The Proofs of Legendre’s Conjecture and Three Related Conjectures

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Abstracts

In this paper, we are going to prove Legendre’s Conjecture: There is a prime number between $n^2$ and $(n + 1)^2$ for every positive integer $n$. We will also prove three related conjectures. The method that we use is to analyze a binomial coefficient. It has been developed from the method of analyzing a central binomial coefficient that was used by Paul Erdős to prove Bertrand’s postulate - Chebyshev’s theorem.

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1. Introduction

Legendre’s Conjecture was proposed by Andrien-Marie Legendre [1]. The conjecture is one of Legendre’s problems (1912) on prime numbers. It states that there is a prime number between $n^2$ and $(n + 1)^2$ for every positive integer $n$.

In this paper, we will prove Legendre’s Conjecture by analyzing the binomial coefficient $\binom{\lambda n}{n}$ where $\lambda \geq 3$ is an integer. It is developed from the method that was used by Paul Erdős [2] to prove Bertrand’s postulate - Chebyshev’s theorem [3].

In Section 1, we will define the prime number factorization operator and clarify some terms and concepts. In Section 2, we will derive some lemmas. In Section 3, we will develop a theorem to be used in the proofs of the conjectures in the later sections. In Section 4, we will prove Legendre’s conjecture, and in Section 5, we will prove Oppermann’s conjecture [4], Brocard’s conjecture [5], and Andrica’s conjecture [6].

**Definition:** $\Gamma_{a \geq p > b} \{\binom{\lambda n}{n}\}$ denotes the prime factorization operator of $\binom{\lambda n}{n}$. It is the product of the prime numbers in the decomposition of $\binom{\lambda n}{n}$ in the range of $a \geq p > b$. In this operator, $p$ is a prime number, $a$ and $b$ are real numbers, and $\lambda n \geq a \geq p > b \geq 1$.

It has some properties:

- It is always true that $\Gamma_{a \geq p > b} \{\binom{\lambda n}{n}\} \geq 1$ — (1.1)
- If there is no prime number in $\Gamma_{a \geq p > b} \{\binom{\lambda n}{n}\}$, then $\Gamma_{a \geq p > b} \{\binom{\lambda n}{n}\} = 1$, or vice versa,
- If $\Gamma_{a \geq p > b} \{\binom{\lambda n}{n}\} = 1$, then there is no prime number in $\Gamma_{a \geq p > b} \{\binom{\lambda n}{n}\}$.

For example, when $\lambda = 5$ and $n = 4$, $\Gamma_{16 \geq p > 10} \{\binom{20}{4}\} = 13^0 \cdot 11^0 = 1$. No prime number 13 or 11 is in $\binom{20}{4}$ in the range of $16 \geq p > 10$.

If there is at least one prime number in $\Gamma_{a \geq p > b} \{\binom{\lambda n}{n}\}$, then $\Gamma_{a \geq p > b} \{\binom{\lambda n}{n}\} > 1$, or vice versa,
- If $\Gamma_{a \geq p > b} \{\binom{\lambda n}{n}\} > 1$, then there is at least one prime number in $\Gamma_{a \geq p > b} \{\binom{\lambda n}{n}\}$.

For example, when $\lambda = 5$ and $n = 4$, $\Gamma_{20 \geq p > 16} \{\binom{20}{4}\} = 19 \cdot 17 > 1$. Prime numbers 19 and 17 are in $\binom{20}{4}$ in the range of $20 \geq p > 16$.

Let $v_p(n)$ be the $p$-adic valuation of $n$, the exponent of the highest power of $p$ that divides $n$. Similar to Paul Erdős’ paper [2], we define $R(p)$ by the inequalities $p^R(p) \leq \lambda n < p^{R(p)+1}$, and determine the $p$-adic valuation of $\binom{\lambda n}{n}$.

\[
v_p \left( \binom{\lambda n}{n} \right) = v_p((\lambda n)!)-v_p((\lambda-1)n)!)-v_p(n!)=\sum_{i=1}^{R(p)} \left( \left\lfloor \frac{\lambda n}{p^i} \right\rfloor - \left\lfloor \frac{(\lambda-1)n}{p^i} \right\rfloor - \left\lfloor \frac{n}{p^i} \right\rfloor \right) \leq R(p)
\]

because for any real numbers $a$ and $b$, the expression of $\lfloor a + b \rfloor - \lfloor a \rfloor - \lfloor b \rfloor$ is 0 or 1.

Thus, if $p$ divides $\binom{\lambda n}{n}$, then \( v_p \left( \binom{\lambda n}{n} \right) \leq R(p) \leq \log_p(\lambda n) \), or $p^{v_p(\binom{\lambda n}{n})} \leq R(p) \leq \lambda n$ — (1.4)

And if $\lambda n \geq p > \sqrt{\lambda n}$, then $0 \leq v_p \left( \binom{\lambda n}{n} \right) \leq R(p) \leq 1$ — (1.5)
Let $\pi(n)$ be the number of distinct prime numbers less than or equal to $n$. Among the first six consecutive natural numbers are three prime numbers 2, 3 and 5. Then, for each additional six consecutive natural numbers, at most one can add two prime numbers, $p \equiv 1 \pmod{6}$ and $p \equiv 5 \pmod{6}$. Thus, $\pi(n) \leq \left\lceil \frac{n}{3} \right\rceil + 2$ for some of $n \equiv 1 \pmod{6}$ and $n \equiv 5 \pmod{6}$ are not prime numbers, as the number counts increase, $\pi(n)$ reduces from $\left\lceil \frac{n}{3} \right\rceil + 2$.

