Proof of 16 Formulas Barnes function

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Abstract
I have already published several months ago in the papers "Values of Barnes Function" and "Another Values of Barnes Function and Formulas" in total 16 conjectural formulas that I find with unusual methods. So, in this article, I write the proof of 16 formulas.

1 Definition
The Barnes function is defined as the following Weierstrass product:

\[ G(1 + z) = (2\pi)^{\frac{z}{2}} e^{-\frac{z(1+z)}{2}} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right)^{k} e^{-z + \frac{z^2}{2k}} \]  

(1)

where gamma is the Euler-Mascheroni constant. The following properties of G are well-known.

2 Properties
\[ G(1) = 1 \]  

(2)

\[ G(1 + z) = G(z) \Gamma(z) \]  

(3)

\[ \log(G(1 + z)) = \frac{z \log(2\pi)}{2} - \frac{z(1 + z)}{2} + z \log(\Gamma(1 + z)) - \int_{0}^{z} \log(\Gamma(t + 1)) \, dt \]  

(4)

\[ \int_{0}^{z} \log(\Gamma(t + 1)) \, dt = \frac{z \log(2\pi)}{2} - \frac{z(1 + z)}{2} + z \log(\Gamma(1 + z)) - \log(G(z)) - \log(\Gamma(z)) \]  

(5)

3 Introduction
I need 5 relations:
\[ \zeta^{(1)}(-1, z) = \zeta^{(1)}(-1) - \log(G(z)) + (z - 1) \log(\Gamma(z)) \quad (6) \]

where \( \zeta^{(1)}(-1, z) \) is the first derivative of Hurwitz Zeta at \( z \). (\( z \) is a positive real)

The Adamchik-Miller’s relation (7) for \( \zeta^{(1)}(1 - 2n, \frac{h}{k}) \):

\[
\frac{(\psi(2n) - \log(2\pi k)) B_{2n}(h/k)}{2n} \cdot \frac{(\psi(2n) - \log(2\pi)) B_{2n}}{2n k^{2n}} + \frac{(-1)^{n+1} \pi}{(2\pi k)^{2n}} \sum_{r=1}^{k-1} \sin \left( \frac{2\pi rh}{k} \right) \psi^{(2n-1)} \left( \frac{r}{k} \right)
\]

\[ + \frac{2 (-1)^{n+1} (2n - 1)!}{(2\pi k)^{2n}} \sum_{r=1}^{k-1} \cos \left( \frac{2\pi rh}{k} \right) \zeta^{(1)} \left( 2n, \frac{r}{k} \right) + \frac{\zeta^{(1)}(1 - 2n)}{k^{2n}} \]

where

\( B_{2n}(h/k) \) is Bernoulli polynomial at \( h/k \) \quad (8).

Here \( h \) and \( k \) both positive integer.

\( B_{2n} \) is Bernoulli numbers. \quad (9)

\( \psi^{(2n-1)} \left( \frac{r}{k} \right) \) is the polygamma function order \( 2n-1 \) at \( r/k \).

But here, in this study, \( n=1 \) and just we have the trigamma function. \quad (10)

The Adamchik-Miller’s relation is very complicated but I remark with this formula, I can make connection between two Barnes G-Function or if I use the relation (6) a connection between two expressions of first derivative of Hurwitz Zeta. Of course, we must choose two parameters \( a \) and \( b \) correctly.

So the principle is simple: just I evaluate closed form of \( \zeta^{(1)}(-1, a) + \zeta^{(1)}(-1, b) \) or \( \zeta^{(1)}(-1, a) - \zeta^{(1)}(-1, b) \) and in particular I can evaluate the complicated second sum.

The relation

\[ \sum_{r=1}^{k-1} \zeta^{(1)} \left( s, \frac{r}{k} \right) \]

And we know that this sum = \( \zeta^{(1)}(s) (k^s - 1) + k^s \zeta(s) \log(k) \) \quad (11)
Here, in this study, \( s = 2 \).

Remember the value: 
\[
\zeta^{(1)}(2) = \frac{\pi^2 \gamma}{6} + \frac{\pi^2 \log(2)}{6} - \frac{\pi^2}{6} + 2\pi^2 \zeta^{(1)}(-1) + \frac{\pi^2 \log(\pi)}{6}
\]

The integral
\[
\int_0^z \pi t \cot(\pi t) \, dt
\]
And we know that this integral = \( z \log(2\pi) + \log \left( \frac{\Gamma(1-z)}{\Gamma(1+z)} \right) \)  

(12)

integral originally due to Kinkelin.

