THE LIMIT OF A STRATEGIC MAPPING OF A RECURSIVE FIBONACCI SEQUENCE.

Sebastian Thomas
Manipal Academy of Higher Education, Manipal-576104.
Mail: sebastianthomas351@gmail.com

ABSTRACT: Let F_1,F_2,F_3,..........F_n represent the sequence of Fibonacci elements. Let us define $\mathcal{F}$ to be the **parent set** of all Fibonacci elements. G and G' are the subsets of $\mathcal{F}$ such that G is a given set of consecutive Fibonacci elements of finite order k and G' is defined to be a shift on G of l degrees, where $l \in N$. Let $R = \min(r_1, r_2, ...)$ denote the set of remainders obtained such that $r_n \in F$. For a given G of order k, we show that a strategic mapping operator ϕ: $(G \times G) \longrightarrow R$ defined by §: $\phi(g \otimes_{g'} h) = r$, where $(G \times G)$ represents the Cartesian product and $g, h \in G$, $g' \in G'$. The strategic map ϕ exists upto $(l+1)^0$ transition, with its limit as $L\left(F_{n+(l+1)}\right)$ thereof. We consider a special introductory case of $|G|$, $|G'|$=4 to illustrate the results and thereby proving the "Fundamental Theorem of limit of a strategic map of Fibonacci sequence[Thomas Theorem] and its consequences".

1 Introduction

The theory of 'Strategic mapping' derives its core ideas from Order(O) of an element of a cyclic group $(G, *)$, where it is defined to be the least positive integer m such that $a^m = e$ and $G =< a >$, $a \in G$ is the generator and e is the identity element of G. The theory intends to explore novel properties of a recursive Fibonacci sequence by broadening its scope to incorporate the ideas of Group theory, 'prey-predation models', 'sequential interaction' of the elements to understand the behavior of the sequence.

The **idea** of this theory is to consider for any fixed F_m and a collection of consecutive Fibonacci elements $F_n \in G(\subset \mathcal{F})$ where $[m = n$ or $m \neq n]$, find the largest possible $F_{n+l} \in G'(\subset \mathcal{F})$ such that when the strategic map operator ϕ is applied to every element in the Cartesian product $H = G \times G$, it yields the set of smallest possible remainders $r \in \mathcal{F}$. This resonates with the idea of a generator of a cyclic group, where the operator ϕ enables in generating the set of smallest possible reminders($r \in \mathcal{F}$) which happens to be the set of all Fibonacci elements. To illustrate this, let us consider for $m = 2$ and $n = 2, 3, 4, 5 \ldots$, we have $F_m = \{2\}$ and $F_n = \{2, 3, 5, 8, 13, \ldots\}$, where both F_m and F_n are elements of G. The Cartesian product $G \times G$ in this case is given by {(2,2),(2,3),(2,5),...}.By applying the strategic map operator ϕ, we can observe $\phi(2 \otimes_3 2) = 1[r_1]$, $\phi(2 \otimes_5 3) = 1[r_2]$, $\phi(2 \otimes_8 5) = 2[r_2]$, $\phi(2 \otimes_{13} 8) = 3[r_3]$, $\phi(2 \otimes_{21} 13) = 5[r_4] \ldots$.. G' represents the largest possible F_{n+l} for every element in H given by $G' = \{3, 5, 8, 13 \ldots\}$. Thus, ϕ generates a collection of remainders, $R \in \mathcal{F}$ which happens to be Fibonacci elements.

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* Sebastian Thomas.

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The **need** for discussing the limit of a strategic map ϕ is to understand the importance of 'sequential analysis' of the interaction between the elements of G defined by ϕ and how the strategic map results in the formation two different **classes** - [2]={2, 5, 13, 34,...} and [3]={3, 8, 21, 55,...} of Fibonacci elements owing to the similarities in properties exhibited by the entities in these two classes which will be discussed in the subsequent sections of the article. As a pre-requisite to reading this paper, the author seek the reader(s) not to develop any existing presumptions regarding the terminologies used in the study as its meaning is being discussed in the forthcoming section of the paper.

