

Proof of 16 Formulas Barnes function

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Abstract

I have already published several months ago in the papers "Values of Barnes Function" and "Another Values of Barnes Function and Formulas" in total 16 conjectural formulas that I find with unusual methods. So, in this article, I write the proof of 16 formulas.

1 Definition

The Barnes function is defined as the following Weierstrass product:

$$G(1+z) = (2\pi)^{\frac{z}{2}} e^{-\frac{z(1+z)}{2} - \frac{\gamma z^2}{2}} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right)^k e^{-z + \frac{z^2}{2k}} \quad (1)$$

where gamma is the Euler-Mascheroni constant.

The following properties of G are well-known.

2 Properties

$$G(1) = 1 \quad (2)$$

$$G(1+z) = G(z)\Gamma(z) \quad (3)$$

$$\log(G(1+z)) = \frac{z \log(2\pi)}{2} - \frac{z(1+z)}{2} + z \log(\Gamma(1+z)) - \int_0^z \log(\Gamma(t+1)) dt \quad (4)$$

$$\int_0^z \log(\Gamma(t+1)) dt = \frac{z \log(2\pi)}{2} - \frac{z(1+z)}{2} + z \log(\Gamma(1+z)) - \log(G(z)) - \log(\Gamma(z)) \quad (5)$$

3 Introduction

I need 4 relations:

$$\zeta^{(1)}(-1, z) = \zeta^{(1)}(-1) - \log(G(z)) + (z - 1) \log(\Gamma(z)) \quad (6) \text{ where}$$

$\zeta^{(1)}(-1, z)$ is the first derivative of Hurwitz Zeta at z . (z is a positiv real)

The Adamchik-Miller's relation (7) for $\zeta^{(1)}(1 - 2n, \frac{h}{k})$:

$$\begin{aligned} & \frac{(\psi(2n) - \log(2\pi k)) B_{2n}(h/k)}{2n} - \frac{(\psi(2n) - \log(2\pi)) B_{2n}}{2n k^{2n}} + \frac{(-1)^{n+1} \pi}{(2\pi k)^{2n}} \sum_{r=1}^{k-1} \sin\left(\frac{2\pi r h}{k}\right) \Psi^{(2n-1)}\left(\frac{r}{k}\right) \\ & + \frac{2(-1)^{n+1} (2n-1)!}{(2\pi k)^{2n}} \sum_{r=1}^{k-1} \cos\left(\frac{2\pi r h}{k}\right) \zeta^{(1)}\left(2n, \frac{r}{k}\right) + \frac{\zeta^{(1)}(1-2n)}{k^{2n}} \end{aligned}$$

where

$B_{2n}(h/k)$ is Bernoulli polynomial at h/k (8). Here h and k both positiv integer.

B_{2n} is Bernoulli numbers. (9)

$\Psi^{(2n-1)}\left(\frac{r}{k}\right)$ is the polygamma function order $2n-1$ at r/k . But here, in this study, $n=1$ and just we have the trigamma function. (10)

The Adamchik-Miller's relation is very complicated but I remark with this formula, I can make connection between two Barnes G-Function or if I use the relation (6) a connection between two expressions of first derivative of Hurwitz Zeta. Of course, we must choose two parameters a and b correctly.

So the principle is simple: just I evaluate closed form of $\zeta^{(1)}(-1, a) + \zeta^{(1)}(-1, b)$ or $\zeta^{(1)}(-1, a) - \zeta^{(1)}(-1, b)$ and in particular I can evaluate the complicated second sum.

The relation

$$\sum_{r=1}^{k-1} \zeta^{(1)}\left(s, \frac{r}{k}\right)$$

And we know that this sum = $\zeta^{(1)}(s) (k^s - 1) + k^s \zeta(s) \log(k)$ (11)

Here, in this study, $s=2$.

