Hamiltonian Instability and the Classical-to-Quantum Transition

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Abstract

The mechanism of Arnold diffusion (AD) describes the dynamic instability of nearly-integrable Hamiltonian systems. Here we argue that AD leads to action quantization for classical systems having an infinite number of degrees of freedom. Planck’s constant emerges as long-time value of the action differential applied to large ensembles of oscillators in near equilibrium conditions.

Key words: Arnold diffusion, Hamiltonian chaos, Planck constant, action quantization, classical to quantum transition.

The formalism of action-angle variables applies to generic Hamiltonian systems undergoing periodic motion, such as harmonic oscillation or Kepler
rotation. It consists of replacing the generalized coordinates and momenta using the transformation [1-3]

\[(q, p) \rightarrow (\theta, I)\]  \hspace{1cm} (1)

Action-angle variables are canonically conjugate and introduced through the generating function [3]

\[S(q, I) = \int q \cdot p(q, H) dq\]  \hspace{1cm} (2)

such that

\[I = \frac{1}{2\pi} \cdot \int_c p(q, H) dq = I(H)\] \hspace{1cm} (3a)

\[\theta = \frac{\partial S(q, I)}{\partial I}\] \hspace{1cm} (3b)

Since \(I\) is a cyclic variable, the corresponding drift of (2) per each period of \(I\) amounts to [1]

\[\Delta S = 2\pi I\] \hspace{1cm} (4)
Consider a nearly-integrable periodic system with $n$ degrees of freedom defined by the Hamiltonian [4-6]

$$H = H_0(I) + \varepsilon H_1(I, \theta, \varepsilon)$$  \hspace{1cm} (5)

where $0 \leq \varepsilon \leq \varepsilon_0$ is a small perturbation parameter and $H_0(I)$ is the unperturbed Hamiltonian, taken to be fully integrable in the limit $\varepsilon = 0$. The frequency of the unperturbed motion is determined by

$$\omega(I) = \frac{\partial H_0}{\partial I}$$  \hspace{1cm} (6)

For $\varepsilon \ll \varepsilon_0$, the equations of motion read

$$\dot{i} = -\frac{\partial H(\theta, I)}{\partial \theta} = 0$$  \hspace{1cm} (7)

$$\dot{\theta} = \frac{\partial H(\theta, I)}{\partial I} = \omega(I)$$  \hspace{1cm} (8)

in which $I, \theta \in \mathbb{R}^n$. The solution of (7)-(8) lies on invariant $n$-tori residing in the phase-space of dimension $\mathbb{R}^{2n}$. For $n \leq 2$, all solutions are stable since 2-tori confine trajectories on a 3-dimensional energy surface. This is no longer the case for $n \geq 3$ where, according to the Arnold diffusion conjecture, the
action of nearly-integrable systems changes by $O(1)$ over a sufficiently long time. In particular, assuming that

$$\left| H_0 \right| < c_1, \quad \left| H_1 \right| < c_1$$  \hspace{1cm} (9)$$

where $c_1$ is a positive constant, and taking the unperturbed Hamiltonian to represent a quasi-convex function of the action variable, the following condition holds [6, 8]

$$\delta I = |I(t) - I(0)| < C_1 \varepsilon^{1/2n}$$  \hspace{1cm} (10)$$

over sufficiently long-times satisfying

$$0 \leq t \leq \exp\left(\frac{C_2^{-1}}{\varepsilon^{1/2n}}\right)$$  \hspace{1cm} (11)$$

In (10) and (11), $C_1, C_2$ are also positive constants. By (4), the corresponding drift in action is given by

$$\delta(\Delta S) = 2\pi \delta I < O(C_1 \varepsilon^{1/2n})$$  \hspace{1cm} (12)$$

Normalizing (12) to $c_1$ confirms that the drift in action is of $O(1)$, which naturally replicates the process of action quantization for $n >> 1$. It follows...
that Planck’s constant may be mapped to the long-time value of (12) applied to large ensembles of oscillators in near equilibrium conditions. Stated differently, the transition from classical to quantum behavior is expected to occur when

\[ 0 \leq t \leq \exp\left(\frac{C^2}{\epsilon^{1/2n}}\right); \; n > 1 \]  

(13)

leading to

\[ \delta(\Delta S) = 2\pi \delta I = O(1) \]  

(14)

These findings fall in line with the approach taken in [7].

References


4. http://authors.library.caltech.edu/13561/1/KALbams08.pdf


