On the continuity of a functor

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Abstract

We examine three definitions of continuity of a functor and search for conditions that may fix one of them as general in the sense of including the previous ones. The first definition conceives continuity in terms of inverse systems and their inverse limits. The second definition was developed in the context of abstract Shape theory and considers continuity relative to a fixed functor K and in a general framework that doesn't depend on the concept of limit. The third definition also considers continuity relative to a fixed functor K, but employs concurrently the concept of limit of a functor. We show how a modification of the third definition allow us to set it as the most general one.

1 Introduction

In category theory there is a standard definition for the limit of a functor $T : \mathcal{C} \to \mathcal{D}$ that consists of a pair $(\lim T, \lambda^T)$ with $\lim T \in \mathrm{Obj}_{\mathcal{D}}$ and $\lambda^T : (\lim T)_{\mathcal{C}} \dot{\to} T$ a natural transformation satisfying a certain universal property (see §4.3). In contrast with the limit concept, continuity has been defined in different ways, which are not apparently equivalent. In what follows we focus our attention on a brief discussion of some definitions in order to illustrate which elements they have in common and how they differ from each other.

It seems the first attempt to define continuity goes back to the work of Eilenberg and Steenrod [1] (from now on referred to as ES). Considering the category $\mathcal{A}$ of topological pairs (X, A) and maps (between pairs) they characterized a continuous homology functor H_q as the one satisfying $H_q \lim \{(X_m, A_m), p_{mn}\} \simeq \lim H_q\{(X_m, A_m), p_{mn}\}$ for every inverse system $\{(X_m, A_m), p_{mn}\}$ in $\mathcal{A}$ (there is a similar definition for the cohomology functor H^q). Here, by $\simeq$ it is understood the existence of a natural equivalence. A carefull notice of this definition reveals that the limit is being taken on the inverse systems $\{(X_m, A_m), p_{mn}\}$ and $\{(H_qX_m, H_qA_m), H_q(p_{mn})\}$ rather than on the functor H_q itself in the sense of attributing to H_q a pair like $(\lim H_q, \lambda^{H_q})$.

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Another definition of continuity similar to the ES definition was employed by W. Holsztynski in his construction of a purely categorical version of Borsuk's Shape theory [2]. Like the ES approach, Holztynski's construction still relies on inverse systems, but it is not restricted to the categories of topological pairs and maps as used by ES. It assumes quite arbitrary categories (except for some technical aspects that restrict the type of categories being used, which plays an essential role in Holzstynsky's Shape theory).

It was also in the context of Shape theory, more precisely in the categorical construction given by Bacon [3] and by Cordier and Porter [4], that the concept of continuity became free from the framework of inverse systems. Their treatment of continuity doesn't use the definition of limit and it is valid in the context of arbitrary categories. In order to establish the continuity of a functor $T : \mathcal{C} \to \mathcal{D}$ they relied on a class of functors $K : \mathcal{B} \to \mathcal{C}$ in terms of which it is developed the concept of $K-$continuity for the functor T. The continuity of T is seen as a particular case of $K-$continuity for $K = 1_{\mathcal{C}}$.

It was only with K. Hofmann study of the categorical foundations of topological algebras [5] that the concept of continuity was built upon the standard concept of the limit of a functor. Like Bacon's approach, given a functor $T : \mathcal{C} \to \mathcal{D}$ Hofmann developed the concept of continuity relative to a functor $K : \mathcal{B} \to \mathcal{C}$ assuming it exists the limits of K and TK. Then $K-$continuity of T is given by means of an isomorphism $T(\lim K) \simeq \lim TK$, which has a form ressembling the ES continuity condition, except that here $\lim K$ and $\lim TK$ refer to the standard definition of the limit of a functor rather than to the limit of inverse systems.

The purpose of our study is to examine the definitions of continuity given by Holsztynski, Bacon-Cordier-Porter and Hofmann and search if one of them may be considered as the most general one in the sense of including the two previous ones as particular cases. One should notice that the concept of continuity we consider is a kind of "intrinsic" concept since it doesn't rely on any topology one may ascribe to the sets $\text{Morf}_{\mathcal{C}}(C, C')$, $\text{Morf}_{\mathcal{D}}(D, D')$ that would allow us to characterize $T : \text{Morf}_{\mathcal{C}}(C, C') \to \text{Morf}_{\mathcal{D}}(TC, TC')$ as continuous in the topological sense.