Remember the value: $\zeta^{(1)}(2)=$

$$\frac{\pi^2\gamma}{6} + \frac{\pi^2 \log(2)}{6} - \frac{\pi^2}{6} + 2\pi^2\zeta^{(1)}(-1) + \frac{\pi^2 \log(\pi)}{6}$$

4 About the $\log(G(1/5))$, $\log(G(2/5))$, $\log(G(3/5))$ and $\log(G(4/5))$

First case

I find $\log(G(4/5))$ easily: just I use relation (4) with $z=-1/5$ and we obtain $\log(G(4/5))$ in terms of

$$\left(\int_0^{-\frac{1}{5}} \log(\Gamma(t+1)) dt \right)$$

Second case

Now I search $\log(G(1/5))$, I evaluate the closed form of $\zeta^{(1)}(-1, \frac{4}{5}) - \zeta^{(1)}(-1, \frac{1}{5})$

And I use 2 times the Adamchik-Miller's relation ($n=1$), we have a very long expression but if you factorize the term $\frac{2(-1)^{n+1}(2n-1)!}{(2\pi k)^{2n}}$

And if you consider only the part $\sum_{r=1}^{5-1} \cos(\frac{2\pi r*4}{5}) \zeta^{(1)}(2n, \frac{r}{5}) - (\sum_{r=1}^{5-1} \cos(\frac{2\pi r*1}{5}) \zeta^{(1)}(2n, \frac{r}{5}))$

Directly we have 0

And then the closed form of $\zeta^{(1)}(-1, \frac{4}{5}) - \zeta^{(1)}(-1, \frac{1}{5})$ is

$$\frac{(-5 - \sqrt{5}) \Psi^{(1)}(\frac{1}{5}) - 2\sqrt{5} \Psi^{(1)}(\frac{2}{5}) + 4\pi^2 (\sqrt{5} + 1)}{50\sqrt{10 + 2\sqrt{5}} \pi}$$

I use the relation (6) with $z=1/5$ and $z=4/5$ and I obtain $\log(G(1/5))$ in terms of

$$\left(\int_0^{-\frac{1}{5}} \log(\Gamma(t+1)) dt \right), \Psi^{(1)}\left(\frac{1}{5}\right), \Psi^{(1)}\left(\frac{2}{5}\right)$$

Third case

Now I search $\log(G(2/5))$, I evaluate the closed form of $\zeta^{(1)}(-1, \frac{1}{5}) + \zeta^{(1)}(-1, \frac{2}{5})$

And I use 2 times the Adamchik-Miller's relation (n=1), we have a very long expression but if you factorize the term $\frac{2(-1)^{n+1}(2n-1)!}{(2\pi k)^{2n}}$

And if you consider only the part $\sum_{r=1}^{5-1} \cos(\frac{2\pi r*1}{5}) \zeta^{(1)}(2n, \frac{r}{5}) + \sum_{r=1}^{5-1} \cos(\frac{2\pi r*2}{5}) \zeta^{(1)}(2n, \frac{r}{5})$

We have

$$\cos(\frac{2\pi}{5}) \zeta^{(1)}(2, \frac{1}{5}) - \cos(\frac{\pi}{5}) \zeta^{(1)}(2, \frac{2}{5}) - \cos(\frac{\pi}{5}) \zeta^{(1)}(2, \frac{3}{5}) + \cos(\frac{2\pi}{5}) \zeta^{(1)}(2, \frac{4}{5}) - \cos(\frac{\pi}{5}) \zeta^{(1)}(2, \frac{1}{5}) + \cos(\frac{2\pi}{5}) \zeta^{(1)}(2, \frac{2}{5}) + \cos(\frac{2\pi}{5}) \zeta^{(1)}(2, \frac{3}{5}) - \cos(\frac{\pi}{5}) \zeta^{(1)}(2, \frac{4}{5})$$

I can simplify

$$-\frac{\zeta^{(1)}(2, \frac{1}{5})}{2} - \frac{\zeta^{(1)}(2, \frac{2}{5})}{2} - \frac{\zeta^{(1)}(2, \frac{3}{5})}{2} - \frac{\zeta^{(1)}(2, \frac{4}{5})}{2}$$

