \quad (r, \mu_\delta) = 1 \quad (3.2.7)$$

Then, we obtain:

$$c = \mu_c \cdot rad(c) = \mu_c \cdot r \cdot rad(\delta) = \delta(\delta^2 - 3X) = \mu_\delta \cdot rad(\delta)(\delta^2 - 3X) \implies r \cdot \mu_c = \mu_\delta(\delta^2 - 3X) \quad (3.2.8)$$

- We suppose $\mu_c = \mu_\delta \implies r = \delta^2 - 3X = (\mu_c \cdot rad(\delta))^2 - 3X$. As $\delta < \delta^2 - 3X \implies r > \delta \implies rad(c) > r > (\mu_c \cdot rad(\delta) = A \cdot rad^n(c) \cdot rad(\delta)) \implies 1 > A \cdot rad^{n-1}(\delta)$, then the contradiction.

- We suppose $\mu_c < \mu_\delta$. As $rad(a) = \delta - 1 = \mu_\delta rad(\delta) - 1$, we obtain:

$$rad(a) > \mu_c \cdot rad(\delta) - 1 > 0 \implies rad(ac) > c \cdot rad(\delta) - rad(c) > 0$$

As $c = 1 + a$ and we consider the cases $c > rad(ac)$, then:

$$c > rad(ac) > c \cdot rad(\delta) - rad(c) > 0 \implies c > c \cdot rad(\delta) - rad(c) > 0 \implies 1 > rad(\delta) - \frac{rad(c)}{c} > 0, \quad rad(\delta) \geq 2 \implies \text{The contradiction} \quad (3.2.9)$$

- We suppose $\mu_c > \mu_\delta$. In this case, from the equation (3.2.8) and as $(r, \mu_\delta) = 1$, it follows we can write:

$$\begin{aligned} \mu_c &= \mu_1 \cdot \mu_2, \quad \mu_1, \mu_2 > 1, \\ c &= \mu_c rad(c) = \mu_1 \cdot \mu_2 \cdot rad(\delta) \cdot r = \delta \cdot (\delta^2 - 3X), \\ \text{We do a choice so that } \mu_2 &= \mu_\delta, \quad r \cdot \mu_1 = \delta^2 - 3X \implies \delta = \mu_2 \cdot rad(\delta). \end{aligned}$$

** 1- We suppose $(\mu_1, \mu_2) \neq 1$, then $\exists c_{j_0}$ so that $c_{j_0} | \mu_1$ and $c_{j_0} | \mu_2$. But $\mu_\delta = \mu_2 \implies c_{j_0}^2 | \delta$. From $3X = \delta^2 - r\mu_1 \implies c_{j_0} | 3X \implies c_{j_0} | X$ or $c_{j_0} = 3$.

- If $c_{j_0} | (X = rad(a))$, it follows the contradiction with $(c, a) = 1$.

- If $c_{j_0} = 3$. We have $r\mu_1 = \delta^2 - 3X = \delta^2 - 3(\delta - 1) \implies \delta^2 - 3\delta + 3 - r\mu_1 = 0$. As $3|\mu_1 \implies \mu_1 = 3^k \mu'_1, 3 \nmid \mu'_1, k \geq 1$, we obtain:

$$\delta^2 - 3\delta + 3(1 - 3^{k-1}r\mu'_1) = 0 \quad (3.2.10)$$

** 1-1- We consider the case $k > 1 \implies 3 \nmid (1 - 3^{k-1}r\mu'_1)$. Let us recall the Eisenstein criterion [6]:

Theorem 3.2.1. (Eisenstein Criterion) Let $f = a_0 + \dots + a_n X^n$ be a polynomial $\in \mathbb{Z}[X]$. We suppose that $\exists p$ a prime number so that $p \nmid a_n, p|a_i, (0 \leq i \leq n-1)$, and $p^2 \nmid a_0$, then f is irreducible in \mathbb{Q} .

We apply Eisenstein criterion to the polynomial $R(Z)$ given by:

$$R(Z) = Z^2 - 3Z + 3(1 - 3^{k-1}r\mu'_1) \quad (3.2.11)$$

then:

$$-3 \nmid 1, -3|(-3), -3|3(1 - 3^{k-1}r\mu'_1), \text{ and } -3^2 \nmid 3(1 - 3^{k-1}r\mu'_1).$$

It follows that the polynomial $R(Z)$ is irreducible in \mathbb{Q} , then, the contradiction with $R(\delta) = 0$.

** 1-2- We consider the case $k = 1$, then $\mu_1 = 3\mu'_1$ and $(\mu'_1, 3) = 1$, we obtain:

$$\delta^2 - 3\delta + 3(1 - r\mu'_1) = 0 \quad (3.2.12)$$

** 1-2-1- We consider that $3 \nmid (1 - r\mu'_1)$, we apply the same Eisenstein criterion to the polynomial $R'(Z)$ given by:

$$R'(Z) = Z^2 - 3Z + 3(1 - r\mu'_1)$$

and we find a contradiction with $R'(\delta) = 0$.

** 1-2-2- We consider that:

$$3|(1 - r\mu'_1) \implies r\mu'_1 - 1 = 3^i h, i \geq 1, 3 \nmid h, h \in \mathbb{N}^* \quad (3.2.13)$$

δ is an integer root of the polynomial $R'(Z)$:

$$R'(Z) = Z^2 - 3Z + 3(1 - r\mu'_1) = 0 \quad (3.2.14)$$

The discriminant of $R'(Z)$ is:

$$\Delta = 3^2 + 3^{i+1} \times 4h$$

As the root δ is an integer, it follows that $\Delta = t^2 > 0$ with t a positive integer. We obtain:

$$\Delta = 3^2(1 + 3^{i-1} \times 4h) = t^2 \quad (3.2.15)$$

$$\implies 1 + 3^{i-1} \times 4h = q^2 > 1, q \in \mathbb{N}^* \quad (3.2.16)$$

We can write the equation (3.2.12) as :

$$\delta(\delta - 3) = 3^{i+1}h \implies 3^3 \mu'_1 \frac{rad(\delta)}{3} \cdot (\mu'_1 rad(\delta) - 1) = 3^{i+1}h \implies \quad (3.2.17)$$

$$\mu'_1 \frac{rad(\delta)}{3} \cdot (\mu'_1 rad(\delta) - 1) = h \quad (3.2.18)$$

We obtain $i = 2$ and $q^2 = 1 + 12h = 1 + 4\mu'_1 rad(\delta)(\mu'_1 rad(\delta) - 1)$. Then, q satisfies :

$$q^2 - 1 = 12h = 4\mu'_1 rad(\delta)(\mu'_1 rad(\delta) - 1) \implies \quad (3.2.19)$$

$$\frac{(q-1)}{2} \cdot \frac{(q+1)}{2} = 3h = (\mu'_1 rad(\delta) - 1) \cdot \mu'_1 rad(\delta) \implies \quad (3.2.20)$$

$$q - 1 = 2\mu'_1 rad(\delta) - 2 \quad (3.2.21)$$

$$q + 1 = 2\mu'_1 rad(\delta) \quad (3.2.22)$$

It follows that $(q = x, 1 = y)$ is a solution of the Diophantine equation:

$$x^2 - y^2 = N \quad (3.2.23)$$

with $N = 4\mu'_1 rad(\delta)(\mu'_1 rad(\delta) - 1) = 12h > 0$. Let $Q(N)$ be the number of the solutions of (3.2.23) and $\tau(N)$ is the number of suitable factorization of N , then we announce the following result concerning the solutions of the Diophantine equation (3.2.23) (see theorem 27.3 in [7]):

- If $N \equiv 2 \pmod{4}$, then $Q(N) = 0$.

- If $N \equiv 1$ or $N \equiv 3 \pmod{4}$, then $Q(N) = [\tau(N)/2]$.

- If $N \equiv 0 \pmod{4}$, then $Q(N) = [\tau(N/4)/2]$.

$[x]$ is the integral part of x for which $[x] \leq x < [x] + 1$.

As $N = 4\mu'_1 rad(\delta)(\mu'_1 rad(\delta) - 1) \implies N \equiv 0 \pmod{4} \implies Q(N) = [\tau(N/4)/2]$. As $(q, 1)$ is a couple of solutions of the Diophantine equation (3.2.23), then $\exists d, d'$ positive integers with $d > d'$ and $N = d \cdot d'$ so that :

$$d + d' = 2q \quad (3.2.24)$$

$$d - d' = 2 \cdot 1 = 2 \quad (3.2.25)$$

** 1-2-2-1 As $N > 1$, we take $d = N$ and $d' = 1$. It follows:

$$\begin{cases} N + 1 = 2q \\ N - 1 = 2 \end{cases} \implies N = 3 \implies \text{then the contradiction with } N \equiv 0 \pmod{4}.$$

** 1-2-2-2 Now, we consider the case $d = 2\mu'_1 rad(\delta)(\mu'_1 rad(\delta) - 1)$ and $d' = 2$. It follows:

$$\begin{cases} 2\mu'_1 rad(\delta)(\mu'_1 rad(\delta) - 1) + 2 = 2q \\ 2\mu'_1 rad(\delta)(\mu'_1 rad(\delta) - 1) - 2 = 2 \end{cases} \implies 2\mu'_1 rad(\delta)(\mu'_1 rad(\delta) - 1) = q + 1$$

As $q + 1 = 2\mu'_1 rad(\delta)$, we obtain $\mu'_1 rad(\delta) = 2$, then the contradiction with $3|d$.

