Null Algebra

Extension 1

Robert S Miller, rmille4612@hotmail.com

Resolving instances of the feedback value $\frac{0}{0}$ in polynomial fractions with radicals

1.1

When evaluating an expression which contains a radical which goes to the feedback value $\frac{0}{0}$, as the input approaches a specific value you are normally taught to use the conjugate method to evaluate the expression.

Consider the following equation:

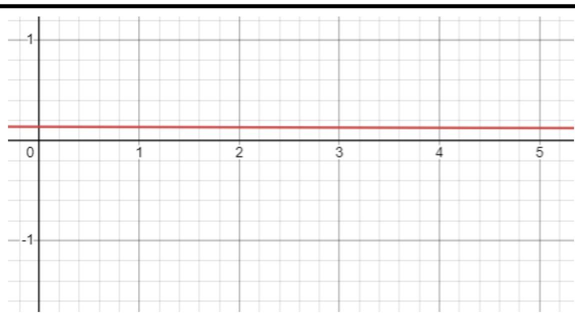
1.1.a

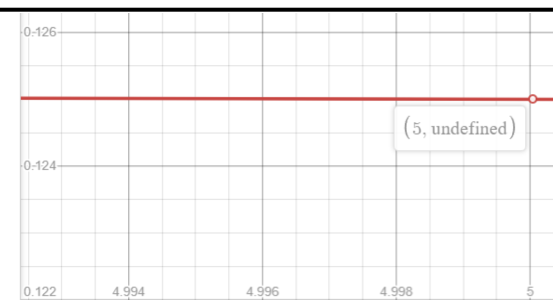
$$\frac{\sqrt{x+11}-4}{x-5}$$

If you attempt direct substitution for the value of $x=5$ the equation will go to $\frac{0}{0}$. Yet the value appears to approach $\frac{1}{8}$.

1.1.b

$$\lim_{x \to 5} \frac{\sqrt{x+11}-4}{x-5} \quad = \quad \frac{1}{8} \quad = \quad 0.125$$





The standard method of completing the evaluation is to multiply the numerator and the denominator by the complex conjugate of the radial expression. In this case that value is $\frac{\sqrt{x+11}+4}{\sqrt{x+11}+4}$. This is equivalent to multiplying the equation by 1 and so does not change its value.

$$\frac{\sqrt{x+11}-4}{x-5} \cdot \frac{\sqrt{x+11}+4}{\sqrt{x+11}+4} \quad = \quad \frac{x+11-16}{(x-5)\left(\sqrt{x+11}+4\right)} \quad = \quad \frac{1}{\left(\sqrt{x+11}+4\right)}$$

Evaluating the limit as x approaches 5 will provide $\frac{1}{8}$

1.1.c

$$\lim_{x \to 5} \frac{1}{\sqrt{x+11}+4} = \frac{1}{8}$$

If you examine the two graphs you'll find that: $\frac{\sqrt{x+11}-4}{x-5} \equiv \frac{1}{\sqrt{x+11}+4}$

From the perspective of Null Algebra we refer to such expressions as *Reciprocal-Conjugates*. The polynomial component $x-5$ in this example is a key element, referred to as the *Supplement.* When you see expressions of the form:

1.1.d

$$\lim_{x \to d} \frac{\sqrt{x \pm a}-b}{x \pm c} = \frac{0}{0} \text{ or } \lim_{x \to d} \frac{x \pm c}{\sqrt{x \pm a}-b} = \frac{0}{0}$$ where a, b, c and d are of the set of $\mathbb{R}$

there shall exist an equivalent expression which is obtained by substituting $(x \pm c)=1$ and then reciprocating the polynomial containing the radical, replacing it with its complex conjugate. See the following example expressions below:

1.1.e

$$If \ \lim_{x \to d} \frac{\sqrt{x+a}+b}{x+c} = \frac{0}{0} \ then \ \frac{\sqrt{x+a}+b}{x+c} \equiv \frac{1}{\sqrt{x+a}-b}$$

$$If \ \lim_{x \to d} \frac{x+c}{\sqrt{x+a}+b} = \frac{0}{0} \ then \ \frac{x+c}{\sqrt{x+a}+b} \equiv \sqrt{x+a}-b$$

This is a nice feature to know but we should also be able to replace the appropriate instances of the expression which are generating the feedback value with *null math naught* η_0. To do that you must be able to clearly identify which linear elements are responsible for creating its occurrence. If you attempt the application of η_0 for the input elements in the below example you will not obtain the correct result.

1.1.f

For: $\lim_{x \to 5} \frac{\sqrt{x+11}-4}{x-5} = \frac{1}{8} = 0.125$

$$\frac{\sqrt{x+11}-4}{x-5} \qquad \frac{\sqrt{\eta_0+11}-4}{\eta_0-5} = \frac{\sqrt{11}-4}{-5} = \frac{3.3166-4}{-5} \approx 0.1367 \neq 0.125$$

Clearly these inputs are not the values responsible for the creation of the feedback value $\frac{0}{0}$. Some other form of the equation must exist which contains a different form of the inputs. This form must already be present in the equation and simply need obtained via some simplification. Indeed that is the case here. Though it is not immediately noticeable the denominator is actually a quadratic and the numerator a simple polynomial. In order to simplify the expression we will use *u*-substitution on the radical.

1.1.g

For: $\frac{\sqrt{x+11}-4}{x-5}$ let: $u=\sqrt{x+11} \longrightarrow x=u^2-11$

$$\frac{\sqrt{x+11}-4}{x-5} \rightarrow \frac{u-4}{u^2-11-5} \rightarrow \frac{u-4}{u^2-16} \rightarrow \frac{u-4}{(u+4)(u-4)}$$

The solving method from here forward is identical to that in Null Algebra: The Math of Division by Zero and The Negative Radical section 2.c.6 Factoring, pages 130 to 132 (*https://vixra.org/abs/2103.0131*). We may now apply the instances of η_0 to the input values responsible for creating the occurrence of the feedback value $\frac{0}{0}$.

1.1.h

$$\frac{u-4}{(u+4)(u-4)} \longrightarrow \frac{\eta_0-4}{(u+4)(\eta_0-4)} \longrightarrow \frac{-4}{(\sqrt{x+11}+4)(-4)} \longrightarrow \frac{1}{\sqrt{x+11}+4}$$

As detailed in Null Algebra: The Math of Division by Zero and The Negative Radical this applies to instances where there are no canceling terms between the numerator and denominator as well. It only happens that in this example there exists canceling terms that accept η_0. The substitution process shows that not only was $\frac{\sqrt{x+11}-4}{x-5}$ un-simplified to but it is in fact identical to $\frac{1}{\sqrt{x+11}+4}$. The radical component, $\sqrt{x+a}=u$ will always be the point for substitution when dealing with polynomial fractions containing a radical with input values in it domain resulting in the feedback value. Further it shouldn't surprise you this ends up as a fraction containing a quadratic.

1.1.i

$$\frac{\sqrt{x+11}-4}{x-5}=\frac{u-4}{u^2-16}$$

The very fact this example contains a radical expression $\sqrt{x+11}$ implies that whatever is inside the radical bar is squared regardless of form it takes. If something is squared in an equation it contains a quadratic element. For Null Algebra naught substitutions to work in instances which generate $\frac{0}{0}$ the various elements need to be simplified to their linear components. Until this is done you cannot correctly identify which elements are producing the occurrence of the feedback value.

1.2—Consider the following additional example:

1.2.a

$$\lim_{x\to -10} \frac{x+10}{\sqrt{x+14}-2} = 4$$

A direct substitution of $x=-10$ will result in $\frac{0}{0}$. The substitutions discussed here will show that

1.2.b

$f(x)=\frac{x+10}{\sqrt{x+14}-2}$ is a *reciprocal-conjugate* equivalent to $\sqrt{x+14}+2$.

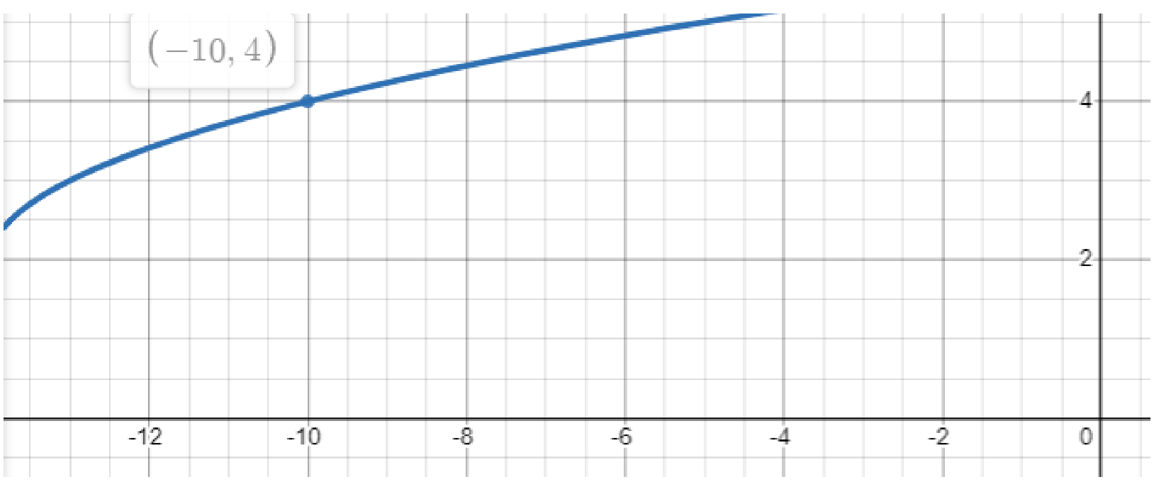
Let $u=\sqrt{x+14}$ and thereby $x=u^2-14$

The equation becomes:

1.2.c

$$\frac{x+10}{\sqrt{x+14}-2} = \frac{(u^2-14)+10}{u-2} = \frac{u^2-4}{u-2} = \frac{(u+2)(u-2)}{u-2} = u+2 = \sqrt{x+14}+2$$

You can clearly see that the function actually equals 4 when at $x=-10$. Its also clear that, although there are canceling terms, the usage of null math naught is still applicable and will result in the same solution.



1.3—Here are two more examples:

1) Single radical in the numerator

1.3.a

$$\frac{\sqrt{x+33}-5}{x+8} \qquad u=\sqrt{x+33} \qquad x=u^2-33$$

For $x=-8$ $$\frac{\sqrt{x+33}-5}{x+8} \longrightarrow \frac{u-5}{u^2-33-8} \longrightarrow \frac{u-5}{u^2-25} \longrightarrow \frac{u-5}{(u+5)(u-5)}$$

Resolves to: $$\frac{1}{u+5} \qquad \frac{1}{\sqrt{x+33}+5}$$

1.3.b

for: $\lim\limits_{x\to-8}\dfrac{\sqrt{x+33}-5}{x+8} \equiv \lim\limits_{x\to-8}\dfrac{1}{\sqrt{x+33}+5} = \dfrac{1}{10}$

2) Single radical in the denominator

1.3.c

$$\frac{x-2}{\sqrt{x+2}-2} \qquad u=\sqrt{x+2} \qquad x=u^2-2$$

1.3.d

For $x=2$ $$\frac{x-2}{\sqrt{x+2}-2} = \frac{u^2-2-2}{u-2} = \frac{u^2-4}{u-2} = \frac{(u+2)(u-2)}{u-2} = u+2$$

$$= \sqrt{x+2}+2$$

2.1—Reciprocal-Conjugate polynomial fractions with Radicals in the Numerator and Denominator:
First consider the following limit:

2.1.a

$$\lim_{x\to 9}\frac{\sqrt{x+7}-4}{\sqrt{x+40}-7}$$

When $x=9$ the function equal $\frac{0}{0}$. Its obvious there should be something that cancels with substitutions on the radicals, but which radical do you use for the substitution? Regardless of which one you pick you'll be left with things which cannot be canceled, you'll be unable to identify the linear polynomial components which are responsible for the instance of $\frac{0}{0}$. It may not be obvious at first glance but this implies a one-to-one ratio exists in this expression which is also going to $\frac{0}{0}$ when $x=9$, but isn't written in the given equation above, *2.1.a*. It is these components which are necessary to allow substitutions with the polynomial components containing radicals. In expressions that look like this example the value of the input which causes the feedback value is the clue. If the value $x=a$ produces an output value of $\frac{0}{0}$ then you will need to multiply the numerator and denominator each by $(x-a)$which is the *Supplement* for both the numerator and the denominator. Doing this you see immediately why its necessary and how it will permit the substitution process to get to the solution.