Our work is organized as follows. In section 2 we give a brief description of Holsztynski's definition of continuity of a functor relative to inverse systems. In section 3 we present the elements that were used by Bacon, Cordier and Porter to develop their concept of continuity, which doesn't depend on any previous limit concept. This approach is able to include the Holzstynski's treatment if we restrict our attention to inverse systems. In section 4 we present the construction given by Hofmann, who defined continuity using the concept of limit of a functor. This definition also includes Hozstysnki's treatment when restricted to inverse systems. In addition, we show how this definition allows for a modification that includes the Bacon-Cordier-Porter definition.

A word about notation. All functors we deal with are covariant. Given a functor $F : \mathcal{B} \to \mathcal{C}$ sometimes we write F_{ob} and F_{mo} to denote its action on the objects and morphisms of $\mathcal{B}$. A morphism $u \in \text{Morf}_{\mathcal{B}}(B, B')$ is written as $u : B \overset{\mathcal{B}}{\to} B'$. Whenever we treat with inverses systems $\{X_\alpha, p_{\alpha\beta}\}_M$, M

is a pre-ordered set where the indexes run. When we write relations like $p_\alpha = p_{\alpha\beta}\, p_\beta$, $u_\alpha = p_\alpha h, \ldots$ it is assumed they are valid $\forall \alpha \in M, \forall \beta \in M$, observing that $\alpha \leq \beta$ whenever it appears in $p_{\alpha\beta}$, therefore, for ease of notation we omit this information. Finally, we follow the convention to write natural transformations putting a dot over the arrow, e.g. $\eta : F \dot{\to} G$ denotes a natural transformation between functors F and G.

2 Holsztynski's definition of continuity

In this section we review the concept of continuity as used by Holsztynski and that it is restricted to inverse systems over a category. Its content is stated in the following definition:

2.1 Def.: Continuity for inverse systems
Let $T : \mathcal{C} \to \mathcal{D}$ be a functor. T is said *continuous for inverse systems* iff

$$T \varprojlim \{X_\alpha, p_{\alpha\beta}\}_M \simeq \varprojlim T\{X_\alpha, p_{\alpha\beta}\}_M \tag{1}$$

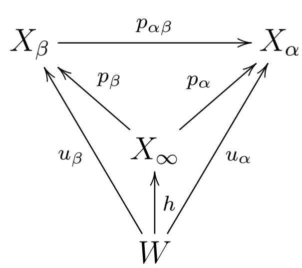
$\forall \{X_\alpha, p_{\alpha\beta}\}_M$ inverse system on $\mathcal{C}$. ■

2.2 Remark: We recall that the limit of an inverse system $\{X_\alpha, p_{\alpha\beta}\}_M$ on a category $\mathcal{C}$, denoted by $\varprojlim \{X_\alpha, p_{\alpha\beta}\}_M$, is a terminal object in the category $\mathrm{inv}\{X_\alpha, p_{\alpha\beta}\}_M$, i.e. it is an object $\{p_\alpha : X_\infty \overset{\mathcal{C}}{\to} X_\alpha\}_M \in \mathrm{Obj}_{\,\mathrm{inv}\{X_\alpha, p_{\alpha\beta}\}_M}$ satisfying

i. $p_\alpha = p_{\alpha\beta}\, p_\beta$

ii. *The universal property for inverse systems*:
$\forall \{u_\alpha : W \overset{\mathcal{C}}{\to} X_\alpha\}_M \in \mathrm{Obj}_{\,\mathrm{inv}\{X_\alpha, p_{\alpha\beta}\}_M}$ with $u_\alpha = p_{\alpha\beta}\, u_\beta$, $\exists! \eta : W \overset{\mathcal{C}}{\to} X_\infty$ with $u_\alpha = p_\alpha\, \eta$.