** 1-2-2-3 Now, we consider the case $d = \mu'_1 rad(\delta)(\mu'_1 rad(\delta) - 1)$ and $d' = 4$. It follows:

$$\begin{cases} \mu'_1 rad(\delta)(\mu'_1 rad(\delta) - 1) + 4 = 2q \\ \mu'_1 rad(\delta)(\mu'_1 rad(\delta) - 1) - 4 = 2 \end{cases} \implies \mu'_1 rad(\delta)(\mu'_1 rad(\delta) - 1) = 6$$

As $\mu'_1 rad(\delta) > 0 \implies \mu'_1 rad(\delta) = 3 \implies \mu'_1 = 1, rad(\delta) = 3$ and $q = 5$. From $q^2 = 1 + 12h$, we obtain $h = 2$. Using the relation (3.2.13) $r\mu'_1 - 1 = 3^i h$ as $\mu'_1 = 1, i = 2, h = 2$, it gives $r - 1 = 9h = 18$. As δ is the positive root of the equation (3.2.12):

$$Z^2 - 3Z + 3(1 - r) = 0 \implies \delta = 9 = 3^2$$

But $\delta = 1 + X = 1 + rad(a) \implies rad(a) = 8 = 2^3$, then the contradiction.

** 1-2-2-4 Now, let c_{j_0} be a prime integer so that $c_{j_0} | rad\delta$, we consider the case $d = \mu'_1 \frac{rad(\delta)}{c_{j_0}} (\mu'_1 rad(\delta) - 1)$ and $d' = 4c_{j_0}$. It follows:

$$\begin{cases} \mu'_1 \frac{rad(\delta)}{c_{j_0}} (\mu'_1 rad(\delta) - 1) + 4c_{j_0} = 2q \\ \mu'_1 \frac{rad(\delta)}{c_{j_0}} (\mu'_1 rad(\delta) - 1) - 4c_{j_0} = 2 \end{cases} \implies \mu'_1 \frac{rad(\delta)}{c_{j_0}} (\mu'_1 rad(\delta) - 1) = 2(1 + 2c_{j_0}) \implies$$

Then the contradiction as the left member is greater than the right member $2(1 + 2c_{j_0})$.

** 1-2-2-5 Now, we consider the case $d = 4\mu'_1 rad(\delta)$ and $d' = (\mu'_1 rad(\delta) - 1)$. It follows:

$$\begin{cases} 4\mu'_1 rad(\delta) + (\mu'_1 rad(\delta) - 1) = 2q \\ 4\mu'_1 rad(\delta) - (\mu'_1 rad(\delta) - 1) = 2 \end{cases} \implies 3\mu'_1 rad(\delta) = 1 \implies \text{Then the contradiction.}$$

** 1-2-2-6 Now, we consider the case $d = 2\mu'_1 rad(\delta)$ and $d' = 2(\mu'_1 rad(\delta) - 1)$. It follows:

$$\begin{cases} 2\mu'_1 rad(\delta) + 2(\mu'_1 rad(\delta) - 1) = 2q \implies 2\mu'_1 rad(\delta) - 1 = q \\ 2\mu'_1 rad(\delta) - 2(\mu'_1 rad(\delta) - 1) = 2 \implies 2 = 2 \end{cases}$$

It follows that this case presents no contradictions a priori.

** 1-2-2-7 $\mu'_1 rad(\delta)$ and $\mu'_1 rad(\delta) - 1$ are coprime, let $\mu'_1 rad(\delta) - 1 = \prod_{j=1}^{j=J} \lambda_j^{\gamma_j}$, we consider the case $d = 2\lambda_{j'} \mu'_1 rad(\delta)$ and $d' = 2 \frac{\mu'_1 rad(\delta) - 1}{\lambda_{j'}}$. It follows:

$$\begin{cases} 2\lambda_{j'} \mu'_1 rad(\delta) + 2 \frac{\mu'_1 rad(\delta) - 1}{\lambda_{j'}} = 2q \\ 2\lambda_{j'} \mu'_1 rad(\delta) - 2 \frac{\mu'_1 rad(\delta) - 1}{\lambda_{j'}} = 2 \end{cases}$$

** 1-2-2-7-1 We suppose that $\gamma_{j'} = 1$. We consider the case $d = 2\lambda_{j'} \mu'_1 rad(\delta)$ and $d' = 2 \frac{\mu'_1 rad(\delta) - 1}{\lambda_{j'}}$. It follows:

$$\begin{cases} 2\lambda_{j'} \mu'_1 rad(\delta) + 2 \frac{\mu'_1 rad(\delta) - 1}{\lambda_{j'}} = 2q \\ 2\lambda_{j'} \mu'_1 rad(\delta) - 2 \frac{\mu'_1 rad(\delta) - 1}{\lambda_{j'}} = 2 \end{cases} \implies 4\lambda_{j'} \mu'_1 rad(\delta) = 2(q + 1) \implies 2\lambda_{j'} \mu'_1 rad(\delta) = q + 1$$

But from the equation (3.2.22), $q + 1 = 2\mu'_1 rad(\delta)$, then $\lambda_{j'} = 1$, it follows the contradiction.

** 1-2-2-7-2 We suppose that $\gamma_{j'} \geq 2$. We consider the case $d = 2\lambda_{j'}^{\gamma_{j'}-r'_{j'}} \mu'_1 rad(\delta)$ and $d' = 2\frac{\mu'_1 rad(\delta) - 1}{\lambda_{j'}^{r'_{j'}}$. It follows:

$$\left\{ \begin{array}{l} 2\lambda_{j'}^{\gamma_{j'}-r'_{j'}} \mu'_1 rad(\delta) + 2\frac{\mu'_1 rad(\delta) - 1}{\lambda_{j'}^{r'_{j'}}} = 2q \\ 2\lambda_{j'}^{\gamma_{j'}-r'_{j'}} \mu'_1 rad(\delta) - 2\frac{\mu'_1 rad(\delta) - 1}{\lambda_{j'}^{r'_{j'}}} = 2 \end{array} \right. \implies 4\lambda_{j'}^{\gamma_{j'}-r'_{j'}} \mu'_1 rad(\delta) = 2(q+1)$$

$$\implies 2\lambda_{j'}^{\gamma_{j'}-r'_{j'}} \mu'_1 rad(\delta) = q+1$$

As above, it follows the contradiction. It is trivial that the other cases for more factors $\prod_{j''} \lambda_{j''}^{\gamma_{j''}-r'_{j''}}$ give also contradictions.

** 1-2-2-8 Now, we consider the case $d = 4(\mu'_1 rad(\delta) - 1)$ and $d' = \mu'_1 rad(\delta)$, we have $d > d'$. It follows:

$$\left\{ \begin{array}{l} 4(\mu'_1 rad(\delta) - 1) + \mu'_1 rad(\delta) = 2q \implies 5\mu'_1 rad(\delta) = 2(q+2) \\ 4(\mu'_1 rad(\delta) - 1) - \mu'_1 rad(\delta) = 2 \implies \mu'_1 rad(\delta) = 2 \end{array} \right. \implies \left\{ \begin{array}{l} \text{Then the contradiction as} \\ 3|\delta. \end{array} \right.$$

** 1-2-2-9 Now, we consider the case $d = 4u(\mu'_1 rad(\delta) - 1)$ and $d' = \frac{\mu'_1 rad(\delta)}{u}$, where $u > 1$ is an integer divisor of $\mu'_1 rad(\delta)$. We have $d > d'$ and:

$$\left\{ \begin{array}{l} 4u(\mu'_1 rad(\delta) - 1) + \frac{\mu'_1 rad(\delta)}{u} = 2q \\ 4u(\mu'_1 rad(\delta) - 1) - \frac{\mu'_1 rad(\delta)}{u} = 2 \end{array} \right. \implies 2u(\mu'_1 rad(\delta) - 1) = \mu'_1 rad(\delta)$$

Then the contradiction as $\mu'_1 rad(\delta)$ and $(\mu'_1 rad(\delta) - 1)$ are coprime.

