A Decomposition Formula for Third-Order Real Antisymmetric Matrices

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Abstract

A decomposition formula for an antisymmetric matrix $A_\omega \in A_3(\mathbb{R})$ is provided, where its axial vector is expressed as $\omega = M\nu$, with $M$ symmetric and $\nu \in \mathbb{R}^3$. The proof is based mainly on vector projection through Frobenius inner product. In the end, a vectorial identity involving cross product is proved as a corollary of the decomposition formula.

Keywords Antisymmetric Matrices · Cross Product · Frobenius Inner Product

1 Introduction

Let $A_3(\mathbb{R}) = \{A \in M_3(\mathbb{R}) : A = -A^T\}$ be the set of third-order real antisymmetric matrices, where $M_3(\mathbb{R})$ is the vector space of square real matrices of order 3. Then $A_3$ is a vector subspace of $M_3$. In fact, given two antisymmetric matrices $A_1, A_2 \in A_3$, it is easy to show the closure with respect to sum:

$$A_1 + A_2 = -A_1^T - A_2^T = -(A_1 + A_2)^T$$

Similarly, for any given $\lambda \in \mathbb{R}$, we can show the closure with respect to multiplication by a scalar:

$$\lambda A_1 = -\lambda A_1^T = -(\lambda A_1)^T$$

Proposition 1.1 $A_3$ has canonical base $B = \{E_1, E_2, E_3\}$, where:

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$
Proof: Any $A \in A_3$ can be expressed as a linear combination of $E_1, E_2, E_3$. In fact:

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{bmatrix} = a_{12}E_1 + a_{13}E_2 + a_{23}E_3$$

therefore $A_3 = \text{Span}(E_1, E_2, E_3)$. Now consider the two following linear combinations:

$$A = \gamma_1' E_1 + \gamma_2' E_2 + \gamma_3' E_3$$
$$A = \gamma_1'' E_1 + \gamma_2'' E_2 + \gamma_3'' E_3$$

By definition, we know that any antisymmetric matrix $A \in A_3$ is such that $A = -A^T$, therefore $A + A^T = 0$. In light of this, we can write:

$$A + A^T = (\gamma_1' E_1 + \gamma_2' E_2 + \gamma_3' E_3) + (\gamma_1'' E_1 + \gamma_2'' E_2 + \gamma_3'' E_3)^T =$$

$$= (\gamma_1' - \gamma_1'') E_1 + (\gamma_2' - \gamma_2'') E_2 + (\gamma_3' - \gamma_3'') E_3 = 0$$

The latter is satisfied if and only if $\gamma_i' = \gamma_i''$ for $i = 1, 2, 3$, which means that there is a unique linear combination to express $A$, hence $\{E_1, E_2, E_3\}$ is a set of linearly independent vectors. Therefore, $B = \{E_1, E_2, E_3\}$ is a base of $A_3$.

An immediate consequence of this is that $\dim(A_3) = 3$. Antisymmetric matrices are useful to express cross products in terms of matrix-vector products. In fact, given two vectors $a, b \in \mathbb{R}^3$, their cross product $a \times b$ can be expressed as:

$$a \times b = A_a b$$  \hspace{1cm} (1)$$

where $A_a$ is antisymmetric. Given $(a_1, a_2, a_3)$ the coordinates of $a$, the matrix $A_a$ reads as follows:

$$A_a = \begin{bmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{bmatrix}$$  \hspace{1cm} (2)$$

Given any antisymmetric matrix, it is always possible to associate it with a vector $a \in \mathbb{R}^3$, which is called axial vector.

Let us now consider the following set of antisymmetric matrices:

$$X_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \hspace{1cm} X_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \hspace{1cm} X_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
Similarly to the last result, it can be easily shown that $B' = \{X_1, X_2, X_3\}$ is a basis of $A_3$, and that given an axial vector $\omega = (\omega_1, \omega_2, \omega_3)$, it is always possible to write its associated antisymmetric matrix $A_\omega$ simply as:

$$A_\omega = \omega_1 X_1 + \omega_2 X_2 + \omega_3 X_3$$

(3)