2.1.b

$$\frac{(x-9)\sqrt{x+7}-4}{(x-9)\sqrt{x+40}-7}=\frac{(x-9)}{\sqrt{x+40}-7}\cdot\frac{\sqrt{x+7}-4}{(x-9)}$$

Before going further with this solution by the substitution method lets solve it in the traditional algebraic way. Besides it should be obvious how the substitution method will end as both pieces on the right side of the equation are reciprocal-conjugates.

Traditional algebra requires you to multiply the numerator and then denominator by the complex conjugate of first the numerator, and then the denominator in a special way. We begin by addressing the numerator complex conjugate.

2.1.c

$$\frac{\sqrt{x+7}-4}{\sqrt{x+40}-7}\cdot\frac{\sqrt{x+7}+4}{\sqrt{x+7}+4}$$

The process now is to multiply out the numerator but leave the denominator completely untouched.

2.1.d

$$\frac{x+7-16}{\sqrt{x+40}-7}\cdot\frac{1}{\sqrt{x+7}+4}=\frac{x-9}{\sqrt{x+40}-7}\cdot\frac{1}{\sqrt{x+7}+4}$$

Now the numerator and denominator are multiplied by the complex conjugate of the denominator. You will then multiply out the denominator but leave the numerator alone.

2.1.d

$$\frac{x-9}{\sqrt{x+40}-7}\cdot\frac{1}{\sqrt{x+7}+4}\cdot\frac{\sqrt{x+40}+7}{\sqrt{x+40}+7} = \frac{x-9}{x+40-49}\cdot\frac{\sqrt{x+40}+7}{\sqrt{x+7}+4}$$

$$= \frac{(x-9)\sqrt{x+40}+7}{(x-9)\sqrt{x+7}+4}$$

$$= \frac{\sqrt{x+40}+7}{\sqrt{x+7}+4}$$

2.2—This process is a little complicated. Observe now we shall return to the place we left off with the Null Algebra process above. The function *f*(*x*) goes to $\frac{0}{0}$ when $x=9$. Because both the numerator and denominator contain a radical polynomial we first multiply the numerator and denominator by x minus the value which generates the feedback value. In this example that is $(x-9)$. We can then separate out the individual reciprocal-conjugates and address them separately.

2.2.a

$$\frac{(x-9)\sqrt{x+7}-4}{(x-9)\sqrt{x+40}-7} = \frac{(x-9)}{\sqrt{x+40}-7}\cdot\frac{\sqrt{x+7}-4}{(x-9)}$$

From here we can easily perform substitution on each of these reciprocal-conjugates on the right side of the equation, and then multiply them together.

2.2.b

For $\frac{(x-9)}{\sqrt{x+40}-7}$ Let: $u=\sqrt{x+40}$ $x=u^2-40$

Then: $\frac{(x-9)}{\sqrt{x+40}-7} = \frac{u^2-40-9}{u-7} = \frac{u^2-49}{u-7} = \frac{(u-7)(u+7)}{u-7}$

$u+7= \sqrt{x+40}+7$

2.2.c

For $\frac{\sqrt{x+7}-4}{(x-9)}$ Let: $u=\sqrt{x+7}$ $x=u^2-7$

Then: $\frac{\sqrt{x+7}-4}{(x-9)} = \frac{u-4}{u^2-7-9} = \frac{u-4}{u^2-16} = \frac{u-4}{(u+4)(u-4)}$

$\frac{1}{u+4} = \frac{1}{\sqrt{x+7}+4}$

They are then simply multiplied together:

2.2.d

$$\sqrt{x+40}+7\cdot\frac{1}{\sqrt{x+7}+4} = \frac{\sqrt{x+40}+7}{\sqrt{x+7}+4}$$

This provides that:

2.2.e

$$\lim_{x\to 9}\frac{\sqrt{x+7}-4}{\sqrt{x+40}-7} \equiv \lim_{x\to 9}\frac{\sqrt{x+40}+7}{\sqrt{x+7}+4} = 1.75$$

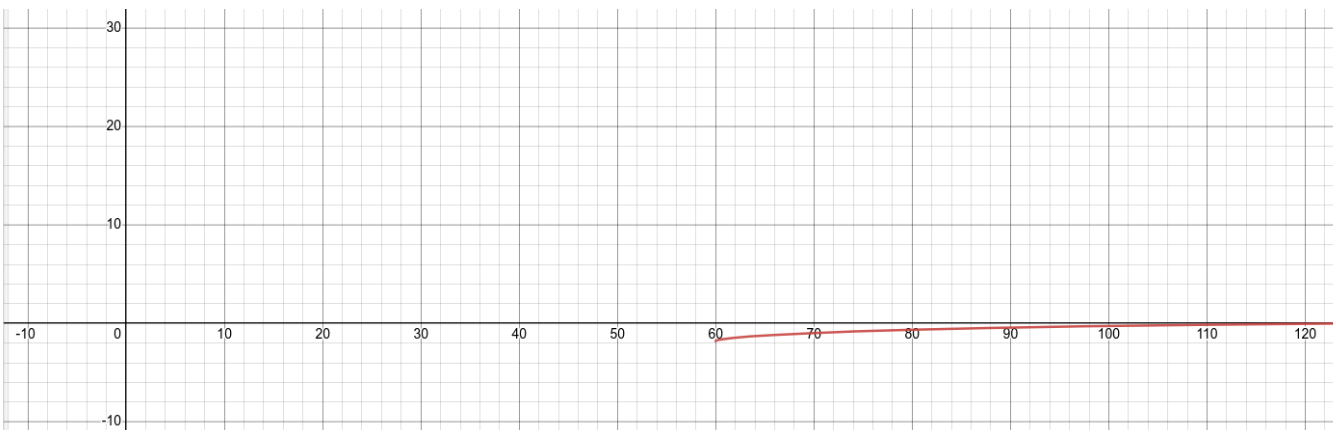
3.1—This procedure is usable in instances which involve the occurrence of the feedback value $\frac{0}{0}$ after the resolution of negative roots. Observe the following example:

Negative Roots Example 1:

3.1.a

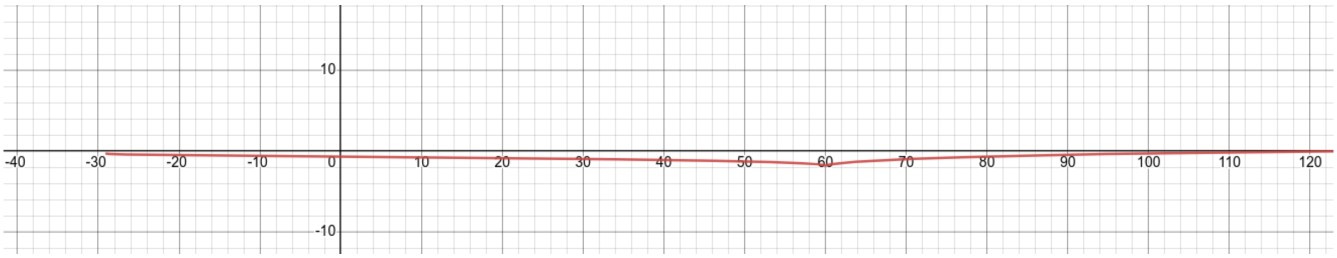
$$f(x)=\frac{\sqrt{x-60}-8}{\sqrt{x+29}-5}$$

We'll first examine the various aspects of this equation within the auspices of traditional mathematics. The Range is $\mathbb{R}$ but the domain is limited. The domain for the input is $x\geq 60$. At any value lower the entire function becomes unusable within traditional algebra as the numerator would then contain negative arguments inside the radical bar. The function is graphed below.

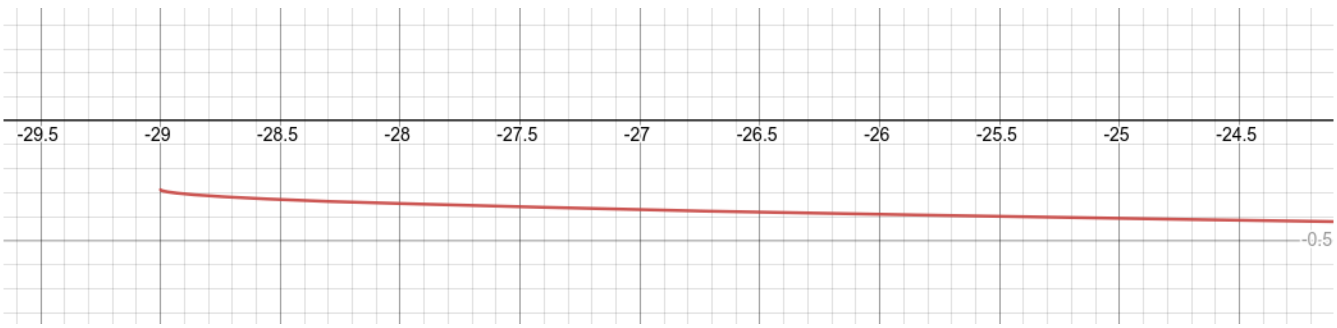


The extended domain will have to include values which generate negative roots. Before looking at them consider the traditional domain does not include any instance which would generate the feedback value. That means for this section of the graph there are no holes, no discontinuities and no exotic resolutions for output values. This means the methods for solving and obtaining the actual value reached for the output at the place of the $\frac{0}{0}$ are unnecessary. The outputs of the negative radicals are plus-and-minus numbers which resolve to their positive magnitude at place of occurrence. On traditional graphing utilities this can be simulated by placing the argument inside the radical bar, inside absolute value brackets.

We first extend the domain of the numerator by allowing the graph to include the values from $-29 \leq x < 60$. The graph will again halt at $x = -29$ as anything less than this value will begin generating negative arguments for the radical bar in the denominator.



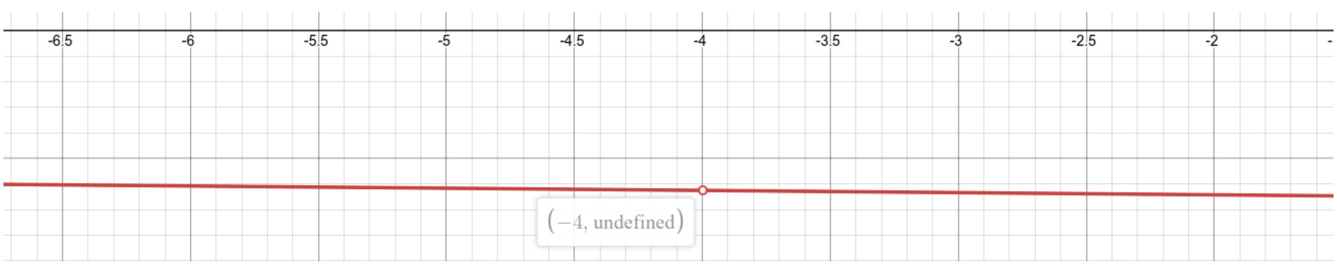
There are now two issues in the graph. First, the point of discontinuity has moved from $x < 60$ to $x < 29$.



The other is there is indeed now an instance of the feedback value at $x = -4$.

<u>3.1.b</u>

$$\lim_{x \to -4} \frac{\sqrt{x-60}-8}{\sqrt{x+29}-5} \to \frac{\sqrt{-4-60}-8}{\sqrt{-4+29}-5} \to \frac{\sqrt{-64}-8}{\sqrt{25}-5} \to \frac{\oplus 8-8}{5-5} \to \frac{\hat{8}-8}{0} \to \frac{0}{0}$$



This value will need resolved by the methods shown above, either by traditional multiplication by complex conjugates or by substitution. Either attempt will result in the same value but there will be some additional considerations. Before exploring those considerations we will now extend the rest of the domain to $\mathbb{R}$. This is done by including values for $x < 29$. In the denominator placing the argument of the radical bar in absolute values will allow the graphing utility to graph it. At $x = -54$ the denominator will reach zero, with the output itself resolving to 0.