4 About the \( \log(G(1/5)), \log(G(2/5)), \log(G(3/5)) \) and \( \log(G(4/5)) \)

First case

I find \( \log(G(4/5)) \) easily: just I use relation (4) with \( z = -1/5 \) and we obtain \( \log(G(4/5)) \) in terms of

\[
\left( \int_0^{-\frac{1}{5}} \log(\Gamma(t+1)) \, dt \right)
\]

Second case

Now I find \( \log(G(1/5)) \) with the Kinkelin’s integral: the software MAPLE give me a closed form in terms of complex dilogarithm and I can simplify the expression in terms of trigamma function and we have

\[
\int_0^{\frac{1}{5}} \pi t \cot(\pi t) \, dt
\]
equals to

\[
\frac{\log(2)}{10} + \frac{\log(5)}{20} - \frac{\log(\sqrt{5} + 1)}{10} + \frac{(5 + \sqrt{5}) \Psi^{(1)}(\frac{1}{5}) + 2\sqrt{5} \Psi^{(1)}(\frac{2}{5}) - 4\pi^2 (\sqrt{5} + 1)}{50\pi \sqrt{10 + 2\sqrt{5}}}
\]

And I use the relation (12) with \( z = 1/5 \) and I obtain \( \log(G(1/5)) \) in terms of

\[
\left( \int_0^{-\frac{1}{5}} \log(\Gamma(t+1)) \, dt \right), \Psi^{(1)}\left(\frac{1}{5}\right), \Psi^{(1)}\left(\frac{2}{5}\right)
\]
**Third case**

Now I search $\log(G(2/5))$, I evaluate the closed form of $\zeta^{(1)}(-1, \frac{1}{5}) + \zeta^{(1)}(-1, \frac{2}{5})$

And I use 2 times the Adamchik-Miller’s relation ($n=1$), we have a very long expression but if you factorize the term $\frac{2(-1)^{n+1}(2n-1)!}{(2\pi)^{2n}}$

And if you consider only the part $\sum_{r=1}^{5-1} \cos\left(\frac{2\pi r}{5}\right) \zeta^{(1)}(2r, \frac{1}{5}) + \sum_{r=1}^{5-1} \cos\left(\frac{2\pi r}{5}\right) \zeta^{(1)}(2r, \frac{2}{5})$

We have

$\cos\left(\frac{2\pi}{5}\right) \zeta^{(1)}\left(2, \frac{1}{5}\right) - \cos\left(\frac{4\pi}{5}\right) \zeta^{(1)}\left(2, \frac{2}{5}\right) - \cos\left(\frac{6\pi}{5}\right) \zeta^{(1)}\left(2, \frac{3}{5}\right) - \cos\left(\frac{8\pi}{5}\right) \zeta^{(1)}\left(2, \frac{4}{5}\right) - \cos\left(\frac{2\pi}{5}\right) \zeta^{(1)}\left(2, \frac{1}{5}\right) + \cos\left(\frac{4\pi}{5}\right) \zeta^{(1)}\left(2, \frac{2}{5}\right) + \cos\left(\frac{6\pi}{5}\right) \zeta^{(1)}\left(2, \frac{3}{5}\right) + \cos\left(\frac{8\pi}{5}\right) \zeta^{(1)}\left(2, \frac{4}{5}\right)$

I can simplify

$-\frac{\zeta^{(1)}\left(2, \frac{1}{5}\right)}{2} - \frac{\zeta^{(1)}\left(2, \frac{2}{5}\right)}{2} - \frac{\zeta^{(1)}\left(2, \frac{3}{5}\right)}{2} - \frac{\zeta^{(1)}\left(2, \frac{4}{5}\right)}{2}$

And now I use the relation (11) and finally

$-12\zeta^{(1)}(2) - \frac{25\pi^2 \log(5)}{12}$

So I can finish the calcul with the trigamma function’s rules and I have the closed form of $\zeta^{(1)}(-1, \frac{1}{5}) + \zeta^{(1)}(-1, \frac{2}{5})$

We obtain

$-\frac{2\zeta^{(1)}(-1)}{5} - \frac{\log(5)}{120} + \frac{5 + 3\sqrt{5}}{120} \Psi^{(1)}\left(\frac{1}{5}\right) + \frac{-5 + \sqrt{5}}{100\sqrt{10} + 2\sqrt{5}\pi} \Psi^{(1)}\left(\frac{2}{5}\right) - \pi^2 \left(\frac{6\sqrt{5} + 2}{100\sqrt{10} + 2\sqrt{5}\pi}\right)$

