A Galactic Spacetime Model to Resolve the Problem Between Mass Density and Rotation Curve

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Abstract

In the present paper, we introduce a spacetime model where the particle circular motions have the characteristics of the galaxy rotation curves. We calculate the Einstein tensor and analyze the mass density-radius relation. We find that near the core the density-radius relation follows inverse fourth-power law, and near the edge it follows the Schechter function, which just like a luminous mass density profile of a real galaxy. In the other words, our result shows that only general relativity without dark matter might be enough to explain the galaxy rotation curves. Our model predicts that the next correction term of the mass density will be dependent on the rotation velocity.

1 Introduction

The shape of a galaxy usually contains a large disc and a dense core in the center. Astrophysicists already known that the surface brightness in the galactic core follows the de Vaucouleurs law and in the galactic disc it follows exponential decay law. The surface brightness profile can be calculated by integrating the luminous mass density, and it shows that the mass density distribution declines with radius as \( r^{-4} \) near the core [1] and exponentially decays near the edge. However, with Newton’s gravity theory, the above mass density distribution is contradiction with the galaxy rotation curves, which rotation velocity almost independent on radial distance.

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In galaxy scale, that the motions are non-relativistic \( (v \ll c) \) and gravity sources are weak, so people believe that Newtonian theory is a good approximation of the gravity theory. In order to explain the galaxy rotation curves, most researchers considered either there were some invisible dark matter distributions, or the original gravity theory needed to be modified. However, some people were also trying to figure out whether the nonlinearity of general relativity can solve the galaxy rotation curve and mass distribution problem.\cite{2, 3}

In this work, we follow the latter idea and try to analyze whether the nonlinearity of general relativity can resolve the problem between the rotation curve and mass density distribution. In our imagination, the galaxy system can be seen as the grand ocean and the unexpected galaxy rotation curves may just like the strong ocean currents due to the various nonlinear interactions in the water (matter distribution). In such system, perturbation methods would be hard to expose these nonlinear phenomenons.\cite{4} So, in order to get a fully nonlinear but handleable gravitational field, we study a spacetime toy model in which the particle circular motions have the characteristics of the galaxy rotation curves. The surprising result is that due to Einstein equation the corresponding energy density follows the inverse fourth-power near the core and the Schechter function near the edge which just like a luminous mass density profile of a real galaxy. In the other words, our result shows that only general relativity without dark matter may be enough to explain the galaxy rotation curves.

In section 2, we explain how we get the spacetime model, and show the rotation curve behavior in this spacetime. In section 3, we calculate the Einstein tensor of the spacetime model and analyze the energy density to show that the energy density follows the inverse fourth-power near the core and the Schechter function near the edge. Finally, Section 4, we propose some further issues. In Appendix A, we show all Einstein tensor components in a reasonable approximation. In Appendix B, we show a possible improving case for our model.

### 2 Metric and geodesic

In general relativity framework, a spacetime of rotating galaxy can be described by an axially symmetric metric. First, we assume the metric written in cylinder coordinates with
the following form
\[ g_{\mu\nu} = \begin{pmatrix} -\frac{(1 + \alpha(r))c^2}{r^2} & 0 & -\beta(r) & 0 \\ 0 & 1 & 0 & 0 \\ -\beta(r) & 0 & r^2 + \gamma(r) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

, where \( c \) is the light speed constant and \( \alpha(r), \beta(r), \gamma(r) \) are the functions of radius \( r \). In this model, the dynamics are independent with \( z \)-variable. So, in fact, our spacetime model can be seen as an simplified three dimensional effective model. We considered the more real case in the appendix B that included \( z \)-variable, but here we first try to study the simplified case.