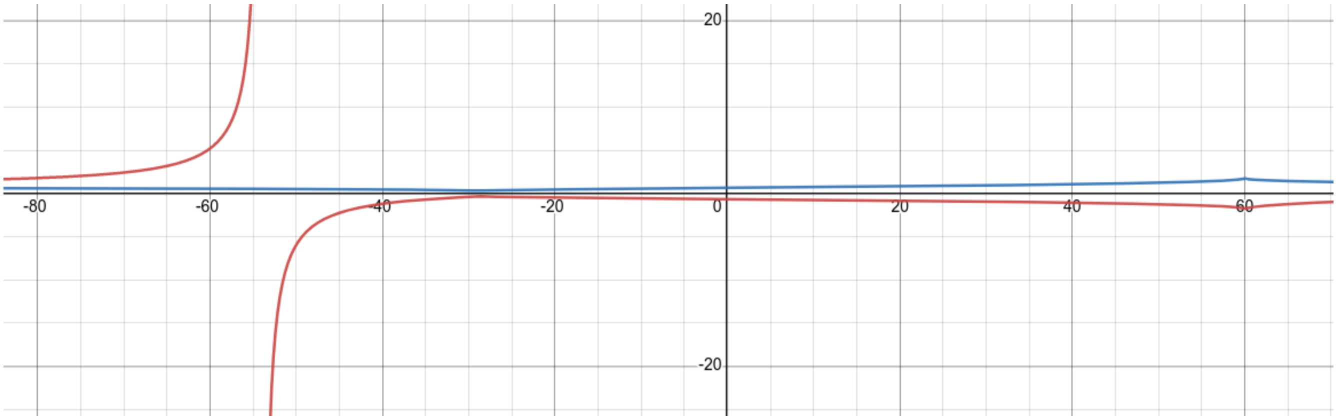
3.1.c

$$\lim_{x \to -54} \frac{\sqrt{x-60}-8}{\sqrt{x+29}-5} \to \frac{\sqrt{-54-60}-8}{\sqrt{-54+29}-5} \to \frac{\sqrt{-114}-8}{\sqrt{-25}-5} \to \frac{\oplus 10.667-8}{\oplus 5-5} \to$$

$$\frac{\hat{1}0.667-8}{0} \to \frac{\hat{2}.667}{0} \to \eta_0 \doteq 0$$

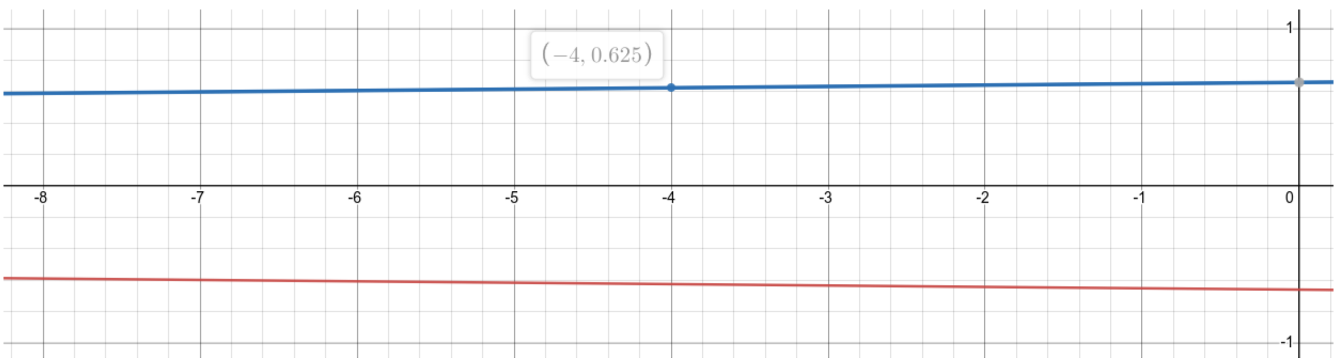
3.2—Recall that the value of the expression is the feedback value for $\lim_{x \to -4} \frac{\sqrt{x-60}-8}{\sqrt{x+29}-5} = \frac{0}{0}$. The progress made thus far would suggest the numerator and denominator are both reciprocal-conjugates and the expression which resolves the instance of the feedback value is obtained by switching their position with a reversal of operator outside the radical bars to $\frac{\sqrt{x+29}+5}{\sqrt{x-60}+8}$. Plotting this equation with the full extended domain will show the graph is not a match. Its shown here below in blue against the original graph.

3.2.a



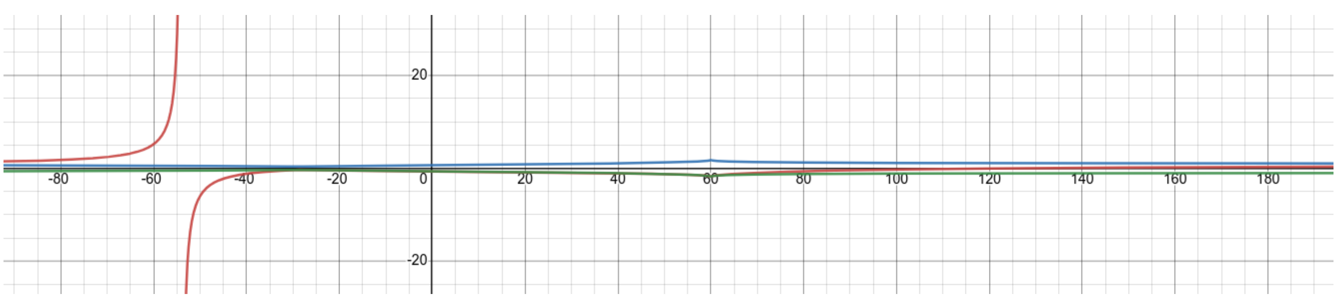
Zooming in on the graph of $\frac{\sqrt{x+29}+5}{\sqrt{x-60}+8}$ at $x=-4$ we see the augmented function implies $f(x)=0.625$. The value is in fact though -0.625. Thus we must explain the differences between the two graphs and resolve the apparent separation of the parts which would match except for a negative, within the domain of $-29 \leq x < 60$.

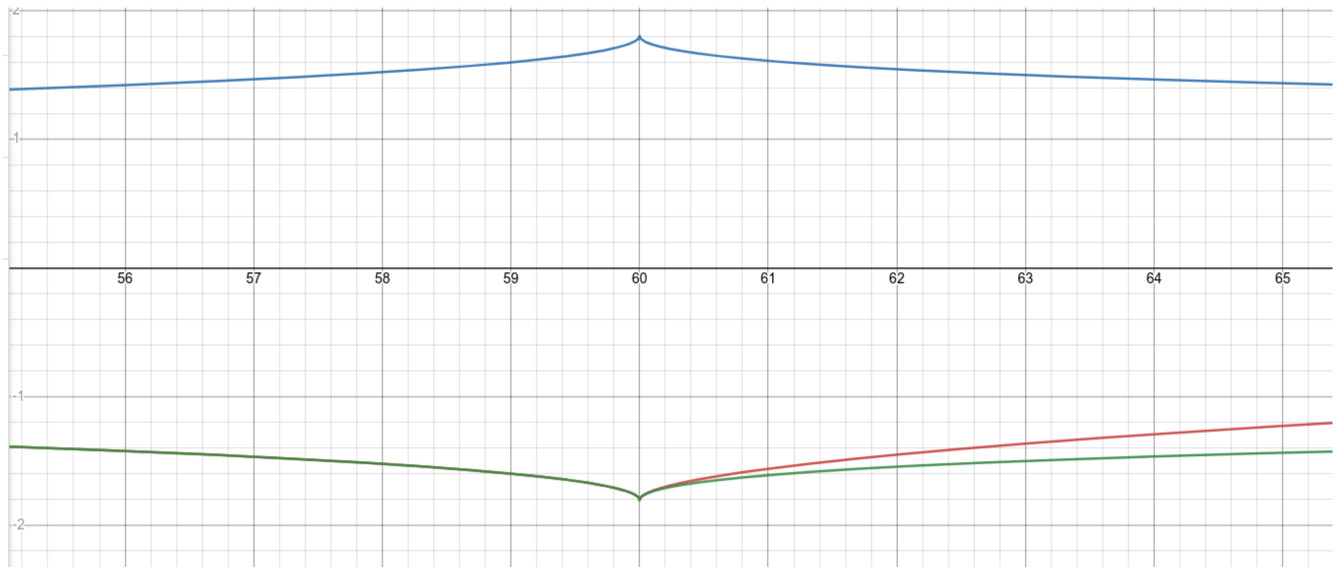
3.2.b



Consider what the graph of $\frac{\sqrt{x+29}+5}{\sqrt{x-60}+8}$ in blue looks like if multiplied by a negative. That graph, $-\frac{\sqrt{x+29}+5}{\sqrt{x-60}+8}$ is shown below in green. The original function is shown still in red.

3.2.c




2
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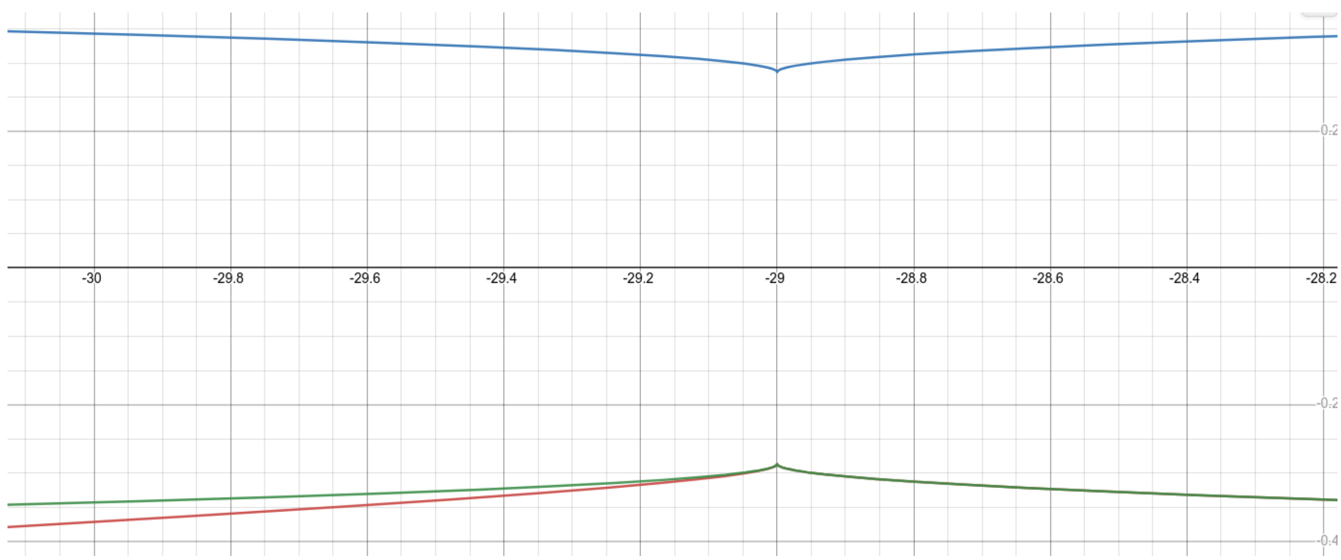
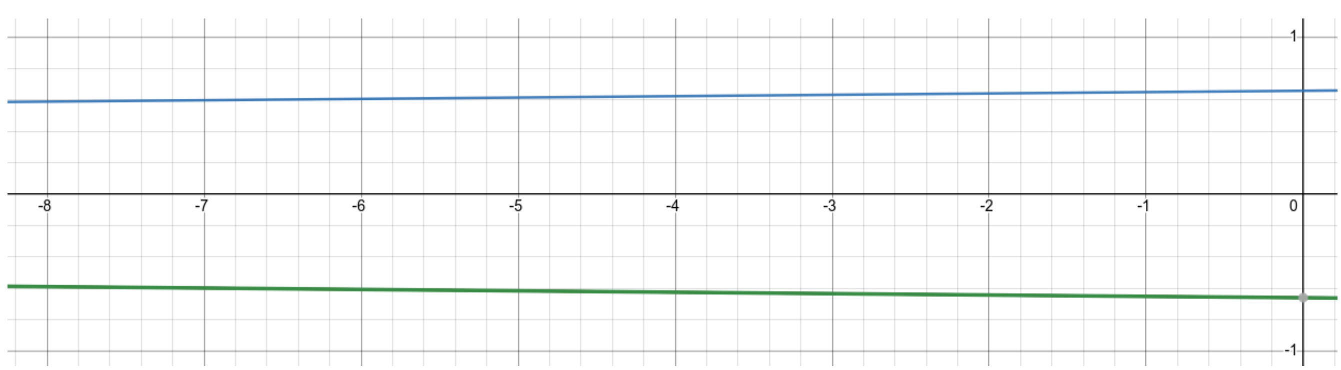
-1
-2


0.2
-30
-29.8
-29.6
-29.4
-29.2
-29
-28.8
-28.6
-28.4
-28.2
-0.2
-0.4


1
-8
-7
-6
-5
-4
-3
-2
-1
0
-1

It can be clearly seen the graph of $-\frac{\sqrt{x+29}+5}{\sqrt{x-60}+8}$ is identical to $\frac{\sqrt{x-60}-8}{\sqrt{x+29}-5}$, over the domain $-29 \leqslant x < 60$. The values of *f*(*x*) at inputs lower than -29 are separated from the remainder of the graph by the second instance of negative roots showing up. The usage of $\sqrt{|x+29|}-5$ will allow you to graph this extended domain for the denominator in a traditional graphing utility. The values $x<29$ do contain instances of a feedback value and do not need resolution by multiplication of complex conjugation or substitution methods any more than do values for the inputs of $x>60$. Values of *x* falling between $x=-29$ and $x=60$ are in a region of the graph which contains the feedback value for this example. It is the outputs for the domain of $-29 \leqslant x < 60$ which require resolution and are represented accurately by the graph of $-\frac{\sqrt{x+29}+5}{\sqrt{x-60}+8}$ as it is only this section of the domain and range which behave in a way that produces an instance of the feedback value and thereby possessing of supplement components which cancel in the process of resolving the equation.