2 Definitions and Terminologies

Definition 1 (Parent set $\mathcal{F}$)**.** *This is an infinite set of Fibonacci elements from which subsets namely G and G' are derived. The subsets G_n=$\{F_n, F_{n+1}, F_{n+2}, F_{n+3} \ldots F_{n+k}\}$ is of finite order k and G' represents the shift on G of l degrees given by G_n^l=$\{F_{n+l}, F_{n+1+l}, F_{n+2+l}, F_{n+3+l} \ldots F_{n+k+l}\}$.*

Definition 2 (l^{th} shift on G)**.** *It is defined by the set G_n^l=$\{F_{n+l}, F_{n+1+l}, F_{n+2+l}, F_{n+3+l} \ldots F_{n+k+l}\}$ of the same order as of G.*

Definition 3 (Strategic map ϕ)**.** *An operator from the non-empty Cartesian product $H = (G \times G)$ into R that assigns each element of H to R, where ϕ is defined by the rule $\phi(g \otimes_{g'} h) = r$ satisfying the following* ***assumptions:***

3.1 *Order of mapping into R*: Consider G of order k. In the first phase, the map considers the sequence of ordered pairs in the order (F_{n+k}, F_{n+k}),(F_{n+k-1}, F_{n+k-1}),(F_{n+k-2}, F_{n+k-2}) $\ldots (F_n, F_n)$ and determines the largest $g' \in G'$ which yields the smallest r for each ordered pair combination. In the second phase, the map considers the sequence of ordered pairs in the order (F_{n+k}, F_{n+k-1}),(F_{n+k-1}, F_{n+k-2}),(F_{n+k-2}, F_{n+k-3}) $\ldots (F_{n-1}, F_n)$ and performs the rule defined by ϕ. This procedure of mapping into R continues till the k^{th} stage in the order (F_{n+k}, F_n).

3.2 For any given G of order k, the order of interaction between any two $g, h \in G$ for the largest possible $g' \in G'$ ensuring the existence of ϕ wherein $g = h$ or both are distinct, is always sequential in nature. The ordered pair (a, b) involving larger numbers exert more influence over other ordered pairs (c, d), in existing upto the last shift of the map such that $(a \times b)$>$(c \times d)$.

3.3 *Behavior of a function*: At any transition, the choice of the same largest $g' \in G'$ for any two ordered pairs $(g, h), (g, k) \in H$ where $g, h, k \in G$ and (g, h)>(g, k) restricts the mapping of (g, k) to R as a consequence of Assumption[3.2].

3.4 Clearly $R \subset F$ and $r_n \leq l$.

The above assumptions will be discussed in great detail while illustrating the theory for $|G| = 4$ case in the forthcoming section of the paper.

Definition 4 (Limit of a strategic map)**.** *The element $F_{n+(l+1)} \in G' \subset (\mathcal{F})$ is said to be the limit(L) of ϕ if it ensures the sustenance of the map upto the $(l+1)^0$ transition for a given non empty set G of order k.[Principle of optimization of a strategic mapping].*

3 Illustration of the theory

Let us consider G=$\{F_n, F_{n+1}, F_{n+2}, F_{n+3}\} \subset \mathcal{F}$ to be the set of four consecutive Fibonacci elements. The condition for ϕ to be well defined is necessary as it is satisfied since the sequence is recursive in nature, the properties that hold for an ordered pair will hold for others too. The Cassini identity of Fibonacci numbers defined by

$$\Delta = \begin{vmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{vmatrix} = \begin{cases} -1 & n = odd \\ 1 & n = even \end{cases}$$

proves the same for any matrices of order 2 as for any $(g, h) \in (G \times G)$, $\{g, h, g', r\}$ represents a combination with its outcome $r \in R$ can be expressed as a square matrix of order 2 with regards to the operator ϕ defined.
Let us examine the patterns exhibited by two consecutive sets namely G_1 and G_2, thereby generalising the result for any G_n of order l.