Hence, with the relation (6), I find $\log(G(2/5))$ in terms of

$\left(\int_{0}^{-\frac{1}{5}} \log(\Gamma(t+1)) \, dt\right), \Psi^{(1)}\left(\frac{1}{5}\right), \Psi^{(1)}\left(\frac{2}{5}\right)$

**Fourth case**

Now I search $\log(G(3/5))$: I repeat the same principle but this time I evaluate the closed form of $\zeta^{(1)}(-1, \frac{2}{5}) - \zeta^{(1)}(-1, \frac{3}{5})$
Finally I have
\[
\frac{(-\sqrt{5} - 5) \Psi^{(1)}\left(\frac{2}{5}\right) + 2\sqrt{5} \Psi^{(1)}\left(\frac{1}{5}\right) - 2\pi^2 (\sqrt{5} - 1)}{50\sqrt{10} + 2\sqrt{5} \pi}
\]

So I have \(\log(G(3/5))\) in terms of
\[
\left( \int_{0}^{-\frac{1}{5}} \log(\Gamma(t + 1)) \, dt \right), \Psi^{(1)}\left(\frac{1}{5}\right), \Psi^{(1)}\left(\frac{2}{5}\right)
\]

5 About the \(\log(G(1/8)), \log(G(3/8)), \log(G(5/8))\) and \(\log(G(7/8))\)

First case

I find \(\log(G(7/8))\) easily: just I use relation (4) with \(z=-1/8\) and we obtain \(\log(G(7/8))\) in terms of
\[
\left( \int_{0}^{-\frac{1}{8}} \log(\Gamma(t + 1)) \, dt \right)
\]

Second case

Now I find \(\log(G(1/8))\) with the Kinkelin’s integral: the software MAPLE give me a closed form in terms of complex dilogarithm and I can simplify the expression in terms of trigamma function and we have
\[
\int_{0}^{\frac{1}{8}} \pi t \cot(\pi t) \, dt
\]
equals to
\[
-\frac{\sqrt{2} K}{4\pi} + \frac{K}{8\pi} - \frac{\pi}{32} - \frac{\pi\sqrt{2}}{32} + \frac{\sqrt{2} \Psi^{(1)}\left(\frac{1}{8}\right)}{64\pi} + \frac{\log(2)}{32} - \frac{\log(1 + \sqrt{2})}{16}
\]
where \(K\) is the Catalan’s constant. (13)

And I use the relation (12) with \(z=1/8\) and I obtain \(\log(G(1/8))\) in terms of
\[
\left( \int_{0}^{-\frac{1}{8}} \log(\Gamma(t + 1)) \, dt \right), \Psi^{(1)}\left(\frac{1}{8}\right)
\]

Third case

Now I search \(\log(G(3/8))\), I evaluate the closed form of \(\zeta^{(1)}(1, -1, \frac{1}{8})\) +
\(\zeta^{(1)}(-1, \frac{3}{8})\)

5
And I use 2 times the Adamchik-Miller’s relation (n=1), we have a very long expression but if you factorize the term $\frac{2(-1)^{n+1}(2n-1)!}{(2\pi k)^{2n}}$

And if you consider only the part $\sum_{r=1}^{8-1} \cos(\frac{2\pi r}{8}) \zeta^{(1)}(2n, \frac{r}{8}) + \sum_{r=1}^{8-1} \cos(\frac{2\pi r}{8}) \zeta^{(1)}(2n, \frac{r}{8})$

We have

$-2\zeta^{(1)}(2, \frac{1}{8})$

I find the value of $\zeta^{(1)}(2, \frac{1}{8})$ with the relation (11) with k=2 and I have $3\zeta^{(1)}(2) + \frac{2\pi^2 \log(2)}{3}$

Finally I have $-6\zeta^{(1)}(2) - \frac{4\pi^2 \log(2)}{3}$

So I can finish the calcul with the trigamma function’s rules and I have the closed form of $\zeta^{(1)}(-1, \frac{1}{8}) + \zeta^{(1)}(-1, \frac{3}{8})$

We obtain

$$-\frac{\zeta^{(1)}(-1)}{16} + \frac{\log(2)}{192} + \frac{\sqrt{2} \Psi^{(1)}(\frac{1}{8})}{64\pi} - \frac{\pi \sqrt{2}}{32} - \frac{\sqrt{2} K}{4\pi} - \frac{\pi}{32}$$