In conclusion, we have found only one case (** 1-2-2-6 above) where there is no contradictions *a priori*. As $\tau(N)$ is large and also $[\tau(N/4)/2]$, it follows the contradiction with $Q(N) \leq 1$ and the hypothesis $(\mu_1, \mu_2) \neq 1$ is false.

** 2- We suppose that $(\mu_1, \mu_2) = 1$.

From the equation $r\mu_1 = \delta^2 - 3X$ and the condition $rad(a) = X > rad^{1.63/1.37}(c) \iff \delta - 1 = X > rad^{1.19}(c)$, we obtain the following inequality:

$$\begin{aligned} \delta - 1 > (r.rad(\delta))^{1.19} &\implies -3(\delta - 1) < -3r.rad(\delta).(r.rad(\delta))^{0.19} \implies \\ r\mu_1 = \delta^2 - 3(\delta - 1) &< (r.rad(\delta))^2 - 3r.rad(\delta).(r.rad(\delta))^{0.19} \implies \\ \mu_1 < r.rad^2(\delta) - 3r.rad(\delta).(r.rad(\delta))^{0.19} &\implies \\ \mu_1 < r.rad^2(\delta) \left(1 - \frac{3}{(r.rad(\delta))^{0.81}} \right) &\quad (3.2.26) \end{aligned}$$

As $a = rad^3(a) < c$, we can write:

$$rad^3(a) < \mu_1 \mu_2 rad(c) < \mu_2 \cdot rad(\delta) \cdot rad^2(c) \left(1 - \frac{3}{(r \cdot rad(\delta))^{0.81}} \right)$$

but $(r, rad(\delta)) = 1$, $r \cdot rad(\delta) \geq 6 \implies (r \cdot rad(\delta))^{0.81} \geq (6^{0.81} \approx 4.26)$ and $\delta = \mu_2 \cdot rad(\delta)$, it follows:

$$rad^3(a) < \mu_1 \mu_2 rad(c) < \mu_2 \cdot rad(\delta) \cdot rad^2(c) \implies rad^3(a) < \delta \cdot rad^2(c) < 1.6 rad(a) \cdot rad^2(c)$$

As $rad(a) > (rad^{1.62/1.37}(c) = rad^{1.19}(c)) \implies rad^{1.19}(c) < rad(a) < 1.27 rad(c)$, then we obtain:

$$rad^{1.19}(c) < 1.27 rad(c) \implies rad(c) < 3.5 \implies rad(c) \leq 3, \text{ but } rad(c) = r \cdot rad(\delta) \geq 6$$

Then the contradiction.

It follows that the case $\mu_c > rad^{2.26}(c) \implies c > rad^{3.26}(c)$ and $a = rad^3(a)$ is impossible.

I-3-2-2- We consider the case $\mu_a = rad^2(a) \implies a = rad^3(a)$ and $c = a + b$. Then, we obtain that $X = rad(a)$ is a solution in positive integers of the equation:

$$X^3 + 1 = \bar{c} \quad (3.2.27)$$

with $\bar{c} = c - b + 1 = a + 1 \implies (\bar{c}, a) = 1$. We obtain the same result as of the case **I-3-2-1-** studied above considering $rad(a) > rad^{\frac{1.63}{1.37}}(\bar{c})$.

I-3-2-3- We suppose $\mu_c > rad^{2.26}(c) \implies c > rad^{3.26}(c)$ and c large and $\mu_a < rad^2(a)$, we consider $c = a + b, b \geq 1$. Then $c = rad^3(c) + h, h > rad^3(c)$, h a positive integer and we can write $a + l = rad^3(a), l > 0$. Then we obtain :

$$rad^3(c) + h = rad^3(a) - l + b \implies rad^3(a) - rad^3(c) = h + l - b > 0 \quad (3.2.28)$$

as $rad(a) > rad^{\frac{1.63}{1.37}}(c)$. We obtain the equation:

$$rad^3(a) - rad^3(c) = h + l - b = m > 0 \quad (3.2.29)$$

Let $X = rad(a) - rad(c)$, then X is an integer root of the polynomial $H(X)$ defined as:

$$H(X) = X^3 + 3rad(ac)X - m = 0 \quad (3.2.30)$$

To resolve the above equation, we denote $X = u + v$, It follows that u^3, v^3 are the roots of the polynomial $G(t)$ given by:

$$G(t) = t^2 - mt - rad^3(ac) = 0 \quad (3.2.31)$$

The discriminant of $G(t)$ is $\Delta = m^2 + 4rad^3(ac) = \alpha^2$, $\alpha > 0$. As $m = rad^3(a) - rad^3(c) > 0$, we obtain that $\alpha = rad^3(a) + rad^3(c) > 0$, then from the expression of the discriminant Δ , it follows that the couple $(\alpha = x, m = y)$ is a solution of the Diophantine equation:

$$x^2 - y^2 = N \quad (3.2.32)$$

with $N = 4rad^3(ac) = 4rad^3(a) \cdot rad^3(c) > 0$. Here, we will use the same method that is given in the above sub-paragraph ** 1-2-2- of the paragraph **I-3-2-1-2-5-**. We have the two

terms $rad^3(a)$ and $rad^3(c)$ coprime. As (α, m) is a couple of solutions of the Diophantine equation (3.2.32) and $\alpha > m$, then $\exists d, d'$ positive integers with $d > d'$ and $N = d.d'$ so that :

$$d + d' = 2\alpha \quad (3.2.33)$$

$$d - d' = 2m \quad (3.2.34)$$

I-3-2-3-1- Let us consider the case $d = 2rad^3(a)$, $d' = 2rad^3(c)$. It follows:

$$\begin{cases} 2rad^3(a) + 2rad^3(c) = 2\alpha \implies \alpha = rad^3(a) + rad^3(c) \\ 2rad^3(a) - 2rad^3(c) = 2m \implies m = rad^3(a) - rad^3(c) \end{cases}$$

It follows that this case presents *a priori* no contradictions.

I-3-2-3-2- Now, we consider for example, the case $d = 4rad^3(a)$ and $d' = rad^3(c) \implies d > d'$. We rewrite the equations (3.2.33-3.2.34):

$$\begin{aligned} 4rad^3(a) + rad^3(c) &= 2(rad^3(a) + rad^3(c)) \implies 2rad^3(a) = rad^3(c) \\ 4rad^3(a) - rad^3(c) &= 2(rad^3(a) - rad^3(c)) \implies 2rad^3(a) = -rad^3(c) \end{aligned}$$

Then the contradiction.

I-3-2-3-3- We consider the case $d = 4rad^3(c)rad^3(a)$ and $d' = 1 \implies d > d'$. We rewrite the equations (3.2.33-3.2.34):

$$\begin{aligned} 4rad^3(c)rad^3(a) + 1 &= 2(rad^3(c) + rad^3(a)) \implies \\ 2(2rad^3(c)rad^3(a) - rad^3(c) - rad^3(a)) &= -1 \implies \text{a contradiction} \\ 4rad^3(c)rad^3(a) - 1 &= 2(rad^3(c) - rad^3(a)) \end{aligned}$$

Then the contradiction.

I-3-2-3-4- Let c_1 be the first factor of $rad(c)$. We consider the case $d = 4c_1rad^3(a)$ and $d' = \frac{rad^3(c)}{c_1} \implies d > d'$. We rewrite the equation (3.2.33):

$$\begin{aligned} 4c_1rad^3(a) + \frac{rad^3(c)}{c_1} &= 2(rad^3(a) + rad^3(c)) \implies \\ 2rad^3(a)(2c_1 - 1) &= \frac{rad^3(c)}{c_1}(2c_1 - 1) \implies 2rad^3(a) = rad^2(c) \cdot \frac{rad(c)}{c_1} \end{aligned}$$

$c_1 = 2$ or not, there is a contradiction with a, c coprime.

The other cases of the expressions of d and d' not coprime so that $N = d.d'$ give also contradictions.

Let $Q(N)$ be the number of the solutions of (3.2.32), as $N \equiv 0 \pmod{4}$, then $Q(N) = [\tau(N/4)/2]$. From the study of the cases above, we obtain that $Q(N) \leq 1$ is $\ll [(\tau(N)/4)/2]$. It follows the contradiction.

Then the cases $\mu_a \leq rad^2(a)$ and $c > rad^{3.26}(c)$ are impossible.

II- We suppose that $rad^{1.63}(c) < \mu_c \leq rad^2(c)$ and $\mu_a > rad^{1.63}(a)$:

II-1- Case $rad(c) < rad(a)$: As $c \leq rad^3(c) = rad^{1.63}(c).rad^{1.37}(c) \implies c < rad^{1.63}(c).rad^{1.37}(a) < rad^{1.63}(ac) < rad^{1.63}(abc) \implies \boxed{c < R^{1.63}}$.

