On some conjectures concerning perfect powers

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Abstract. The starting point of our paper is Kashihara's open problem #30, concerning the sequence A001292 of the OEIS, asking how many terms are perfect squares of integers. We confirm his last conjecture up to the 100128-th term and provide a general theorem which rules out 4/9 of the candidates. Moreover, we formulate a new, intriguing, conjecture involving the sequence A352991 of the OEIS (which includes all the terms of A001292, except the first one). Our conjecture states that all the perfect powers of integers belonging to the sequence A352991 are perfect squares and they cannot be written as higher order perfect powers. This new conjecture has been checked for any integer smaller than 1011112131415161718192021222345 6789 and no counterexample has been found.

Keywords: Perfect power, Perfect square, Conjecture, Integer sequence.

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1 Introduction

In late 2010, the author of this paper found a recreative open problem by Kenichiro Kashihara (see [1], open problem #30, p. 25) concerning the sequence A001292 of the On-Line Encyclopedia of Integer Sequences (OEIS) [2]. Kashihara's problem #30 consists of two independent parts and the author solved the first one quite easily at the time (the complete solution can be found in [6], Section 3.3, pp. 12–15), since it asks to find the probability $0 < p(c) < 1$ that the trailing digit of the generic term of the sequence A001292 is $c \in \{0, 1, 2, \dots, 9\}$ and the formula provided in [6] shows that $p(c) = \frac{11-c}{55}$ for any $c \neq 0$, whereas $p(0) = 0.0\overline{18}$ (e.g., if $c = 7$, then $p(7) = \frac{4}{55} = 0.0\overline{72}$).

In the present paper, we will focus ourselves on the second part of the above mentioned Kashihara's problem #30, asking how many elements of the sequence A001292 are perfect powers, since Kashihara conjectured that there are none.

Now, bearing in mind that a perfect power of an integer $d > 0$ is a natural number $k \geq 2$ such that $a^k = d$, where also a is a positive integer, we could point out that A001292 $(1) = 1$ can be considered as a solution and argue how this disproves the conjecture, but (from here on) we will disregard this special case and assume that we are looking for a nontrivial counterexample to Kashihara's conjecture.

Lastly, Section 3 is devoted to introduce a new, fascinating, conjecture concerning perfect powers of integers which appear in the OEIS sequence A352991 [4, 5].

2 The $\{2, 3, 4, 5, 6, 7, 8\}(\text{mod } 9)$ exclusion criterion

In order to be clear on the invoked OEIS sequences, let us introduce a few useful definitions.

Definition 1. We define the m-th term of the sequence A007908 as A007908$(m) :=$ $123\_\ldots\_(m-1)\_m$, where $m \in \mathbb{Z}^+$.

Definition 2. We define the sequence A001292 of the OEIS as the concatenations (sorted in ascending order) of every cyclic permutation of the elements of the sequence A007908 (e.g., given $m = 3$, A001292(A007908(3)) $= 123, 231, 312$).

Definition 3. We define the sequence A352991 of the OEIS as the concatenation of all the distinct permutations of the first strictly positive m integers, sorted into ascending order (e.g., 12345671089 is a term of the sequence, while 12345670189 does not belong to A352991, even if all the digits of the string $123\ldots910$ appear once and only once, since "10" is missed).

After having checked the first 100128 terms of the sequence A001292 (see Appendix), exploring any exponent at or above 2, we have not found any perfect power, so that Kashihara's conjecture has been verified up to 10^{1235} (i.e., the 100129-th term of A001292 is the smallest cyclic permutation of A007908(448) and is greater than 10^{1235} by construction).

Moreover, we can prove the following Theorem 1, concerning the sequence A352991 which includes every term of A001292.

Theorem 1. For any $m > 1$, A352991(n) cannot be a perfect power of an integer if A352991(n) is a permutation of A007908(m) and $m : m \equiv \{2, 3, 5, 6\}(\text{mod } 9)$.