The full domain and range of $\frac{\sqrt{x-60}-8}{\sqrt{x+29}-5}$ is represented by:

3.2.d

For $x \geq 60$	$f(x)=\frac{\sqrt{x-60}-8}{\sqrt{x+29}-5}$
For $-29 \leqslant x < 60$	$f(x)=-\frac{\sqrt{x+29}+5}{\sqrt{x-60}+8}$
For $x<-29$	$f(x)=\frac{\sqrt{x-60}-8}{\sqrt{x+29}-5}$

Full graph is visible and graphable on a traditional graphing utility using the equation:

$$f(x)=\frac{\sqrt{|x-60|}-8}{\sqrt{|x+29|}-5}$$

This is done only to allow the graphing utility which is very likely not programmed to understand what is occurring with negative square-root arguments to interpret the expression. The actual function remains $f(x)=\frac{\sqrt{x-60}-8}{\sqrt{x+29}-5}$, with the usage of Null Algebra precepts to solve for values of *f*(*x*) when *x* inputs generate negative root arguments.

These considerations included, the domain and range of the given example function are now ℝ.

Unless an equation contains an instance of the feedback value for a given input, it will not contain any reciprocal conjugates even if it contains radical expressions in its numerator, denominator or both. Next lets explore what happens when we perform evaluation geared toward resolving the occurrence of the feedback value of the traditionally defined domain for this example—over the domain of $x \geq 60$.

In this example problem at the point of $x=-4$ we have the occurrence of the feedback value, but is not included in the domain of $x \geq 60$. Over this domain the behavior of the function $f(x)=\frac{\sqrt{x-60}-8}{\sqrt{x+29}-5}$ is such that attempt to resolve the equation via both complex-conjugate multiplication and substitution with the input supplement of $(x+4)$ will not yield anything useful in simplification. It instead makes the equation more complex and does not provide a value for the equation at $x=-4$.

We begin with the traditional method of solving via conjugate multiplication for the domain of $x \geq 60$.

<u>3.2.e</u>

$$\frac{\sqrt{x-60}-8}{\sqrt{x+29}-5} \cdot \frac{\sqrt{x-60}+8}{\sqrt{x-60}+8} \cdot \frac{\sqrt{x+29}+5}{\sqrt{x+29}+5}$$

The numerator complex conjugate is $\sqrt{x-60}+8$ and the complex conjugate of the denominator is $\sqrt{x+29}+5$. For the domain of $x \geq 60$ the process of multiplication by complex conjugate is identical to that in traditional algebra. For the domain of $x<60$ the argument inside the radical bar for the numerator represents an unsolved negative radical. Because $x-60$ in the numerator is inside a radical bar, we must understand that this quantity is squared. It also means since $\sqrt{x-60}$ will produce negative arguments for the radical in the domain of $x<60$, the expression $\sqrt{x-60}$ must be treated as an *i*-multiple, a null-algebra resolved ⊕ number. Thus the multiplication by complex conjugate in the numerator as $(\sqrt{x-60}-8)(\sqrt{x-60}+8)$ is in fact squaring the *i*-multiple polynomial halves:

<u>3.2.f</u>

The value here of $\sqrt{x-60}-8$ for the the domain of $x<60$ is a plus-and-minus number. It is therefore better to view this expression in arrangement most representative of a complex number, $a+bi$.

<u>3.2.g</u>

$$\sqrt{x-60}-8 \equiv -8+\sqrt{x-60}$$

This is the resolved value we get to see in the expression, the value that was resolved to from the up-component of the *i*-multiple. We can undo the resolution by returning to the *up*-form of the expression.

<u>3.2.h</u>

$$-8+\sqrt{x-60} \equiv -8+\sqrt{x\dot{-}60} \equiv -8+\sqrt{x\hat{-}60}$$

This expression being an *i*-multiple has a partner, its down component occurring in a subspace and representative of the complex conjugate half related to complex numbers. This number exits along with the up component but is not seen in the original equation. It nonetheless must be used when squaring the expression $\sqrt{x-60}-8$ over the domain of $x<60$. Its resolved form is also obtained by unresolving the $a+bi$ form of the expression.

3.2.i

$$-8+\sqrt{x-60} \equiv -8+\dot{\sqrt{x-60}} \equiv -8+\left(-\check{\sqrt{x-60}}\right) \equiv -8-\check{\sqrt{x-60}}$$

These tow components must be multiplied by each other when we square the expression $\sqrt{x-60}-8$, the numerator in the equation $\frac{\sqrt{x-60}-8}{\sqrt{x+29}-5}$. Note below the order of the $a+bi$components are shown in their order given in the example equation, as $bi+a$. i.e. $-8+\hat{\sqrt{x-60}} \equiv \hat{\sqrt{x-60}}-8$ and $-8-\check{\sqrt{x-60}} \equiv -\check{\sqrt{x-60}}-8$.

3.2.j

$$\left(\sqrt{x-60}-8\right)^2 = \left(\hat{\sqrt{x-60}}-8\right)\left(-\check{\sqrt{x-60}}-8\right)$$

Note the extra negative will occur when up and down components multiply with each other according to precepts of Null Algebra. The remaining components multiply as normal.

3.2.k

$$= (x-60)-8\cdot\left(\sqrt{x-60}-8\right)+8\cdot\left(\sqrt{x-60}-8\right)+64 = x-60+64$$

For the input $x=-4$the numerator will equal 0.

When $x \geq 60$ in this example the consideration is unnecessary. Attempts to solve through complex conjugate multiplication across the numerator and denominator simply yield a different equation when $x \geq 60$ due to the traditional behavior of the equation in this domain. Nothing cancels out in it, and the graph of new equation is identical to the original equation for input and output values over the traditional domain of $x \geq 60$.

3..2.l

$$\frac{\sqrt{x-60}-8}{\sqrt{x+29}-5} \cdot \frac{\sqrt{x-60}+8}{\sqrt{x-60}+8} \cdot \frac{\sqrt{x+29}+5}{\sqrt{x+29}+5}$$

$$\frac{\sqrt{x+29}+5}{\sqrt{x+29}+5} \cdot \frac{x-124}{\left(\sqrt{x+29}-5\right)\left(\sqrt{x-60}+8\right)}$$

$$\frac{(x-124)\left(\sqrt{x+29}+5\right)}{(x+4)\left(\sqrt{x-60}+8\right)} \text{ which is } \equiv \frac{\sqrt{x-60}-8}{\sqrt{x+29}-5} \text{ for domain } x \geq 60$$

If you try adding in absolute value bars around any or all of the x inputs in this new equation you'll see it does not match the extended domain of the original equation. This equation is identical to the original equation but only for the traditional domain, a domain in which there is no instance of the feedback value. To see the actual value the function reaches at the input generating the feed back value (*in this example it is* $x=-4$) is to use the substitution or complex-conjugate multiplication method of evaluation while including null algebra precepts for the negative root arguments. This will treat the radical bar as a plus-and-minus number in this example and generate an additional negative when the roots square during multiplication. For the domain of $-29 \leq x < 60$ the following method for evaluation is used.

Attempting the substitution method of evaluating the expression will give you the form of the equation used in evaluating *f*(*x*) when $x=-4$ in the domain of $-29\leq x<60$ provided you account for the behavior of the graph within this domain. If you fail to remember to add in a negative when squaring the resolved *i*-multiple portion of the equation (*for this example in this domain it is the* $\sqrt{x-60}$ *component*) it will produce the wrong result. It does this as the substitution method applies only to the domain and range which contain the feedback solution of $\frac{0}{0}$.

<u>3.2.m</u>

$$\frac{\sqrt{x-60}-8}{\sqrt{x+29}-5}$$

Let $\quad u=\sqrt{x-60} \qquad s=\sqrt{x+29}$

$\quad x_u=u^2+60 \qquad x_s=s^2-29$

Supplement: $(x+4)$

The supplement portion is canceled out in the original equation. It has been added back in for each of the respective reciprocal-conjugates.

<u>3.2.n</u>

$$\frac{(x_s+4)(\sqrt{x-60}-8)}{(x_u+4)(\sqrt{x+29}-5)} \rightarrow \frac{(s^2-29+4)(u-8)}{(u^2+60+4)(s-5)} \rightarrow \frac{(s^2-25)(u-8)}{(u^2+64)(s-5)}$$

Note in this next step there is a negative added after factoring out the u^2 element. The substitution method directly associates $\sqrt{x-60}$ to u. This value is a resolved *plus-and-minus* number. Squaring such a number results in negative number. The substitution process provides and instance of u^2 in the u^2+64component shown here above. Because this instance of u^2 resulted from substitution, the expression u^2+64implies a negative will show up when the value of whatever is represented by *u* is squared. We account for that negative when the expression is factored into its linear components which are representative of the separate *up* and *down* portions of the radical expression. i.e. squaring this value will create a negative, and since the substitution process starts off with a squared value, we must assume a negative was canceled when it squared. Thus we must account for a negative when factoring just as when squaring.

<u>3.2.o</u>

$$\rightarrow \quad -\frac{(s+5)(s-5)(u-8)}{(u+8)(u-8)(s-5)} \rightarrow -\frac{(s+5)}{(u+8)} \rightarrow -\frac{\sqrt{x+29}+5}{\sqrt{x-60}+8}$$

Consider this is the same equation you will obtain using multiplication by complex conjugate, while including the null algebra precepts which show the squared root representing a plus-and-minus value generating a negative.

<u>3.2.p</u>

$$\frac{\sqrt{x-60}-8}{\sqrt{x+29}-5}\cdot\frac{\sqrt{x-60}+8}{\sqrt{x-60}+8}\cdot\frac{\sqrt{x+29}+5}{\sqrt{x+29}+5}$$

$$\frac{-[x-60]-64}{(\sqrt{x+29}-5)(\sqrt{x-60}+8)}\cdot\frac{\sqrt{x+29}+5}{\sqrt{x+29}+5}$$

$$-\frac{x-60+64}{\left(\sqrt{x+29}-5\right)\left(\sqrt{x-60}+8\right)}\cdot\frac{\sqrt{x+29}+5}{\sqrt{x+29}+5}$$

$$-\frac{(x+4)\left(\sqrt{x+29}+5\right)}{(x+4)\left(\sqrt{x-60}+8\right)}$$

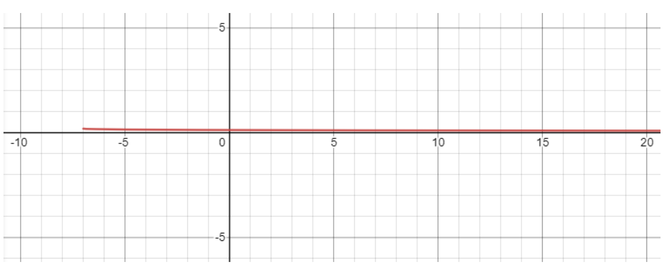
$-\dfrac{\sqrt{x+29}+5}{\sqrt{x-60}+8}$ for domain of $-29 \leqslant x < 60.$

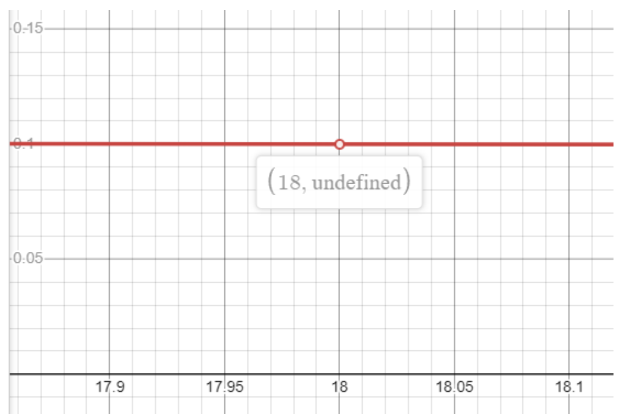
4.0—Below are several additional examples of the solving process.