And now I use the relation (11) and finally

$$-12\zeta^{(1)}(2) - \frac{25\pi^2 \log(5)}{12}$$

So I can finish the calcul with the trigamma function's rules and I have the closed form of $\zeta^{(1)}(-1, \frac{1}{5}) + \zeta^{(1)}(-1, \frac{2}{5})$

We obtain

$$-\frac{2\zeta^{(1)}(-1)}{5} - \frac{\log(5)}{120} + \frac{(5 + 3\sqrt{5}) \Psi^{(1)}(\frac{1}{5}) + (-5 + \sqrt{5}) \Psi^{(1)}(\frac{2}{5}) - \pi^2 (6\sqrt{5} + 2)}{100\sqrt{10 + 2\sqrt{5}}\pi}$$

Hence, with the relation (6), I find $\log(G(2/5))$ in terms of

$$\left(\int_0^{-\frac{1}{5}} \log(\Gamma(t+1)) dt \right), \Psi^{(1)}\left(\frac{1}{5}\right), \Psi^{(1)}\left(\frac{2}{5}\right)$$

Fourth case

Now I search $\log(G(3/5))$: I repeat the same principle but this time I evaluate the closed form of $\zeta^{(1)}(-1, \frac{2}{5}) - \zeta^{(1)}(-1, \frac{3}{5})$

Finally I have

$$\frac{(-\sqrt{5} - 5) \Psi^{(1)}(\frac{2}{5}) + 2\sqrt{5} \Psi^{(1)}(\frac{1}{5}) - 2\pi^2 (\sqrt{5} - 1)}{50\sqrt{10 + 2\sqrt{5}}\pi}$$

So I have $\log(G(3/5))$ in terms of

$$\left(\int_0^{-\frac{1}{5}} \log(\Gamma(t+1)) dt \right), \Psi^{(1)}\left(\frac{1}{5}\right), \Psi^{(1)}\left(\frac{2}{5}\right)$$

5 About the $\log(G(1/8))$, $\log(G(3/8))$, $\log(G(5/8))$ and $\log(G(7/8))$

First case

I find $\log(G(7/8))$ easily: just I use relation (4) with $z=-1/8$ and we obtain $\log(G(7/8))$ in terms of

$$\left(\int_0^{-\frac{1}{8}} \log(\Gamma(t+1)) dt \right)$$

Second case

Now I search $\log(G(1/8))$, I evaluate the closed form of $\zeta^{(1)}(-1, \frac{7}{8}) - \zeta^{(1)}(-1, \frac{1}{8})$

And I use 2 times the Adamchik-Miller's relation ($n=1$), we have a very long expression but if you factorize the term $\frac{2(-1)^{n+1}(2n-1)!}{(2\pi k)^{2n}}$

$$\text{And if you consider only the part } \sum_{r=1}^{8-1} \cos\left(\frac{2\pi r*7}{8}\right) \zeta^{(1)}\left(2n, \frac{r}{8}\right) - \left(\sum_{r=1}^{8-1} \cos\left(\frac{2\pi r*1}{8}\right) \zeta^{(1)}\left(2n, \frac{r}{8}\right)\right)$$

Directly we have 0

And then the closed form of $\zeta^{(1)}(-1, \frac{7}{8}) - \zeta^{(1)}(-1, \frac{1}{8})$ is

$$\frac{-\sqrt{2} \Psi^{(1)}\left(\frac{1}{8}\right) + (2\pi^2 + 16K) \sqrt{2} + 2\pi^2 - 8K}{64\pi}$$

where K is the Catalan's constant (12)

I use the relation (6) with $z=7/8$ and $z=1/8$ and I obtain $\log(G(1/8))$ in terms of

$$\left(\int_0^{-\frac{1}{8}} \log(\Gamma(t+1)) dt \right), \Psi^{(1)}\left(\frac{1}{8}\right)$$