II-2- Case $rad(a) < rad(c) < rad^{\frac{1.63}{1.37}}(a)$:

As $c \leq rad^3(c) \leq rad^{1.63}(c).rad^{1.37}(c) \implies c < rad^{1.63}(c).rad^{1.63}(a) < rad^{1.63}(abc) \implies \boxed{c < R^{1.63}}$.

II-3- Case $rad^{\frac{1.63}{1.37}}(a) < rad(c)$:

II-3-1- We suppose $rad^{1.63}(a) < \mu_a \leq rad^{2.26}(a) \implies a \leq rad^{1.63}(a).rad^{1.63}(a) \implies a < rad^{1.63}(a).rad^{1.37}(c) \implies c = a + b < 2a < 2rad^{1.63}(a).rad^{1.63}(c) < rad^{1.63}(abc) \implies c < R^{1.63} \implies \boxed{c < R^{1.63}}$.

II-3-2- We suppose $\mu_a > rad^{2.26}(a) \implies a > rad^{3.26}(a)$ and $\mu_c \leq rad^2(c)$. Using the same method as it was explicated in the paragraphs **I-3-2-** (permuting a, c see in Appendix **II'-3-2-**), we arrive at a contradiction. It follows that the cases $\mu_c \leq rad^2(c)$ and $\mu_a > rad^{2.26}(a)$ are impossible.

Case $\mu_a > rad^{1.63}(a)$ and $\mu_c > rad^{1.63}(c)$:

Taking into account the cases studied above, it remains to see the following two cases:

- $\mu_c > rad^2(c)$ and $\mu_a > rad^{1.63}(a)$,
- $\mu_a > rad^2(a)$ and $\mu_c > rad^{1.63}(c)$.

III- We suppose $\mu_c > rad^2(c)$ and $\mu_a > rad^{1.63}(a) \implies c > rad^3(c)$ and $a > rad^{2.63}(a)$. We can write $c = rad^3(c) + h$ and $a = rad^3(a) + l$ with h a positive integer and $l \in \mathbb{Z}$.

III-1- We suppose $rad(c) < rad(a)$. We obtain the equation:

$$rad^3(a) - rad^3(c) = h - l - b = m > 0 \quad (3.2.35)$$

Let $X = rad(a) - rad(c)$, from the above equation, X is a real root of the polynomial:

$$H(X) = X^3 + 3rad(ac)X - m = 0 \quad (3.2.36)$$

As above, to resolve (3.2.36), we denote $X = u + v$, It follows that u^3, v^3 are the roots of the polynomial $G(t)$ given by :

$$G(t) = t^2 - mt - rad^3(ac) = 0 \quad (3.2.37)$$

The discriminant of $G(t)$ is:

$$\Delta = m^2 + 4rad^3(ac) = \alpha^2, \quad \alpha > 0 \quad (3.2.38)$$

As $m = rad^3(a) - rad^3(c) > 0$, we obtain that $\alpha = rad^3(a) + rad^3(c) > 0$, then from the equation (3.2.38), it follows that $(\alpha = x, m = y)$ is a solution of the Diophantine equation:

$$x^2 - y^2 = N \quad (3.2.39)$$

with $N = 4rad^3(ac) > 0$. Let $Q(N)$ be the number of the solutions of (3.2.39) and $\tau(N)$ is the number of suitable factorization of N , and using the same method as in the paragraph I-3-2-3- above, we obtain a contradiction.

III-2- We suppose $rad(a) < rad(c)$. We obtain the equation:

$$rad^3(c) - rad^3(a) = b + l - h = m > 0 \quad (3.2.40)$$

Let X be the variable $X = rad(c) - rad(a)$, we use the similar calculations as in the paragraph above I-3-2-3- permuting c, a , we find a contradiction.

It follows that the case $\mu_c > rad^2(c)$ and $\mu_a > rad^{1.63}(a)$ is impossible.

IV - We suppose $\mu_a > rad^2(a)$ and $\mu_c > rad^{1.63}(c)$, we obtain $a > rad^3(a)$ and $c > rad^{2.63}(c)$. We can write $a = rad^3(a) + h$ and $c = rad^3(c) + l$ with h a positive integer and $l \in \mathbb{Z}$.

The calculations are similar to those in the cases of the paragraph III. We obtain a contradiction.

It follows that the case $\mu_c > rad^{1.63}(c)$ and $\mu_a > rad^2(a)$ is impossible.

All possible cases are discussed. □

We can state the following important theorem:

Theorem 3.2.2. *Let a, b, c positive integers relatively prime with $c = a + b$, then $c < rad^{1.63}(abc)$.*

From the theorem above, we can announce also:

Corollary 3.2.2.1. *Let a, b, c positive integers relatively prime with $c = a + b$, then the conjecture $c < rad^2(abc)$ is true.*

Appendix

II'-3-2- We suppose $\mu_a > rad^{2.26}(a) \implies a > rad^{3.26}(a)$.

II'-3-2-1- We consider the case $\mu_c = rad^2(c) \implies c = rad^3(c)$ and $c = a + 1$. Then, we obtain that $Y = rad(c)$ is a solution in positive integers of the equation:

$$Y^3 - 1 = a \quad (3.2.41)$$

II'-3-2-1-1- We suppose that $a = rad^n(a)$ with $n \geq 4$, we obtain the equation:

$$rad^3(c) - rad^n(a) = 1 \quad (3.2.42)$$

But the solutions of the Catalan equation [5] $x^p - y^q = 1$ where the unknowns x, y, p and q take integer values, all ≥ 2 , has only one solution $(x, y, p, q) = (3, 2, 2, 3)$, but the solution of the equation (3.2.42) are $(rad(c) = 3, rad(a) = 2, 3 \neq 2, n \geq 4)$, it follows the contradiction with $n \geq 4$ and the case $a = rad^n(a), n \geq 4$ is to reject.

II'-3-2-1-2- In the following, we will study the cases $\mu_a = A.rad^n(a)$ with $rad(a) \nmid A, n \geq 0$. The above equation (3.2.41) can be written as :

$$(Y - 1)(Y^2 + Y + 1) = a \quad (3.2.43)$$

Let δ one divisor of a so that :

$$Y - 1 = \delta \quad (3.2.44)$$

$$Y^2 + Y + 1 = \frac{a}{\delta} = m = \delta^2 + 3Y \quad (3.2.45)$$

We recall that $rad(c) > rad^{1.63}_{1.37}(a)$.

II'-3-2-1-2-1- We suppose $\delta = l.rad(a)$. We have $\delta = l.rad(a) < a = \mu_a.rad(a) \implies l < \mu_a$. As δ is a divisor of a , then l is a divisor of $\mu_a, \frac{a}{\delta} = \frac{\mu_a.rad(a)}{l.rad(a)} = \frac{\mu_a}{l} = m = \delta^2 + 3Y$, then $\mu_a = l.m$. From $\mu_a = l(\delta^2 + 3Y)$, we obtain:

$$m = l^2rad^2(a) + 3rad(c) \implies 3rad(c) = m - l^2rad^2(a)$$

A' - Case $3|m \implies m = 3m', m' > 1$: As $\mu_a = ml = 3m'l \implies 3|rad(a)$ and $(rad(a), m')$ not coprime. We obtain:

$$rad(c) = m' - l^2rad(a). \frac{rad(a)}{3}$$

It follows that a, c are not coprime, then the contradiction.

B' - Case $m = 3 \implies \mu_a = 3l \implies a = 3lrad(a) = 3\delta = \delta(\delta^2 + 3Y) \implies \delta^2 = 3(1 - Y) = -3\delta < 0$, then the contradiction.

II'-3-2-1-2-2- We suppose $\delta = l.rad^2(a), l \geq 2$. If $n = 0$ then $\mu_a = A$ and from the equation above (3.2.45):

$$m = \frac{a}{\delta} = \frac{\mu_a.rad(a)}{lrad^2(a)} = \frac{A.rad(a)}{lrad^2(a)} = \frac{A}{lrad(a)} \implies rad(a)|A$$

It follows the contradiction with the hypothesis above $rad(a) \nmid A$.

II'-3-2-1-2-3- We suppose $\delta = lrad^2(a), l \geq 2$ and in the following $n > 0$. As $m = \frac{a}{\delta} = \frac{\mu_a.rad(a)}{lrad^2(a)} = \frac{\mu_a}{lrad(a)}$, if $lrad(a) \nmid \mu_a$ then the case is to reject. We suppose $lrad(a)|\mu_a \implies \mu_a = m.lrad(a)$, with $m, rad(a)$ not coprime, then $\frac{a}{\delta} = m = \delta^2 + 3rad(c)$.

