For Students of Geometric Algebra: Demonstrating the Equivalence of Different Formulas for Rotating 3D Vectors

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Abstract

As an aid to teachers and students of introductory-level Geometric Algebra, we address a doubt that an inquiring student might (and should) ask: "Are GA's various formulas for rotating a vector truly equivalent?" Here, we use GA identities to prove the equivalence of two rotation formulas. Rather than merely present the proof, we first review the relevant identities, then formulate and explore reasonable conjectures that lead to the conclusion. Readers are encouraged to examine, in addition, a recent publication by Verhoeff ([1]) that contrasts "GA" rotation methods with methods that use matrices or classical geometry.

1 Two "GA" Formulas for Rotating a Vector in 3D: Are They Equivalent?

An important feature of GA is its ability to express and manipulate rotations conveniently. For example, the vector $\mathbf{v}'$ in Fig. 1 can be written as

$$\mathbf{v}' = \left[e^{-\hat{\mathbf{B}}\frac{\theta}{2}}\right] \mathbf{v} \left[e^{\hat{\mathbf{B}}\frac{\theta}{2}}\right]. \tag{1.1}$$

This formulation of rotations is especially useful when we must find the result of two or more successive rotations.

Macdonald's ([2], p. 89) derivation of Eq. (1.1) is presented and explained in the Appendix (A) of the present document. Macdonald begins by expressing

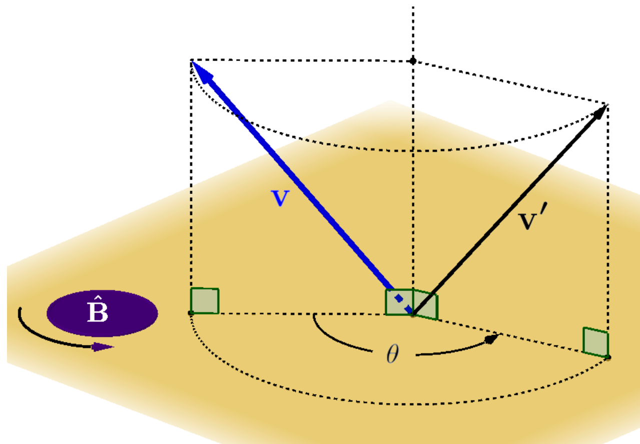

Figure 1: Vector $\mathbf{v}'$ is the vector that results from rotating $\mathbf{v}$ through the angle θ about an axis that is perpendicular to the bivector $\hat{\mathbf{B}}$, and in the same sense of rotation as $\hat{\mathbf{B}}$.

$\mathbf{v}$ as the sum of its components parallel and perpendicular to $\hat{\mathbf{B}}$ (Fig. 2). Then, Macdonald notes that although the vertical component is unaffected by the rotation, the parallel component becomes $\mathbf{v}_{\parallel}e^{\hat{\mathbf{B}}\theta}$. Therefore, $\mathbf{v}'$ can also be written as

$$\begin{aligned}\mathbf{v}' &= \mathbf{v}_{\perp} + \mathbf{v}_{\parallel}\left[e^{\hat{\mathbf{B}}\frac{\theta}{2}}\right] \\ &= \mathbf{v}_{\perp} + \mathbf{v}_{\parallel}\left[\cos\theta + \hat{\mathbf{B}}\sin\theta\right] \\ &= \mathbf{v}_{\perp} + \mathbf{v}_{\parallel}\cos\theta + \mathbf{v}_{\parallel}\hat{\mathbf{B}}\sin\theta. \end{aligned} \tag{1.2}$$

How might we demonstrate that Eqs. (1.1) and (1.2) are equivalent? We begin by expanding Eq. (1.1):

$$\begin{aligned}\mathbf{v}' &= \left[\cos\frac{\theta}{2} - \hat{\mathbf{B}}\sin\frac{\theta}{2}\right]\mathbf{v}\left[\cos\frac{\theta}{2} + \hat{\mathbf{B}}\sin\frac{\theta}{2}\right] \\ &= \mathbf{v}\cos^2\frac{\theta}{2} + \mathbf{v}\hat{\mathbf{B}}\cos\frac{\theta}{2}\sin\frac{\theta}{2} - \hat{\mathbf{B}}\mathbf{v}\sin\frac{\theta}{2}\cos\frac{\theta}{2} - \hat{\mathbf{B}}\mathbf{v}\hat{\mathbf{B}}\sin^2\frac{\theta}{2}. \end{aligned} \tag{1.3}$$

To make further progress, we need to review a bit.