**Definition 1.1** Given two real square matrices of order $n$ $A$, $B$, the Frobenius inner product is a bilinear form $\langle \cdot, \cdot \rangle_F : M_n(\mathbb{R}) \times M_n(\mathbb{R}) \to \mathbb{R}$ defined as:

$$\langle A, B \rangle_F = \text{Tr}(A^T B)$$

The norm induced by this product is given by:

$$\|A\|_F = \sqrt{\langle A, A \rangle_F}$$

**Proposition 1.2** $B' = \{X_1, X_2, X_3\}$ is an orthogonal basis with respect to the Frobenius inner product.

**Proof:** We have to prove that $\langle X_i, X_j \rangle_F = 0$, for $i, j = 1, 2, 3, i \neq j$. It is straightforward to see that multiplying each row of $X_i^T$ with the correspondent column of $X_j$ (i.e. first row with first column, second row with second column, and so on), one gets a null-diagonal matrix, hence the product is identically zero for any $i \neq j$, proving the statement. □

To conclude with, we can report the following theorem on vector projection [1] applied to antisymmetric matrices expressed with respect to $B' = \{X_1, X_2, X_3\}$.

**Theorem 1.1** Given $C \in A_3(\mathbb{R})$ and the orthogonal basis $B' = \{X_1, X_2, X_3\}$ with respect to the Frobenius inner product, it holds that:

$$C = c_1 X_1 + c_2 X_2 + c_3 X_3$$

where

$$c_i = \frac{\langle C, X_i \rangle_F}{\langle X_i, X_i \rangle_F}$$

(4)

are called Fourier’s coefficients.

2 **Decomposition Formula**

**Theorem 2.1** Given two axial vector $\nu, \omega \in \mathbb{R}^3$, where $\omega$ is expressible as $\omega = M \nu$ with $M$ symmetric, the following equality holds:

$$A_\omega = \text{Tr}(M) A_\nu - 2\text{Asym}(MA_\nu)$$

(5)

where $A_\nu, A_\omega$ are the antisymmetric matrices associated to the axial vectors $\nu, \omega$ respectively, and $\text{Asym}(MA_\nu)$ is the antisymmetric part of $MA_\nu$. 

3
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Adding and subtracting it to (6), one has:

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Let us show that \( I \) corresponds to \( \text{Tr}(M) A \). In fact:

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First of all, let us observe that we can remove \( \sigma \). In fact, for \( i = j \), the term \( m_{ij} \nu_i - m_{ii} \nu_j = 0 \), hence we can simply put:

We want to find out who \( C \) is. Since \( \text{Tr}(M) A \nu \) is antisymmetric, \( C \) must be forcedly antisymmetric in order to enforce (6) and have \( A \sigma \in \mathcal{A}_3(\mathbb{R}) \). Let us observe from (8) that the components of \( C \) are obtained from some linear operation between \( M \) and \( \nu \). We cannot choose \( C = A M \nu \) because it already appears at the left-hand member of (7), so a hint
for $C$ would be:

$$C = \lambda \text{Asym}(MA_{\nu})$$

with $\lambda$ opportunely chosen. Observe that this intuition makes sense since the components of $C$ would consist of a sum of addenda where each of them is a product of some $m_{ij}$ multiplying some $\nu_i$ (eventually with a shifted sign), as predicated by (8). In addition, taking the antisymmetric part will ensure the requirement of antisymmetry of $C$. Also this choice is well-defined because:

$$(MA_{\nu})^T = A_{\nu}^T M^T = -A_{\nu} M$$

which means $MA_{\nu}$ is neither symmetric nor antisymmetric. Moreover:

$$\text{Asym}(MA_{\nu}) = \frac{1}{2} \left[ MA_{\nu} - (MA_{\nu})^T \right] = \frac{1}{2} \left[ MA_{\nu} + A_{\nu} M \right] = \frac{1}{2} \left[ A_{\nu} M + MA_{\nu} \right] = \frac{1}{2} \left[ A_{\nu} M - M^T A_{\nu}^T \right] = \frac{1}{2} \left[ A_{\nu} M - (A_{\nu} M)^T \right] = \text{Asym}(A_{\nu} M)$$

In order to show this intuition is actually true, we will take $C = \lambda \text{Asym}(MA_{\nu})$, project it on $B' = \{X_1, X_2, X_3\}$, and check if the projection coefficients are actually corresponding to the components of $C$ as expressed in (8). Before continuing, we need to introduce the following lemma.