C' - Case $m = 1 = a/\delta \implies \delta^2 + 3rad(c) = 1$, then the contradiction.

D' - Case $m = 3$, we obtain $3(1 - rad(c)) = \delta^2 \implies \delta^2 < 0$. Then the contradiction.

E' - Case $m \neq 1, 3$, we obtain: $3rad(c) = m - l^2rad^4(a) \implies rad(a)$ and $rad(c)$ are not coprime. Then the contradiction.

II'-3-2-1-2-4- We suppose $\delta = l.rad^n(a), l \geq 2$ with $n \geq 3$. From $a = \mu_a.rad(a) = lrad^n(a)(\delta^2 + 3rad(c))$, we denote $m = \delta^2 + 3rad(c) = \delta^2 + 3Y$.

F' - As seen above (paragraphs C', D'), the cases $m = 1$ and $m = 3$ give contradictions, it follows the reject of these cases.

G' - Case $m \neq 1, 3$. Let q be a prime that divides m (q can be equal to m), it follows $q|\mu_a \implies q = a_{j_0} \implies a_{j_0}|\delta^2 \implies a_{j_0}|3rad(c)$. Then $rad(a)$ and $rad(c)$ are not coprime. It follows the contradiction.

II'-3-2-1-2-5- We suppose $\delta = \prod_{j \in J_1} a_j^{\beta_j}$, $\beta_j \geq 1$ with at least one $j_0 \in J_1$ with:

$$\beta_{j_0} \geq 2, \quad rad(a) \nmid \delta \quad (3.2.46)$$

We can write:

$$\delta = \mu_\delta \cdot rad(\delta), \quad rad(a) = r \cdot rad(\delta), \quad r > 1, \quad (r, rad(\delta)) = 1 \implies (r, \mu_\delta) = 1 \quad (3.2.47)$$

Then, we obtain:

$$\begin{aligned} a = \mu_a \cdot rad(a) &= \mu_a \cdot r \cdot rad(\delta) = \delta(\delta^2 + 3Y) = \mu_\delta \cdot rad(\delta)(\delta^2 + 3Y) \implies \\ r \cdot \mu_a &= \mu_\delta(\delta^2 + 3Y) \end{aligned} \quad (3.2.48)$$

- We suppose $\mu_a = \mu_\delta \implies r = \delta^2 + 3Y = (\mu_a \cdot rad(\delta))^2 + 3Y$. As $\delta < \delta^2 + 3Y \implies r > \delta \implies rad(a) > r > (\mu_a \cdot rad(\delta) = A \cdot rad^n(a) \cdot rad(\delta)) \implies 1 > A \cdot rad^{n-1}(\delta)$, then the contradiction.

- We suppose $\mu_a < \mu_\delta$. As $rad(c) = \mu_\delta rad(\delta) + 1$, we obtain:

$$rad(c) > \mu_a \cdot rad(\delta) + 1 > 0 \implies rad(ac) > a \cdot rad(\delta) + rad(a) > 0$$

As $c = 1 + a$ and we consider the cases $c > rad(ac)$, then:

$$\begin{aligned} c > rad(ac) > a \cdot rad(\delta) + rad(a) > 0 &\implies a + 1 \geq a \cdot rad(\delta) + rad(a) > 0 \implies \\ a \geq a \cdot rad(\delta) + rad(\delta) &\implies 1 \geq rad(\delta) + \frac{rad(a)}{a} > 0, \quad rad(\delta) \geq 2 \implies \text{The contradiction} \end{aligned}$$

- We suppose $\mu_a > \mu_\delta$. In this case, from the equation (3.2.8) and as $(r, \mu_\delta) = 1$, it follows we can write:

$$\mu_a = \mu_1 \cdot \mu_2, \quad \mu_1, \mu_2 > 1 \quad (3.2.49)$$

$$a = \mu_a \cdot rad(a) = \mu_1 \cdot \mu_2 \cdot r \cdot rad(\delta) = \delta \cdot (\delta^2 + 3Y) \quad (3.2.50)$$

$$\text{so that } r \cdot \mu_1 = \delta^2 + 3Y, \quad \mu_2 = \mu_\delta \implies \delta = \mu_2 \cdot rad(\delta) \quad (3.2.51)$$

** 1- We suppose $(\mu_1, \mu_2) \neq 1$, then $\exists a_{j_0}$ so that $a_{j_0}|\mu_1$ and $a_{j_0}|\mu_2$. But $\mu_\delta = \mu_2 \implies a_{j_0}^2|\delta$. From $3Y = r\mu_1 - \delta^2 \implies a_{j_0}|3Y \implies a_{j_0}|Y$ or $a_{j_0} = 3$.

- If $a_{j_0}|(Y = rad(c))$, it follows the contradiction with $(c, a) = 1$.

- If $a_{j_0} = 3$. We have $r\mu_1 = \delta^2 + 3Y = \delta^2 + 3(\delta + 1) \implies \delta^2 + 3\delta + 3 - r\mu_1 = 0$. As $3|\mu_1 \implies \mu_1 = 3^k \mu'_1, 3 \nmid \mu'_1, k \geq 1$, we obtain:

$$\delta^2 + 3\delta + 3(1 - 3^{k-1}r\mu'_1) = 0 \quad (3.2.52)$$

** 1-1- We consider the case $k > 1 \implies 3 \nmid (1 - 3^{k-1}r\mu'_1)$. Let us recall the Eisenstein criterion [6]

Theorem 3.2.3. (Eisenstein Criterion) Let $f = a_0 + \dots + a_n X^n$ be a polynomial $\in \mathbb{Z}[X]$. We suppose that $\exists p$ a prime number so that $p \nmid a_n$, $p \mid a_i$, ($0 \leq i \leq n-1$), and $p^2 \nmid a_0$, then f is irreducible in \mathbb{Q} .

We apply Eisenstein criterion to the polynomial $R(Z)$ given by:

$$R(Z) = Z^2 + 3Z + 3(1 - 3^{k-1}r\mu'_1) \quad (3.2.53)$$

then:

$$-3 \nmid 1, -3 \mid (+3), -3 \mid 3(1 - 3^{k-1}r\mu'_1), \text{ and } -3^2 \nmid 3(1 - 3^{k-1}r\mu'_1).$$

It follows that the polynomial $R(Z)$ is irreducible in \mathbb{Q} , then, the contradiction with $R(\delta) = 0$.

** 1-2- We consider the case $k = 1$, then $\mu_1 = 3\mu'_1$ and $(\mu'_1, 3) = 1$, we obtain:

$$\delta^2 + 3\delta + 3(1 - r\mu'_1) = 0 \quad (3.2.54)$$

** 1-2-1- We consider that $3 \nmid (1 - r\mu'_1)$, we apply the same Eisenstein criterion to the polynomial $R'(Z)$ given by:

$$R'(Z) = Z^2 + 3Z + 3(1 - r\mu'_1)$$

and we find a contradiction with $R'(\delta) = 0$.

** 1-2-2- We consider that:

$$3 \mid (1 - r\mu'_1) \implies r\mu'_1 - 1 = 3^i \cdot h, \quad i \geq 1, \quad 3 \nmid h, \quad h \in \mathbb{N}^* \quad (3.2.55)$$

δ is an integer root of the polynomial $R'(Z)$:

$$R'(Z) = Z^2 + 3Z + 3(1 - r\mu'_1) = 0 \quad (3.2.56)$$

The discriminant of $R'(Z)$ is:

$$\Delta = 3^2 + 3^{i+1} \times 4h$$

As the root δ is an integer, it follows that $\Delta = t^2 > 0$ with t a positive integer. We obtain:

$$\Delta = 3^2(1 + 3^{i-1} \times 4h) = t^2 \quad (3.2.57)$$

$$\implies 1 + 3^{i-1} \times 4h = q^2 > 1, \quad q \in \mathbb{N}^* \quad (3.2.58)$$

As $\mu_\delta = \mu_2$ and $3 \mid \mu_2 \implies \mu_2 = 3\mu'_2$, then we can write the equation (3.2.54) as :

$$\delta(\delta + 3) = 3^{i+1} \cdot h \implies 3^3 \mu'_2 \frac{rad(\delta)}{3} \cdot (\mu'_2 rad(\delta) + 1) = 3^{i+1} \cdot h \implies \quad (3.2.59)$$

$$\mu'_2 \frac{rad(\delta)}{3} \cdot (\mu'_2 rad(\delta) + 1) = h \quad (3.2.60)$$