The Lagrangian of a test particle in a curved spacetime can be written as
\[ \mathcal{L} = \frac{1}{2} m g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \]

, where \( m \) is the particle mass and \( \tau \) is the proper time. We focus on the radial \( r \)-direction Euler-Lagranian equation for our model
\[ \frac{1}{m} \left( \frac{\partial \mathcal{L}}{\partial r} - \frac{d}{d\tau} \frac{\partial \mathcal{L}}{\partial \dot{r}} \right) = -\frac{1}{2} \alpha' c^2 \dot{r}^2 - \beta' \dot{r} \dot{\phi} + \frac{1}{2} (2r + \gamma') \dot{\phi}^2 - \ddot{r} = 0 \]

(1)

where the dot symbol \( \dot{\ } \) denotes the operator \( \frac{d}{d\tau} \) and the prime symbol \( \prime \) denotes the operator \( \frac{\partial}{\partial r} \). If the particle orbit is very close to a perfect circle, we may take the radial acceleration approximating to zero \( \ddot{r} \approx 0 \). With the approximation and equation (1), we can derive the circular velocity for the rest observer in far away:
\[ v = r \frac{d\phi}{dt} = \frac{r \beta' \pm \sqrt{(r \beta')^2 + (2r + \gamma')(c^2 r^2 \alpha')}}{(2r + \gamma')} \]

(2)

We take the positive sign in the equation (2). If we want the circular velocity almost independent on radial distance in a certain region, a simple approach is taking a smooth box function \( \beta' = b \left( \frac{1}{1 + e^{-\frac{r-w}{s}}} - \frac{1}{1 + e^{-\frac{r-w}{e + s}}} \right) \), so \( \beta = bs \ln \left( \frac{1 + e^{-\frac{r-w}{s}}}{e^{-\frac{r-w}{s}} + e^{s}} \right) \), where \( b, w, s \) are constants.

* Other approaches to achieve the constant velocity are very difficult. Because, \( \gamma(r) \) is the correction term of \( r^2 \), so \( \gamma' \ll r \) in the far region. Thus, the suitable approach for our demand is to choose the \( \beta \) function.

For the remained two functions \( \alpha(r) \) and \( \gamma(r) \), we try to study some simple cases. First, if \( \alpha(r) = 0, \gamma(r) = 0 \), then Einstein field equation shows that the energy density \( T_{00} \) contain
negative energy. After trying hard, we found the $\gamma(r)$ can be chosen as the same form with $\beta(r)$, i.e. $\gamma(r) \propto \beta(r)$. Thus, our rotating spacetime model is

$$g_{\mu\nu} = \begin{pmatrix}
-c^2 & 0 & -bs \ln \left( \frac{1+e^{r/s}}{e^{r/s}+e^{s}} \right) & 0 \\
0 & 1 & 0 & 0 \\
-bs \ln \left( \frac{1+e^{r/s}}{e^{r/s}+e^{s}} \right) & 0 & r^2 + qs \ln \left( \frac{1+e^{r/s}}{e^{r/s}+e^{s}} \right) & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}$$

, where $q$ is a constant, i.e. $\alpha(r) = 0$ and $\gamma(r) = \frac{q}{b} \beta(r)$. The line element can be written as

$$ds^2 = -c^2 \left( dt + \frac{\beta(r)}{c^2} d\phi \right)^2 + dr^2 + \left( r^2 + \frac{q}{b} \beta(r) + \frac{\beta(r)^2}{c^2} \right) d\phi^2 + dz^2 \quad (3)$$

Note that $\beta(r) < 0$ and as $r \to 0$, $\beta(r) \to -bw$, so we need $q < \frac{b^2}{c^2}w$ to avoid the signature changing. By equation (2), we can draw the rotation curve of this spacetime in the following figure.

![Rotation Curve](image)

Figure 1: A rotation curve was calculated by equation (2). And the figure showed the effects of these constants. Here we taken $b = 150 \text{ km/s}$ ($\approx 5 \times 10^{-15} \text{ kpc/s}$), $w = 60 \text{ kpc}$, $s = 2.5 \text{ kpc}$, $q = 10^{-4}$.

So, three of the parameters in the metric can be easily determined by the observed rotation curve: (i) the maximum velocity is $b$, (ii) the beginning rising ratio is about $\frac{b}{4s}$, (iii) the maximum radius is about $w$. Thus, $w > 2s$ is a basic assumption for the existence of the constant rotation velocity.
3 Energy density