2 Some GA and Trigonometric Identities

For any vector $\mathbf{v}$ and any unit bivector $\hat{\mathbf{B}}$,

1. The multiplicative inverse of $\hat{\mathbf{B}}$ (written $\hat{\mathbf{B}}^{-1}$) is $\text{-}\hat{\mathbf{B}}$

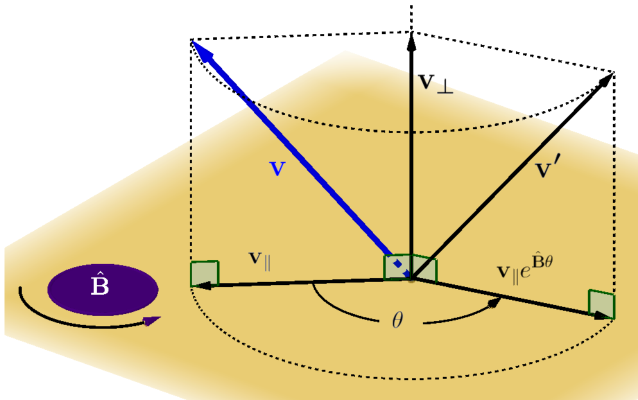

Figure 2: The vectors $\mathbf{v}_\parallel$ and $\mathbf{v}_\perp$ are (respectively) $\mathbf{v}$'s components parallel and perpendicular to $\hat{\mathbf{B}}$. The vertical component is unaffected by the rotation, but the parallel component becomes $\mathbf{v}_\parallel e^{\hat{\mathbf{B}}\theta}$.

2. $\hat{\mathbf{B}} \cdot \mathbf{v} = -\mathbf{v} \cdot \hat{\mathbf{B}}$
3. $\hat{\mathbf{B}} \wedge \mathbf{v} = \mathbf{v} \wedge \hat{\mathbf{B}}$
4. $\mathbf{v}\hat{\mathbf{B}} = \mathbf{v} \cdot \hat{\mathbf{B}} + \mathbf{v} \wedge \hat{\mathbf{B}}$
5. $\hat{\mathbf{B}}\mathbf{v} = \hat{\mathbf{B}} \cdot \mathbf{v} + \hat{\mathbf{B}} \wedge \mathbf{v} = -\mathbf{v} \cdot \hat{\mathbf{B}} + \mathbf{v} \wedge \hat{\mathbf{B}}$.
6. The components of $\mathbf{v}$ parallel to and perpendicular to $\hat{\mathbf{B}}$ are
   (a) $\mathbf{v}_\parallel = \left(\mathbf{v} \cdot \hat{\mathbf{B}}\right)\hat{\mathbf{B}}^{-1} = \left(\mathbf{v} \cdot \hat{\mathbf{B}}\right)\left(-\hat{\mathbf{B}}\right)$
   (b) $\mathbf{v}_\perp = \left(\mathbf{v} \wedge \hat{\mathbf{B}}\right)\hat{\mathbf{B}}^{-1} = \left(\mathbf{v} \wedge \hat{\mathbf{B}}\right)\left(-\hat{\mathbf{B}}\right)$.
7. From 3, 4, and 5 (above),
   (a) $\hat{\mathbf{B}}\mathbf{v} = -\mathbf{v}\hat{\mathbf{B}} + 2\mathbf{v} \wedge \hat{\mathbf{B}}$
   (b) $\hat{\mathbf{B}}\mathbf{v} = \mathbf{v}\hat{\mathbf{B}} - 2\mathbf{v} \cdot \hat{\mathbf{B}}$.
8. From trigonometry,
   (a) $2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} = \sin\alpha$
   (b) $\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2} = \cos\alpha$.