We summarize conditions i, ii in the commutative diagram below



We have that $\{TX_\alpha, T(p_{\alpha\beta})\}_M$ is an inverse system in $\mathcal{D}$, then the continuity condition given in (1) is equivalent to $\{T(p_\alpha) : TX_\infty \overset{\mathcal{D}}{\to} TX_\alpha\}_M \simeq \varprojlim \{TX_\alpha, T(p_{\alpha\beta})\}_M$ that may be expressed by conditions analogue to i, ii:

$$T(p_\alpha) = T(p_{\alpha\beta}) T(p_\beta) \tag{2}$$

$\forall \{u_\alpha : W \overset{\mathcal{D}}{\to} TX_\alpha\}_M \in \mathrm{Obj}_{\,\mathrm{inv}\{TX_\alpha, T(p_{\alpha\beta})\}_M}$ with $u_\alpha = T(p_{\alpha\beta})\, u_\beta$, $\exists! \eta : W \overset{\mathcal{D}}{\to} TX_\infty$ such that
$$u_\alpha = T(p_\alpha)\, \eta \tag{3}$$

For further use in sections 3 and 4, we write down Holzstynski's definition of projection:

2.3: Def.: Projection

Let $\mathcal{B}$ and $\mathcal{C}$ be categories with $\mathrm{Obj}_{\mathcal{B}} = \mathrm{Obj}_{\mathcal{C}}$. A functor $K : \mathcal{B} \to \mathcal{C}$ is called a projection iff $K(B) = B, \forall B \in \mathrm{Obj}_{\mathcal{B}}$, and $K : \mathrm{Morf}_{\mathcal{B}}(B, B') \to \mathrm{Morf}_{\mathcal{C}}(B, B')$ is surjective $\forall B, B' \in \mathrm{Obj}_{\mathcal{B}}$. ■

Then, when $K : \mathcal{B} \to \mathcal{C}$ is a projection it becomes implicit that we are dealing with categories $\mathcal{B}$ and $\mathcal{C}$ with $\mathrm{Obj}_{\mathcal{B}} = \mathrm{Obj}_{\mathcal{C}}$.

3 Bacon-Cordier-Porter's definition of continuity

Here we review the concept of continuity developed by Bacon, Cordier and Porter. The concept is referred to a fixed functor K and for this reason is called $K-$continuity. Before presenting it we need to introduce some preliminary concepts (see [4]).

3.1 Def.: Comma category $C \downarrow K$

Let $K : \mathcal{B} \to \mathcal{C}$ be a functor and $C \in \mathrm{Obj}_{\mathcal{C}}$. The comma category of $K-$objects under C is the category $C \downarrow K$ defined as follows:

$\mathrm{Obj}_{C\downarrow K} := \{(f, B) \mid B \in \mathrm{Obj}_{\mathcal{B}}, f \in \mathrm{Morf}_{\mathcal{C}}(C, KB)\}$

$\mathrm{Morf}_{C\downarrow K}\big((f, B), (f', B')\big) := \{h : B \overset{\mathcal{B}}{\to} B' \mid f' = K(h)f\}$ ■

Then, what we concretely define as $h : (f, B) \overset{C\downarrow K}{\to} (f', B')$ is in fact a morphism $h : B \overset{\mathcal{B}}{\to} B'$. Therefore we have that $\mathrm{Morf}_{C\downarrow K}\big((f, B), (f', B')\big) \subset \mathrm{Morf}_{\mathcal{B}}(B, B')$.

3.2 Def.: Codomain functor $\delta^{C\downarrow K}$

Let $K : \mathcal{B} \to \mathcal{C}$ be a functor, $C \in \mathrm{Obj}_{\mathcal{C}}$ and $C \downarrow K$ be the comma category of $K-$objects under C. We define the codomain functor $\delta^{C\downarrow K} : C \downarrow K \to \mathcal{B}$ as follows

$$\begin{aligned} \delta^{C\downarrow K}_{ob} : \mathrm{Obj}_{C\downarrow K} &\to \mathrm{Obj}_{\mathcal{B}} \\ (f, B) &\to \delta^{C\downarrow K}_{ob}(f, B) := B \end{aligned}$$

$$\begin{aligned} \delta^{C\downarrow K}_{mo} : \mathrm{Morf}_{C\downarrow K}\big((f, B), (f', B')\big) &\to \mathrm{Morf}_{\mathcal{B}}(B, B') \\ h &\to \delta^{C\downarrow K}_{mo}(h) := h \end{aligned}$$ ■

3.3 Def.: $Func(C \downarrow K, D \downarrow TK)$

Let $K : \mathcal{B} \to \mathcal{C}$ and $T : \mathcal{C} \to \mathcal{D}$ be functors and $C \in \mathrm{Obj}_{\mathcal{C}}$, $D \in \mathrm{Obj}_{\mathcal{D}}$. We define $Func(C \downarrow K, D \downarrow TK)$ as the class having for elements functors $V : C \downarrow K \to D \downarrow TK$ such that $\delta^{D\downarrow TK} \circ V = \delta^{C\downarrow K}$. ■

From this condition, $Func(C \downarrow K, D \downarrow TK)$ may be characterized in terms of a map V^* as we see in the next result.