Third case

Now I search $\log(G(3/8))$, I evaluate the closed form of $\zeta^{(1)}(-1, \frac{1}{8}) + \zeta^{(1)}(-1, \frac{3}{8})$

And I use 2 times the Adamchik-Miller's relation (n=1), we have a very long expression but if you factorize the term $\frac{2(-1)^{n+1}(2n-1)!}{(2\pi k)^{2n}}$

And if you consider only the part $\sum_{r=1}^{8-1} \cos\left(\frac{2\pi r*1}{8}\right) \zeta^{(1)}\left(2n, \frac{r}{8}\right) + \sum_{r=1}^{8-1} \cos\left(\frac{2\pi r*3}{8}\right) \zeta^{(1)}\left(2n, \frac{r}{8}\right)$

We have

$$-2\zeta^{(1)}\left(2, \frac{1}{2}\right)$$

I find the value of $\zeta^{(1)}\left(2, \frac{1}{2}\right)$ with the relation (11) with k=2 and I have $3\zeta^{(1)}(2) + \frac{2\pi^2 \log(2)}{3}$

Finally I have $-6\zeta^{(1)}(2) - \frac{4\pi^2 \log(2)}{3}$

So I can finish the calcul with the trigamma function's rules and I have the closed form of $\zeta^{(1)}\left(-1, \frac{1}{8}\right) + \zeta^{(1)}\left(-1, \frac{3}{8}\right)$

We obtain

$$-\frac{\zeta^{(1)}(-1)}{16} + \frac{\log(2)}{192} + \frac{\sqrt{2} \Psi^{(1)}\left(\frac{1}{8}\right)}{64\pi} - \frac{\pi\sqrt{2}}{32} - \frac{\sqrt{2} K}{4\pi} - \frac{\pi}{32}$$

Hence, with the relation (6), I find $\log(G(3/8))$ in terms of

$$\left(\int_0^{-\frac{1}{8}} \log(\Gamma(t+1)) dt \right)$$

Fourth case

Now I search $\log(G(5/8))$: I repeat the same principle but this time I evaluate the closed form of $\zeta^{(1)}\left(-1, \frac{3}{8}\right) - \zeta^{(1)}\left(-1, \frac{5}{8}\right)$

Finally I have

$$-\frac{K}{8\pi} - \frac{\pi\sqrt{2}}{32} - \frac{\pi}{32} - \frac{\sqrt{2} K}{4\pi} + \frac{\sqrt{2} \Psi^{(1)}\left(\frac{1}{8}\right)}{64\pi}$$

So I have $\log(G(5/8))$ in terms of

$$\left(\int_0^{-\frac{1}{8}} \log(\Gamma(t+1)) dt \right), \Psi^{(1)}\left(\frac{1}{8}\right)$$

6 About the $\log(G(1/10))$, $\log(G(3/10))$, $\log(G(7/10))$ and $\log(G(9/10))$

It's easy: the duplication formula (13) is well-known:

$$G(2z)$$

is equals to

$$e^{-\frac{1}{4}A^3 2^{2z^2-3z+\frac{11}{12}} \pi^{\frac{1}{2}-z}} G(z) G\left(z + \frac{1}{2}\right)^2 G(1+z)$$

where A is the Glaisher-Kinkelin's constant (14)

If $z=3/5$, I have directly $\log(G(1/10))$ in terms of

$$\left(\int_0^{-\frac{1}{5}} \log(\Gamma(t+1)) dt \right), \Psi^{(1)}\left(\frac{1}{5}\right), \Psi^{(1)}\left(\frac{2}{5}\right)$$

If $z=4/5$, I have directly $\log(G(3/10))$ in terms of

$$\left(\int_0^{-\frac{1}{5}} \log(\Gamma(t+1)) dt \right), \Psi^{(1)}\left(\frac{1}{5}\right), \Psi^{(1)}\left(\frac{2}{5}\right)$$