We obtain $i = 2$ and $q^2 = 1 + 12h = 1 + 4\mu'_2 rad(\delta)(\mu'_2 rad(\delta) + 1)$. Then, q satisfies :

$$q^2 - 1 = 12h = 4\mu'_2 rad(\delta)(\mu'_2 rad(\delta) + 1) \implies \quad (3.2.61)$$

$$\frac{(q-1)}{2} \cdot \frac{(q+1)}{2} = 3h = \mu'_2 rad(\delta)(\mu'_2 rad(\delta) + 1). \implies \quad (3.2.62)$$

$$q + 1 = 2\mu'_2 rad(\delta) + 2 \quad (3.2.63)$$

$$q - 1 = 2\mu'_2 rad(\delta) \quad (3.2.64)$$

It follows that $(q = x, 1 = y)$ is a solution of the Diophantine equation:

$$x^2 - y^2 = N \quad (3.2.65)$$

with $N = 4\mu'_2 rad(\delta)(\mu'_2 rad(\delta) + 1) = 12h > 0$. Let $Q(N)$ be the number of the solutions of (3.2.65) and $\tau(N)$ is the number of suitable factorization of N , then we announce the following result concerning the solutions of the Diophantine equation (3.2.65) (see theorem 27.3 in [7]):

- If $N \equiv 2 \pmod{4}$, then $Q(N) = 0$.
 - If $N \equiv 1$ or $N \equiv 3 \pmod{4}$, then $Q(N) = [\tau(N)/2]$.
 - If $N \equiv 0 \pmod{4}$, then $Q(N) = [\tau(N/4)/2]$.
- $[x]$ is the integral part of x for which $[x] \leq x < [x] + 1$.

As $N = 4\mu'_2 rad(\delta)(\mu'_2 rad(\delta) + 1) \implies N \equiv 0 \pmod{4} \implies Q(N) = [\tau(N/4)/2]$. As $(q, 1)$ is a couple of solutions of the Diophantine equation (3.2.65), then $\exists d, d'$ positive integers with $d > d'$ and $N = d.d'$ so that :

$$d + d' = 2q \quad (3.2.66)$$

$$d - d' = 2.1 = 2 \quad (3.2.67)$$

** 1-2-2-1 As $N > 1$, we take $d = N$ and $d' = 1$. It follows:

$$\begin{cases} N + 1 = 2q \\ N - 1 = 2 \end{cases} \implies N = 3 \implies \text{then the contradiction with } N \equiv 0 \pmod{4}.$$

** 1-2-2-2 Now, we consider the case $d = 2\mu'_2 rad(\delta)(\mu'_2 rad(\delta) + 1)$ and $d' = 2$. It follows:

$$\begin{cases} 2\mu'_2 rad(\delta)(\mu'_2 rad(\delta) + 1) + 2 = 2q \\ 2\mu'_2 rad(\delta)(\mu'_2 rad(\delta) + 1) - 2 = 2 \end{cases} \implies \mu'_2 rad(\delta)(\mu'_2 rad(\delta) + 1) = q - 1$$

As $q - 1 = 2\mu'_2 rad(\delta)$, we obtain $\mu'_2 rad(\delta) = 1$, then the contradiction.

** 1-2-2-3 Now, we consider the case $d = \mu'_2 rad(\delta)(\mu'_2 rad(\delta) + 1)$ and $d' = 4$. It follows:

$$\begin{cases} \mu'_2 rad(\delta)(\mu'_2 rad(\delta) + 1) + 4 = 2q \\ \mu'_2 rad(\delta)(\mu'_2 rad(\delta) + 1) - 4 = 2 \end{cases} \implies \mu'_2 rad(\delta)(\mu'_2 rad(\delta) + 1) = 6$$

As $\mu'_2 rad(\delta) \geq 2 \implies \mu'_2 rad(\delta) = 2 \implies \mu'_2 = 1 \implies \mu_2 = 3 = \mu_\delta$ and $rad(\delta) = 2$ but $3 \nmid 2$, then the contradiction.

** 1-2-2-4 Now, let a_{j_0} be a prime integer so that $a_{j_0} | rad\delta$, we consider the case $d = \mu'_2 \frac{rad(\delta)}{a_{j_0}} (\mu'_2 rad(\delta) + 1)$ and $d' = 4a_{j_0}$. It follows:

$$\begin{cases} \mu'_2 \frac{rad(\delta)}{a_{j_0}} (\mu'_2 rad(\delta) + 1) + 4a_{j_0} = 2q \\ \mu'_2 \frac{rad(\delta)}{a_{j_0}} (\mu'_2 rad(\delta) + 1) - 4a_{j_0} = 2 \end{cases} \implies \mu'_2 \frac{rad(\delta)}{a_{j_0}} (\mu'_2 rad(\delta) + 1) = 2(1 + 2a_{j_0}) \implies$$

Then the contradiction as the left member is greater than the right member $2(1 + 2a_{j_0})$.

** 1-2-2-5 Now, we consider the case $d = 4\mu'_2 rad(\delta)$ and $d' = (\mu'_2 rad(\delta) + 1)$. It follows:

$$\begin{cases} 4\mu'_2 rad(\delta) + (\mu'_2 rad(\delta) + 1) = 2q \\ 4\mu'_2 rad(\delta) - (\mu'_2 rad(\delta) + 1) = 2 \end{cases} \implies 3\mu'_2 rad(\delta) = 3 \implies \text{Then the contradiction.}$$

** 1-2-2-6 Now, we consider the case $d = 2(\mu'_2 rad(\delta) + 1)$ and $d = 2\mu'_2 rad(\delta)$. It follows:

$$\begin{cases} 2(\mu'_2 rad(\delta) + 1) + 2\mu'_2 rad(\delta) = 2q \implies 2\mu'_2 rad(\delta) + 1 = q \\ 2(\mu'_2 rad(\delta) + 1) - 2\mu'_2 rad(\delta) = 2 \implies 2 = 2 \end{cases}$$

It follows that this case presents no contradictions *a priori*.

** 1-2-2-7 $\mu'_2 rad(\delta)$ and $\mu'_2 rad(\delta) + 1$ are coprime, let $\mu'_2 rad(\delta) + 1 = \prod_{j=1}^{j=J} \lambda_j^{\gamma_j}$, we consider the

case $d = 2\lambda_{j'} \mu'_2 rad(\delta)$ and $d' = 2 \frac{\mu'_2 rad(\delta) + 1}{\lambda_{j'}}$. It follows:

$$\begin{cases} 2\lambda_{j'} \mu'_2 rad(\delta) + 2 \frac{\mu'_2 rad(\delta) + 1}{\lambda_{j'}} = 2q \\ 2\lambda_{j'} \mu'_2 rad(\delta) - 2 \frac{\mu'_2 rad(\delta) + 1}{\lambda_{j'}} = 2 \end{cases}$$

** 1-2-2-7-1 We suppose that $\gamma_{j'} = 1$. We consider the case $d = 2\lambda_{j'} \mu'_2 rad(\delta)$ and $d' = 2 \frac{\mu'_2 rad(\delta) + 1}{\lambda_{j'}}$. It follows:

$$\begin{cases} 2\lambda_{j'} \mu'_2 rad(\delta) + 2 \frac{\mu'_2 rad(\delta) + 1}{\lambda_{j'}} = 2q \\ 2\lambda_{j'} \mu'_2 rad(\delta) - 2 \frac{\mu'_2 rad(\delta) + 1}{\lambda_{j'}} = 2 \end{cases} \implies 4\lambda_{j'} \mu'_2 rad(\delta) = 2(q + 1) \implies 2\lambda_{j'} \mu'_2 rad(\delta) = q + 1$$

But from the equation (3.2.22), $q + 1 = 2\mu'_2 rad(\delta)$, then $\lambda_{j'} = 1$, it follows the contradiction.

** 1-2-2-7-2 We suppose that $\gamma_{j'} \geq 2$. We consider the case $d = 2\lambda_{j'}^{\gamma_{j'} - r_{j'}} \mu'_2 rad(\delta)$ and $d' = 2 \frac{\mu'_2 rad(\delta) + 1}{\lambda_{j'}^{r_{j'}}$. It follows:

$$\begin{cases} 2\lambda_{j'}^{\gamma_{j'} - r_{j'}} \mu'_2 rad(\delta) + 2 \frac{\mu'_2 rad(\delta) + 1}{\lambda_{j'}^{r_{j'}}} = 2q \\ 2\lambda_{j'}^{\gamma_{j'} - r_{j'}} \mu'_2 rad(\delta) - 2 \frac{\mu'_2 rad(\delta) + 1}{\lambda_{j'}^{r_{j'}}} = 2 \end{cases} \implies 4\lambda_{j'}^{\gamma_{j'} - r_{j'}} \mu'_2 rad(\delta) = 2(q + 1)$$

$$\implies 2\lambda_{j'}^{\gamma_{j'} - r_{j'}} \mu'_2 rad(\delta) = q + 1$$

As above, it follows the contradiction. It is trivial that the other cases for more factors $\prod_{j''} \lambda_{j''}^{\gamma_{j''} - r_{j''}}$ give also contradictions.