4.1.a

Numerator reciprocal conjugate expression with feedback value in standard domain.

$\dfrac{\sqrt{x+7}-5}{x-18}$ Extended domain not shown in this example. Domain is $x \geqslant -7$. Feedback value at $x=18$.





Solve by setting the radical bar in the numerator to *u*. Solve for *x* and make substitutions. Simplify and re-substitute for *u*.

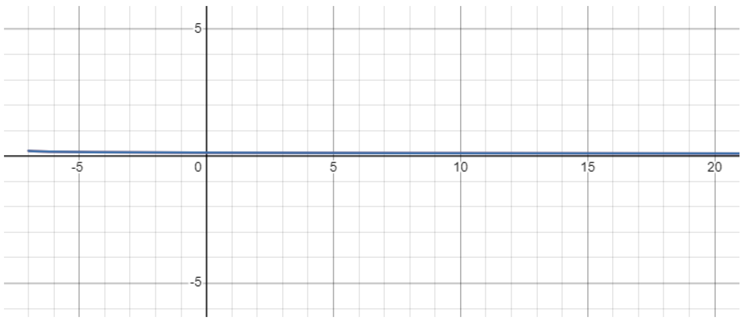
4.1.b

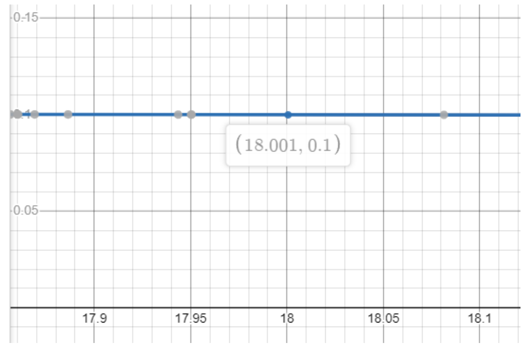
Let $u=\sqrt{x+7}$ and $x=u^2-7$. Supplement is $x-18$.

$$\frac{\sqrt{x+7}-5}{x-18} = \frac{u-5}{u^2-7-18} = \frac{u-5}{u^2-25} = \frac{u-5}{(u+5)(u-5)}$$

$$= \frac{\eta_0-5}{(u+5)(\eta_0-5)} = \frac{1}{u+5} = \frac{1}{\sqrt{x+7}+5}$$

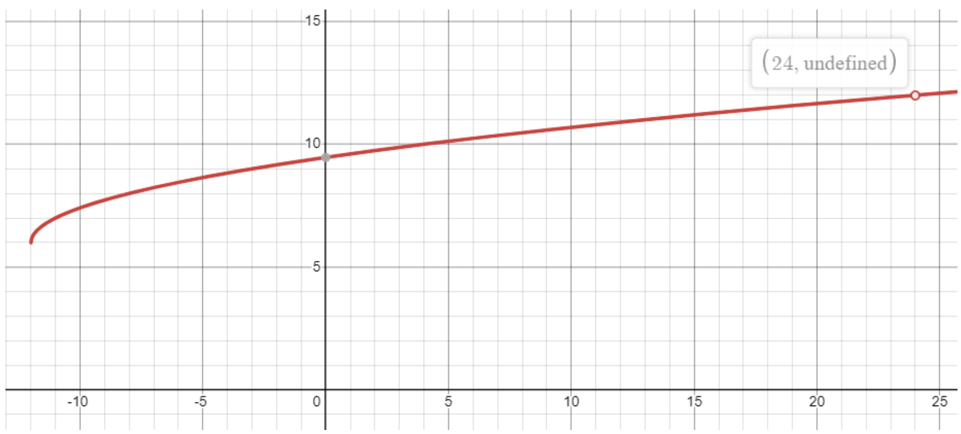
Graphs of $\dfrac{1}{\sqrt{x+7}+5}$





4.2.a.) Denominator reciprocal conjugate expression with feedback value in standard domain.

$\dfrac{x-24}{\sqrt{x+12}-6}$ Extended domain not shown in this example. Domain is $x \geqslant -12$. Feedback value at $x=24$.



Solve by setting the radical bar in the denominator to *u*. Solve for *x* and make substitutions. Simplify and re-substitute for *u*.

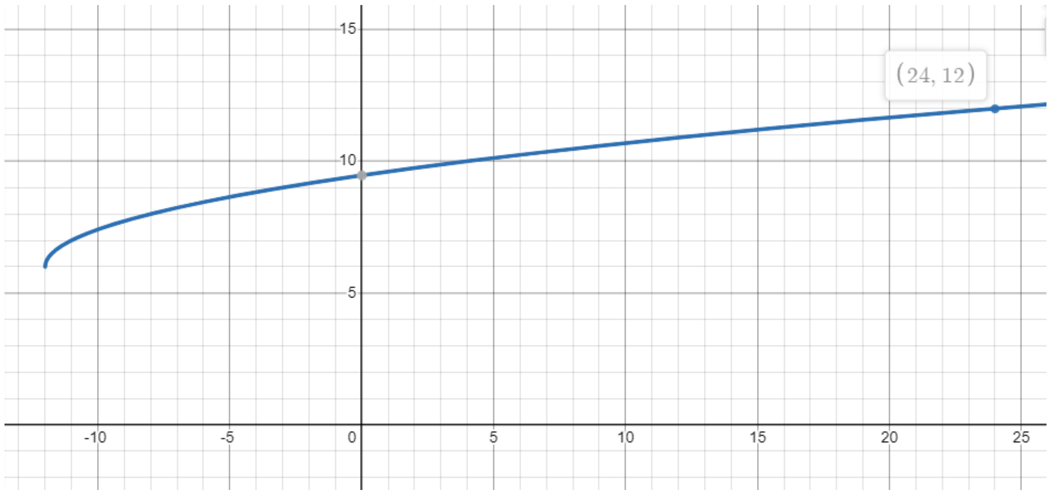
<u>4.2.b</u>

Let $u=\sqrt{x+12}$ and $x=u^2-12$. Supplement is $x-24$

$$\frac{x-24}{\sqrt{x+12}-6} = \frac{u^2-12-24}{u-6} = \frac{u^2-36}{u-6} = \frac{(u+6)(u-6)}{u-6}$$

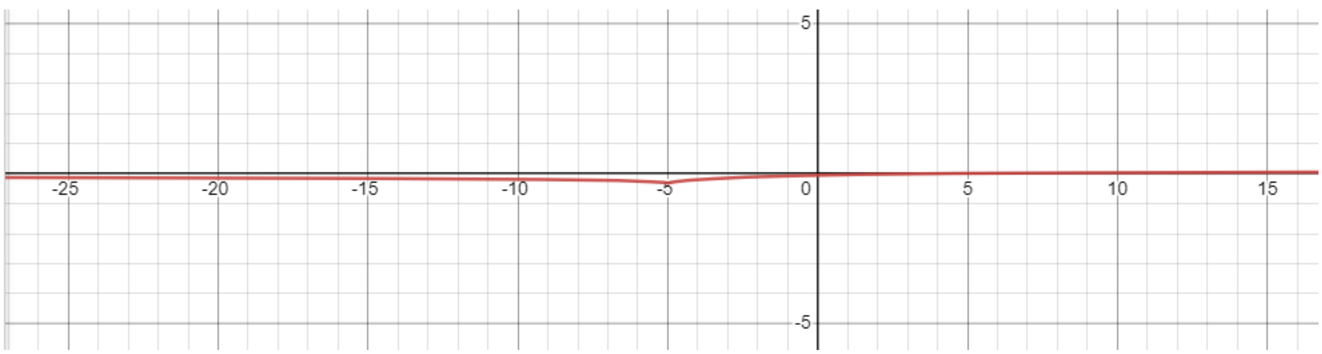
$$= \frac{(u+6)(\eta_0-6)}{\eta_0-6} = u+6 = \sqrt{x+12}+6$$

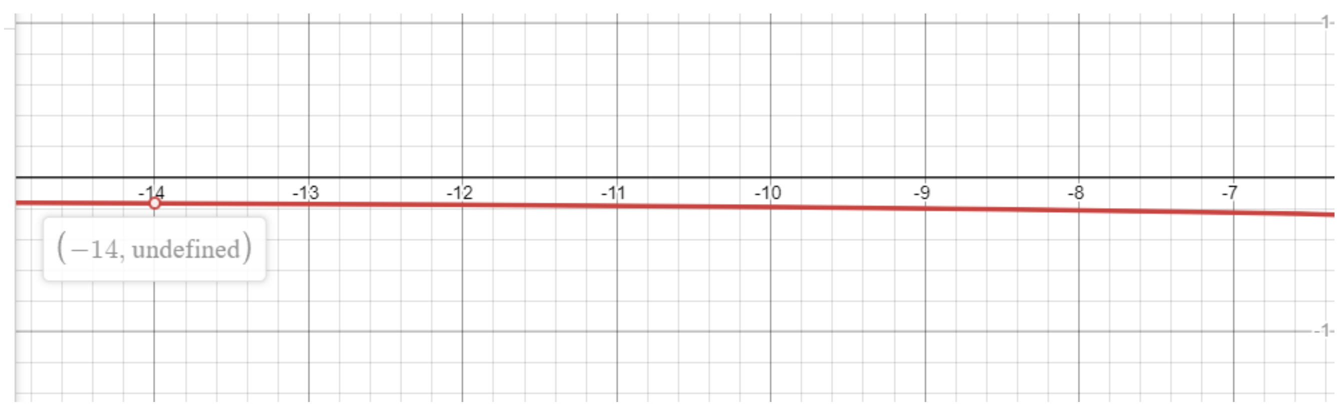
Graph of $\sqrt{x+12}+6$



4.3.a.) Numerator reciprocal conjugate with Feedback value occurring in extended domain of numerator radical.

$\dfrac{\sqrt{x+5}-3}{x+14}$ Extended domain for values $x \leq -5$. Feedback value $\dfrac{0}{0}$ at $x=-14$.





4.3.b

Let $u=\sqrt{x+5}$. Solving for x account for the inclusion of a negative in squaring of the u element. Distributive the negative, factor, simplify and solve.