3 Demonstration of the Equivalence of Our Two Expressions for $\mathbf{v}'$

After reviewing the identities in Section 2, several possible possible routes suggest themselves. For example, we can combine the two $\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}$ terms in Eq. 1.3 to obtain

$$\mathbf{v}' = \mathbf{v}\cos^2\frac{\theta}{2} + \left(\mathbf{v}\hat{\mathbf{B}} - \hat{\mathbf{B}}\mathbf{v}\right)\sin\frac{\theta}{2}\cos\frac{\theta}{2} - \hat{\mathbf{B}}\mathbf{v}\hat{\mathbf{B}}\sin^2\frac{\theta}{2}.$$

Now, from point 7b, we see that $\mathbf{v}\hat{\mathbf{B}} - \hat{\mathbf{B}}\mathbf{v} = 2\mathbf{v} \cdot \hat{\mathbf{B}}$. Therefore,

$$\begin{aligned}
\mathbf{v}' &= \mathbf{v}\cos^2\frac{\theta}{2} + 2\mathbf{v}\cdot\hat{\mathbf{B}}\sin\frac{\theta}{2}\cos\frac{\theta}{2} - \hat{\mathbf{B}}\mathbf{v}\hat{\mathbf{B}}\sin^2\frac{\theta}{2} \\
&= \mathbf{v}\cos^2\frac{\theta}{2} + \mathbf{v}\cdot\hat{\mathbf{B}}\left[2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right] - \hat{\mathbf{B}}\mathbf{v}\hat{\mathbf{B}}\sin^2\frac{\theta}{2} \\
&= \mathbf{v}\cos^2\frac{\theta}{2} + \mathbf{v}\cdot\hat{\mathbf{B}}\sin\theta - \hat{\mathbf{B}}\mathbf{v}\hat{\mathbf{B}}\sin^2\frac{\theta}{2}.
\end{aligned} \tag{3.1}$$

We now have a $\sin\theta$ term in Eq. (3.1), just as we do in Eq. (1.2). We can demonstrate the equality of those two terms (i.e, that the $\mathbf{v}_\parallel\hat{\mathbf{B}}$ in Eq. (1.2) is the same thing as the $\mathbf{v}\cdot\hat{\mathbf{B}}$ in Eq. (3.1)) by noting that $\mathbf{v}_\parallel = \left(\mathbf{v}\cdot\hat{\mathbf{B}}\right)\hat{\mathbf{B}}^{-1}$, so that $\mathbf{v}_\parallel\hat{\mathbf{B}} = \left[\left(\mathbf{v}\cdot\hat{\mathbf{B}}\right)\hat{\mathbf{B}}^{-1}\right]\hat{\mathbf{B}} = \mathbf{v}\cdot\hat{\mathbf{B}}$.

What to do with the factor $\hat{\mathbf{B}}\mathbf{v}\hat{\mathbf{B}}$ in Eq. (3.1) may not be clear. One idea is to "reverse" the product $\hat{\mathbf{B}}\mathbf{v}$, using identities 7a and 7b, in order to express $\hat{\mathbf{B}}\mathbf{v}\hat{\mathbf{B}}$ in terms of either $\hat{\mathbf{B}}\cdot\mathbf{v}$ or $\hat{\mathbf{B}}\wedge\mathbf{v}$. Using Identity 7a,

$$\begin{aligned}
\hat{\mathbf{B}}\mathbf{v}\hat{\mathbf{B}} &= \left[-\mathbf{v}\hat{\mathbf{B}} + 2\mathbf{v}\wedge\hat{\mathbf{B}}\right]\hat{\mathbf{B}} \\
&= -\mathbf{v}\hat{\mathbf{B}}\hat{\mathbf{B}} + 2\left(\mathbf{v}\wedge\hat{\mathbf{B}}\right)\hat{\mathbf{B}} \\
&= \mathbf{v} + 2\left(\mathbf{v}\wedge\hat{\mathbf{B}}\right)\hat{\mathbf{B}}.
\end{aligned} \tag{3.2}$$