Thanks to the computer algebra system (Mathematica packages xAct)[5], that helps us to calculate the Einstein tensor. First, we focus on the $G_{tt}$ component:

$$G_{tt} = R_{tt} - \frac{1}{2} g_{tt} R$$

$$= \frac{b^2 c^2 e^{2q} (e^{2\beta} - 1)^2}{2 (1 + e^{2\beta})^2 (e^{2\beta} + e^{2\gamma})^2 (r^2 c^2 + \frac{q}{b} c^2 \beta(r) + \beta(r))^2}$$

$$+ \frac{c^2}{4 s (1 + e^{2\beta})^2 (e^{2\beta} + e^{2\gamma})^2 (r^2 c^2 + \frac{q}{b} c^2 \beta(r) + \beta(r))^2} \left\{ - \frac{c^2}{b} \beta(r) \left[ 2 c^2 q \left( 2 s (1 + e^{2\beta})^2 (e^{2\beta} + e^{2\gamma})^2 - q \beta \left( e^{2\beta} - 1 \right) \left( e^{2\beta} - e^{2\gamma} \right) \right) \right.$$

$$\left. - b^2 e^{2\beta} \left( e^{2\beta} - 1 \right) \left( q s e^{2\beta} \left( e^{w/s} - 1 \right) + 4 r \left( 2 s e^{2\beta} + 2 q e^{2\gamma/s} + e^{2\beta} (2 s - r) + e^{2\beta} (2 s + r) \right) \right) \right]\right.$$

$$\left. \beta(r) \left[ 2 c^2 \left( 2 s (1 + e^{2\beta})^2 (e^{2\beta} + e^{2\gamma})^2 - 3 q e^{2\beta} \left( e^{2\beta} - 1 \right) \left( e^{2\beta} - e^{2\gamma} \right) \right) - b^2 s e^{2\beta} \left( e^{2\beta} - 1 \right)^2 \right] \right.$$

$$+ 4 b \beta(r)^3 e \left( e^{2\beta} - e^{2\gamma} \right)$$

$$+ c^2 e^{2\beta} \left( e^{2\beta} - 1 \right) \left[ c^2 q \left( q s e^{2\beta} \left( e^{2\beta} - 1 \right) + 2 r \left( 2 s e^{2\beta} + 2 q e^{2\gamma/s} + e^{2\beta} (2 s - r) + e^{2\beta} (2 s + r) \right) \right) \right.$$

$$\left. + 2 c^2 q r^2 e^{2\beta} - 3 b^2 s r^2 e^{2\beta} \left( e^{2\beta} - 1 \right) \right] \right\}$$

With the Einstein field equation $G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$, we look around $r \approx w$ to check the positive energy condition. Note that when $r \to w$ we have $\beta(r) \to 0$. With the condition $q < \frac{b^2}{c^2} w$, we find the maximum contributions come from the last line in the above equation, i.e. the blue words. If the energy is positive, the sum of the coefficients of blue words must be positive, thus we have $q > \frac{3}{2} \frac{b^2}{c^2} s$. So, the reasonable range of $q$ is $\frac{3 b^2}{2 c^2} s < q < \frac{b^2}{c^2} w$.

In order to get a clear formula, we take reasonable approximation $s \ll w$, $r \ll c$ and low velocity comparing to light speed $b \ll c$, so $q \propto \frac{b^2}{c} s \ll s$, and $\frac{b}{c} w < r$. In this approximation, some functions can be ignored due to $|\beta(r)| \approx b r \ll c r$ if there exist other large values, for example $r^2 c^2 + \frac{q}{b} c^2 \beta(r) + \beta(r)^2 \approx r^2 c^2$. In this approximation, the energy density can be approximated as

$$T_{tt} \approx \frac{c^4}{8\pi G} \frac{1}{4 s \left( e^{- \frac{r-w}{s}} + 1 \right)^2} \left\{ \left( 2 c^2 q r^2 + s \left[ 4 c^2 q r - b^2 r^2 \right] e^{- \frac{r-w}{s}} \right) e^{- \frac{r-w}{s}} \right.$$

$$\left. - 4 b \beta(r) \left( \frac{c^2 q s}{b} \left( 1 + e^{- \frac{r-w}{s}} \right)^2 - b r^2 e^{- \frac{r-w}{s}} e^{- \frac{w}{s}} \right) \right\}$$

(4)

Then, we can further consider two subregions:
• Near edge region $w < r$, so $e^{-\frac{r}{s}} \to 0$, $\beta(r) \to 0$, then the leading term is

$$T_{tt} \approx \frac{e^6}{16\pi G} \frac{q e^w}{s} \frac{e^{-\frac{r}{s}}}{r^2}$$