3.1 Case 1. For $G_1 = \{F_n, F_{n+1}, F_{n+2}, F_{n+3}\}$, when n is odd.

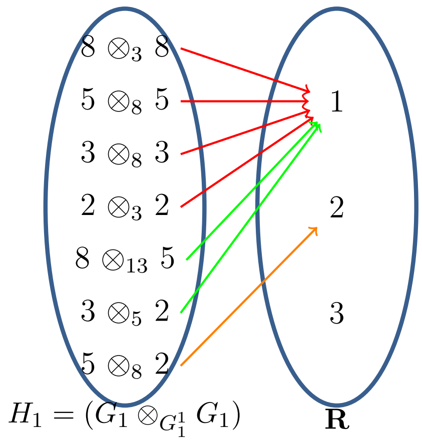
For $n = 3$, we have $G_1 = \{2, 3, 5, 8\}$.

1^0 **transition:** G_1^1= {3,5,8,13} $\subset F$ is the first shift($l = 1$) of G_1. The elements of $H_1 = (G_1 \times G_1)$ is given by $\{(8, 8), (5, 5), (3, 3), (2, 2), (8, 5), (5, 8), (5, 3), (3, 5), (3, 2), (2, 3), (8, 3), (3, 8), (5, 2), (2, 5), (8, 2), (2, 8)\}$.

In the **first phase**, the strategic map ϕ determines the largest possible elements of G_1^1 that maps each of the elements $\{(8, 8), (5, 5), (3, 3), (2, 2)\}$ in order to R. Under the assumption [2.1], 3 is the largest possible element in G_1^1 for which $\phi(8 \otimes_3 8) = 1$ and it maps $(8, 8) \in H_1$ to $1 \in R$. Sequentially, we have $\phi(5 \otimes_8 5) = 1$ maps $(5, 5)$ to 1, $(3, 3)$ is mapped to 1 and $(2, 2)$ is mapped to 1.

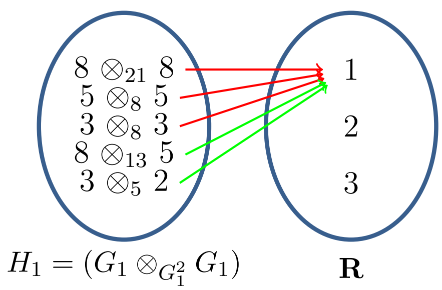
In the **second phase**, the strategic map ϕ determines the largest possible elements of G_1^1 that maps each of the elements $\{(8, 5), (5, 8), (5, 3), (3, 5), (3, 2), (2, 3)\}$ in order to R. Under the assumption [2.1], 13 is the largest possible element in G_1^1 for which $\phi(8 \otimes_{13} 5) = 1$ and it maps $(8, 5) \in H_1$ to $1 \in R$. Sequentially, $(3, 2)$ is mapped to 1 by $\phi(3 \otimes_5 2) = 1$. In this case, following assumptions [2.1 and 2.2], there is no element $g' \in G_1^1$ for which $(5, 3)$ can be mapped to R. Even though, $\phi(5 \otimes_{13} 3) = 2$ is feasible, this is not possible as in Phase-2, 13 was the largest possible element in G' for which $(8, 5)$ was mapped to 1. Also the same element $g(= 5)$ in different ordered pairs $(8, 5), (5, 3) \in H_1$ is mapped to two different images namely 1 and 2 in R respectively for the same choice of $g'(= 13)$ defined by the operator ϕ, violating assumption[2.1] where $(8 \times 5 = 40) > (5 \times 3 = 15)$ restricts the mapping of $(5, 3)$ to R for the same choice of largest possible $g'(= 13)$.

Following the same analogy, in the **third phase** the strategic map defined by $\phi(5 \otimes_8 2) = 2$ maps $(2, 5) \in H_1$ to $2 \in R$. It does not map $(3, 8)$ as $(g' = 21)$ was the largest element for which $(8, 8)$ was mapped to 1 in Phase-1. Thus, the sequential nature of ϕ with the assumptions governs the mapping from H_1 to R. As a result, the last element in H_1 to be mapped to R is $(5, 2)$.