The "reversal" that's based upon Identity 7b gives

$$\begin{aligned}
\hat{\mathbf{B}}\mathbf{v}\hat{\mathbf{B}} &= \left[\mathbf{v}\hat{\mathbf{B}} - 2\mathbf{v}\cdot\hat{\mathbf{B}}\right]\hat{\mathbf{B}} \\
&= \mathbf{v}\hat{\mathbf{B}}\hat{\mathbf{B}} - 2\left(\mathbf{v}\cdot\hat{\mathbf{B}}\right)\hat{\mathbf{B}} \\
&= -\mathbf{v} - 2\left(\mathbf{v}\cdot\hat{\mathbf{B}}\right)\hat{\mathbf{B}}.
\end{aligned} \tag{3.3}$$

Is either Eq. (3.2) or (3.3) useful to us? Our goal is to prove that Eqs. (1.2) and (1.3) are equivalent, so let's examine the way in which each of those equations formulates the rotation. In Eq. (1.2), the rotation formula is expressed in terms of $\mathbf{v}_\perp$ and $\mathbf{v}_\parallel$. Therefore, we should find out what happens if we express the results of Eqs. (3.2) and (3.3) in terms of $\mathbf{v}_\perp$ and $\mathbf{v}_\parallel$. In the case of Eq. (3.2),

$$\begin{aligned}
\hat{\mathbf{B}}\mathbf{v}\hat{\mathbf{B}} &= \mathbf{v} + 2\left(\mathbf{v}\wedge\hat{\mathbf{B}}\right)\hat{\mathbf{B}} \\
&= \mathbf{v}_\perp + \mathbf{v}_\parallel - 2\left[\left(\mathbf{v}\wedge\hat{\mathbf{B}}\right)\hat{\mathbf{B}}^{-1}\right] \\
&= \mathbf{v}_\perp + \mathbf{v}_\parallel - 2\mathbf{v}_\perp \\
&= \mathbf{v}_\parallel - \mathbf{v}_\perp.
\end{aligned}$$

Similarly, from Eq. (3.3),

$$\begin{aligned}\hat{\mathbf{B}}\mathbf{v}\hat{\mathbf{B}} &= -\mathbf{v} - 2\left(\mathbf{v}\cdot\hat{\mathbf{B}}\right)\hat{\mathbf{B}}\\ &= -\mathbf{v}_\perp - \mathbf{v}_\parallel + 2\left[\left(\mathbf{v}\cdot\hat{\mathbf{B}}\right)\hat{\mathbf{B}}^{-1}\right]\\ &= -\mathbf{v}_\perp - \mathbf{v}_\parallel + 2\mathbf{v}_\parallel\\ &= \mathbf{v}_\parallel - \mathbf{v}_\perp.\end{aligned}$$

Thus, $\hat{\mathbf{B}}\mathbf{v}\hat{\mathbf{B}} = \mathbf{v}_\parallel - \mathbf{v}_\perp$. In retrospect, we could have obtained that same result directly:

$$\begin{aligned}\hat{\mathbf{B}}\mathbf{v}\hat{\mathbf{B}} &= \left[\hat{\mathbf{B}}\cdot\mathbf{v} + \hat{\mathbf{B}}\wedge\mathbf{v}\right]\hat{\mathbf{B}}\\ &= \left[-\mathbf{v}\cdot\hat{\mathbf{B}} + \mathbf{v}\wedge\hat{\mathbf{B}}\right]\hat{\mathbf{B}}\\ &= \left(-\mathbf{v}\cdot\hat{\mathbf{B}}\right)\hat{\mathbf{B}} + \left(\mathbf{v}\wedge\hat{\mathbf{B}}\right)\hat{\mathbf{B}}\\ &= \left(\mathbf{v}\cdot\hat{\mathbf{B}}\right)\left(-\hat{\mathbf{B}}\right) - \left(\mathbf{v}\wedge\hat{\mathbf{B}}\right)\left(-\hat{\mathbf{B}}\right)\\ &= \mathbf{v}_\parallel - \mathbf{v}_\perp.\end{aligned}$$

