Proof of Primality Using the Set S

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Abstract

This paper provides a proof of primality using a set construction S. Specifically, it demonstrates that a positive integer $p > 1$ is prime if and only if $p \notin S$. The set S is defined in terms of prime divisors of p and is constructed by considering natural numbers within certain bounds. Examples are provided to illustrate the application of this criterion for both prime and composite numbers, including $p = 121$.

Theorem

A positive integer $p > 1$ is prime if and only if $p \notin S$, where the set S is defined as:

$$S = \{a(1+n) \mid a \text{ is a prime}, \ a \leq \sqrt{p}, \ n \in \mathbb{N}, \ 2 \leq n \leq \frac{p}{a} - 1\}.$$

Proof

Definition of the Set S

Let $p > 1$ be a positive integer. The set S is constructed as follows:

$$S = \{a + an \mid a \text{ is a prime}, \ a \leq \sqrt{p}, \ n \in \mathbb{N}, \ 2 \leq n \leq \frac{p}{a} - 1\}.$$

Each element of S takes the form $a(1+n)$, where:

- a is a prime divisor of p with $a \leq \sqrt{p}$,
- n is a natural number such that $2 \leq n \leq \frac{p}{a} - 1$.

Case 1: p is Composite

Assume p is composite. Then p has a proper divisor a such that $1 < a \leq \sqrt{p}$. Let $b = \frac{p}{a}$, so b is also an integer, and $b > a$. Observe that:

$$\frac{p}{a} - 1 = b - 1 \quad \implies \quad 2 \leq n = b - 1 \leq \frac{p}{a} - 1.$$

Thus, n is a valid natural number in the range $2 \leq n \leq \frac{p}{a} - 1$. For this n, the element $a(1+n)$ satisfies:

$$a(1+n) = a\left(1 + \frac{p}{a} - 1\right) = p.$$

Hence, $p \in S$.

Case 2: p is Prime

Assume p is prime. Then p has no divisors other than 1 and p. For any prime $a \leq \sqrt{p}$, the term $\frac{p}{a}$ is not an integer, as a does not divide p. Therefore:

$$\frac{p}{a} - 1 \notin \mathbb{N} \quad \implies \quad n \notin \mathbb{N}.$$

This means no valid n exists in the range $2 \leq n \leq \frac{p}{a} - 1$, so p cannot be expressed as $a(1+n)$ for any prime $a \leq \sqrt{p}$. Thus, $p \notin S$.

Conclusion

From the two cases, we conclude that:

$$p \text{ is prime} \iff p \notin S.$$

Examples

Example 1: $p = 15$

Let $p = 15$. The primes $a \leq \sqrt{15}$ are $\{2, 3\}$.

- For $a = 2$, $\frac{p}{a} - 1 = 6.5$. The range $2 \leq n \leq 6$ yields:

  $S = \{2(1+2), 2(1+3), 2(1+4), 2(1+5), 2(1+6)\} = \{6, 8, 10, 12, 14\}$.

  Since $15 \notin S$, we proceed to the next a.

- For $a = 3$, $\frac{p}{a} - 1 = 4$. The range $2 \leq n \leq 4$ yields:

$$S = \{3(1+2), 3(1+3), 3(1+4)\} = \{9, 12, 15\}.$$

Since $15 \in S$, we conclude that $p = 15$ is composite.

Example 2: $p = 13$

Let $p = 13$. The primes $a \leq \sqrt{13}$ are $\{2, 3\}$.

- For $a = 2$, $\frac{p}{a} - 1 = 5.5$. The range $2 \leq n \leq 5$ yields:

$$S = \{2(1+2), 2(1+3), 2(1+4), 2(1+5)\} = \{6, 8, 10, 12\}.$$

Since $13 \notin S$, we proceed to the next a.

- For $a = 3$, $\frac{p}{a} - 1 = 3.33$. The range $2 \leq n \leq 3$ yields:

$$S = \{3(1+2), 3(1+3)\} = \{9, 12\}.$$

Since $13 \notin S$, we conclude that $p = 13$ is prime.

Example 3: $p = 121$

Let $p = 121$. The primes $a \leq \sqrt{121}$ are $\{2, 3, 5, 7, 11\}$.

- For $a = 2$, $\frac{p}{a} - 1 = 59.5$. The range $2 \leq n \leq 59$ yields:

$$S = \{2(1+2), 2(1+3), \ldots, 2(1+59)\} = \{6, 8, 10, \ldots, 120\}.$$

Since $121 \notin S$, we proceed to the next a.

- For $a = 3$, $\frac{p}{a} - 1 = 39.33$. The range $2 \leq n \leq 39$ yields:

$$S = \{3(1+2), 3(1+3), \ldots, 3(1+39)\} = \{9, 12, 15, \ldots, 120\}.$$

Since $121 \notin S$, we proceed to the next a.

- For $a = 5$, $\frac{p}{a} - 1 = 23.2$. The range $2 \leq n \leq 23$ yields:

$$S = \{5(1+2), 5(1+3), \ldots, 5(1+23)\} = \{15, 20, 25, \ldots, 120\}.$$

Since $121 \notin S$, we proceed to the next a.

- For $a = 7$, $\frac{p}{a} - 1 = 16.2857$. The range $2 \leq n \leq 16$ yields:

  $$S = \{7(1+2), 7(1+3), \ldots, 7(1+16)\} = \{21, 28, 35, \ldots, 119\}.$$

  Since $121 \notin S$, we proceed to the next a.

- For $a = 11$, $\frac{p}{a} - 1 = 10$. The range $2 \leq n \leq 10$ yields:

  $$S = \{11(1+2), 11(1+3), \ldots, 11(1+10)\} = \{33, 44, 55, 66, 77, 88, 99, 110, 121\}.$$

  Since $121 \in S$, we conclude that $p = 121$ is composite.