Proof. By definition, A007908(m) [3] cannot be a perfect power if $123\_\ldots\_(m-1)\_m$ is divisible by 3 and it is not divisible by 3^2. Thus, from the well-known divisibility by 3 and 9 criteria, $m :$ $\left(3 \mid \sum_{j=1}^{m} j\right) \wedge \left(3^2 \nmid \sum_{j=1}^{m} j\right)$ is a sufficient, but not necessary, condition for letting us disregard any permutation of $123\_\ldots\_(m-1)\_m$ (i.e., given m, if a generic permutation of A007908(m) is divisible by 3 and is not congruent to $0(\text{mod } 9)$, then all the permutations of A007908(m) are divisible by 3 once and only once, since the commutativity property holds for addition).

It follows that, for any $n \in \mathbb{Z}^+$, A352991(n) cannot be a perfect power if it is a permutation of the string $123\_\ldots\_(m-1)\_m$, where m is such that A134804(m) is divisible by 3. Therefore, the residue modulo 9 of every perfect power belonging to A352991 cannot be 2 or 3 or 5 or 6, and this concludes the proof of Theorem 1. □

Corollary 1. Kashihara's conjecture is true for the concatenation of any cyclic permutation of A007908(m), where $m : (m \equiv \{2, 3, 5, 6\}(\text{mod } 9) \vee m < 448)$.

Proof. We observe that A001292 is a subsequence of A352991 [2, 4]. By invoking Theorem 1, we can state that every perfect power candidate has to be the concatenation of a (cyclic) permutation of A007908(m), where m is such that $m \equiv \{0, 1, 4, 7, 8\}(\text{mod } 9)$. On the other hand, all the remaining terms up to $99\_100\_101\_\ldots\_445\_446\_447\_1\_2\_3\_\ldots\_96\_97\_98$ have been directly checked (see Appendix for details) and no perfect power has been found.

Therefore, Corollary 1 confirms Kashihara's conjecture for any term of A001292 such that m is congruent to $\{2, 3, 5, 6\}(\text{mod } 9)$ or $m \leq 447$. □

Corollary 2. $\nexists n :$ A353025$(n) \equiv \{2, 3, 4, 5, 6, 7, 8\}(\text{mod } 9)$ and a term of A001292 cannot be a perfect power of an integer if the sum of its digits is not congruent to $\{0, 1\}(\text{mod } 9)$.

Proof. Trivially, $10 \equiv 1 (\mathrm{mod}\ 9)$ and also $(1+0) \equiv 1 (\mathrm{mod}\ 9)$, so that any positive integer is congruent modulo 9 to its digit sum.

Since from Theorem 1 it follows that every term of the sequence A353025 [5] is a special permutation of A007908(m) which is characterized by $m \equiv \{0, 1, 4, 7, 8\} (\mathrm{mod}\ 9)$, in order to prove Corollary 2, it is sufficient to note that

$$\sum_{j=1}^{m} j \equiv \begin{cases} 0 (\mathrm{mod}\ 9) & \text{if } m : m \equiv 0 (\mathrm{mod}\ 9) \\ 1 (\mathrm{mod}\ 9) & \text{if } m : m \equiv 1 (\mathrm{mod}\ 9) \\ 1 (\mathrm{mod}\ 9) & \text{if } m : m \equiv 4 (\mathrm{mod}\ 9) \\ 1 (\mathrm{mod}\ 9) & \text{if } m : m \equiv 7 (\mathrm{mod}\ 9) \\ 0 (\mathrm{mod}\ 9) & \text{if } m : m \equiv 8 (\mathrm{mod}\ 9) \end{cases}. \tag{1}$$

□

Remark 1. A well-known property of integers is that every perfect power which is congruent modulo 5 to 0 is also necessarily congruent to $\{0, 25, 75\} (\mathrm{mod}\ 100)$, while if a perfect power is congruent modulo 10 to 6, then its second last digit is odd.

Thus, we are free to combine these additional constraints with Corollary 2 in order to reduce the number of perfect power candidates among the terms of A352991.

3 The conjecture of the perfect squares of A352991

In the first half of April 2022, playing with Kashihara's conjecture, a more general (and maybe more interesting) conjecture arose, it is as follows.