For $n \geq 24$, $\pi(n) \leq \left\lceil \frac{n}{3} \right\rceil + 1 \leq \frac{n}{3} + 1$ \hfill (1.6)

From the prime number decomposition,

when $n > \sqrt[6]{\lambda n}$, \( \binom{\lambda n}{n} = \Gamma_{\lambda n} \times p \cdot n \{ \binom{(\lambda n)!}{n! \cdot ((\lambda - 1)n)!} \cdot \Gamma_{\sqrt[6]{\lambda n}} \times p \{ \binom{(\lambda n)!}{n! \cdot ((\lambda - 1)n)!} \} \)

when $n \leq \sqrt[6]{\lambda n}$, \( \binom{\lambda n}{n} \leq \Gamma_{\lambda n} \times p \cdot n \{ \binom{(\lambda n)!}{n! \cdot ((\lambda - 1)n)!} \} \cdot \Gamma_{\sqrt[6]{\lambda n}} \times p \{ \binom{(\lambda n)!}{n! \cdot ((\lambda - 1)n)!} \} \)

Thus, \( \binom{\lambda n}{n} \leq \Gamma_{\lambda n} \times p \cdot n \{ \binom{(\lambda n)!}{n! \cdot ((\lambda - 1)n)!} \} \) since all prime numbers in $n!$ do not appear in the range of $\lambda n \geq p > n$.

Referring to (1.5), $\Gamma_{\lambda n} \times p \{ \binom{(\lambda n)!}{n! \cdot ((\lambda - 1)n)!} \} \leq \prod_{n \geq p} \ p$. It has been proved [7] that for $n \geq 3$, $\prod_{n \geq p} \ p < 2^{2n-3}$. Thus, for $n \geq 3$, $\Gamma_{\lambda n} \times p \{ \binom{(\lambda n)!}{n! \cdot ((\lambda - 1)n)!} \} \leq \prod_{n \geq p} \ p < 2^{2n-3}$.

Referred to (1.4) and (1.6), $\Gamma_{\sqrt[6]{\lambda n}} \times p \{ \binom{(\lambda n)!}{n! \cdot ((\lambda - 1)n)!} \} \leq (\lambda n)^{\sqrt[6]{\lambda n}} + 1$ when $\sqrt[6]{\lambda n} \geq 24$.

Thus, for $n \geq 3$ and $\sqrt[6]{\lambda n} \geq 24$, $\binom{\lambda n}{n} < \Gamma_{\lambda n} \times p \{ \binom{(\lambda n)!}{n! \cdot ((\lambda - 1)n)!} \} \leq 2^{2n-3} \cdot (\lambda n)^{\sqrt[6]{\lambda n}} + 1$ \hfill (1.7)

2. Lemmas

**Lemma 1:** If a real number $x \geq 3$, then $\frac{2(2x-1)}{x-1} > \left( \frac{x}{x-1} \right)^x$ \hfill (2.1)

**Proof:**

Let $f_1(x) = \frac{2(2x-1)}{x-1}$, then $f_1'(x) = \frac{2(x-1)(2x-1)' - 2(2x-1)(x-1)'}{(x-1)^2} = \frac{-2}{(x-1)^2} < 0$.

Thus $f_1(x)$ is a strictly decreasing function for $x > 1$.

Since $f_1(3) = 5$ and $\lim_{x \to \infty} f_1(x) = 4$, for $x \geq 3$, we have $5 \geq f_1(x) = \frac{2(2x-1)}{x-1} \geq 4$. \hfill (2.1.1)

Let $f_2(x) = \left( \frac{x}{x-1} \right)^x$, then $f_2'(x) = \left( \left( \frac{x}{x-1} \right)^x \right)' = \left( e^{x \cdot \ln \frac{x}{x-1}} \right)' = e^{x \cdot \ln \frac{x}{x-1}} \cdot \left( x \cdot \ln \frac{x}{x-1} \right)$.

Thus, $f_2'(x) = \left( \frac{x}{x-1} \right)^x \cdot \left( \ln \frac{x}{x-1} + x \cdot \left( \ln \frac{x}{x-1} \right)' \right) = \left( \frac{x}{x-1} \right)^x \cdot \left( \ln \frac{x}{x-1} + x \cdot \ln \frac{x}{x-1} \cdot \frac{x-1-x}{x(x-1)^2} \right)$ \hfill (2.1.2)
In (2.1.2), \( \frac{1}{x-1} = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \frac{1}{x^5} + \frac{1}{x^6} + \cdots \) — (2.1.3)

Using the formula: \( \ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots \), we have

\[
\ln \frac{x}{x-1} = \ln \frac{1}{1 + \frac{1}{x}} = -\ln \left(1 + \frac{1}{x}\right) = \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{3x^3} + \frac{1}{4x^4} + \frac{1}{5x^5} + \frac{1}{6x^6} + \cdots
\] — (2.1.4)

Thus for \( x \geq 3 \), \( \ln \frac{x}{x-1} - \frac{1}{x-1} < 0 \) — (2.1.5)

Since \( \left(\frac{x}{x-1}\right)^x \) is a positive number for \( x \geq 3 \), \( f_2'(x) = \left(\frac{x}{x-1}\right)^x \cdot \left(\ln \frac{x}{x-1} - \frac{1}{x-1}\right) < 0 \). — (2.1.6)

Thus \( f_2(x) \) is a strictly deceasing function for \( x \geq 3 \).

