On the calculation of a polynomial approximation of numerical table values using matrix algebra

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Abstract

In this communication, we propose a computational method involving matrix algebra to approximate numerical values of a given table by a second degree polynomial.

1 Application of matrix algebra to approximate tabulated values by a second degree polynomial

In Engineering practice we encounter the problem of programming in a micro-controller a table consisting of a large number of numerical values. Depending on the size of the table, i.e. the amount of its elements, programming can be a tedious task to complete. Therefore, in this communication we propose to use matrix algebra to approximate the table values by the coefficients of a second degree polynomial, an approximation resulting in sufficient accuracy. To demonstrate the matrix algebra employed, out of lay–out reasons, we limit the size of a given non–symmetrical table to 3×4. The formulas presented here are by no means restricted to symmetrical or non–symmetrical tables, i.e. the formulas can be applied to a table of any given size.
Define

$$\vec{z} = \mathbf{M} \cdot \vec{c} \Rightarrow \mathbf{M}^T \cdot \vec{z} = \mathbf{M}^T \cdot \mathbf{M} \cdot \vec{c}$$
$$\vec{c} = \left(\mathbf{M}^T \cdot \mathbf{M}\right)^{-1} \cdot \mathbf{M}^T \cdot \vec{z} \text{ for det} \left(\mathbf{M}^T \cdot \mathbf{M}\right) \neq 0$$

The components of the vector $\vec{c}$ denote the coefficients $c_{p,q}$ of a polynomial $p(x, y)$ with variables x, y

$$p(x, y) := \sum_{p+q \leq n} c_{p,q} x^p y^q \quad n, p, q \in \mathbb{N}$$

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To demonstrate the application of matrix algebra, we write the given table consisting of three rows and four columns as a matrix with elements $z_{i,j}$:

	y_1	y_2	y_3	y_4
x_1	$z_{1,1}$	$z_{1,2}$	$z_{1,3}$	$z_{1,4}$
x_2	$z_{2,1}$	$z_{2,2}$	$z_{2,3}$	$z_{2,4}$
x_3	$z_{3,1}$	$z_{3,2}$	$z_{3,3}$	$z_{3,4}$

The numerical figures in the table are approximated by the second degree polynomial $(n = 2)$

$$p\left(x_r, y_k\right) = c_{0,0} + c_{1,0} \cdot x_r + c_{0,1} \cdot y_k + c_{2,0} \cdot x_r^2 + c_{1,1} \cdot x_r y_k + c_{0,2} \cdot y_k^2$$

Now, we write all matrix elements of the given table as a vector $\vec{z}$

$$\vec{z} = \left(z_{1,1}, z_{2,1}, z_{3,1}, z_{1,2}, z_{2,2}, z_{3,2}, \cdots, z_{1,4}, z_{2,4}, z_{3,4}\right)^T$$

and we write the matrix elements of $\mathbf{M}$ as

$$\mathbf{M} = \begin{pmatrix} 1 & x_1 & y_1 & x_1^2 & x_1 y_1 & y_1^2 \\ 1 & x_2 & y_1 & x_2^2 & x_2 y_1 & y_1^2 \\ 1 & x_3 & y_1 & x_3^2 & x_3 y_1 & y_1^2 \\ 1 & x_1 & y_2 & x_1^2 & x_1 y_2 & y_2^2 \\ 1 & x_2 & y_2 & x_2^2 & x_2 y_2 & y_2^2 \\ 1 & x_3 & y_2 & x_3^2 & x_3 y_2 & y_2^2 \\ 1 & x_1 & y_3 & x_1^2 & x_1 y_3 & y_3^2 \\ 1 & x_2 & y_3 & x_2^2 & x_2 y_3 & y_3^2 \\ 1 & x_3 & y_3 & x_3^2 & x_3 y_3 & y_3^2 \\ 1 & x_1 & y_4 & x_1^2 & x_1 y_4 & y_4^2 \\ 1 & x_2 & y_4 & x_2^2 & x_2 y_4 & y_4^2 \\ 1 & x_3 & y_4 & x_3^2 & x_3 y_4 & y_4^2 \end{pmatrix}$$

Solving the matrix equation

$$\vec{c} = \left(\mathbf{M}^T \cdot \mathbf{M}\right)^{-1} \cdot \mathbf{M}^T \cdot \vec{z}$$

results in the vector $\vec{c}$ with its sought for coefficients

$$\vec{c} = \left(c_{0,0}, c_{1,0}, c_{0,1}, c_{2,0}, c_{1,1}, c_{0,2}\right)^T$$