(5)

This is a Schechter function.\(^\dagger\)

• Near core region $r < w$, so $e^{-\frac{r-w}{s}} \gg 1$, $\beta(r) \approx b(r-w)$, then

$$T_{tt} \approx \frac{c^4}{32\pi G} \left( 4c^2qw \frac{1}{r^4} - b^2 \frac{1}{r^2} \right)$$

(6)

The first term gives us the inverse fourth-power law. The second term predicts that the next correction term of the mass density will be dependent on the rotation velocity square. This relation may be checked by the further observed data.

Other components of the Einstein tensor in the same approximation can be seen in the appendix A. Here, we just focus on the mass density $T_{tt}/c^2$.

The diagram of the mass density is showing in the following figure:

![Diagram of the mass density](image)

Figure 2: The log-linear figure shows the mass density-radius relations in different approximations. Here we take $b = 5 \times 10^{-15}$ kpc/s, $w = 60$ kpc, $s = 2.5$ kpc, $q = 10^{-4}$.

One can see that the shape of the mass density follows the galaxy luminous mass distribution law, but the magnitude ($\approx 10^{-15} M_\odot \cdot kpc^{-3}$) is too small comparing to real one ($\approx 10^5 M_\odot \cdot kpc^{-3}$). This is because that the energy momentum tensor in Einstein field

\(^\dagger\)If we don’t drop out $\beta(r)$ then we can get a modified Schechter function

$$T_{tt} \approx \frac{e^6}{16\pi G} \left( \frac{q e^w}{s} \frac{e^{-\frac{r}{s}}}{r^2} - 2c^2 q \frac{b}{r} \beta(r) \right)$$

And we find that the next leading term of the mass density will be dependent on the rotation velocity.
equation is not directly including the gravitational sources, just like Schwarzschild solution has the vanished energy-momentum tensor. However, near a large gravitational source the ambient gas or radiation density would be larger than the small one. So, although our mass density underestimates the whole mass but it is able to reflect the whole mass distribution.

After suitable parameter choosing, all components of the energy-momentum tensor in our model can be positive, except the $z$-direction stress $T_{zz}$. This is because that we didn’t really consider the distribution of $z$-direction, i.e. our metric is independent with $z$-variable. In the appendix B, we have tried to include $z$-variable into the metric, the negativity problem has been improved, but the equations became too complicated and hard to analyze. However, if we ignore the $z$-direction and consider this spacetime as a three dimensional effective model, then in qualitatively the simple model tells us that general relativity may be enough to resolve the problem between mass distribution and rotation curve!

4 Discussion

In this work, we introduced a spacetime model that just can resolve the problem between mass distribution and rotation curve. After this study, we found that general relativity may be necessary to understand the large scale system even for low velocities and weak gravitational sources. However, some interesting issues for this model shoud be considered, although we did not carefully study yet.

First, as showing in section 3 the mass density in our theory is quite small comparing to a real galaxy mass density. The reason is that energy-momentum tensor usually underestimates the whole gravitational sources. But we could use other theory to estimate the total mass for our galaxy spacetime model, for example: ADM mass or quasi-local mass.

Second, the stability of our model still needs to be analyzed. This could be a quite interesting and hard issue. The further study may open a door to answer why after evolution most galaxy systems have the similar shapes.

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A  Einstein tensor components

Here, we list all nonvanish Einstein tensor components for the spacetime of the equation (3) in the reasonable approximation $s \ll w$, $s \ll r$, $b \ll c$, $q \propto \frac{b^2}{c^2}s \ll s$, and $\frac{c}{b}w < r$.