Since \( f_2(3) = 3.375 \) and \( \lim_{x \to \infty} f_2(x) = e \approx 2.718 \), for \( x \geq 3 \), \( 3.375 \geq f_2(x) = \left(\frac{x}{x-1}\right)^x \geq e \) — (2.1.7)

Since for \( x \geq 3 \), \( f_1(x) \) has a lower bound of 4 and \( f_2(x) \) has an upper bound of 3.375,

\[
f_1(x) = \frac{2(2x-1)}{x-1} > f_2(x) = \left(\frac{x}{x-1}\right)^x\text{ is proven.}
\] — (2.1.8)

**Lemma 2:** For \( n \geq 2 \) and \( \lambda \geq 3 \), \( \left(\frac{\lambda^n}{n}\right) > \frac{\lambda^{\lambda n - \lambda + 1}}{n(\lambda - 1)(\lambda - 1)n - \lambda + 1} \) — (2.2)

**Proof:**

When \( \lambda \geq 3 \) and \( n = 2 \), \( \left(\frac{\lambda^n}{n}\right) = \left(\frac{\lambda^2}{2}\right) = \frac{2\lambda(2\lambda - 1)(2\lambda - 2)!}{2(2\lambda - 2)!} = \lambda(2\lambda - 1) \) — (2.2.1)

\[
\frac{\lambda^{\lambda n - \lambda + 1}}{n(\lambda - 1)(\lambda - 1)n - \lambda + 1} = \frac{\lambda^{2\lambda - \lambda + 1}}{2(\lambda - 1)^2(\lambda - 1)n - \lambda + 1} = \frac{\lambda(\lambda - 1)}{2} \cdot \left(\frac{\lambda}{\lambda - 1}\right)^\lambda
\] — (2.2.2)

In (2.1) when \( x = \lambda \geq 3 \), we have \( \frac{2(2\lambda - 1)}{\lambda - 1} > \left(\frac{\lambda}{\lambda - 1}\right)^\lambda \) — (2.2.3)

Since \( \frac{\lambda(\lambda - 1)}{2} \) is a positive number for \( \lambda \geq 3 \), referring to (2.2.1) and (2.2.2), when \( \frac{\lambda(\lambda - 1)}{2} \) multiplies to both sides of (2.2.3), we have

\[
\left(\frac{\lambda(\lambda - 1)}{2}\right) \left(\frac{2(2\lambda - 1)}{\lambda - 1}\right) = \lambda(2\lambda - 1) = \left(\frac{\lambda^n}{n}\right) > \left(\frac{\lambda(\lambda - 1)}{2}\right) \left(\frac{\lambda}{\lambda - 1}\right)^\lambda = \frac{\lambda^{\lambda n - \lambda + 1}}{n(\lambda - 1)(\lambda - 1)n - \lambda + 1}
\]

Thus, \( \left(\frac{\lambda^n}{n}\right) > \frac{\lambda^{\lambda n - \lambda + 1}}{n(\lambda - 1)(\lambda - 1)n - \lambda + 1} \) when \( \lambda \geq 3 \) and \( n = 2 \). — (2.2.4)

By induction on \( n \), when \( \lambda \geq 3 \), if \( \left(\frac{\lambda^n}{n}\right) > \frac{\lambda^{\lambda n - \lambda + 1}}{n(\lambda - 1)(\lambda - 1)n - \lambda + 1} \) is true for \( n \), then for \( n + 1 \), we have

\[
\left(\frac{\lambda(n+1)}{n+1}\right) = \left(\frac{\lambda^n + \lambda}{n+1}\right) = \frac{(\lambda n + \lambda)(\lambda n + \lambda - 1) \cdots (\lambda n + 2)(\lambda n + 1)}{(\lambda n + \lambda - 1)(\lambda n + \lambda - 2) \cdots (\lambda - 1)} \cdot \left(\frac{\lambda^n}{n}\right)
\]

\[
\left(\frac{\lambda(n+1)}{n+1}\right) > \frac{(\lambda n + \lambda)(\lambda n + \lambda - 1) \cdots (\lambda n + 2)(\lambda n + 1)}{(\lambda n + \lambda - 1)(\lambda n + \lambda - 2) \cdots (\lambda - 1)} \cdot \frac{\lambda^{\lambda n - \lambda + 1}}{n(\lambda - 1)(\lambda - 1)n - \lambda + 1}
\]

\[
\left(\frac{\lambda(n+1)}{n+1}\right) > \frac{(\lambda n + \lambda)(\lambda n + \lambda - 1) \cdots (\lambda n + 2)}{(\lambda n + \lambda - 1)(\lambda n + \lambda - 2) \cdots (\lambda - 1)} \cdot \frac{\lambda^{\lambda n - \lambda + 1}}{n(\lambda - 1)(\lambda - 1)n - \lambda + 1}
\]
Notice \( \frac{\lambda n+1}{n} > \lambda \), and

\[
\frac{(\lambda n+\lambda)(\lambda n+\lambda-1)\cdots(\lambda n+2)}{(\lambda n+\lambda-1)(\lambda n+\lambda-2)\cdots(\lambda n+1)} > \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}
\]

because

\[
\frac{\lambda n+\lambda}{\lambda n+\lambda-1} > \frac{\lambda}{\lambda-1}, \quad \frac{\lambda n+\lambda-1}{\lambda n+\lambda-2} > \frac{\lambda}{\lambda-1}, \quad \cdots \quad \frac{\lambda n+2}{\lambda n+1} > \frac{\lambda}{\lambda-1}.
\]

Thus

\[
\frac{\lambda (n+1)}{n+1} \cdot \frac{\lambda}{1} \cdot \frac{1}{(n+1)} \cdot \frac{\lambda n-\lambda+1}{(\lambda(n-\lambda-1)(\lambda-1)n-\lambda+1)} = \frac{\lambda (n+1)}{n(n-\lambda)(\lambda-1)n-\lambda+1}
\]

From (2.2.4) and (2.2.5), we have for \( n \geq 2 \) and \( \lambda \geq 3 \),

\[
\frac{\lambda n}{n} > \frac{\lambda (n+1)}{n(n-\lambda)(\lambda-1)n-\lambda+1}
\]

Thus, Lemma 2 is proven.