Our reason for expressing $\hat{\mathbf{B}}\mathbf{v}\hat{\mathbf{B}}$ in terms of $\mathbf{v}_\parallel$ and $\mathbf{v}_\perp$ was to see whether we could transform Eq. (3.1) into something more like Eq. (1.2). So, let's replace $\hat{\mathbf{B}}\mathbf{v}\hat{\mathbf{B}}$ with $\mathbf{v}_\parallel - \mathbf{v}_\perp$ in Eq. (3.1):

$$\mathbf{v}' = \mathbf{v}\cos^2\frac{\theta}{2} + \mathbf{v}\cdot\hat{\mathbf{B}}\sin\theta - \left(\mathbf{v}_\parallel - \mathbf{v}_\perp\right)\sin^2\frac{\theta}{2}. \tag{3.4}$$

Now we can see the terms $\cos^2\frac{\theta}{2}$ and $\sin^2\frac{\theta}{2}$ might be combined per the double-angle formulas (Identities in 8) if we write $\mathbf{v}$ as $\mathbf{v}_\parallel - \mathbf{v}_\perp$ in the $\cos^2$ term of Eq. (3.4):

$$\mathbf{v}' = \left(\mathbf{v}_\parallel - \mathbf{v}_\perp\right)\cos^2\frac{\theta}{2} + \mathbf{v}\cdot\hat{\mathbf{B}}\sin\theta - \left(\mathbf{v}_\parallel - \mathbf{v}_\perp\right)\sin^2\frac{\theta}{2}.$$

The rest is simple:

$$\begin{aligned}\mathbf{v}' &= \mathbf{v}_\perp\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right) + \mathbf{v}_\parallel\left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right) + \mathbf{v}\cdot\hat{\mathbf{B}}\sin\theta\\ &= \mathbf{v}_\perp + \mathbf{v}_\parallel\cos\theta + \mathbf{v}\cdot\hat{\mathbf{B}}\sin\theta,\end{aligned}$$

which is the same as Eq. (1.2).

References

[1] T. Verhoeff, Comparison of Methods for Rotating a Point in R^3: A Case Study, https://doi.org/10.48550/arXiv.2504.04286.

[2] A. Macdonald, *Linear and Geometric Algebra* (First Edition), CreateSpace Independent Publishing Platform (Lexington, 2012).

A Macdonald's derivation of $\mathbf{v}' = \left[e^{-\hat{\mathbf{B}}\frac{\theta}{2}}\right] \mathbf{v} \left[e^{\hat{\mathbf{B}}\frac{\theta}{2}}\right]$

Macdonald ([2], p. 89) begins the derivation of this formula by expressing $\mathbf{v}$ as the sum of its components parallel and perpendicular to $\hat{\mathbf{B}}$ (Fig. 2):

$$\mathbf{v} = \mathbf{v}_\perp + \mathbf{v}_\parallel.$$

Then, Macdonald notes that although the vertical component is unaffected by the rotation, the parallel component becomes $\mathbf{v}_\parallel e^{\hat{\mathbf{B}}\theta}$. That is, after rotation, the vector $\mathbf{v}$ becomes $\mathbf{v}'$ (Fig. 2), which is

$$\mathbf{v}' = \mathbf{v}_\perp + \mathbf{v}_\parallel e^{\hat{\mathbf{B}}\theta}$$

Next, as preparation for later steps, Macdonald introduces the "$\hat{\mathbf{B}}\frac{\theta}{2}$" exponents:

$$\mathbf{v}' = \mathbf{v}_\perp \underbrace{e^{-\hat{\mathbf{B}}\frac{\theta}{2}} e^{\hat{\mathbf{B}}\frac{\theta}{2}}}_{=1} + \mathbf{v}_\parallel e^{\hat{\mathbf{B}}\frac{\theta}{2}} e^{\hat{\mathbf{B}}\frac{\theta}{2}}$$

The next step might need some explanation:

$$\mathbf{v}' = e^{-\hat{\mathbf{B}}\frac{\theta}{2}} \mathbf{v}_\perp e^{\hat{\mathbf{B}}\frac{\theta}{2}} + e^{-\hat{\mathbf{B}}\frac{\theta}{2}} \mathbf{v}_\parallel e^{\hat{\mathbf{B}}\frac{\theta}{2}}. \tag{A.1}$$