Hence, with the relation (6), I find log(G(3/8)) in terms of

$$\left(\int_{0}^{\frac{1}{8}} \log(\Gamma(t+1)) \, dt\right)$$

**Fourth case**

Now I search log(G(5/8)): I repeat the same principle but this time I evaluate the closed form of $\zeta^{(1)}(-1, \frac{3}{8}) - \zeta^{(1)}(-1, \frac{5}{8})$

Finally I have

$$-\frac{K}{8\pi} - \frac{\pi \sqrt{2}}{32} - \frac{\pi}{32} - \frac{\sqrt{2} K}{4\pi} + \frac{\sqrt{2} \Psi^{(1)}(\frac{1}{8})}{64\pi}$$

So I have log(G(5/8)) in terms of

$$\left(\int_{0}^{\frac{1}{8}} \log(\Gamma(t+1)) \, dt\right), \psi^{(1)}(\frac{1}{8})$$
6 About the log(G(1/10)), log(G(3/10)),
log(G(7/10)) and log(G(9/10))

It’s easy: the duplication formula (14) is well-known:

\[ G(2z) \]

is equals to

\[ e^{-\frac{1}{4}} A^{32z^2 - 3z + \frac{1}{12}} \pi^{\frac{1}{2}} z^2 G\left(z + \frac{1}{2}\right)^2 G(1 + z) \]

where A is the Glaisher-Kinkelin constant’s \(^{(15)}\)

If \(z=3/5\), I have directly log(G(1/10)) in terms of

\[ \left( \int_{0}^{\frac{1}{5}} \log(\Gamma(t + 1)) \, dt \right), \psi^{(1)}\left(\frac{1}{5}\right), \psi^{(1)}\left(\frac{2}{5}\right) \]

If \(z=4/5\), I have directly log(G(3/10)) in terms of

\[ \left( \int_{0}^{\frac{1}{5}} \log(\Gamma(t + 1)) \, dt \right), \psi^{(1)}\left(\frac{1}{5}\right), \psi^{(1)}\left(\frac{2}{5}\right) \]

If \(z=1/5\), I have directly log(G(7/10)) in terms of

\[ \left( \int_{0}^{\frac{1}{5}} \log(\Gamma(t + 1)) \, dt \right), \psi^{(1)}\left(\frac{1}{5}\right), \psi^{(1)}\left(\frac{2}{5}\right) \]

If \(z=2/5\), I have directly log(G(9/10)) in terms of

\[ \left( \int_{0}^{\frac{1}{5}} \log(\Gamma(t + 1)) \, dt \right), \psi^{(1)}\left(\frac{1}{5}\right), \psi^{(1)}\left(\frac{2}{5}\right) \]

7 About the log(G(1/12)), log(G(5/12)),
log(G(7/12)) and log(G(11/12))

First case

I find log(G(11/12)) easily: just I use relation (4) with \(z=-1/12\) and we obtain log(G(11/12)) in terms of

\[ \left( \int_{0}^{\frac{1}{12}} \log(\Gamma(t + 1)) \, dt \right) \]
Second case

Now I find $\log(G(1/12))$ with the Kinkelin’s integral: the software MAPLE give me a closed form in terms of complex dilogarithm and I can simplify the expression in terms of trigamma function and we have

$$
\int_0^{\frac{1}{12}} \pi \cot(\pi t) \, dt
$$
equals to

$$\sqrt{3} \psi^{(1)}\left(\frac{1}{12}\right) - \frac{\pi \sqrt{3}}{72} + \frac{K}{3\pi} - \frac{\log(1 + \sqrt{3})}{12} + \frac{\log(2)}{24}
$$

And I use the relation (12) with $z=1/12$ and I obtain $\log(G(1/12))$ in terms of

$$Z_{1/12} - \log(\Gamma(t+1)) + \psi^{(1)}\left(\frac{1}{12}\right)
$$

If you prefer, we can use the trigamma identity $\psi^{(1)}\left(\frac{1}{12}\right) = 10\psi^{(1)}\left(\frac{1}{3}\right) + 2\pi^2 \sqrt{3} - \frac{8\pi^2}{3} + 40K$

Third case

Now I search $\log(G(5/12))$, I evaluate the closed form of $\zeta^{(1)}(-1, \frac{1}{12}) + \zeta^{(1)}(-1, \frac{5}{12})$

And I use 2 times the Adamchik-Miller’s relation $(n=1)$, we have a very long expression but if you factorize the term $\frac{2(-1)^n + (2n-1)!}{(2\pi k)^{2n}}$

And if you consider only the part $\sum_{r=1}^{12-1} \cos\left(\frac{2\pi r+1}{12}\right) \zeta^{(1)}\left(2n, \frac{r}{12}\right) + \sum_{r=1}^{12-1} \cos\left(\frac{2\pi r+5}{12}\right) \zeta^{(1)}\left(2n, \frac{r}{12}\right)$