3. A Prime Number between \((\lambda - 1)n\) and \(\lambda n\) when \(n \geq (\lambda - 2) \geq 24\)

Proposition:

For \( n \geq (\lambda - 2) \geq 24 \), there exists at least a prime number \( p \) such that \((\lambda - 1)n < p \leq \lambda n\). — (3.1)

Proof:

When \( n \geq (\lambda - 2) \geq 24 \), in \( \Gamma_{\lambda n \geq p > n} \left\{ \frac{\lambda n!}{(\lambda-1)n!} \right\} \), \( p \geq n+1 > \sqrt{(n+2)n} > [\sqrt[n]{n}] \). Referring to (1.5), we have

\[ 0 \leq \nu_p \left( \Gamma_{\lambda n \geq p > n} \left\{ \frac{\lambda n!}{(\lambda-1)n!} \right\} \right) \leq R(p) \leq 1. \]

And \( n \geq (\lambda - 2) \geq [\sqrt[n]{n}] \geq 24 \).

\[
\Gamma_{\lambda n \geq p > n} \left\{ \frac{\lambda n!}{(\lambda-1)n!} \right\} = \Gamma_{\lambda n \geq p > (\lambda-1)n} \left\{ \frac{\lambda n!}{(\lambda-1)n!} \right\} \cdot \prod_{i=1}^{i=2} \left( \Gamma_{\lambda n \geq p > (\lambda-1)n} \left\{ \frac{\lambda n!}{(\lambda-1)n!} \right\} \right)
\]

In every distinct prime number \( p \) in these ranges, the numerator \( \lambda n! \) has the product of \( p \cdot 2p \cdot 3p \cdots ip \). The denominator \((\lambda-1)n!\) also has the same product of \( i! \cdot p \). Thus, they cancel to each other in \( \frac{\lambda n!}{(\lambda-1)n!} \).

Referring to (1.2), \( \prod_{i=1}^{i=2} \left( \Gamma_{\lambda n \geq p > (\lambda-1)n} \left\{ \frac{\lambda n!}{(\lambda-1)n!} \right\} \right) = 1 \).

Therefore, when \( n \geq \lambda - 2 \geq 24 \),

\[
\Gamma_{\lambda n \geq p > n} \left\{ \frac{\lambda n!}{(\lambda-1)n!} \right\} = \Gamma_{\lambda n \geq p > (\lambda-1)n} \left\{ \frac{\lambda n!}{(\lambda-1)n!} \right\} \cdot \prod_{i=1}^{i=2} \left( \Gamma_{\lambda n \geq p > (\lambda-1)n} \left\{ \frac{\lambda n!}{(\lambda-1)n!} \right\} \right) = \prod_{i=1}^{\lambda = 3} \left( \Gamma_{\lambda n \geq p > (\lambda-1)n} \left\{ \frac{\lambda n!}{(\lambda-1)n!} \right\} \right).
\]

Referring to (1.7), \( \frac{\lambda n!}{n} < \Gamma_{\lambda n \geq p > n} \left\{ \frac{\lambda n!}{(\lambda-1)n!} \right\} \cdot 2^{2n-3} \cdot \left(\lambda n\right)^{\sqrt[n]{n}} \). Applying this inequality to (2.2), when \( n \geq (\lambda - 2) \geq 24 \), we have

\[
\frac{\lambda n-\lambda+1}{n(n-\lambda)(\lambda-1)n-\lambda+1} < \left(\frac{\lambda n}{\lambda n-\lambda+1} \right)^{\lambda n-\lambda+1} \cdot 2^{2n-3} \cdot \left(\lambda n\right)^{\sqrt[n]{n}} + 1.
\]

\[
\frac{\lambda n^2}{n(n-\lambda)(\lambda-1)n-\lambda+1} < \Gamma_{\lambda n \geq p > n} \left\{ \frac{\lambda n!}{(\lambda-1)n!} \right\} \cdot 2^{2n-3} \cdot \left(\lambda n\right)^{\sqrt[n]{n}} + 1. \]

Since \( (\lambda n)^{\sqrt[n]{n}} + 1 > 1 \) and \( 2^{2n-3} > 1 \),
\[ f(x, y) = \frac{2y^2 \cdot \left( \frac{y-1}{4} \right) \cdot e^{(x-1)}}{(xy)^{\frac{3}{3}+2}} \] where both \( x \) and \( y \) are positive real numbers.  

When \( x = (y - 2) \), \( f_3(x, y) = \frac{2(x+2)^2 \cdot \left( \frac{x+1}{4} \right) \cdot e^{(x-1)}}{(x(x+2))^{\frac{3}{3}+2}} > f_4(x) = \frac{2(x+2)^2 \cdot \left( \frac{x+1}{4} \right) \cdot e^{(x-1)}}{(x(x+2))^{\frac{3}{3}+2}} \)  

\[ f_4'(x) = f_4(x) \cdot \left( \frac{2}{x+2} + \ln \left( \frac{x+1}{4} \right) + \frac{4}{3} - \frac{2}{x+1} - \frac{1}{3} \ln(x \cdot (x + 2)) - \frac{7}{3x} - \frac{5}{3(x+2)} \right) = f_4(x) \cdot f_5(x) \]

where \( f_5(x) = \frac{2}{x+2} + \ln \left( \frac{x+1}{4} \right) + \frac{4}{3} - \frac{2}{x+1} - \frac{1}{3} \ln(x \cdot (x + 2)) - \frac{7}{3x} - \frac{5}{3(x+2)} \)

\[ f_5'(x) = \frac{4x+6}{(x+1)^2(x+2)^2} + \frac{7}{3x} + \frac{5}{3(x+2)^2} > 0. \] Thus, \( f_5(x) \) is a strictly increasing function for \( x \geq 1 \).