2 Example

In the following we apply the formulas derived above to approximate the numerical values of a given symmetrical Mollier–table. The elements of this table denote relative humidity values as a function of a column vector $x_1 = 1, x_2 = 2, x_3 = 3$ which denotes the temperature difference in Celsius between the wet and the dry bulb and a row vector $y_1 = 15, y_2 = 18, y_3 = 20$ that denotes the different temperatures of the environment in Celsius.
Assume that the Mollier–table $z_{i,j}$ is given as a matrix with each three rows and columns:

	y_1	y_2	y_3
x_1	$z_{1,1}$	$z_{1,2}$	$z_{1,3}$
x_2	$z_{2,1}$	$z_{2,2}$	$z_{2,3}$
x_3	$z_{3,1}$	$z_{3,2}$	$z_{3,3}$

=

	15	18	20
1	90	80	71
2	91	82	73
3	91	83	75

We now approximate the figures in the table by the second degree polynomial $(n = 2)$

$$p(x_r, y_k) = c_{0,0} + c_{1,0} \cdot x_r + c_{0,1} \cdot y_k + c_{2,0} \cdot x_r^2 + c_{1,1} \cdot x_r y_k + c_{0,2} \cdot y_k^2$$

Denote vector $\vec{z}$ by

$$\vec{z} = (z_{1,1}, z_{2,1}, z_{3,1}, z_{1,2}, z_{2,2}, z_{3,2}, \cdots, z_{1,3}, z_{2,3}, z_{3,3})^T$$

and the matrix $\mathbf{M}$ as

$$\mathbf{M} = \begin{pmatrix} 1 & x_1 & y_1 & x_1^2 & x_1 y_1 & y_1^2 \\ 1 & x_2 & y_1 & x_2^2 & x_2 y_1 & y_1^2 \\ 1 & x_3 & y_1 & x_3^2 & x_3 y_1 & y_1^2 \\ 1 & x_1 & y_2 & x_1^2 & x_1 y_2 & y_2^2 \\ 1 & x_2 & y_2 & x_2^2 & x_2 y_2 & y_2^2 \\ 1 & x_3 & y_2 & x_3^2 & x_3 y_2 & y_2^2 \\ 1 & x_1 & y_3 & x_1^2 & x_1 y_3 & y_3^2 \\ 1 & x_2 & y_3 & x_2^2 & x_2 y_3 & y_3^2 \\ 1 & x_3 & y_3 & x_3^2 & x_3 y_3 & y_3^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 15 & 1 & 15 & 225 \\ 1 & 2 & 15 & 4 & 30 & 225 \\ 1 & 3 & 15 & 9 & 45 & 225 \\ 1 & 1 & 18 & 1 & 18 & 324 \\ 1 & 2 & 18 & 4 & 36 & 324 \\ 1 & 3 & 18 & 9 & 54 & 324 \\ 1 & 1 & 20 & 1 & 20 & 400 \\ 1 & 2 & 20 & 4 & 40 & 400 \\ 1 & 3 & 20 & 9 & 60 & 400 \end{pmatrix}$$

Solving the matrix equation

$$\vec{c} = \left(\mathbf{M}^T \cdot \mathbf{M}\right)^{-1} \cdot \mathbf{M}^T \cdot \vec{z}$$

results in the vector $\vec{c}$ with its sought for coefficients

$$\vec{c} = (97.4503, -14.6140, 0.3433, 0.1667, 0.2895, -0.0111)^T$$

Comparing the approximated table A to the given one, we get

	15	18	20
1	90	80	71
2	91	82	73
3	91	83	75

vs.

	15	18	20
1	89.9942	80.2222	70.7836
2	90.7924	81.8889	73.3187
3	91.2135	82.8889	74.8977

and as an indication of accuracy we calculate the difference between the approximated and the given table:

$\Delta =$

	15	18	20
1	0.0058	−0.2222	0.2164
2	0.2076	0.1111	−0.3187
3	−0.2135	0.1111	0.1023

2.1 Matlab program

```
%
% Mollier table written as vector z
%
z = [90 ; 80 ; 71 ; 91 ; 82 ; 73 ; 91 ; 83 ; 75 ];
%
% Matrix M as a function of the column vector x and
% the row vector y
%
M = [1 1 15 1 15 225
     1 2 15 4 30 225
     1 3 15 9 45 225
     1 1 18 1 18 324
     1 2 18 4 36 324
     1 3 18 9 54 324
     1 1 20 1 20 400
     1 2 20 4 40 400
     1 3 20 9 60 400];
%
% calculation of coefficients of vector c
%
c = inv(M'*M)*M'*z
%
% calculation and result of approximation vector A to
% the Mollier table
%
A = M*c
%
% difference between given and calculated figures
%
Delta = z - A
```

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