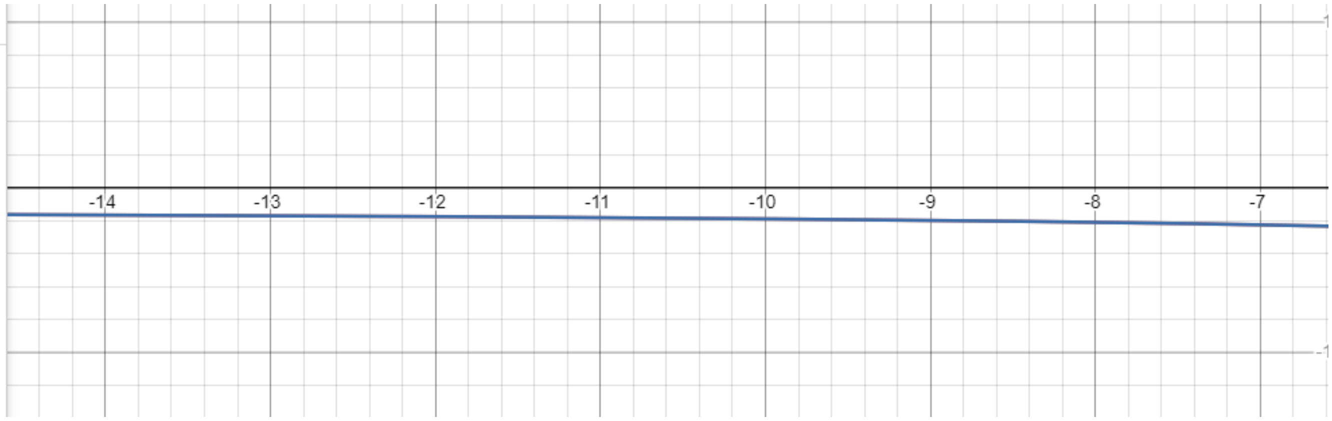
Let $x=-u^2-5$

$$\frac{\sqrt{x+5}-3}{x+14} = \frac{u-3}{-u^2-5+14} = -\frac{u-3}{u^2+5-14} = -\frac{u-3}{u^2-9}$$

$$= -\frac{u-3}{(u+3)(u-3)} = -\frac{\eta_0-3}{(u+3)(\eta_0-3)} = -\frac{1}{u+3}$$

$$= -\frac{1}{\sqrt{x+5}+3}$$

$\frac{\sqrt{x+5}-3}{x+14}$ is used to represent the full equation with the extended domain graphable in a traditional graphing utility by using $\frac{\sqrt{|x+5|}-3}{x+14}$. The graph of the equation with the resolved solution replacing the feedback value is $-\frac{1}{\sqrt{x+5}+3}$, which is valid over the domain of $x\leqslant-5$. It is only over this extended domain of $x\leqslant-5$ where the graph behaves in a way to generate the feedback value. Graph of $\frac{\sqrt{x+5}-3}{x+14}$ is shown below in red for domain of $\mathbb{R}$, over lapped with $-\frac{1}{\sqrt{x+5}+3}$ for domain of $x\leqslant-5$.



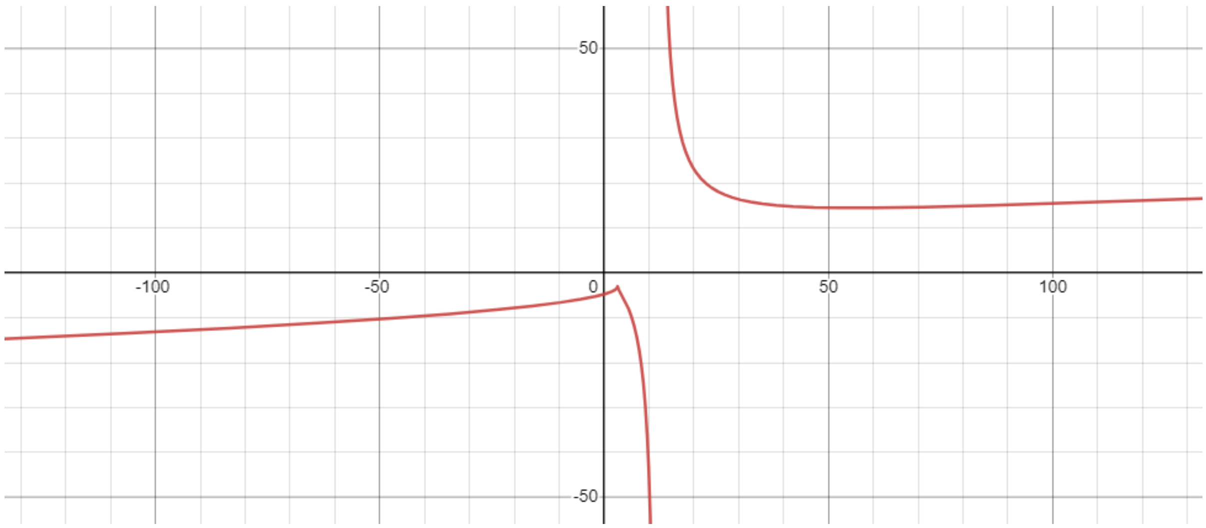
4.4.a) Denominator reciprocal conjugate with Feedback value occurring in extended domain of denominator radical.

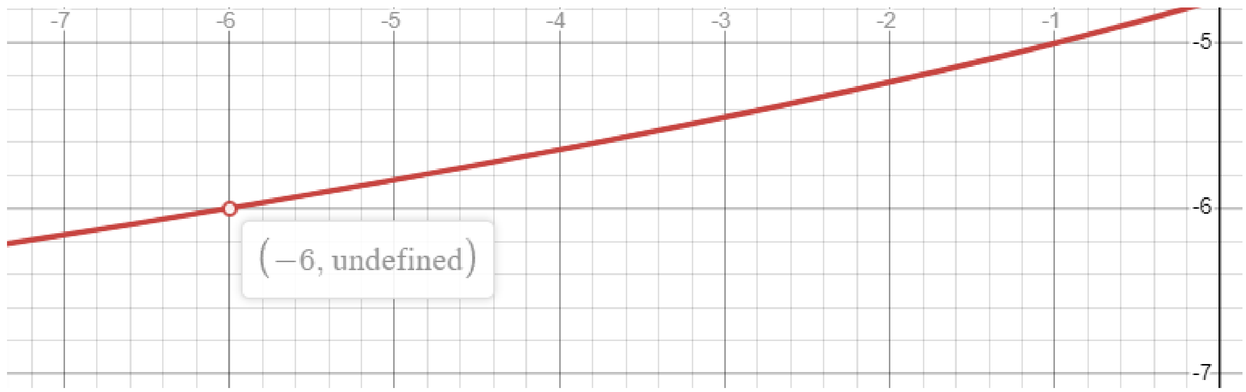
$\frac{x+6}{\sqrt{x-3}-3}$ Division by 0 at $x=12$ resolvable to 0. Extended domain for $x \leq 3$.

Feedback value at $x=-6$.

<u>4.4.b</u>
Let $u=\sqrt{x-3}$. In solving for x account for a negative in squaring of the u element in the extended domain.

Let $x=-u^2+3$
Supplement is $x+6$





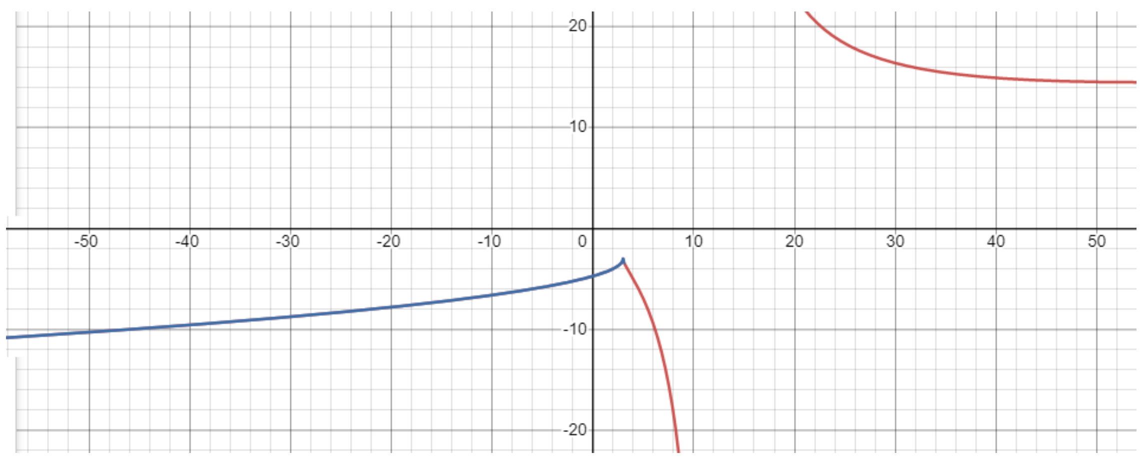
4.4.c

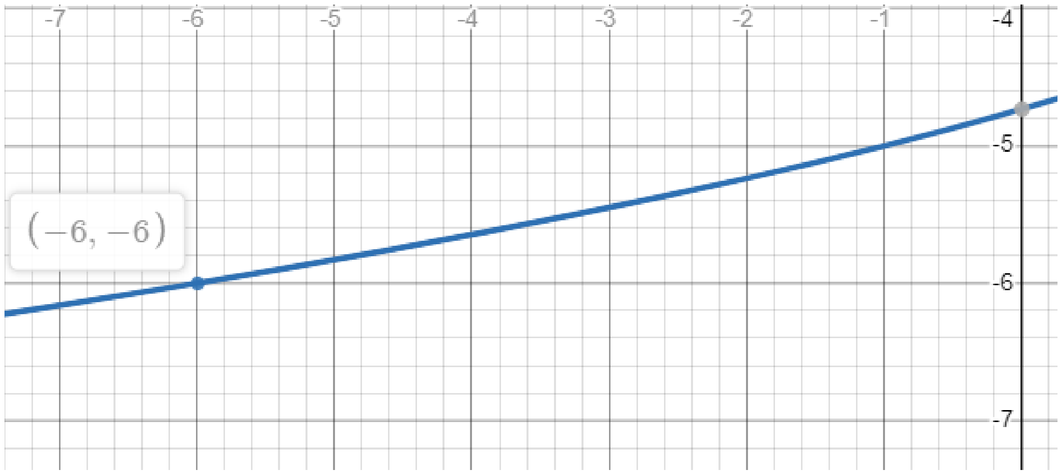
$$\frac{x+6}{\sqrt{x-3}-3} = \frac{-u^2+3+6}{u-3} = -\left(\frac{u^2-3-6}{u-3}\right) = \left(-\frac{u^2-9}{u-3}\right)$$

$$= -\left(\frac{(u+3)(u-3)}{u-3}\right) = -\left(\frac{(u+3)(\eta_0-3)}{\eta_0-3}\right) = -(u+3)$$

$$= -u-3 = -\sqrt{x-3}-3$$

$\frac{x+6}{\sqrt{x-3}-3}$ is used to represent the full equation with the extended domain graphable in a traditional graphing utility by using, $\frac{x+6}{\sqrt{|x-3|}-3}$. The graph of the equation with the resolved solution replacing the feedback value is, $-\sqrt{x-3}-3$which is valid over the domain of $x \leqslant 3$. It is only over this extended domain of $x \leqslant 3$ where the graph behaves in a way to generate the feedback value. The graph of $\frac{x+6}{\sqrt{x-3}-3}$ is shown below in red for domain of $\mathbb{R}$, over lapped with $-\sqrt{x-3}-3$ for domain of $x \leqslant 3$.

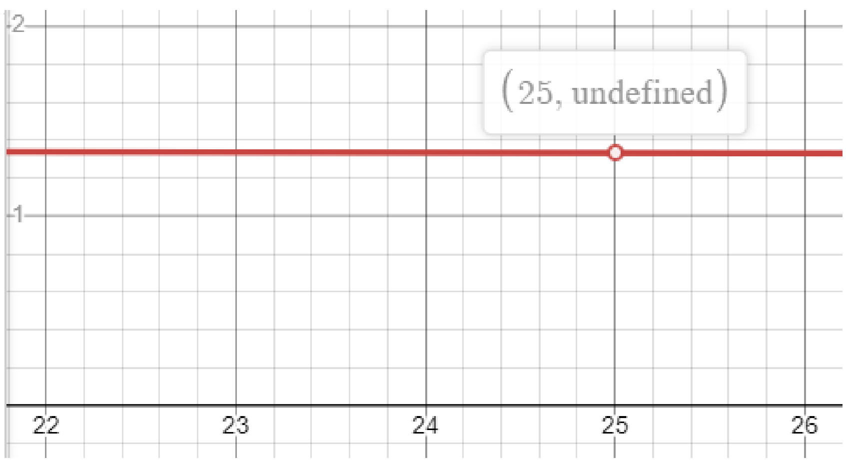




4.5.a) Reciprocal Conjugates in numerator and denominator with feedback value occurring in standard domain of both reciprocal conjugates.

$\frac{\sqrt{x+56}-9}{\sqrt{x+119}-12}$ Extended domain not included in graph.

Standard Domain for $x \geq -56$. Feedback value at $x = 25$.