3.4 Res.: The condition $\delta^{D\downarrow TK} \circ V = \delta^{C\downarrow K}$ fixes the form of $V \in Func(C \downarrow K, D \downarrow TK)$ as follows

$$\begin{aligned} V_{ob} : \mathrm{Obj}_{C\downarrow K} &\to \mathrm{Obj}_{D\downarrow TK} \\ (f, B) &\to V_{ob}(f, B) := (V^*(f), B) \end{aligned}$$

$$V_{mo} : \mathrm{Morf}_{C\downarrow K}\big((f,B),(f',B')\big) \to \mathrm{Morf}_{D\downarrow TK}\big((V^*(f),B),(V^*(f'),B')\big)$$
$$h \to V_{mo}(h) = h$$

where

$$V^* : \cup_{B\in \mathrm{Obj}_{\mathcal{B}}} \mathrm{Morf}_{\mathcal{C}}(C,KB) \to \cup_{B\in \mathrm{Obj}_{\mathcal{B}}} \mathrm{Morf}_{\mathcal{D}}(D,TKB)$$
$$f : C \overset{\mathcal{C}}{\to} KB \to V^*(f) : D \overset{\mathcal{D}}{\to} TKB$$

satisfies

$$\forall f : C \overset{\mathcal{C}}{\to} KB,\ \forall f' : C \overset{\mathcal{C}}{\to} KB',\ \forall h : B \overset{\mathcal{B}}{\to} B' \in \mathrm{Morf}_{C\downarrow K}\big((f,B),(f',B') : V^*(f') = TK(h)V^*(f) \quad (4)$$

Proof: It follows straightforwardly from the condition $\delta^{D\downarrow TK} \circ V = \delta^{C\downarrow K}$. ■

3.5 Remark: Given f, f' the condition $f' = K(h)f$ imposes a restriction on the form of $h : B \overset{\mathcal{B}}{\to} B'$. Similarly, the other condition $V^*(f') = TK(h)V^*(f)$ imposes a restriction on the form of V^*. It may happen that the latter condition follows from the former one, but it is not necessary.

3.6 Def.: $\delta_T : C \downarrow K \to TC \downarrow TK$

Let $K : \mathcal{B} \to \mathcal{C}$ and $T : \mathcal{C} \to \mathcal{D}$ be functors. We define the functor $\delta_T : C \downarrow K \to TC \downarrow TK$ as

$$\delta_{T\,ob} : \mathrm{Obj}_{C\downarrow K} \to \mathrm{Obj}_{TC\downarrow TK}$$
$$(f,B) \to \delta_{T\,ob}(f,B) := (T(f),B)$$

$$\delta_{T\,mo} : \mathrm{Morf}_{C\downarrow K}\big((f,B),(f',B')\big) \to \mathrm{Morf}_{TC\downarrow TK}\big((T(f),B),(T(f'),B')\big)$$
$$h \to \delta_{T\,mo}(h) := h \ ■$$

Our next definition associates to every morphism $g : D \overset{\mathcal{D}}{\to} TC$ an induced functor between comma categories.

3.7 Def.: Let $K : \mathcal{B} \to \mathcal{C}$ and $T : \mathcal{C} \to \mathcal{D}$ be functors. Given a morphism $g : D \overset{\mathcal{D}}{\to} TC$, it induces a functor between comma categories $g^* : TC \downarrow TK \to D \downarrow TK$ defined as follows

$$g^*_{ob} : \mathrm{Obj}_{TC\downarrow TK} \to \mathrm{Obj}_{D\downarrow TK}$$
$$(w,B) \to g^*_{ob}(w,B) := (wg,B)$$

$$g^*_{mo} : \mathrm{Morf}_{TC\downarrow TK}\big((w,B),(w',B')\big) \to \mathrm{Morf}_{D\downarrow TK}\big((wg,B),(w'g,B')\big)$$
$$u \to g^*_{mo}(u) := u \ ■$$

We are now equipped to define $K-$continuity of a functor according to Bacon-Cordier-Porter.