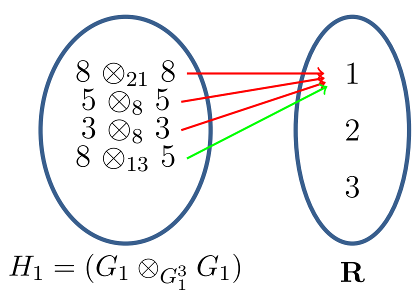


2^0 **transition:** G_1^2= {5,8,13,21} $\subset \mathfrak{F}$ is the second shift of G_1. In the 2^0 transition, ϕ determines the largest possible elements in G_1^2 for which the elements in H_1 can be mapped to R.

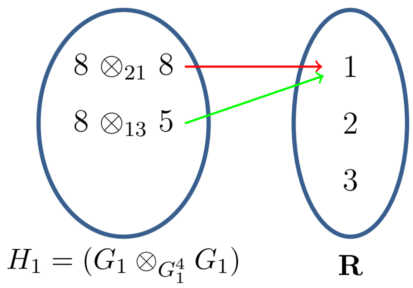
In the first phase, $(2, 2)$ is not mapped to R as there exists no largest possible element in G_1^2 for which $2 \otimes_{g_1'} 2 = r \in \mathfrak{F}$. Following the sequential nature of ϕ, the last element in H_1 to be mapped to R is $(3, 2)$.



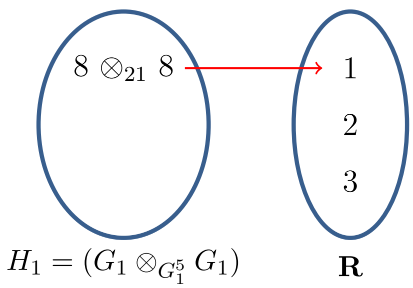
3^0 **transition:** G_1^3= {8,13,21,34} $\subset \mathfrak{F}$ is the third shift of G_1. Following the sequential nature of ϕ, the last element in H_1 to be mapped to R is $(8, 5)$.



4^0 **transition:** G_1^4= {13,21,34,55} $\subset \mathfrak{F}$ is the fourth shift of G_1. Following the sequential nature of ϕ and its assumptions, the last element in H_1 to be mapped to R is $(8, 5)$.



5^0 **transition:** G_1^5= {21,34,55,89} $\subset \mathcal{F}$ is the fifth shift of G_1. Following the sequential nature of ϕ and its assumptions, the last element in H_1 to be mapped to R is $(8, 8)$.

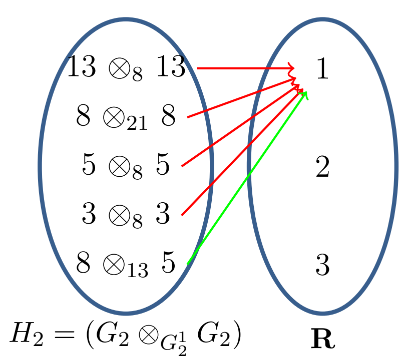


Limit of the strategic map ϕ for G_1: Clearly ϕ vanishes at the 6^0 transition with the limit of the strategic map ϕ being $21(F_{n+7})$ when $n \geq 3$ is odd for the given $G_1 = \{F_n = 2, F_{n+1} = 3, F_{n+2} = 5, F_{n+3} = 8\}$ of order $l = 4$ at the 5^0 transition.

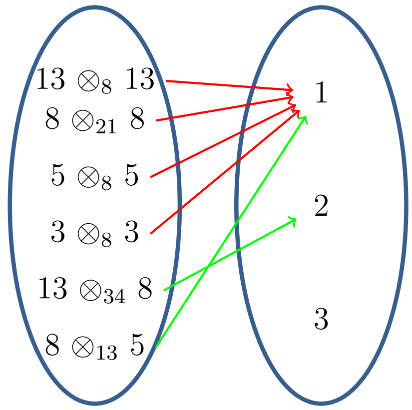
3.2 Case 2. For $G_2 = \{F_n, F_{n+1}, F_{n+2}, F_{n+3}\}$, when n is even.