Why is this result correct? First, let's consider $\mathbf{v}_\parallel e^{\hat{\mathbf{B}}\frac{\theta}{2}}$. Note that for any vector $\mathbf{u}$ that is parallel to a bivector $\mathbf{M}$, $\mathbf{u}\mathbf{M} = -\mathbf{M}\mathbf{u}$. Therefore,

$$\begin{aligned}
\mathbf{v}_\parallel e^{\hat{\mathbf{B}}\frac{\theta}{2}} &= \mathbf{v}_\parallel \left[\cos\frac{\theta}{2} + \hat{\mathbf{B}} \sin\frac{\theta}{2}\right] \\
&= \mathbf{v}_\parallel \cos\frac{\theta}{2} + \left(\mathbf{v}_\parallel \hat{\mathbf{B}}\right) \sin\frac{\theta}{2} \\
&= \mathbf{v}_\parallel \cos\frac{\theta}{2} + \left(-\hat{\mathbf{B}} \mathbf{v}_\parallel\right) \sin\frac{\theta}{2} \\
&= \mathbf{v}_\parallel \cos\frac{\theta}{2} + \hat{\mathbf{B}} \mathbf{v}_\parallel \left(-\sin\frac{\theta}{2}\right) \\
&= \left[\cos\frac{\theta}{2} - \hat{\mathbf{B}} \sin\frac{\theta}{2}\right] \mathbf{v}_\parallel \\
&= e^{-\hat{\mathbf{B}}\frac{\theta}{2}} \mathbf{v}_\parallel.
\end{aligned}$$

In contrast, for any vector $\mathbf{p}$ that is perpendicular to a bivector $\mathbf{M}$, $\mathbf{p}\mathbf{M} =$

Mp. Thus,

$$
\begin{aligned}
\mathbf{v}_\perp e^{-\hat{\mathbf{B}}\frac{\theta}{2}} &= \mathbf{v}_\perp \left[\cos\frac{\theta}{2} - \hat{\mathbf{B}}\sin\frac{\theta}{2}\right] \\
&= \mathbf{v}_\perp \cos\frac{\theta}{2} - \left(\mathbf{v}_\perp \hat{\mathbf{B}}\right)\sin\frac{\theta}{2} \\
&= \mathbf{v}_\perp \cos\frac{\theta}{2} - \left(\hat{\mathbf{B}}\mathbf{v}_\perp\right)\sin\frac{\theta}{2} \\
&= \mathbf{v}_\perp \cos\frac{\theta}{2} + \hat{\mathbf{B}}\mathbf{v}_\perp \left(-\sin\frac{\theta}{2}\right) \\
&= \left[\cos\frac{\theta}{2} - \hat{\mathbf{B}}\sin\frac{\theta}{2}\right]\mathbf{v}_\perp \\
&= e^{-\hat{\mathbf{B}}\frac{\theta}{2}}\mathbf{v}_\perp .
\end{aligned}
$$

Returning now to Eq. (A.1), the rest is straightforward:

$$
\begin{aligned}
\mathbf{v}' &= e^{-\hat{\mathbf{B}}\frac{\theta}{2}}\mathbf{v}_\perp e^{\hat{\mathbf{B}}\frac{\theta}{2}} + e^{-\hat{\mathbf{B}}\frac{\theta}{2}}\mathbf{v}_{\|} e^{\hat{\mathbf{B}}\frac{\theta}{2}} \\
&= e^{-\hat{\mathbf{B}}\frac{\theta}{2}}\left[\mathbf{v}_\perp + \mathbf{v}_{\|}\right] e^{\hat{\mathbf{B}}\frac{\theta}{2}} \\
&= \left[e^{-\hat{\mathbf{B}}\frac{\theta}{2}}\right]\mathbf{v}\left[e^{\hat{\mathbf{B}}\frac{\theta}{2}}\right].
\end{aligned}
$$