When \( x = 7 \), \( f_5(x) = \frac{2}{7+2} + \ln \left( \frac{7+1}{4} \right) + \frac{4}{3} - \frac{2}{7+1} - \frac{1}{3} \ln(7 + 2) - \frac{7}{21} - \frac{5}{27} > 0. \) Thus, for \( x \geq 7 \), \( f_5(x) > 0 \). Then, \( f_4'(x) = f_4(x) \cdot f_5(x) > 0. \) Thus, \( f_4(x) \) is a strictly increasing function for \( x \geq 7 \).

When \( x = 16 \), \( f_4(x) = \frac{2 \cdot (18)^2 \cdot \left( \frac{18}{4} \right)^{15} \cdot e^{15}}{(16 \cdot 18)^{\frac{16+1}{3}+2}} \approx \frac{5.647E+18}{7.167E+18} < 1, \) then for \( x \leq 16 \), \( f_4(x) < 1 \).  

When \( x = 17 \), \( f_4(x) = \frac{2 \cdot (19)^2 \cdot \left( \frac{19}{4} \right)^{16} \cdot e^{16}}{(17 \cdot 19)^{\frac{17+1}{3}+2}} \approx \frac{1.814E+20}{1.185E+20} > 1, \) then for \( x \geq 17 \), \( f_4(x) > 1 \).  

Referring to (3.5), when \( x = (y - 2) \geq 17 \), \( f_3(x, y) > f_4(x) > 1 \).  

From (3.4), \( f_3(x, y) = \frac{2y^2 \cdot \left( \frac{y-1}{4} \right) \cdot e^{(x-1)}}{(xy)^{\frac{3}{3}+2}} \)

\[ \frac{\partial f_3(x, y)}{\partial x} = f_3(x, y) \cdot \left( \ln \left( \frac{y-1}{4} \right) + 1 - \frac{\sqrt{y}}{6x} \cdot \ln(yx) - \frac{\sqrt{y}}{3x^2} - \frac{2}{x} \right) = f_3(x, y) \cdot f_6(x, y) \]  

where \( f_6(x, y) = \ln \left( \frac{y-1}{4} \right) + 1 - \frac{\sqrt{y}}{6x} \cdot \ln(yx) - \frac{\sqrt{y}}{3x^2} - \frac{2}{x} \)

\[ \frac{\partial f_6(x, y)}{\partial x} = \frac{\sqrt{y}}{12x\sqrt{x}} \cdot \ln(y) + \frac{\sqrt{y}}{12x\sqrt{x}} \cdot \ln(x) + \frac{\sqrt{y}}{6x\sqrt{x}} + \frac{\sqrt{y}}{6x\sqrt{x}} + \frac{2}{x^2} > 0 \] when \( x \geq 1 \) and \( y \geq 1 \).

Thus, \( f_6(x, y) \) is a strictly increasing function with respect to \( x \) when \( x \geq 1 \) and \( y \geq 1 \).
When \( x = (y - 2) \geq 7 \), \( f_6(x, y) = \ln \left( \frac{9 - 1}{4} \right) + 1 - \frac{\sqrt{9}}{3\sqrt{7}} \cdot \ln(7) - \frac{\sqrt{9}}{3\sqrt{7}} \cdot \frac{2}{7} > 0 \).

Thus, when \( x \geq (y - 2) \geq 7 \), \( f_6(x, y) > 0 \), then from (3.9), \( \frac{\partial f_3(x, y)}{\partial x} = f_3(x, y) \cdot f_6(x, y) > 0 \).

Thus, \( f_3(x, y) \) is a strictly increasing function with respect to \( x \) when \( x \geq (y - 2) \geq 7 \). — (3.10)

Referring to (3.8), when \( x = (y - 2) \geq 17 \), \( f_3(x, y) > 1 \). Thus, when \( x \geq (y - 2) \geq 17 \), \( f_3(x, y) > 1 \).

Let \( x = n \) and \( y = \lambda \), then when \( n \geq (\lambda - 2) \geq 17 \), \( f_3(n, \lambda) > 1 \). — (3.11)

From (3.2), (3.3) and (3.11), when \( n \geq (\lambda - 2) \geq 24 \),

\[
\Gamma_{\lambda n \geq p > n} \left\{ \frac{(\lambda n)!}{((\lambda - 1)n)!} \right\} = \prod_{i=1}^{\lambda - 1} \left( \Gamma_{\lambda n \geq p > (\lambda - 1)\frac{n}{i}} \left\{ \frac{(\lambda n)!}{((\lambda - 1)n)!} \right\} \right) > f_3(n, \lambda) > 1.
\] — (3.12)

Referring to (3.10), \( \prod_{i=1}^{\lambda - 1} \left( \Gamma_{\lambda n \geq p > (\lambda - 1)\frac{n}{i}} \left\{ \frac{(\lambda n)!}{((\lambda - 1)n)!} \right\} \right) \) is a strictly increasing function with respect to \( n \) when \( n \geq (\lambda - 2) \geq 24 \). — (3.13)

\[
\prod_{i=1}^{\lambda - 1} \left( \Gamma_{\lambda n \geq p > (\lambda - 1)\frac{n}{i+1}} \left\{ \frac{(\lambda n)!}{((\lambda - 1)n)!} \right\} \right) = \Gamma_{\lambda n \geq p > (\lambda - 1)n} \left\{ \frac{(\lambda n)!}{((\lambda - 1)n)!} \right\} \cdot \prod_{i=1}^{\lambda - 2} \left( \Gamma_{\lambda n \geq p > (\lambda - 1)\frac{n}{i+1}} \left\{ \frac{(\lambda n)!}{((\lambda - 1)n)!} \right\} \right) \geq 1.
\]