For n=4, we have $G_2 = \{3, 5, 8, 13\}$.

1^0 **transition:** G_2^1= {5,8,13,21} $\subset \mathcal{F}$ is the first shift($l = 1$) of G_2. The elements of $H_2 = (G_2 \times G_2)$ is given by $\{(13, 13), (8, 8), (5, 5), (3, 3), (13, 8), (8, 13), (8, 5), (5, 8), (5, 3), (3, 5), (13, 5), (5, 13), (8, 3), (3, 8), (13, 3), (3, 13)\}$. Following the sequential nature of ϕ and its assumptions, the last element to be mapped is $(8, 15) \in H_2$ to $1 \in R$. The subsequent transitions are obtained.

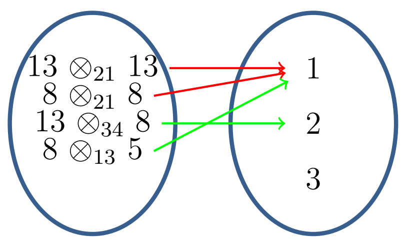


2^0 **transition:** $G_2^2 = \{8,13,21,34\} \subset \mathcal{F}$ is the second shift of G_2.



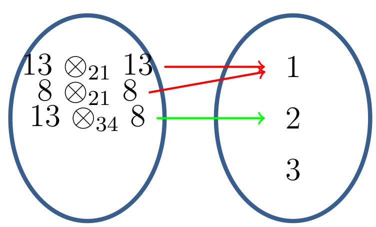
$H_2 = (G_2 \otimes_{G_2^2} G_2)$ **R**

3^0 **transition:** $G_2^3 = \{13,21,34,55\} \subset \mathcal{F}$ is the third shift of G_2.



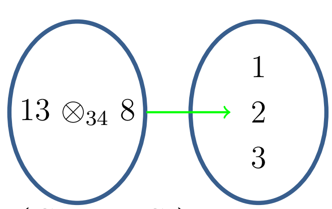
$H_2 = (G_2 \otimes_{G_2^3} G_2)$ **R**

4^0 **transition:** $G_2^4 = \{21,34,55,89\} \subset \mathcal{F}$ is the fourth shift of G_2.



$H_2 = (G_2 \otimes_{G_2^4} G_2)$ **R**

5^0 **transition:** $G_2^5 = \{34,55,89,144\} \subset \mathcal{F}$ is the fifth shift of G_2.



H$=(G_2 \otimes_{G_2^5} G_2)$ **R**

Limit of the strategic map ϕ for G_2: Clearly ϕ vanishes at the 6^0 transition with the limit of the strategic map ϕ being $34(F_{n+7})$ when $n \geq 3$ is even for the given $G_2 = \{F_n = 3, F_{n+1} = 5, F_{n+2} = 8, F_{n+3} = 13\}$ of order $l = 4$ at the 5^0 transition.

4. Results

Result 4.1. *The strategic map ϕ for a given $|G|$ and its shifts $|G'|= 4$,the map exists up to the 5^0 transition.*

Proof. Consider G_n=$\{F_n, F_{n+1}, F_{n+2}, F_{n+3}\}$ and its l transitions given by G_n^l=$\{F_{n+l}, F_{n+1+l}, F_{n+2+l}, F_{n+3+l}\} \subset F$ and $n \geqslant 3$.

Case I: For any G_n=$\{F_n, F_{n+1}, F_{n+2}, F_{n+3}\}$ of order 4 such that n is odd, clearly the ordered pair (F_{n+3}, F_{n+3}) yields the highest product amongst other combinations ensuring ϕ to sustain for the maximum transitions.We have to prove that F_{n+3}^2= $(F_{n+5}.F_{n+1})$+1. The result holds for n=1. In fact, it holds for any n=k which is guaranteed by the Cassini identity of Fibonacci numbers. Using elementary transformations $R_1 \to R_1 - R_2$ twice and $C_2 \to C_2 - C_1$ we establish, $\begin{vmatrix} F_{k+3} & F_{k+5} \\ F_{k+1} & F_{k+3} \end{vmatrix} = -(-1)^k = 1$ ($\because k$ is odd).