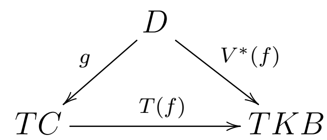
3.8 Def.: $K-$continuity

Let $K : \mathcal{B} \to \mathcal{C}$ and $T : \mathcal{C} \to \mathcal{D}$ be functors. We say that T is $K-$continuous at $C \in \mathrm{Obj}_{\mathcal{C}}$ iff $\forall D \in \mathrm{Obj}_{\mathcal{D}}$, $\forall V \in Func(C \downarrow K, D \downarrow TK)$, $\exists! g : D \overset{\mathcal{D}}{\to} TC$ such that $V = g^*\,\delta_T$. ■

This condition is equivalent to the form given below:

$$\forall D \in \mathrm{Obj}_{\mathcal{D}},\ \forall V^* : \cup_{B\in \mathrm{Obj}_{\mathcal{B}}} \mathrm{Morf}_{\mathcal{C}}(C,KB) \to \cup_{B\in \mathrm{Obj}_{\mathcal{B}}} \mathrm{Morf}_{\mathcal{D}}(D,TKB) \ \text{ satisfying (4)}$$
$$\exists! g : D \overset{\mathcal{D}}{\to} TC,\ \forall f : C \overset{\mathcal{C}}{\to} KB :\ V^*(f) = T(f)g \quad (5)$$

where the last relation is summarized in the commutative diagram



We say that T is $K-$continuous if T is $K-$continuous $\forall C \in \text{Obj}_{\mathcal{C}}$.

3.9 Remark: As a consistency check, we observe that the association of g to V^* being independent of the f allows the form of $V^*(f)$ given in (5) to satisfy condition (4).

We define continuity as a particular case of $K-$continuity as follows. Let us assume $\mathcal{B} = \mathcal{C}$ and $K = 1_{\mathcal{C}}$, then $K-$continuity becomes continuity as defined below:

3.10 Def.: Continuous Functor

Let $T : \mathcal{C} \to \mathcal{D}$ be a functor. T is continuous in $C \in \text{Obj}_{\mathcal{C}}$ iff $\forall D \in \text{Obj}_{\mathcal{D}}$, $\forall V \in Func(C \downarrow 1_{\mathcal{C}}, D \downarrow T)$, $\exists! g : D \overset{\mathcal{D}}{\to} TC$, $V = g_T^* \, \delta_T$.

This condition is equivalent to

$$\forall D \in \text{Obj}_{\mathcal{D}},\ \forall V^* : \cup_{C^* \in \text{Obj}_{\mathcal{C}}} \text{Morf}_{\mathcal{C}}(C, C^*) \to \cup_{C^* \in \text{Obj}_{\mathcal{C}}} \text{Morf}_{\mathcal{D}}(D, TC^*)\ \text{ satisfying}$$
$$\forall f : C \overset{\mathcal{C}}{\to} C^*,\ \forall f' : C \overset{\mathcal{C}}{\to} C^{*\prime},\ \forall h : C^* \overset{\mathcal{C}}{\to} C^{*\prime} \in \text{Morf}_{C\downarrow K}\big((f, C^*), (f', C^{*\prime})\big) : V^*(f') = T(h)V^*(f)$$
$$\exists! g : D \overset{\mathcal{D}}{\to} TC,\ \forall f : C \overset{\mathcal{C}}{\to} C^* : V^*(f) = T(f)g \ \blacksquare$$

We now examine how the concept of $K-$continuity implies continuity relative to inverse systems as given in §2.1. Before that we notice that if we consider $\{p_\alpha : X_\infty \overset{\mathcal{C}}{\to} X_\alpha\}_M = \varprojlim \{X_\alpha, p_{\alpha\beta}\}_M$ we may consider a particular type of projection $K : \mathcal{B} \to \mathcal{C}$ such that $K\big(\text{Morf}_{C\downarrow K}\big((p_\beta, X_\beta), (p_\alpha, X_\alpha)\big)\big) = \{p_{\alpha\beta} : X_\beta \overset{\mathcal{C}}{\to} X_\alpha\}_M$.