Conjecture 1. Let $n \in \mathbb{N} - \{0, 1\}$ be given. We conjecture that if n is such that A352991(n) is a perfect power of an integer, then $\nexists k \in \mathbb{N} - \{0, 1, 2\} :$ A352991$(n) = c^k$, $c \in \mathbb{N}$.

Remark 2. If confirmed, Conjecture 1 would imply that all the perfect powers (greater than 1) in A352991 are perfect squares and only perfect squares (no cube, no square of square, and so forth).

On April 16 2022, a direct search was performed by the author on the first 10^7 terms of the sequence and no counterexample has been found (42 perfect squares only).

A few days later, Aldo Roberto Pessolano, performed a deeper search running the Mathematica codes published in Appendix, without finding any counterexample and thus confirming Conjecture 1 (at least) up to the smallest permutation of A007908(22) (i.e., for any term of A352991 which is greater than 1 and smaller than $10111121314151617181920212223456789$) meanwhile he found 94 distinct perfect squares concatenating all the distinct permutations of A007908(2), A007908(3), …, A007908(15).

Additional open problems. How many perfect squares are there in A352991? Is their number finite?

4 Conclusion

Kashihara's open problem #30 has not been completely solved yet. Even if the first part, concerning the probability that the trailing digit of A001292(n) is $c = 1, 2, \ldots, 9$, was solved by the author a dozen of years ago [6], the second part still needs a proof or a nontrivial counterexample (the smallest candidate has 1236 digits) to the related conjecture.

Moreover, in the present paper, we have introduced a wider conjecture, pertaining the sequence A352991 of the OEIS, which allow us to ask to ourselves why there are so many (maybe infinitely many) perfect squares in A352991 and not a single higher perfect power has been found among all the terms below 10^{34}.

5 Appendix

Aldo Roberto Pessolano helped the author of the present paper by verifying Kashihara's conjecture and Conjecture 1 for a very large number of terms. All the provided Mathematica codes run on the M1 processor of his Apple MacBook Air (2020).

Kashihara's conjecture has been currently tested up to the 100128-th term of A001292 and we confirm that it holds for every perfect power in that range (i.e., the conjecture is true for every integer belonging to the set $\{\text{A001292}(2),\ \text{A001292}(3), \ldots,\ \text{A001292}(100128)\}$). The search reached the term $99\_100\_\ldots\_446\_447\_1\_2\_\ldots\_97\_98 \approx 9.91 \cdot 10^{1232}$ in 28823 seconds (about 8 hours of calculations) and the code is as follows:

```
c = True;
p = Table[Prime[q], {q, 1, 565}];
Do[rn = Range[k];
    n = ToExpression[StringJoin[ToString[#]&/@rn]];
    If[And[Mod[n, 9] != 3, Mod[n, 9] != 6],
        Do[r = RotateLeft[rn, i - 1];
            nk = ToExpression[StringJoin[ToString[#]&/@r]];
            If[IntegerQ[nk^(1/#)],
                Print[nk, " = ", nk^(1/#), "^", #]; c = False; Break[]
            ]&/@p,
        {i, 1, k}]
    ];
    If[c, Print["1..", k, " checked."], Break[]],
{k, 2, 447}]
```

About our investigation on the perfect powers in A352991, Pessolano has recently completed the direct check of every term of A352991 which falls in the interval $(1,\ 98765432212019181716151413121110]$ (see the code below). As expected, the test has not returned any perfect power above 2.

```
z = False;
h = 3;
p = Table[Prime[q], {q, 2, 10}];
q[x_, k_, d_, m_] :=
    (
    y = x^k;
    If[DigitCount[y] == d,
        c = True;
        Do[
            If[Not[StringContainsQ[ToString[x], ToString[i]]],
                c = False; Break[],
                c = True
            ],
        {i, 10, m}],
        c = False
    ];
```

```
        Return[c]
        )
Do[r = Range[k];
        n = ToExpression[StringJoin[ToString[#]&/@r]];
        If[And[Mod[n, 9] != 3, Mod[n, 9] != 6],
                d = DigitCount[n];
                (
                        s = IntegerPart[(10^(IntegerLength[n] - 1))^(1/#)];
                        f = IntegerPart[(10^(IntegerLength[n]))^(1/#)];
                        Do[
                                If[q[x, #, d, k], Print[x, "^", #, " = ", y]; z = True; Break[]],
                        {x, s, f}]
                )&/@p;
                g = 2^h;
                While[g < n,
                        If[q[#, h, d, k], Print[x, "^", h, " = ", y]; z = True; Break[]]
                        &/@{2, 3, 5, 6, 7};
                        h++;
                        g = 2^h
                ]
        ];
        If[z, Break[], Print["1..", k, " checked."]],
{k, 2, 21}]
```