4.5.b

$u = \sqrt{x+56}$ $\qquad x = u^2 - 56$ $\qquad$ Supplement is $x_u - 25$

$s = \sqrt{x+119}$ $\qquad x = s^2 - 119$ $\qquad$ Supplement is $x_s - 25$

Make respective substitutions, simplify and solve. Note that the supplements must be added back into the equation as they are canceled out in the given form.

$$\frac{\sqrt{x+56}-9}{\sqrt{x+119}-12} = \frac{(x_s-25)\left(\sqrt{x_u+56}-9\right)}{(x_u-25)\left(\sqrt{x_s+119}-12\right)} = \frac{\sqrt{x_u+56}-9}{x_u-25}\,\frac{x_s-25}{\sqrt{x_s+119}-12}$$

4.5.c

Solving for x_u component:

$$\frac{\sqrt{x_u+56}-9}{x_u-25} = \frac{u-9}{u^2-56-25} = \frac{u-9}{u^2-81}$$

$$= \frac{u-9}{(u+9)(u-9)} = \frac{\eta_0-9}{(u+9)(\eta_0-9)}$$

$$= \frac{1}{u+9} = \frac{1}{\sqrt{x+56}+9}$$

4.5.d

Solving for x_s component:

$$\frac{x_s-25}{\sqrt{x_s+119}-12} = \frac{s^2-119-25}{s-12} = \frac{s^2-144}{s-12}$$

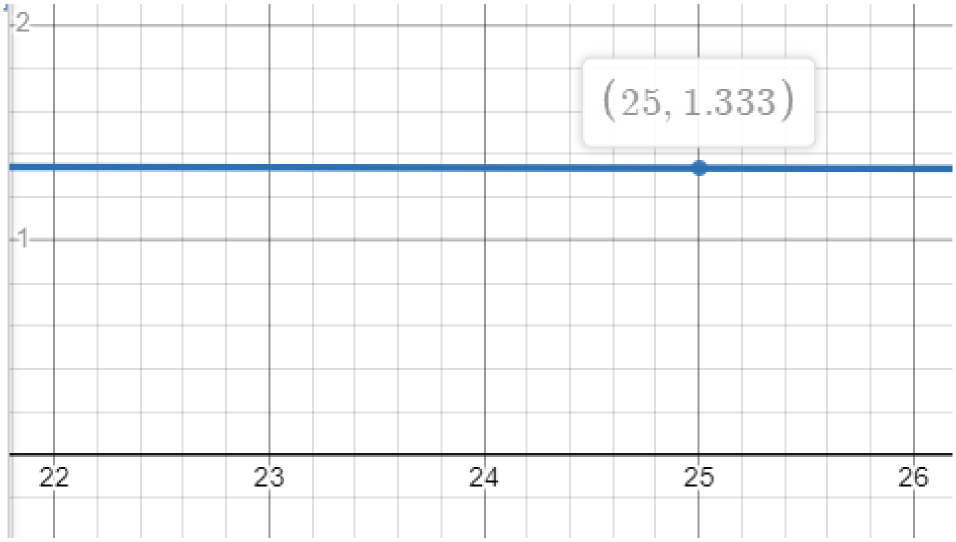
$$= \frac{(s+12)(s-12)}{s-12} = \frac{(s+12)(\eta_0-12)}{\eta_0-12}$$

$$= s+12 = \sqrt{x+119}+12$$

4.5.e

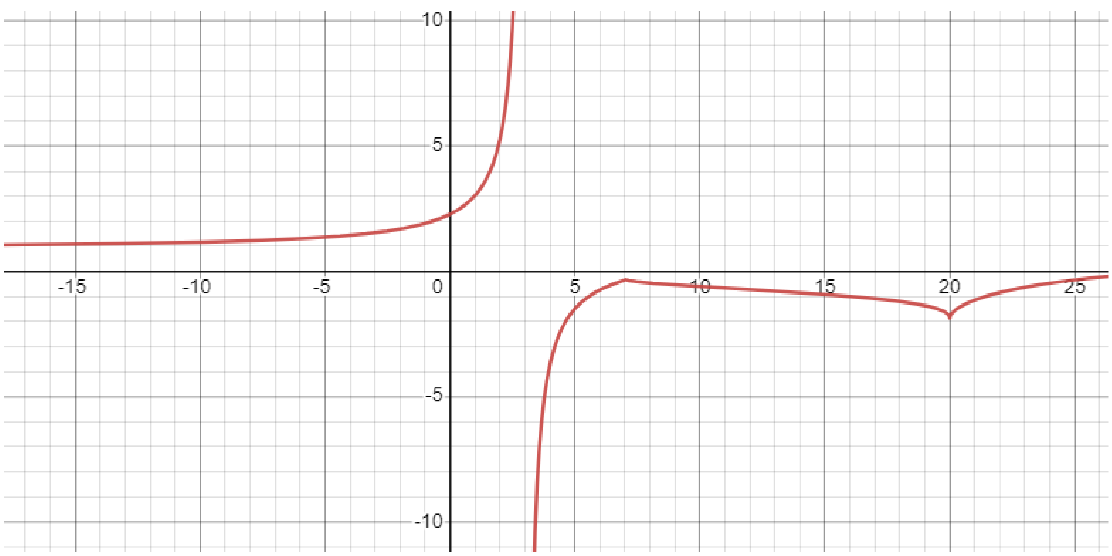
Recombine x_u and x_s components

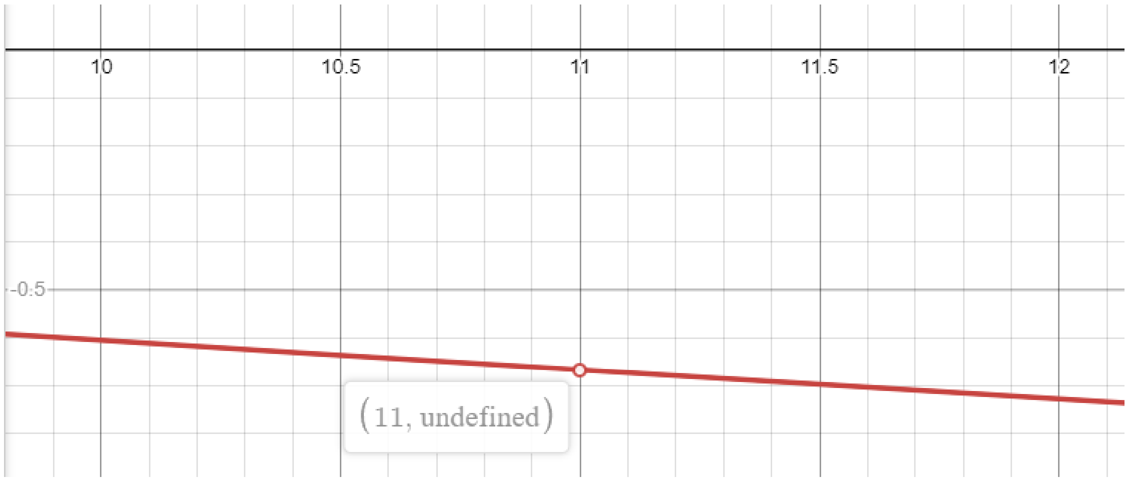
$$\frac{\sqrt{x+56}-9}{\sqrt{x+119}-12} = \frac{\sqrt{x_u+56}-9}{x_u-25}\,\frac{x_s-25}{\sqrt{x_s+119}-12} = \frac{\sqrt{x+119}+12}{\sqrt{x+56}+9}$$



4.6.a.) Reciprocal conjugates in numerator and denominator. Feedback value occurring at $x=11$ in extended domain of numerator but in standard domain of denominator.

$$\frac{\sqrt{x-20}-3}{\sqrt{x-7}-2}$$





The full extended domain of the function is graphable by $\frac{\sqrt{|x-20|}-3}{\sqrt{|x-7|}-2}$, and is shown below. The substitution process to solve for the value of $\frac{0}{0}$ will apply to region of the function which behaves in a manner that produces the feedback value, over the domain of $7 \leq x < 20$. Division by 0 at $x=3$ resolvable to 0.

<u>4.6.b</u>

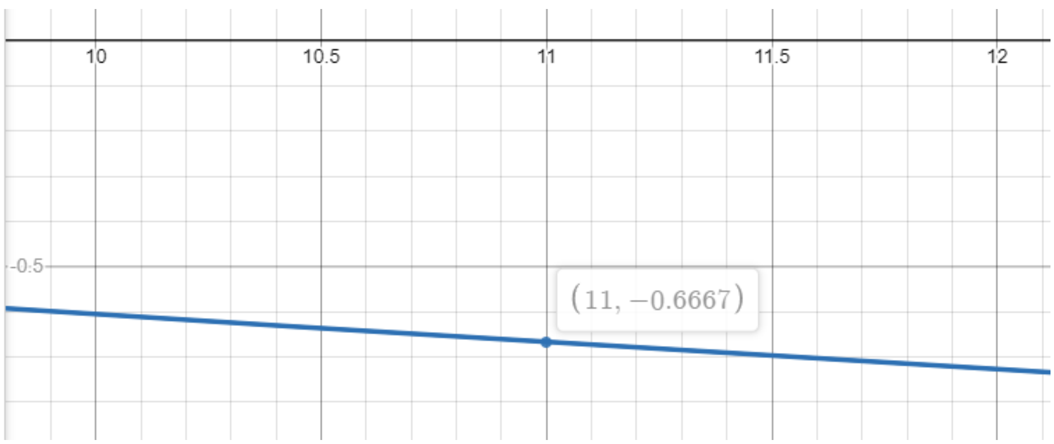
$u=\sqrt{x-20}$ $\quad x_u=-u^2+20$ $\qquad$ Supplement: x_u-11

$s=\sqrt{x-7}$ $\quad x_s=s^2+7$ $\qquad$ Supplement: x_s-11

$$\frac{(x_s-11)(\sqrt{x_u-20}-3)}{(x_u-11)(\sqrt{x_s-7}-2)} = \frac{(s^2+7-11)(u-3)}{(-u^2+20-11)(s-2)} = \frac{(s^2-4)(u-3)}{(-u^2+9)(s-2)} = -\frac{(s^2-4)(u-3)}{(u^2-9)(s-2)}$$

$$= -\frac{(s+2)(s-2)(u-3)}{(u+3)(u-3)(s-2)} = -\frac{(s+2)(\eta_0-2)(\eta_0-3)}{(u+3)(\eta_0-3)(\eta_0-2)}$$

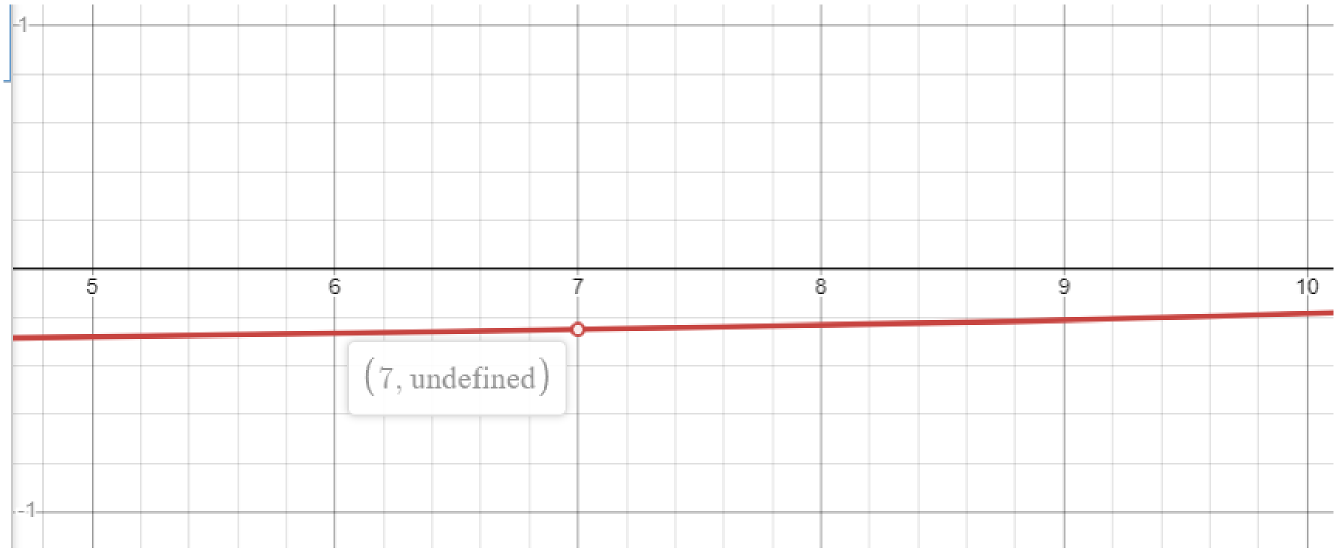
$$= -\frac{s+2}{u+3} = -\frac{\sqrt{x-7}+2}{\sqrt{x-20}+3}$$



4.7.a) Reciprocal conjugates in numerator and denominator. Feedback value occurring at $x=7$ in extended domain of denominator but in standard domain of numerator.

$$\frac{\sqrt{x+57}-8}{\sqrt{x-11}-2}$$

The full extended domain of the function is graphable by $\frac{\sqrt{|x+57|}-8}{\sqrt{|x-11|}-2}$, and is shown below. The substitution process to solve for the value of $\frac{0}{0}$ will apply to region of the function which behaves in a manner that produces the feedback value, over the domain of $-57 \leq x < 11$. Division by 0 at $x=15$ resolvable to 0.