**Lemma 2.1** Given a symmetric matrix $M$ and an axial vector $\nu$ with associated antisymmetric matrix $A_{\nu}$, it holds that:

$$\langle A_{\nu} M + MA_{\nu}, X_i \rangle_F = 2 \langle A_{\nu} M, X_i \rangle_F \quad i = 1, 2, 3$$

where $X_i \in B'$.

**Proof:** Calculate $\langle A_{\nu} M, X_i \rangle_F$ first:

$$\langle A_{\nu} M, X_i \rangle_F = \text{Tr} \left( (MA_{\nu})^T X_i \right) = \text{Tr} \left( A_{\nu}^T M^T X_i \right) = -\text{Tr}(A_{\nu} M X_i)$$

By the commutation property of the trace operator applied to a matrix product, for real square matrices we have $\text{Tr}(AB) = \text{Tr}(BA)$, which allows us to express the Frobenius inner product of two matrices alternatively as:

$$\langle A, B \rangle_F = \langle B, A \rangle_F = \text{Tr}(B^T A) = \text{Tr}(AB^T)$$

Therefore, considering $\langle MA_{\nu}, X_i \rangle_F$:

$$\langle MA_{\nu}, X_i \rangle_F = \text{Tr}(A_{\nu} MX_i^T) = -\text{Tr}(A_{\nu} MX_i) = \langle A_{\nu} M, X_i \rangle_F$$

Therefore:

$$\langle A_{\nu} M + MA_{\nu}, X_i \rangle_F = \langle A_{\nu} M, X_i \rangle_F + \langle MA_{\nu}, X_i \rangle_F = \langle A_{\nu} M, X_i \rangle_F + \langle A_{\nu} M, X_i \rangle_F = 2 \langle A_{\nu} M, X_i \rangle_F$$
which proves the lemma.

Now we can use this lemma to compute the Fourier’s coefficients of $C = \lambda \text{Asym}(MA_\nu)$ along $X_1, X_2, X_3$. We have that:

$$C = \lambda \text{Asym}(MA_\nu) = \frac{\lambda}{2} [A_\nu M + MA_\nu]$$

and

$$c_i = \frac{(C, X_i)_F}{(X_i, X_i)_F} = \frac{\lambda}{2} \frac{(A_\nu M + MA_\nu, X_i)_F}{(X_i, X_i)_F} = \frac{\lambda}{2} \frac{(A_\nu M, X_i)_F}{(X_i, X_i)_F}$$  \hspace{1cm} (10)

It is easy to calculate that $(X_i, X_i)_F = \|X_i\|_F^2 = 2$ for $i = 1, 2, 3$. In fact, take $i = 1$:

$$(X_1, X_1)_F = \text{Tr} \left( \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right) = \text{Tr} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 2$$

It is easy to show that also for $X_2$ and $X_3$, allowing us to rewrite (10) as:

$$c_i = \frac{\lambda}{2} (A_\nu M, X_i)_F$$

which we need to explicit for $i = 1, 2, 3$. Consider $i = 1$:

$$c_1 = \frac{\lambda}{2} (A_\nu M, X_1)_F =$$

$$= \frac{\lambda}{2} \text{Tr} \left( \begin{bmatrix} 0 & \nu_3 & -\nu_2 \\ -\nu_3 & 0 & \nu_1 \\ \nu_2 & -\nu_1 & 0 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{23} \\ m_{13} & m_{23} & m_{33} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}^T \right) =$$

$$= \frac{\lambda}{2} \text{Tr} \left( \begin{bmatrix} 0 & \nu_3 & -\nu_2 \\ -\nu_3 & 0 & \nu_1 \\ \nu_2 & -\nu_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & m_{13} & -m_{12} \\ m_{13} & m_{23} & -m_{22} \\ m_{13} & m_{23} & m_{33} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right) =$$

$$= \frac{\lambda}{2} \left( -m_{13}\nu_3 + m_{33}\nu_1 - m_{12}\nu_2 + m_{22}\nu_1 \right) =$$

$$= \frac{\lambda}{2} \left( m_{22} + m_{33} \right) \nu_1 - m_{12}\nu_2 - m_{12}\nu_3$$  \hspace{1cm} (11)