Refrering to (1.1), \( \Gamma_{\lambda n \geq p > (\lambda - 1)n} \left\{ \frac{(\lambda n)!}{((\lambda - 1)n)!} \right\} \geq 1 \) and \( \prod_{i=1}^{\lambda - 2} \left( \Gamma_{\lambda n \geq p > (\lambda - 1)\frac{n}{i+1}} \left\{ \frac{(\lambda n)!}{((\lambda - 1)n)!} \right\} \right) \geq 1 \).

Refrering to (3.12), at least one of these two parts is greater than 1.

If \( \Gamma_{\lambda n \geq p > (\lambda - 1)n} \left\{ \frac{(\lambda n)!}{((\lambda - 1)n)!} \right\} > 1 \), then refrering to (1.3), there exists at least a prime number \( p \) such that \( (\lambda - 1)n < p \leq \lambda n \).

If \( \prod_{i=1}^{\lambda - 2} \left( \Gamma_{\lambda n \geq p > (\lambda - 1)\frac{n}{i+1}} \left\{ \frac{(\lambda n)!}{((\lambda - 1)n)!} \right\} \right) = 1 \), then \( \Gamma_{\lambda n \geq p > (\lambda - 1)n} \left\{ \frac{(\lambda n)!}{((\lambda - 1)n)!} \right\} > 1 \). — (3.15)

If \( \prod_{i=1}^{\lambda - 2} \left( \Gamma_{\lambda n \geq p > (\lambda - 1)\frac{n}{i+1}} \left\{ \frac{(\lambda n)!}{((\lambda - 1)n)!} \right\} \right) > 1 \), then at least one factor \( \Gamma_{\lambda n \geq p > (\lambda - 1)\frac{n}{i+1}} \left\{ \frac{(\lambda n)!}{((\lambda - 1)n)!} \right\} > 1 \).

When a factor \( \Gamma_{\lambda n \geq p > (\lambda - 1)\frac{n}{i+1}} \left\{ \frac{(\lambda n)!}{((\lambda - 1)n)!} \right\} > 1 \), let integer \( m = (i+1)n \), then \( m > n \).

Refrering to (3.13), \( \Gamma_{\lambda m \geq p > (\lambda - 1)m} \left\{ \frac{(\lambda m)!}{((\lambda - 1)m)!} \right\} = \Gamma_{\lambda n \geq p > (\lambda - 1)n} \left\{ \frac{(\lambda n)!}{((\lambda - 1)n)!} \right\} \cdot \prod_{i=1}^{\lambda - 1} \left( \Gamma_{\lambda n \geq p > (\lambda - 1)\frac{n}{i+1}} \left\{ \frac{(\lambda n)!}{((\lambda - 1)n)!} \right\} \right) > 1 \). — (3.16)

\[
\frac{(\lambda n)!}{((\lambda - 1)n)!} = (\lambda n)(\lambda n - 1)(\lambda n - 2) \cdots ((\lambda - 1)n + 1). \] This product has \( n \) factors.

\[
\frac{(\lambda m)!}{((\lambda - 1)m)!} = (\lambda m)(\lambda m - 1)(\lambda m - 2) \cdots ((\lambda - 1)m + 1). \] This product has \( m = (i+1)n \) factors.

Since \( m = (i+1)n \), \( \frac{(\lambda m)!}{((\lambda - 1)m)!} \) contains the factors of \( \frac{(\lambda n)!}{((\lambda - 1)n)!} \cdot (i + 1)^n \).

\[
\frac{(\lambda m)!}{((\lambda - 1)m)!} = \Gamma_{\lambda m \geq p \cdot \lambda n} \left\{ \frac{(\lambda m)!}{((\lambda - 1)m)!} \right\} \cdot \Gamma_{\lambda n \geq p > (\lambda - 1)n} \left\{ \frac{(\lambda n)!}{((\lambda - 1)n)!} \right\} \cdot \prod_{i=1}^{\lambda - 1} \left( \Gamma_{\lambda n \geq p > (\lambda - 1)\frac{n}{i+1}} \left\{ \frac{(\lambda n)!}{((\lambda - 1)n)!} \right\} \right) > 1 \). Notice that \((i + 1) \leq (\lambda - 1)\). Prime numbers in \((i + 1)\) are not in the range of \(\lambda n \geq p > (\lambda - 1)n\).
Since prime numbers in $\Gamma_{\lambda m \geq p} \lambda n \left\{ \frac{(\lambda m)!}{(\lambda - 1)m)!} \right\}$, $\Gamma_{\lambda n \geq p} (\lambda - 1)n \left\{ (i + 1)^n \right\}$, and $\Gamma_{(\lambda - 1)n \geq p} (\frac{\lambda m)!}{(\lambda - 1)m)!} \right\}$ are not in the range of $\lambda n \geq p > (\lambda - 1)n$, $\Gamma_{\lambda n \geq p} (\lambda - 1)n \left\{ \frac{(\lambda m)!}{(\lambda - 1)m)!} \right\} = 1$, $\Gamma_{\lambda n \geq p} (\lambda - 1)n \left\{ (i + 1)^n \right\} = 1$, and $\Gamma_{(\lambda - 1)n \geq p} (\frac{\lambda m)!}{(\lambda - 1)m)!} \right\} = 1$.