4.7.b

Let $u=\sqrt{x+57}$ with $x_u=u^2-57$ Supplement: x_u-7

Let $s=\sqrt{x-11}$with $x_s=-s^2+11$ Supplement: x_s-7

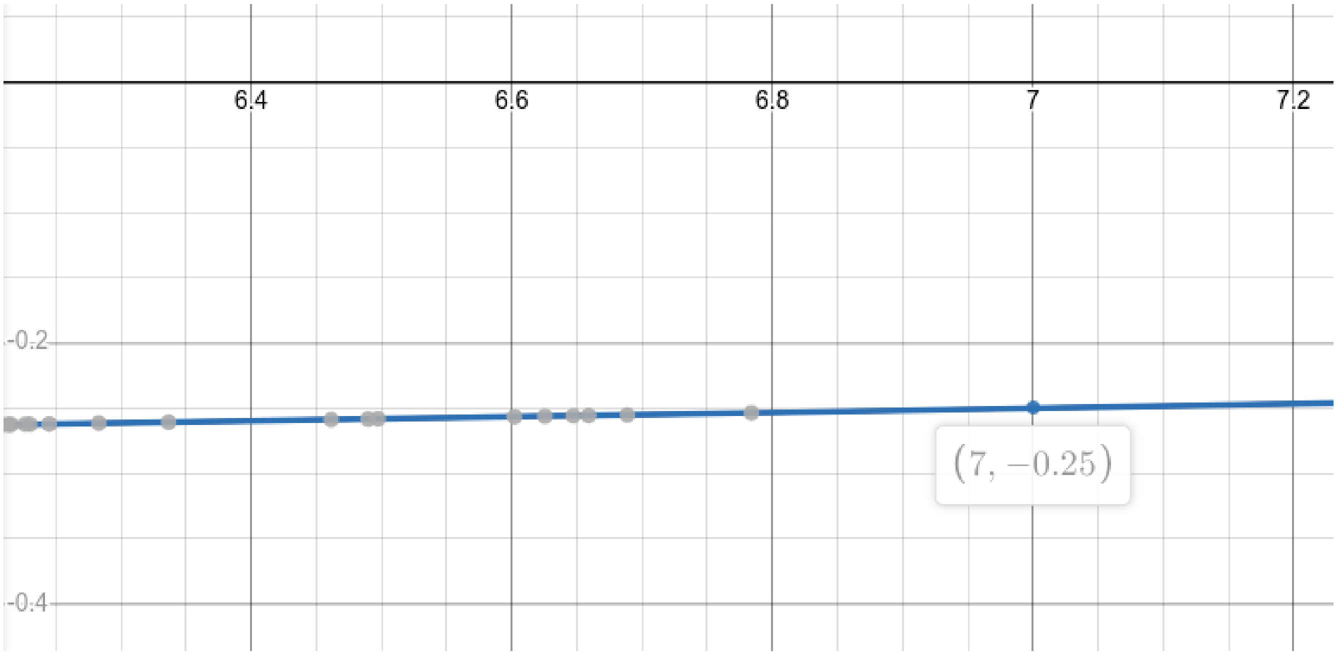
The $x_s$element accounts for the squaring of the s substitution containing negative root arguments, showing the inclusion of the resulting negative from squaring.

$$\frac{\sqrt{x+57}-8}{\sqrt{x-11}-2} = \frac{(x_s-7)(\sqrt{x+57}-8)}{(x_u-7)(\sqrt{x-11}-2)} = \frac{(-s^2+11-7)(u-8)}{(u^2-57-7)(s-2)} = \frac{(-s^2+4)(u-8)}{(u^2-64)(s-2)}$$

$$= -\frac{(s^2-4)(u-8)}{(u^2-64)(s-2)} = -\frac{(s+2)(s-2)(u-8)}{(u+8)(u-8)(s-2)} = -\frac{(s+2)(\eta_0-2)(\eta_0-8)}{(u+8)(\eta_0-8)(\eta_0-2)}$$

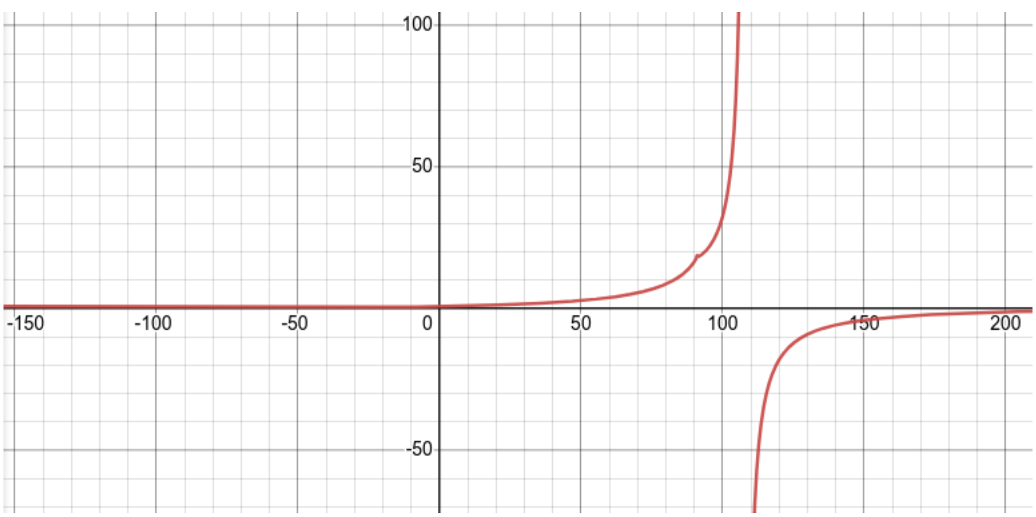
$$= -\frac{s+2}{u+8} = -\frac{\sqrt{x-11}+2}{\sqrt{x+57}+8}$$

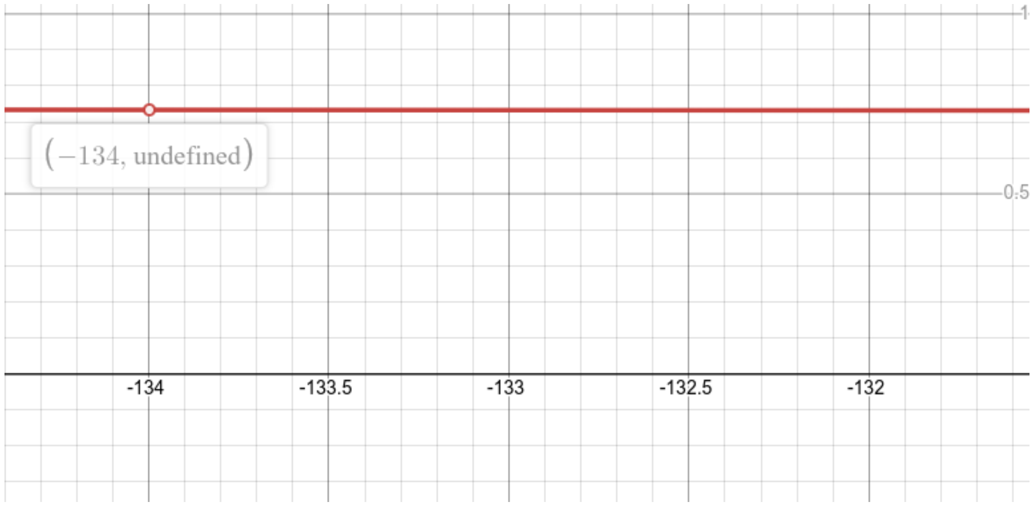
For $x=7$ $f(x)=-0.25$



4.8.a) Reciprocal conjugates in numerator and denominator. Feedback value occurring at $x = -134$ in extended domain of numerator and denominator.

$$\frac{\sqrt{x-91}-15}{\sqrt{x+13}-11}$$





4.8.b

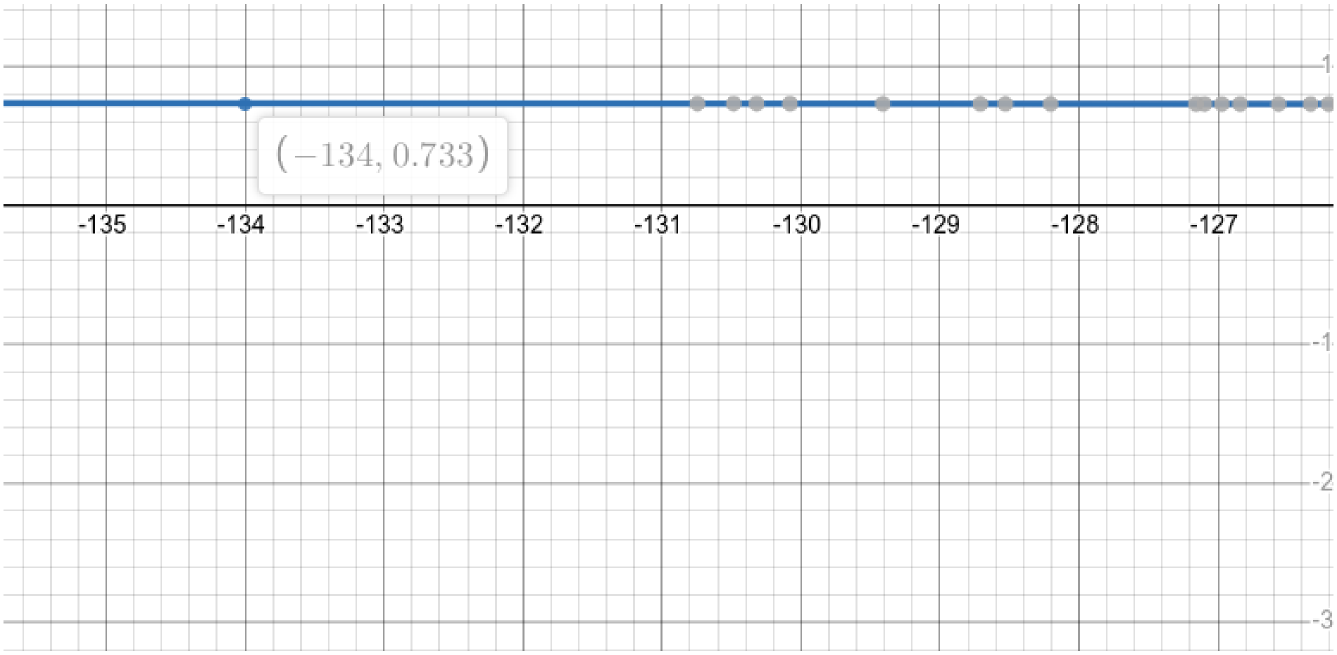
Let $u = \sqrt{x-91}$ $\quad x_u = -u^2 + 91$ $\quad$ Supplement: $x + 134$

Let $s = \sqrt{x+13}$ $\quad x_s = -s^2 - 13$ $\quad$ Supplement: $x + 134$

$$\frac{\sqrt{x-91}-15}{\sqrt{x+13}-11} = \frac{(x_s+134)(\sqrt{x-91}-15)}{(x_u+134)(\sqrt{x+13}-11)} = \frac{([-s^2-13]+134)(u-15)}{([-u^2+91]+134)(s-11)} = \frac{([s^2+13]-134)(u-15)}{([u^2-91]-134)(s-11)}$$

$$= \frac{(s^2-121)(u-15)}{(u^2-225)(s-11)} = \frac{(s+11)(s-11)(u-15)}{(u+15)(u-15)(s-11)} = \frac{(s+11)(\eta_0-11)(\eta_0-15)}{(u+15)(\eta_0-15)(\eta_0-11)} = \frac{(s+11)}{(u+15)}$$

$$= \frac{(\sqrt{x+13}+11)}{(\sqrt{x-91}+15)}$$


(−134, 0.733)
-135
-134
-133
-132
-131
-130
-129
-128
-127