If $z=1/5$, I have directly $\log(G(7/10))$ in terms of

$$\left(\int_0^{-\frac{1}{5}} \log(\Gamma(t+1)) dt \right), \Psi^{(1)}\left(\frac{1}{5}\right), \Psi^{(1)}\left(\frac{2}{5}\right)$$

If $z=2/5$, I have directly $\log(G(9/10))$ in terms of

$$\left(\int_0^{-\frac{1}{5}} \log(\Gamma(t+1)) dt \right), \Psi^{(1)}\left(\frac{1}{5}\right), \Psi^{(1)}\left(\frac{2}{5}\right)$$

7 About the $\log(G(1/12))$, $\log(G(5/12))$, $\log(G(7/12))$ and $\log(G(11/12))$

First case

I find $\log(G(11/12))$ easily: just I use relation (4) with $z=-1/12$ and we obtain $\log(G(11/12))$ in terms of

$$\left(\int_0^{-\frac{1}{12}} \log(\Gamma(t+1)) dt \right)$$

Second case

Now I search $\log(G(1/12))$, I evaluate the closed form of $\zeta^{(1)}(-1, \frac{11}{12}) - \zeta^{(1)}(-1, \frac{1}{12})$

And I use 2 times the Adamchik-Miller's relation (n=1), we have a very long expression but if you factorize the term $\frac{2(-1)^{n+1}(2n-1)!}{(2\pi k)^{2n}}$

And if you consider only the part $\sum_{r=1}^{12-1} \cos(\frac{2\pi r*11}{12}) \zeta^{(1)}(2n, \frac{r}{12}) - (\sum_{r=1}^{12-1} \cos(\frac{2\pi r*1}{12}) \zeta^{(1)}(2n, \frac{r}{12}))$

Directly we have 0

And then the closed form of $\zeta^{(1)}(-1, \frac{11}{12}) - \zeta^{(1)}(-1, \frac{1}{12})$ is

$$\frac{-48K - 3\sqrt{3} \Psi^{(1)}(\frac{1}{3}) + 2\sqrt{3} \pi^2}{144\pi}$$

And I use the relation (6) with $z=1/12$ and $z=11/12$ and I obtain $\log(G(1/12))$ in terms of

$$\left(\int_0^{-\frac{1}{12}} \log(\Gamma(t+1)) dt \right), \Psi^{(1)}\left(\frac{1}{12}\right)$$

If you prefer, we can use the trigamma identity $\Psi^{(1)}(\frac{1}{12}) = 10\Psi^{(1)}(\frac{1}{3}) + 2\pi^2\sqrt{3} - \frac{8\pi^2}{3} + 40K$

Third case

Now I search $\log(G(5/12))$, I evaluate the closed form of $\zeta^{(1)}(-1, \frac{1}{12}) + \zeta^{(1)}(-1, \frac{5}{12})$

And I use 2 times the Adamchik-Miller's relation (n=1), we have a very long expression but if you factorize the term $\frac{2(-1)^{n+1}(2n-1)!}{(2\pi k)^{2n}}$

And if you consider only the part $\sum_{r=1}^{12-1} \cos(\frac{2\pi r*1}{12}) \zeta^{(1)}(2n, \frac{r}{12}) + \sum_{r=1}^{12-1} \cos(\frac{2\pi r*5}{12}) \zeta^{(1)}(2n, \frac{r}{12})$

We have

$$\zeta^{(1)}(2, \frac{1}{6}) - \zeta^{(1)}(2, \frac{1}{3}) - 2\zeta^{(1)}(2, \frac{1}{2}) - \zeta^{(1)}(2, \frac{2}{3}) + \zeta^{(1)}(2, \frac{5}{6})$$