Referring to (3.16), $\Gamma_{\lambda n \geq p} (\lambda - 1)n \left\{ \frac{(\lambda m)!}{(\lambda - 1)m)!} \right\} = \Gamma_{\lambda n \geq p} (\lambda - 1)n \left\{ (\frac{(\lambda n)!}{(\lambda - 1)n)!} \right\} > 1$.

Thus, if $\prod_{i=1}^{\lambda - 2} \left( \frac{(\lambda n)!}{(\lambda - 1)n)!} \frac{(\lambda n)!}{(\lambda - 1)n)!} \right) > 1$, then $\Gamma_{\lambda n \geq p} (\lambda - 1)n \left\{ (\frac{(\lambda n)!}{(\lambda - 1)n)!} \right\} > 1$, and there exists at least a prime number $p$ such that $(\lambda - 1)n < p \leq \lambda n$. — (3.17)

Combining (3.13), (3.14), (3.15), and (3.17), we have proven the Proposition, (3.1):

For $n \geq (\lambda - 2) \geq 24$, there exists at least a prime number $p$ such that $(\lambda - 1)n < p \leq \lambda n$.

It becomes a theorem: Theorem (3.1).

4. The Proof of Legendre's Conjecture

Legendre's Conjecture states that there is a prime number between $n^2$ and $(n + 1)^2$ for every positive integer $n$. — (4.1)

Proof:

Referring to Theorem (3.1), for integers $j \geq k \geq 2 \geq 24$, there exists at least a prime number $p$ such that $j(k - 1) < p \leq jk$. — (4.2)

When $k = j + 1 \geq 26$, then $j = k - 1 \geq 25$

Applying $k = j + 1$ into (4.2), then $j^2 < p \leq j(j + 1) < (j + 1)^2$

Let $n = j \geq 25$, then we have $n^2 < p < (n + 1)^2$. — (4.3)

For $1 \leq n \leq 24$, we have a table, Table 1, that shows Legendre's conjecture valid. — (4.4)

| Table 1: For $1 \leq n \leq 24$, there is a prime number between $n^2$ and $(n + 1)^2$. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| $n^2$ | 1 | 4 | 9 | 16 | 25 | 36 | 49 | 64 | 81 | 100 | 121 | 144 |
| $p$ | 3 | 5 | 11 | 19 | 29 | 41 | 53 | 67 | 83 | 103 | 127 | 149 |
| $(n + 1)^2$ | 4 | 9 | 16 | 25 | 36 | 49 | 64 | 81 | 100 | 121 | 144 | 169 |
| $n$ | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 |
| $n^2$ | 169 | 196 | 225 | 256 | 289 | 324 | 361 | 400 | 441 | 484 | 529 | 576 |
| $p$ | 173 | 199 | 229 | 263 | 307 | 331 | 373 | 409 | 449 | 491 | 541 | 587 |
| $(n + 1)^2$ | 196 | 225 | 256 | 289 | 324 | 361 | 400 | 441 | 484 | 529 | 576 | 625 |

Combining (4.3) and (4.4), we have proven Legendre's conjecture.
Extension of Legendre’s conjecture

There are at least two prime numbers, \( p_n \) and \( p_m \), between \( j^2 \) and \( (j + 1)^2 \) for every positive integer \( j \) such that \( j^2 < p_n \leq j(j+1) \) and \( j(j+1) < p_m < (j + 1)^2 \) where \( p_n \) is the \( n \)th prime number, \( p_m \) is the \( m \)th prime number, and \( m \geq n + 1 \). — (4.5)

**Proof:**

Referring to Theorem (3.1), for integers \( j \geq k - 2 \geq 24 \), there exists at least a prime number \( p \) such that \( j(k - 1) < p \leq jk \).

When \( k - 1 = j \geq 25 \), then \( j(k - 1) = j^2 < p_n \leq jk = j(j+1) \). Thus, there is at least a prime number \( p_n \) such that \( j^2 < p_n \leq j(j+1) \) when \( j = k - 1 \geq 25 \).

When \( j = k - 2 \geq 25 \), then \( k = j + 2 \). Thus, \( j(k - 1) = j(j+1) < p_m \leq jk = j(j+2) < (j + 1)^2 \). Thus, there is at least another prime number \( p_m \) such that \( j(j+1) < p_m < (j + 1)^2 \) when \( j = k - 2 \geq 25 \).

Thus, when \( j \geq 25 \), there are at least two prime numbers \( p_n \) and \( p_m \) between \( j^2 \) and \( (j + 1)^2 \) such that \( j^2 < p_n \leq j(j+1) < p_m < (j + 1)^2 \) where \( m \geq n + 1 \) for \( p_m > p_n \). — (4.6)

For \( 1 \leq j \leq 24 \), we have a table, Table 2, that shows (4.5) valid. — (4.7)

**Table 2:** For \( 1 \leq j \leq 24 \), there are 2 prime numbers such that \( j^2 < p_n \leq j(j+1) < p_m < (j + 1)^2 \).

<table>
<thead>
<tr>
<th>( j )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<td>9</td>
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<td>67</td>
<td>83</td>
<td>103</td>
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</tr>
<tr>
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<td>6</td>
<td>12</td>
<td>20</td>
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<td>42</td>
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<td>( j^2 )</td>
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<td>240</td>
<td>272</td>
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<td>380</td>
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<tr>
<td>( (j + 1)^2 )</td>
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<td>256</td>
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<td>324</td>
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<td>400</td>
<td>441</td>
<td>484</td>
<td>529</td>
<td>576</td>
<td>625</td>
</tr>
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Combining (4.6) and (4.7), we have proven (4.5). It becomes a theorem: Theorem (4.5).
5. The Proofs of Three Related Conjectures

**Oppermann’s conjecture** was proposed by Ludvig Oppermann [4] in March 1877. It states that for every integer \( x > 1 \), there is at least one prime number between \( x(x - 1) \) and \( x^2 \), and at least another prime between \( x^2 \) and \( x(x + 1) \).