** 1-2-2-8 Now, we consider the case $d = 4(\mu'_2 rad(\delta) + 1)$ and $d' = \mu'_2 rad(\delta)$, we have $d > d'$. It follows:

$$\begin{cases} 4(\mu'_2 rad(\delta) + 1) + \mu'_2 rad(\delta) = 2q \Rightarrow 5\mu'_2 rad(\delta) = 2(q + 2) \\ 4(\mu'_2 rad(\delta) + 1) - \mu'_2 rad(\delta) = 2 \Rightarrow \mu'_2 rad(\delta) = 2 \end{cases} \Rightarrow \begin{cases} \text{Then the contradiction as} \\ 3|\delta. \end{cases}$$

** 1-2-2-9 Now, we consider the case $d = 4u(\mu'_2 rad(\delta) + 1)$ and $d' = \frac{\mu'_2 rad(\delta)}{u}$, where $u > 1$ is an integer divisor of $\mu'_2 rad(\delta)$. We have $d > d'$ and:

$$\begin{cases} 4u(\mu'_2 rad(\delta) + 1) + \frac{\mu'_2 rad(\delta)}{u} = 2q \\ 4u(\mu'_2 rad(\delta) + 1) - \frac{\mu'_2 rad(\delta)}{u} = 2 \end{cases} \implies 2u(\mu'_2 rad(\delta) + 1) = \mu'_2 rad(\delta) + 1 \implies 2u = 1$$

Then the contradiction.

In conclusion, we have found only one case (** 1-2-2-6 above) where there is no contradictions *a priori*. As $\tau(N)$ is large and also $\lceil \tau(N/4)/2 \rceil$, it follows the contradiction with $Q(N) \leq 1$ and the hypothesis $(\mu_1, \mu_2) \neq 1$ is false.

** 2- We suppose that $(\mu_1, \mu_2) = 1$.

We recall that $rad(c) = Y > rad^{1.63/1.37}(a)$, $\delta + 1 = Y$, $rad(a) = r \cdot rad(\delta)$, $(r, rad(\delta)) = 1$, $\delta = \mu_2 rad(\delta)$ and $r\mu_1 = \delta^2 + 3X$, it follows:

$$U(\delta) = \delta^2 + 3\delta + 3 - r\mu_1 = 0 \quad (3.2.68)$$

** 2-1- We suppose $3|(3 - r\mu_1)$ and $3^2 \nmid (3 - r\mu_1)$, then we use the Eisenstein criterion [6] to the polynomial $U(\delta)$ given by the equation (3.2.68), and the contradiction.

** 2-2- We suppose $3|(3 - r\mu_1)$ and $3^2|(3 - r\mu_1)$. From $3|(3 - r\mu_1) \implies 3|r\mu_1 \implies 3|r$ or $3|\mu_1$.
 - If $3|r \implies (3, rad\delta) = 1 \implies 3 \nmid \delta$. Then the contradiction with $3|\delta^2$ by the equation (3.2.68).
 - If $3|\mu_1 \implies 3 \nmid \mu_2 \implies 3 \nmid \delta$, it follows the contradiction with $3|\delta^2$ by the equation (3.2.68).

** 2-3- We suppose $3 \nmid (3 - r\mu_1) \implies 3 \nmid r\mu_1 \implies 3 \nmid r$ and $3 \nmid \mu_1$. From the equation (3.2.68), $U(\delta) = 0 \implies r\mu_1 \equiv \delta^2 \pmod{3}$, as δ^2 is a square then $\delta^2 \equiv 1 \pmod{3} \implies r\mu_1 \equiv 1 \pmod{3}$, but this result is not all verified. Then the contradiction.

It follows that the case $\mu_a > rad^{2.26}(a) \implies a > rad^{3.26}(a)$ and $c = rad^3(c)$ is impossible.

II'-3-2-2- We consider the case $\mu_c = rad^2(c) \implies c = rad^3(c)$ and $c = a + b$. Then, we obtain that $Y = rad(c)$ is a solution in positive integers of the equation:

$$Y^3 + 1 = \bar{c} \quad (3.2.69)$$

with $\bar{c} = a + b + 1 = c + 1 \implies (\bar{c}, c) = 1$. We obtain the same result as of the case **I-3-2-1-** studied above considering $rad(\bar{c}) > rad^{\frac{1.63}{1.37}}(c)$.

II'-3-2-3- We suppose $\mu_a > rad^{2.26}(a) \implies a > rad^{3.26}(a)$ and c large and $\mu_c < rad^2(c)$, we consider $c = a + b, b \geq 1$. Then $a = rad^3(a) + h, h > 0, h$ a positive integer and we can write $c + l = rad^3(c), l > 0$. As $rad(c) > rad^{\frac{1.63}{1.37}}(a) \implies rad(c) > rad(a) \implies h + l + b = m > 0$, it follows:

$$rad^3(c) - l = rad^3(a) + h + b > 0 \implies rad^3(c) - rad^3(a) = h + l + b = m > 0 \quad (3.2.70)$$

We obtain the same result (a contradiction) as of the case **I-3-2-3-** studied above considering $rad(c) > rad^{\frac{1.63}{1.37}}(a)$. Then, this case is to reject.

Then the cases $\mu_c \leq rad^2(c)$ and $a > rad^{3.26}(a)$ are impossible.

Bibliography

- [1] M. Waldschmidt: On the abc Conjecture and some of its consequences, presented at The 6th World Conference on 21st Century Mathematics, Abdus Salam School of Mathematical Sciences (ASSMS), Lahore (Pakistan), March 6-9, (2013).
- [2] B. De Smit: <https://www.math.leidenuniv.nl/~desmit/abc/>. Accessed December 2020.
- [3] P. Mihăilescu: *Around ABC*, European Mathematical Society Newsletter, N° 93, (2014), 29-34.
- [4] A. Nitaj: Aspects expérimentaux de la conjecture *abc*. Séminaire de Théorie des Nombres de Paris(1993-1994), London Math. Soc. Lecture Note Ser., Vol n°235, (1996), 145-156, Cambridge Univ. Press.
- [5] P. Mihăilescu: *Primary cyclotomic units and a proof of Catalan's Conjecture'*, Journal für die Reine und Angewandte Mathematik, Vol. 2004, Issue 572, (2004), 167-195. <https://doi.org/10.1515/crll.2004.048>
- [6] C. Touibi: *Algèbre Générale* (in French), Cérès Editions, Tunis, (1996).
- [7] B.M. Stewart B.M: *Theory of Numbers*, 2^{sd} edition, (1964), The Macmillan Compagny, N.Y.

Chapter 4

Is The *abc* Conjecture True?

Abstract

In this paper, we consider the *abc* conjecture. As the conjecture $c < rad^2(abc)$ is true, then we give the proof of the *abc* conjecture for $\epsilon \geq 1$ and for the case $\epsilon \in]0, 1[$, we consider that the *abc* conjecture is false, from the proof, we arrive in a contradiction.

Résumé

Dans cet article, nous considérons la conjecture *abc*. Comme la conjecture $c < rad^2(abc)$ est vraie, nous donnons la preuve que la conjecture *abc* est vraie pour $\epsilon \geq 1$ et pour les cas $\epsilon \in]0, 1[$, supposant que la conjecture est fausse nous arrivons à une contradiction.

4.1 Introduction and notations

Let a positive integer $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as :

$$a = \prod_i a_i^{\alpha_i} = rad(a) \cdot \prod_i a_i^{\alpha_i - 1} \quad (4.1.1)$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \implies a = \mu_a \cdot rad(a) \quad (4.1.2)$$

The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Oesterlé of Pierre et Marie Curie University (Paris 6) [1]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

Conjecture 4.1.1. (*abc* Conjecture): For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :

$$c < K(\epsilon) \cdot rad^{1+\epsilon}(abc) \quad (4.1.3)$$

where K is a constant depending only of ϵ .

The idea to try to write a paper about this conjecture was born after the publication in September 2018, of an article in *Quanta* magazine about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [3]. The difficulty to find a proof of the *abc* conjecture is due to the incomprehensibility how the prime factors are organized in c giving a, b with $c = a + b$. So, I will give a simple proof that can be understood by undergraduate students.

We know that numerically, $\frac{\text{Log}c}{\text{Log}(\text{rad}(abc))} \leq 1.629912$ [1]. A conjecture was proposed that $c < \text{rad}^2(abc)$ [4]. It is the key to resolve the *abc* conjecture. In the following, as the conjecture $c < \text{rad}^2(abc)$ holds (chapter 3), I propose an elementary proof of the *abc* conjecture.

4.2 The Proof of the *abc* conjecture

Proof. We note $R = \text{rad}(abc)$ in the case $c = a + b$ or $R = \text{rad}(ac)$ in the case $c = a + 1$.