I have successively $\sum_{r=1}^{6-1} \zeta^{(1)}(2n, \frac{r}{6}) = 6\pi^2 \log(2) + 6\pi^2 \log(3) + 35\zeta^{(1)}(2)$

is equals to $\zeta^{(1)}\left(2, \frac{1}{6}\right) + \zeta^{(1)}\left(2, \frac{1}{3}\right) + \zeta^{(1)}\left(2, \frac{1}{2}\right) + \zeta^{(1)}\left(2, \frac{2}{3}\right) + \zeta^{(1)}\left(2, \frac{5}{6}\right)$

And $\zeta^{(1)}\left(2, \frac{1}{6}\right) + \zeta^{(1)}\left(2, \frac{5}{6}\right) = 35\zeta^{(1)}(2) + 6\pi^2 \log(6) - \zeta^{(1)}\left(2, \frac{1}{3}\right) - \zeta^{(1)}\left(2, \frac{1}{2}\right) - \zeta^{(1)}\left(2, \frac{2}{3}\right)$

And $\sum_{r=1}^{3-1} \zeta^{(1)}\left(2n, \frac{r}{3}\right) = 8\zeta^{(1)}(2) + \frac{3\pi^2 \log(3)}{2}$

is equals to $\zeta^{(1)}\left(2, \frac{1}{3}\right) + \zeta^{(1)}\left(2, \frac{2}{3}\right)$

I have $\zeta^{(1)}\left(2, \frac{1}{6}\right) + \zeta^{(1)}\left(2, \frac{5}{6}\right) = \frac{16\pi^2 \log(2)}{3} + \frac{9\pi^2 \log(3)}{2} + 24\zeta^{(1)}(2)$

Finally I have $4\pi^2 \log(2) + 3\pi^2 \log(3) + 10\zeta^{(1)}(2)$

So I can finish the calcul with the trigamma function's rules and I have the closed form of $\zeta^{(1)}\left(-1, \frac{1}{12}\right) + \zeta^{(1)}\left(-1, \frac{5}{12}\right)$

We obtain

$$\frac{\zeta^{(1)}(-1)}{12} + \frac{K}{3\pi} + \frac{\log(3)}{288}$$

Hence, with the relation (6), I find $\log(G(5/12))$ in terms of

$$\left(\int_0^{-\frac{1}{12}} \log(\Gamma(t+1)) dt \right), \Psi^{(1)}\left(\frac{1}{12}\right)$$

Fourth case

Now I search $\log(G(7/12))$: I repeat the same principle but this time I evaluate the closed form of $\zeta^{(1)}\left(-1, \frac{5}{12}\right) - \zeta^{(1)}\left(-1, \frac{7}{12}\right)$

Finally I have

$$-\frac{\sqrt{3} \Psi^{(1)}\left(\frac{1}{3}\right)}{48\pi} + \frac{\pi\sqrt{3}}{72} + \frac{K}{3\pi}$$

So I have $\log(G(7/12))$ in terms of

$$\left(\int_0^{-\frac{1}{12}} \log(\Gamma(t+1)) dt \right)$$

Conclusion:

The 16 formulas Barnes G-function are proved.

I remark that this paper is complementary with my 2 papers "Values of Barnes function" and "Another values of Barnes function and formulas":

In this article, I prove the 16 formulas but I have no information about the closed form of

$$\left(\int_0^{-\frac{1}{12}} \log(\Gamma(t+1)) dt \right)$$

or

$$\left(\int_0^{-\frac{1}{8}} \log(\Gamma(t+1)) dt \right)$$

or

$$\left(\int_0^{-\frac{1}{5}} \log(\Gamma(t+1)) dt \right)$$

In the 2 papers "Values of Barnes function" and "Another values of Barnes function and formulas", I don't prove the formulas but in the same time, I have more information about the integral log gamma but just I can evaluate some terms, hence it isn't sufficient to obtain a final closed form of the integrals.

The priority is to find closed form of three integrals and the trigamma identity $\Psi^{(1)}\left(\frac{2}{5}\right)$ in terms of $\Psi^{(1)}\left(\frac{1}{5}\right)$.

8 References

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