\[ G_{tt} \approx \frac{1}{4s(e^{-\frac{r-w}{s}} + 1)^2 r^4} \left\{ \left( 2e^2qr^2 + s \left[ 4c^2qr - b^2r^2 \right] e^{-\frac{r-w}{s}} \right) e^{-\frac{r-w}{s}} - 4\beta(r) \left( \frac{c^2qs}{b} \left( 1 + e^{-\frac{r-w}{s}} \right) - br^2 e^{-\frac{r-w}{s}} e^{-\frac{r}{s}} \right) \right\} \]

\[ G_{rr} \approx \frac{e^{2w/s}b^2}{4(e^{w/s} + e^{r/s})^2 r^2 c^2} \]

\[ G_{t\phi} \approx \frac{be^{w/s}}{2(e^{w/s} + e^{r/s})^2 sr} \left( e^{w/s} + e^{r/s} \right) \]

\[ G_{\phi\phi} \approx \frac{be^{-w/s} \beta(r)}{4c^2rs(e^{r/s} + 1)^2(e^{w/s} + e^{r/s})^2} \]

\[ G_{zz} \approx \frac{r e^{w/s} \left( 3b^2sre^{w/s} - 2e^2q \left( 2sre^{w/s} + re^{r/s} \right) \right) + 4\frac{\beta(r)}{b} \left( c^2qs(e^{w/s} + e^{r/s})^2 - b^2re^{w/s} \left( 2sre^{w/s} + re^{r/s} \right) \right)}{4c^2sr^2(e^{w/s} + e^{r/s})^2} \]

As $r \to \infty$, $\beta(r) \to 0$ and we can see that $G_{zz}$ is always negative at the infinity, in fact it is also negative near the center.

B  Adding $z$-variable

To study the spacetime metric with $z$-variable, we consider the following metric

\[ g_{\mu\nu} = \begin{pmatrix} -(1 + q_{t}, f(r, z))c^2 & 0 & -\beta(r, z) & 0 \\ 0 & 1 & 0 & 0 \\ -\beta(r, z) & 0 & r^2 + \frac{q_{z}}{b}\beta(r, z) & 0 \\ 0 & 0 & 0 & 1 + q_{z}, f(r, z) \end{pmatrix} \]

where $\beta(r, z) = bs \ln \left( \frac{1 + e^{r/s}}{e^{r/s} + e^{r}} \right) e^{-\frac{z}{h}}$, $f(r, z) = \frac{1}{e^{\frac{r}{s} + 1}} e^{-\frac{z}{h}}$ and $b, s, w, q_{t}, q_{\phi}, q_{z}, h$ are constants and the new constant $h$ describes the thickness of the galaxy. \(^{\dagger}\) The following figure 3 shows the behaviors of the energy-momentum tensor components.

\(^{\dagger}\)The first factor of $f(r, z) = \frac{1}{e^{\frac{r}{s} + 1}} e^{-\frac{z}{h}}$ is Fermi-Dirac distribution. The value of the distribution is $\approx 1$ as $r < w$ and is $\approx 0$ as $r > w$. Thus, when we fix the value $z$, the metric will reduce to metric (3) in
Figure 3: The log-linear figures show the behaviors of the energy-momentum tensor components. Here we take $b = 5 \times 10^{-15}$ kpc/s, $w = 60$ kpc, $s = 2.5$ kpc, $q_t = 10^{-4}$, $q_\phi = 10^{-4}$, $q_z = 10^{-4}$, $h = 10^{-3}$, $z = 10^{-5}$ kpc. The orange line ($T_{rr}$) in the right figure is cutted near the core, because it quickly drops down to negative value.

The negativity of $T_{zz}$ now is improved, and the shape of the mass density profile is not changed too serious. And, the magnitude of the mass density now is larger due to the finite thickness of the galaxy model. However, the stress $T_{rr}$ have small negative value near the core. Perhaps, there are other nice models or we need some techniques (ex: glue-and-cut methods replace the galaxy core by a black hole geometry) to solve the negative stress issue. Unfortunately, we should mention that the positive energy and positive stress constraints for the parameters are more complicated, and we don’t know the exact conditions yet.

References


And it is necessary to consider the function $g_{zz}$. Because the Euler-Lagrangian equation in $z$-direction

$$g_{zz} \ddot{z} = \left( \dot{z}^2 \frac{\partial g_{\phi\phi}}{\partial z} + z^2 \frac{\partial g_{zz}}{\partial z} + \ldots \right)$$

in the right hand side, the first term will be positive in our case. It means that there is acceleration in $z$-direction so that the objects will escape from the galaxy disk. Thus, we need function $g_{zz}$ and $\frac{\partial}{\partial z} g_{zz} < 0$ and that may avoid the objects escaping.