4.2.1 Case : $\epsilon \geq 1$

As $c < R^2$ is true, we have $\forall \epsilon \geq 1$:

$$c < R^2 \leq R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad \text{with } K(\epsilon) = e, \quad \epsilon \geq 1 \quad (4.2.1)$$

Then the *abc* conjecture is true.

4.2.2 Case: $0 < \epsilon < 1$

For the cases $c < R$, it is trivial that the *abc* conjecture is true. In the following we consider that $c > R$. From the statement of the *abc* conjecture 4.1.1, we want to give a proof that $c < K(\epsilon)R^{1+\epsilon} \implies \text{Log}K(\epsilon) + (1 + \epsilon)\text{Log}R - \text{Log}c > 0$.

For our proof, we proceed by contradiction of the *abc* conjecture. We suppose that the *abc* conjecture is false:

$$\exists \epsilon_0 \in]0, 1[, \forall K(\epsilon) > 0, \quad \exists c_0 = a_0 + b_0; \quad a_0, b_0, c_0 \text{ coprime so that} \\ c_0 > K(\epsilon_0)R_0^{1+\epsilon_0} \quad (4.2.2)$$

We choose the constant $K(\epsilon) = \frac{1}{e\epsilon^2}$. Let :

$$Y_{c_0}(\epsilon) = \frac{1}{\epsilon^2} + (1 + \epsilon)\text{Log}R_0 - \text{Log}c_0, \quad \epsilon \in]0, 1[\quad (4.2.3)$$

From the above explications, if we will obtain $\forall \epsilon \in]0, 1[, Y_{c_0}(\epsilon) > 0 \implies c_0 < K(\epsilon)R_0^{1+\epsilon} \implies c_0 < K(\epsilon_0)R_0^{1+\epsilon_0}$, then the contradiction with (4.2.2).

About the function Y_{c_0} , we have:

$$\lim_{\epsilon \rightarrow 1} Y_{c_0}(\epsilon) = 1 + \text{Log}(R_0^2/c_0) = \lambda > 0 \\ \lim_{\epsilon \rightarrow 0} Y_{c_0}(\epsilon) = +\infty$$

The function $Y_{c_0}(\epsilon)$ has a derivative for $\forall \epsilon \in]0, 1[$, we obtain:

$$Y'_{c_0}(\epsilon) = -\frac{2}{\epsilon^3} + \text{Log}R_0 = \frac{\epsilon^3 \text{Log}R_0 - 2}{\epsilon^3} \tag{4.2.4}$$

$$Y'_{c_0}(\epsilon) = 0 \implies \epsilon = \epsilon' = \sqrt[3]{\frac{2}{\text{Log}R_0}} \in]0, 1[\text{ for } R_0 \geq 8.$$

ϵ	0	ϵ'	1
$Y'(\epsilon)$	-	0	+
$Y(\epsilon)$	$+\infty$	$Y(\epsilon')$	$\lambda > 0$

Figure 4.1: Table of variations

Discussion from the table (Fig.: 4.1):

- If $Y_{c_0}(\epsilon') \geq 0$, it follows that $\forall \epsilon \in]0, 1[, Y_{c_0}(\epsilon) \geq 0$, then the contradiction with $Y_{c_0}(\epsilon_0) < 0 \implies c_0 > K(\epsilon_0)R_0^{1+\epsilon_0}$ and the supposition that the *abc* conjecture is false can not hold. Hence the *abc* conjecture is true for $\epsilon \in]0, 1[$.

- If $Y_{c_0}(\epsilon') < 0 \implies \exists 0 < \epsilon_1 < \epsilon' < \epsilon_2 < 1$, so that $Y_{c_0}(\epsilon_1) = Y_{c_0}(\epsilon_2) = 0$. Then we obtain:

$$c_0 = K(\epsilon_1)R_0^{1+\epsilon_1} = K(\epsilon_2)R_0^{1+\epsilon_2} \tag{4.2.5}$$

We recall the following definition:

Definition 4.2.1. The number ζ is called algebraic number if there is at least one polynomial:

$$l(x) = l_0 + l_1x + \dots + l_mx^m, \quad l_m \neq 0 \tag{4.2.6}$$

with integral coefficients such that $l(\zeta) = 0$, and it is called transcendental if no such polynomial exists.

We consider the equality :

$$c_0 = K(\epsilon_1)R_0^{1+\epsilon_1} \implies \frac{c_0}{R_0} = \frac{\mu_{c_0}}{\text{rad}(a_0b_0)} = e^{\frac{1}{\epsilon_1^2} R_0^{\epsilon_1}} \tag{4.2.7}$$

i) - We suppose that $\epsilon_1 = \beta_1$ is an algebraic number then $\beta_0 = 1/\epsilon_1^2$ and $\alpha_1 = R_0$ are also algebraic numbers. We obtain:

$$\frac{c_0}{R_0} = \frac{\mu_{c_0}}{\text{rad}(a_0b_0)} = e^{\frac{1}{\epsilon_1^2} R_0^{\epsilon_1}} = e^{\beta_0 \cdot \alpha_1^{\beta_1}} \tag{4.2.8}$$

From the theorem (see theorem 3, page 196 in [2]):

Theorem 4.2.1. $e^{\beta_0} \alpha_1^{\beta_1} \dots \alpha_n^{\beta_n}$ is transcendental for any nonzero algebraic numbers $\alpha_1, \dots, \alpha_n, \beta_0, \dots, \beta_n$.

we deduce that the right member $e^{\beta_0} \alpha_1^{\beta_1}$ of (4.2.8) is transcendental, but the term $\frac{\mu_{c_0}}{\text{rad}(a_0 b_0)}$ is an algebraic number, then the contradiction and the case $Y_{c_0}(\epsilon') < 0$ is impossible. It follows $Y_{c_0}(\epsilon') \geq 0$ then the *abc* conjecture is true.

ii) - We suppose that ϵ_1 is transcendental, then $1/(\epsilon_1^2)$ is transcendental. If not, $1/(\epsilon_1^2)$ is an algebraic number and from the definition (4.2.1) above, we find a contradiction. As R_0 is an algebraic number, then $\text{Log} R_0$ is transcendental. We rewrite the equation (4.2.5) as:

$$\frac{c_0}{R_0} = e^{\frac{1}{\epsilon_1^2}} R_0^{\epsilon_1} = e^{\frac{1}{\epsilon_2^2}} R_0^{\epsilon_2} \implies \frac{c_0}{R_0} = e^{\frac{1}{\epsilon_1^2} + \epsilon_1 \text{Log} R_0} = e^{\frac{1}{\epsilon_2^2} + \epsilon_2 \text{Log} R_0} \quad (4.2.9)$$

As e is transcendental and e^x is transcendental, it follows the contradiction with c_0/R_0 an algebraic number. It follows that $Y_{c_0}(\epsilon') \geq 0$ and the *abc* conjecture is true.

Then the proof of the *abc* conjecture is finished. As $c < R^2$ is true, we obtain that $\forall \epsilon > 0$, $\exists K(\epsilon) > 0$, if $c = a + b$ with a, b, c positive integers relatively coprime, then :

$$c < K(\epsilon) \cdot \text{rad}^{1+\epsilon}(abc) \quad (4.2.10)$$

and the constant $K(\epsilon)$ depends only of ϵ .

Q.E.D

Ouf, end of the mystery!

□

4.3 Conclusion

As $c < R^2$ is true, we have given an elementary proof of the *abc* conjecture. We can announce the important theorem:

Theorem 4.3.1. The *abc* conjecture is true:

For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then:

$$c < K(\epsilon) \cdot \text{rad}^{1+\epsilon}(abc) \quad (4.3.1)$$

where K is a constant depending of ϵ .

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Bibliography

- [1] M. Waldschmidt, On the abc Conjecture and some of its consequences, presented at The 6th World Conference on 21st Century Mathematics, Abdus Salam School of Mathematical Sciences (ASSMS), Lahore (Pakistan), March 6-9. (2013)
- [2] A. Baker, Effective Methods in Diophantine Problems. Proceedings of Symposia in Pure Mathematics, Volume XX, 1969 Number Theory Institute. AMS. (1971) 451 pages, pp 195–205.
- [3] K. Kremmerz, Titans of Mathematics Clash Over Epic Proof of ABC Conjecture. The Quanta Newsletter, 20 September 2018. www.quantamagazine.org. (2018)
- [4] P. Mihăilescu, Around ABC. European Mathematical Society Newsletter, N° 93, September 2014, (2014) pp 29–34.
- [5] A. Borisovich Shidlovskii, Transcendental Numbers. Translated from the Russian by Neal Koblitz. (De Gruyter Studies in Mathematics; 12). 488 pages (1989)

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