--- (5.1)

**Proof:**

**Theorem (4.5)** states there are at least two prime numbers, \( p_n \) and \( p_m \), between \( j^2 \) and \( (j + 1)^2 \) for every positive integer \( j \) such that \( j^2 < p_n \leq j(j+1) \) and \( j(j+1) < p_m < (j + 1)^2 \) where \( m \geq n + 1 \) for \( p_m > p_n \).

\( j(j+1) \) is a composite number except \( j = 1 \). Since \( j^2 < p_n \leq j(j+1) \) is valid for every positive integer \( j \), when we replace \( j \) with \( j+1 \), we have \( (j + 1)^2 < p_n < (j+1)(j+2) \).

Thus, we have \( j(j+1) < p_m < (j + 1)^2 < p_v < (j+1)(j+2) \).

--- (5.2)

When \( x > 1 \), then \( x - 1 \geq 1 \). Substitute \( j \) with \( x - 1 \) in (5.2), we have \( x(x - 1) < p_m < x^2 < p_v < x(x + 1) \)

Thus, we have proven Oppermann’s conjecture.

--- (5.3)

**Brocard’s conjecture** is named after Henri Brocard [5]. It states that there are at least 4 prime numbers between \( (p_n)^2 \) and \( (p_{n+1})^2 \), where \( p_n \) is the \( n \)th prime number, for every \( n > 1 \).

--- (5.4)

**Proof:**

**Theorem (4.5)** states there are at least two prime numbers, \( p_n \) and \( p_m \), between \( j^2 \) and \( (j + 1)^2 \) for every positive integer \( j \) such that \( j^2 < p_n \leq j(j+1) \) and \( j(j+1) < p_m < (j + 1)^2 \) where \( m \geq n + 1 \) for \( p_m > p_n \). When \( j > 1 \), \( j(j+1) \) is a composite number. Then **Theorem (4.5)** can be written as \( j^2 < p_n < j(j+1) \) and \( j(j+1) < p_m < (j + 1)^2 \).

In the series of prime numbers: \( p_1=2, \ p_2=3, \ p_3=5, \ p_4=7, \ p_5=11... \) all prime numbers except \( p_1 \) are odd numbers. Their gaps are two or more. Thus when \( n > 1, \ (p_{n+1} - p_n) \geq 2 \).

Thus, we have \( p_n < (p_n + 1) < p_{n+1} \) when \( n > 1 \).

--- (5.5)

Applying **Theorem (4.5)** to (5.5), when \( n > 1 \), we have at least two prime numbers \( p_{m1} \), \( p_{m2} \) in between \( (p_n)^2 \) and \( (p_n + 1)^2 \) such that \( (p_n)^2 < p_{m1} < p_n(p_n+1) < p_{m2} < (p_n + 1)^2 \), and at least two more prime numbers \( p_{m3} \), \( p_{m4} \) in between \( (p_n + 1)^2 \) and \( (p_{n+1})^2 \) such that \( (p_n + 1)^2 < p_{m3} < p_{n+1}(p_{n+1}) < p_{m4} < (p_{n+1})^2 \).

Thus, there are at least 4 prime numbers between \( (p_n)^2 \) and \( (p_{n+1})^2 \) for \( n > 1 \) such that \( (p_n)^2 < p_{m1} < p_n(p_n+1) < p_{m2} < (p_n + 1)^2 < p_{m3} < p_{n+1}(p_{n+1}) < p_{m4} < (p_{n+1})^2 \).

--- (5.6)

Thus, Brocard’s conjecture is proven.
Andrica’s conjecture is named after Dorin Andrica. It is a conjecture regarding the gaps between prime numbers. The conjecture states that the inequality $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$ holds for all $n$ where $p_n$ is the $n$th prime number. If $g_n = p_{n+1} - p_n$ denotes the $n$th prime gap, then Andrica’s conjecture can also be rewritten as $g_n < 2\sqrt{p_n} + 1$. — (5.7)

Proof:

From Theorem (4.5), for every positive integer $j$, there are at least two prime numbers $p_n$ and $p_m$ between $j^2$ and $(j + 1)^2$ such that $j^2 < p_n < j(j+1) < p_m < (j + 1)^2$ where $m \geq n + 1$ for $p_m > p_n$.

Since $m \geq n + 1$, we have $p_m \geq p_{n+1}$.

Thus, we have $j^2 < p_n$. — (5.8)

And $p_{n+1} \leq p_m < (j + 1)^2$. — (5.9)

Since $j$, $p_n$, $p_{n+1}$ and $(j + 1)$ are positive integers,

$j < \sqrt{p_n}$ — (5.10)

And $\sqrt{p_{n+1}} < j + 1$ — (5.11)

Applying (5.10) to (5.11), we have $\sqrt{p_{n+1}} < \sqrt{p_n} + 1$. — (5.12)

Thus, $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$ holds for all $n$ since in Theorem (4.5), $j$ holds for all positive integers.

Using the prime gap to prove the conjecture, from (5.8) and (5.9), we have

$g_n = p_{n+1} - p_n < (j + 1)^2 - j^2 = 2j + 1$. From (5.10), $j < \sqrt{p_n}$.

Thus, $g_n = p_{n+1} - p_n < 2\sqrt{p_n} + 1$. — (5.13)

Thus, Andrica’s conjecture is proven.

6. References