Case II: For any G_n=$\{F_n, F_{n+1}, F_{n+2}, F_{n+3}\}$ of order 4 such that n is even. By Cassini Identity, this time the ordered pair (F_{n+3}, F_{n+2}) yields the feasible highest product amongst other combinations such that $(g \otimes_{g'} h) = r \in R$ which ensures ϕ to sustain for the maximum transitions.We have to prove that $F_{n+2}.F_{n+3}$=$(F_{n+5}.F_n)$+2. Using mathematical induction and Cassini identity, we prove the same. Thus, the result holds. □

Result 4.2. *A strategic map ϕ for a given G of order $l(l \geqslant 2)$ exists up to $(l+1)^0$ transitions with $L = F_{n+(1+l)}$ as its limit,* $\forall n \geqslant 3$.[Thomas Theorem]

Proof. Let us examine the limit of ϕ for a given G= $\{F_n, F_{n+1}, F_{n+2}, F_{n+3}\}$ for different order l and the nature of n thereby generalising the result.

Case 1: It is baseless to consider the set G in which l=1, as it violates assumption(iii).

Case 2: For l=2,3,4 and depending upon the nature of n(even or odd) as discussed in Result 4.1, the application of Casinni identity with the tool of mathematical induction as shown in Result 1 establishes the theorem, $\forall n \geqslant 3$.

Case 3: In general for any l,

From the properties of Cassini - Catalan identities discussed in (Voll, 2010) (Panwar, Singh, & Gupta, 2014) with principles of mathematical induction, one may easily prove that:

1. **If l is odd(even) and n is even(odd)**: Without loss of generality(WLOG),we can conclude that the ordered pair $(F_{n+(l-1)}, F_{n+(l-1)})$ yields the highest product such that for any $F_{n+(l-1)} \in G$, the element $(F_{n+(l-1)} \otimes_L F_{n+(l-1)}) \in H = 1 \in R$ exerts more influence over other combinations say (g, h) to ensure the strategic map ϕ sustains for the maximum number of transitions i.e. $(l+1)^0$. Thus, L=$F_{n+(l+1)}$ is the limit of the strategic map ϕ.

2. **If both l and n are even(odd):** Without loss of generality(WLOG), we can conclude that the ordered pair $(F_{n+(l-1)}, F_{n+(l-2)})$ yields the highest product such that for any $F_{n+(l-1)}, F_{n+(l-2)} \in G$, the element $(F_{n+(l-1)} \otimes_L F_{n+(l-2)}) \in H = 2 \in R$ exerts more influence over other combinations say $(g, h) \in H$ to ensure the strategic map ϕ sustains for the maximum number of transitions i.e. $(l+1)^0$.Thus, L=$F_{n+(l+1)}$ is the limit of the strategic map ϕ.

Hence, in both the cases irrespective of the nature of l and n, the limit of the strategic map ϕ remains $F_{n+(l+1)}$ and it exists up to $(l+1)^0$ transition.

This proves the theorem.

5. Consequences

Corollary 5.1. The GolCaSeb Identity: *Relationship between four consecutive Fibonacci elements* $(F_n) \forall n \geq 6$

$$\lim_{n\to\infty} \sum_{k=3}^{n} \left(\frac{F_{2k+1}}{F_{2k-1}^2} - \frac{F_{2(k+1)}}{F_{2k-1}F_{2k}} \right) = 0$$

Proof. The golden ratio is defined as the limit of the ratios of two successive Fibonacci terms given by,