On the other hand, the following code run on Pessolano's M1 processor for 8408.08 seconds and returned the complete list of the smallest 94 perfect squares belonging to A352991.

```
z = 1;
Do[r = Range[k];
        n = ToExpression[StringJoin[ToString[#]&/@r]];
        If[And[Mod[n, 9] != 3, Mod[n, 9] != 6],
                d = DigitCount[n];
                s = IntegerPart[Sqrt[10^(IntegerLength[n] - 1)]];
                f = IntegerPart[Sqrt[10^(IntegerLength[n])]];
                Do[y = x^2;
                        If[DigitCount[y] == d,
                                c = True;
                                Do[
                                        If[Not[StringContainsQ[ToString[y], ToString[i]]],
                                                c = False
                                        ],
                                {i, 10, k}];
                                If[c, Print[z, " ", y]; z++]
                        ],
                {x, s, f}]
        ],
{k, 2, 13}]
```

These 94 perfect squares correspond to all the perfect powers in (1, 98765432131211110] belonging to A352991, while the next perfect square is

$10111382414519161571236 \approx 1.01 \cdot 10^{22}$ (we observe that 100555369894^2 is a permutation of $123\_\ldots\_16$, as suggested by the statement of Theorem 1).

1	13527684
2	34857216
3	65318724
4	73256481
5	81432576
6	139854276
7	152843769
8	157326849
9	215384976
10	245893761
11	254817369
12	326597184
13	361874529
14	375468129
15	382945761
16	385297641
17	412739856
18	523814769
19	529874361
20	537219684
21	549386721
22	587432169
23	589324176
24	597362481
25	615387249
26	627953481
27	653927184
28	672935481
29	697435281
30	714653289
31	735982641
32	743816529
33	842973156
34	847159236
35	923187456
36	14102987536
37	24891057361
38	27911048356
39	28710591364
40	57926381041
41	59710832164
42	75910168324
43	10135681742311129
44	10145718212113936
45	10273411121318569
46	10391412113852176
47	10694871331152121
48	10713293512411681

49	10947281211113536
50	11013125389146721
51	11038121341751296
52	11053681319247121
53	11213173481106529
54	11213472311091856
55	11213748695310121
56	11214101328395716
57	11291351028471361
58	11318912105421376
59	11328110357491216
60	11361038197125241
61	11613105128317924
62	11831375612104129
63	11867213103954121
64	12131047811153296
65	12210531113617984
66	12291331154108176
67	12311021567131849
68	12371368115129104
69	12511389126371041
70	12598411132110736
71	12741133825910161
72	12859110713124361
73	12861113173295104
74	13101118612573924
75	13318759261211041
76	13751214611018329
77	15113103721812496
78	16213112510379841
79	16798112351131024
80	18132127110314569
81	18351311069274121
82	31329116112107584
83	32121784510113169
84	39811362127511104
85	43139171611081225
86	51371123211048169
87	51611037284113129
88	58911124131067321
89	71121251383691041
90	71289611431311025
91	72511393110124816
92	83761113421105129
93	91384713212510116
94	95641012181133721

In the end, our tests have finally confirmed that all the perfect powers which are smaller than 10^{34} and that belong to the OEIS sequence A352991 are perfect squares (only).

At the present time, Conjecture 1 has been tested for every integer smaller than $10111121314151617181920212223456789$ and no counterexample has been found.

Acknowledgments

We sincerely thank Aldo Roberto Pessolano for having helped us very much with the search of perfect powers of integers belonging to A001292 and A352991, letting us confirm Kashihara's conjecture up to 10^{1235} and Conjecture 1 up to $1.01 \cdot 10^{34}$.

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