We have

$$\zeta^{(1)}\left(2, \frac{1}{6}\right) - \zeta^{(1)}\left(2, \frac{1}{3}\right) - 2\zeta^{(1)}\left(2, \frac{1}{2}\right) - \zeta^{(1)}\left(2, \frac{2}{3}\right) + \zeta^{(1)}\left(2, \frac{5}{6}\right)
$$

I have successively $\sum_{r=1}^{6-1} \zeta^{(1)}\left(2n, \frac{r}{6}\right) = 6\pi^2 \log(2) + 6\pi^2 \log(3) + 35\zeta^{(1)}(2)$

is equals to $\zeta^{(1)}\left(2, \frac{1}{6}\right) + \zeta^{(1)}\left(2, \frac{1}{3}\right) + \zeta^{(1)}\left(2, \frac{1}{2}\right) + \zeta^{(1)}\left(2, \frac{2}{3}\right) + \zeta^{(1)}\left(2, \frac{5}{6}\right)$

And $\zeta^{(1)}\left(2, \frac{1}{6}\right) + \zeta^{(1)}\left(2, \frac{5}{6}\right) = 35\zeta^{(1)}(2) + 6\pi^2 \log(6) - \zeta^{(1)}\left(2, \frac{1}{2}\right) - \zeta^{(1)}\left(2, \frac{2}{3}\right)$

And $\sum_{r=1}^{3-1} \zeta^{(1)}\left(2n, \frac{r}{3}\right) = 8\zeta^{(1)}(2) + \frac{3\pi^2 \log(3)}{2}$
is equals to $\zeta^{(1)}(2, \frac{1}{3}) + \zeta^{(1)}(2, \frac{2}{3})$

I have $\zeta^{(1)}(2, \frac{1}{3}) + \zeta^{(1)}(2, \frac{2}{3}) = \frac{16\pi^2 \log(2)}{3} + \frac{9\pi^2 \log(3)}{2} + 24\zeta^{(1)}(2)$

Finally I have $4\pi^2 \log(2) + 3\pi^2 \log(3) + 10\zeta^{(1)}(2)$

So I can finish the calculus with the trigamma function’s rules and I have the closed form of $\zeta^{(1)}(-1, \frac{1}{12}) + \zeta^{(1)}(-1, \frac{7}{12})$

We obtain

$$\frac{\zeta^{(1)}(-1)}{12} + \frac{K}{3\pi} + \frac{\log(3)}{288}$$

Hence, with the relation (6), I find $\log(G(5/12))$ in terms of

$$\left(\int_0^{-\frac{1}{12}} \log(\Gamma(t + 1)) \, dt\right), \psi^{(1)}\left(\frac{1}{12}\right)$$

**Fourth case**

Now I search $\log(G(7/12))$: I repeat the same principle but this time I evaluate the closed form of $\zeta^{(1)}(-1, \frac{7}{12}) - \zeta^{(1)}(-1, \frac{5}{12})$

Finally I have

$$-\frac{\sqrt{3}\psi^{(1)}(\frac{1}{3})}{48\pi} + \frac{\sqrt{3}}{72} + \frac{K}{3\pi}$$

So I have $\log(G(7/12))$ in terms of

$$\left(\int_0^{-\frac{1}{12}} \log(\Gamma(t + 1)) \, dt\right)$$

**Conclusion:**

The 16 formulas Barnes G-function are proved.

I remark that this paper is complementary with my 2 papers "Values of Barnes function" and "Another values of Barnes function and formulas":
In this article, I prove the 16 formulas but I have no information about the closed form of
\[
\left( \int_{0}^{-\frac{1}{2}} \log(\Gamma(t + 1)) \, dt \right)
\]
or
\[
\left( \int_{0}^{-\frac{1}{8}} \log(\Gamma(t + 1)) \, dt \right)
\]
or
\[
\left( \int_{0}^{-\frac{1}{5}} \log(\Gamma(t + 1)) \, dt \right)
\]

In the 2 papers ”Values of Barnes function” and ”Another values of Barnes function and formulas”, I don’t prove the formulas but in the same time, I have more information about the integral log gamma but just I can evaluate some terms, hence it isn’t sufficient to obtain a final closed form of the integrals.

The priority is to find closed form of three integrals and the trigamma identity \( \Psi^{(1)} \left( \frac{1}{2} \right) \) in terms of \( \Psi^{(1)} \left( \frac{1}{5} \right) \).

8 References


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