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi$$

$$\lim_{n\to\infty} \left[\sum_{k=3}^{n} \left(\frac{F_{2k+1}}{F_{2k-1}^2} - \frac{F_{2(k+1)}}{F_{2k-1}F_{2k}} \right) \right] = \lim_{n\to\infty} \left[\sum_{k=3}^{n} \frac{1}{F_{2k-1}} \left(\frac{F_{2k+1}}{F_{2k}} \frac{F_{2k}}{F_{2k-1}} - \frac{F_{2(k+1)}}{F_{2k+1}} \frac{F_{2k+1}}{F_{2k}} \right) \right]$$

$$= \lim_{n\to\infty} \left[\sum_{k=3}^{n} \frac{1}{F_{2k-1}} \left(\varphi^2 - \varphi^2 \right) \right] = 0$$

□

Definition 5 (A-nary Fibonacci modulus)**.** *Consider a congruence relation defined by* $F_{2k}F_{2k+a} \equiv F_a(modF_{2k+(a+1)})$ *and* $F_{2k+1}F_{2k+a} \equiv F_a(modF_{2k+(a+2)}) \quad \forall k, a \in \mathbb{N}$. *For any Fibonacci ordered pair* $\big(F_{2k}, F_{2k+a)}\big)$ *and* $\big(F_{2k+1}, F_{2k+a}\big)$, *an a-nary Fibonacci modulus is defined by the term* $F_{2k+(a+1)}$ *and* $F_{2k+(a+2)}$ *respectively by the above congruence relation yielding the remainder* F_a*, the* a^{th} *Fibonacci term.*

The following consequences are established by using the matrix form of Cassini and Catalian identities (Panwar et al., 2014). Cassini's identity states that - Considering $F_0 = 1, F_1 = 1$, for any F_n,

$$\det \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = (-1)^{n-1}$$

Corollary 5.2. $F_{2k} \otimes_{F_{2k+1}} F_{2k} = 1 \quad \forall k \in \mathbb{N}$

Proof. Claim: $F_{2k}^2 \equiv 1(modF_{2k+1})$ and let us consider

$$\det \begin{bmatrix} F_{2k} & F_{2k-1} \\ F_{2k+1} & F_{2k} \end{bmatrix}$$

By performing the elementary row operation $R_2 \longrightarrow R_2 - R_1$ once and by considering $n = 2k-1$, we get back to the original form of Cassini's identity i.e.

$$\det \begin{bmatrix} F_{2k} & F_{2k-1} \\ F_{2k-1} & F_{2k} \end{bmatrix} = (-1)^{(2k-2)} = 1.$$

□

Corollary 5.3. $F_{2k} \otimes_{F_{2(k+1)}} F_{2k+1} = 1 \quad \forall k \in \mathbb{N}$

Proof. The proof follows by considering

$$\det \begin{bmatrix} F_{2k} & F_{2k-1} \\ F_{2k+2} & F_{2k+1} \end{bmatrix}$$

and by the repeated application of elementary row operations from (1) □

Corollary 5.4. $F_{2k} \otimes_{F_{2k+3}} F_{2(k+1)} = 2 \quad \forall k \in \mathbb{N}$

Proof. The proof follows by considering

$$\det \begin{bmatrix} F_{2k} & F_{2k-1} \\ F_{2k+3} & F_{2k+2} \end{bmatrix}$$

and by the repeated application of elementary row operation $R_2 \longrightarrow R_2 - R_1$ we obtain

$$\det \begin{bmatrix} F_{2k} & F_{2k-1} \\ F_{2k+2} + F_{2k-1} & F_{2k+1} + F_{2k-2} \end{bmatrix} = \det \begin{bmatrix} F_{2k} & F_{2k-1} \\ F_{2k+1} + F_{2k-1} & F_{2k} + F_{2k-2} \end{bmatrix}$$

on simplifying further,we finally arrive at the form

$$\det \begin{bmatrix} F_{2k} & F_{2k-1} \\ 2F_{2k-1} & 2F_{2k-2} \end{bmatrix} = 2(-1)^{(2k-2)} = 2$$

□

Corollary 5.5. In general, $F_{2k} \otimes_{F_{2k+(a+1)}} F_{2k+a} = F_a$ is independent of $k \ \ \forall k, a \in \mathbb{N}$

Proof. The proof follows from repeated application of elementary row operations for a fixed choice of a. □

□

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