In a similar way, we can find out that:

$$c_2 = \frac{\lambda}{2} \left( m_{11} + m_{33} \right) \nu_2 - m_{12}\nu_1 - m_{23}\nu_3$$  \hspace{1cm} (12)

$$c_3 = \frac{\lambda}{2} \left( m_{11} + m_{22} \right) \nu_3 - m_{12}\nu_1 - m_{23}\nu_2$$  \hspace{1cm} (13)
Now, let us write explicitly the coordinates $C$ as expressed in (8). Still using Einstein’s notation, it reads:

$$C = \left( m_{i1} \nu_i - m_{ii} \nu_1 \right) X_1 + \left( m_{i2} \nu_i - m_{ii} \nu_2 \right) X_2 + \left( m_{i3} \nu_i - m_{ii} \nu_3 \right) X_3 = C_1 = c_1$$

Marking summation explicitly and using $m_{ij} = m_{ji}$, we have:

$$c_1 = \sum_{i=1}^{3} m_{i1} \nu_i - m_{ii} \nu_1 = -(m_{22} + m_{33}) \nu_1 + (m_{12} \nu_2 + m_{13} \nu_3)$$  \hfill (14)$$

$$c_2 = \sum_{i=1}^{3} m_{i2} \nu_i - m_{ii} \nu_2 = -(m_{11} + m_{33}) \nu_2 + (m_{12} \nu_1 + m_{23} \nu_3)$$  \hfill (15)$$

$$c_3 = \sum_{i=1}^{3} m_{i3} \nu_i - m_{ii} \nu_3 = -(m_{11} + m_{22}) \nu_3 + (m_{13} \nu_1 + m_{23} \nu_2)$$  \hfill (16)$$

Thus, (14), (15) and (16) coincide with (11), (12) and (13) respectively for $\lambda = -2$. This allows us to finally express $C$ as:

$$C = -2 \text{Asym}(MA_\nu)$$

Therefore, putting all together in (7), it yields:

$$A_\omega = \text{Tr}(M)A_\nu - 2 \text{Asym}(MA_\nu)$$

□

Since it is always possible to associate an antisymmetric matrix to the axial vector $\omega$ and viceversa, this formula holds as long as the axial vector is expressible as a matrix-vector product through $M$ and $\nu$ ($M$ symmetric). From this decomposition formula, we can immediately deduce the following result.

**Corollary 2.1** Given $a, b \in \mathbb{R}^3$ and a symmetric matrix $M$, the following relationship is true:

$$M(a \times b) = \text{Tr}(M) a \times b - a \times Mb + b \times Ma$$  \hfill (17)$$

**Proof:** Consider $\nu \equiv a$ and $\omega = Ma$. Then, using (5), we have:

$$A_{Ma} = \text{Tr}(M)A_a - 2 \text{Asym}(MA_a) = \text{Tr}(M)A_a - \left[ MA_a - (MA_a)^T \right] =$$

$$= \text{Tr}(M)A_a - MA_a + A_a^T M = \text{Tr}(M)A_a - MA_a - A_a M$$  \hfill (18)$$

Applying $b$ to both members of (18), one gets:

$$A_{Ma} b = \text{Tr}(M)A_a b - MA_a b - A_a Mb$$

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Using (2), we can write further:

\[(Ma) \times b = \text{Tr}(M) \times a - M(a \times b) - a \times (Mb)\]

If we reorganize the members and rewrite \((Ma) \times b = -b \times (Ma)\), we obtain exactly:

\[M(a \times b) = \text{Tr}(M) \times a - a \times Mb + b \times Ma\]

3 Conclusion

In the previous section, we have shown how a generic antisymmetric matrix of axial vector \(\omega\) can be decomposed. While it is always trivial to associate any \(A \in A_3(\mathbb{R})\) with a vector of \(\omega \in \mathbb{R}^3\), it is not obvious how to find \(M\) and \(\nu\) such that \(\omega = M\nu\), under the symmetry constraint of \(M\). Future work may consist of showing the existence of the couple \((M, \nu)\) for any given \(\omega \in \mathbb{R}^3\). Moreover, on the basis of that, one could seek for an optimal procedure of determining a three-dimensional vector \(\omega\) from 9 degrees of freedom (6 accounting for \(M\), and 3 for \(\nu\)). Finally, given the vectorial form of equation (17), one could investigate its prospective applications in fields like Vector Calculus, Differential Geometry and Mechanics.

References