3.11 Res.: Let $K : \mathcal{B} \to \mathcal{C}$ be a projection satisfying $K\big(\text{Morf}_{C\downarrow K}\big((p_\beta, X_\beta), (p_\alpha, X_\alpha)\big)\big) = \{p_{\alpha\beta} : X_\beta \overset{\mathcal{C}}{\to} X_\alpha\}_M$, $\forall \{X_\alpha, p_{\alpha\beta}\}_M$ inverse system in $\mathcal{C}$ with inverse limit $\{p_\alpha : X_\infty \overset{\mathcal{C}}{\to} X_\alpha\}_M = \varprojlim \{X_\alpha, p_{\alpha\beta}\}_M$. If $T : \mathcal{C} \to \mathcal{D}$ is $K-$continuous then T is continuous for inverse systems.

Proof: Let $\{X_\alpha, p_{\alpha\beta}\}_M$ be an inverse system in $\mathcal{C}$ and let us assume there is defined the inverse limit $\{p_\alpha : X_\infty \overset{\mathcal{C}}{\to} X_\alpha\}_M = \varprojlim \{X_\alpha, p_{\alpha\beta}\}_M$. Given the covariant functor $T : \mathcal{C} \to \mathcal{D}$ we have that $\{TX_\alpha, T(p_{\alpha\beta})\}_M$ is an inverse system in $\mathcal{D}$ and $\{T(p_\alpha) : TX_\infty \overset{\mathcal{D}}{\to} TX_\alpha\}_M$ satisfies

$$T(p_\alpha) = T(p_{\alpha\beta})T(p_\beta) \tag{6}$$

Let $\{u_\alpha : W \overset{\mathcal{D}}{\to} TX_\alpha\}_M$ be such that

$$u_\alpha = T(p_{\alpha\beta})u_\beta \tag{7}$$

Now, since $X_\infty \in \text{Obj}_{\mathcal{C}}$ and $W \in \text{Obj}_{\mathcal{D}}$ we have that every $V \in Func(X_\infty \downarrow K, W \downarrow TK)$ is characterized by a map $V^* : \cup_{B \in \text{Obj}_{\mathcal{B}}} \text{Morf}_{\mathcal{C}}(X_\infty, B) \to \cup_{B \in \text{Obj}_{\mathcal{B}}} \text{Morf}_{\mathcal{D}}(W, TB)$ satisfying (4), which reads as

$$\forall f : X_\infty \overset{\mathcal{C}}{\to} B,\ \forall f' : X_\infty \overset{\mathcal{C}}{\to} B',\ \forall h : B \overset{\mathcal{B}}{\to} B' \in \mathrm{Morf}_{C\downarrow K}\big((f,B),(f',B')\big) : V^*(f') = TK(h)V^*(f) \quad (8)$$

Consider now a particular choice for V^* such that $V^*(p_\gamma) = u_\gamma$. That this choice exists it is readily seen for taking $f = p_\beta$, $f' = p_\alpha$ and $h : X_\beta \overset{\mathcal{B}}{\to} X_\alpha \in \mathrm{Morf}_{X_\infty\downarrow K}\big((p_\beta, X_\beta),(p_\alpha, X_\alpha)\big)$ we have $K(h) = p_{\alpha\beta}$ and $V^*(p_\alpha) = T(p_{\alpha\beta})V^*(p_\beta)$, where this last condition is garanteed by (7).

Since V is continuous, from (5) we have that

$$\exists! g : W \overset{\mathcal{D}}{\to} TX_\infty, \forall p_\alpha : X_\infty \overset{\mathcal{C}}{\to} X_\alpha, V^*(p_\alpha) = T(p_\alpha)g$$

i.e.

$$u_\alpha = T(p_\alpha)g\ . \quad (9)$$

From (6), (7) and (9) the family $\{T(p_\alpha) : TX_\infty \overset{\mathcal{D}}{\to} TX_\alpha\}_M$ satisfies the analogue of conditions (2) and (3) relative to the inverse system $\{TX_\alpha, T(p_{\alpha\beta})\}_M$, therefore

$$\{T(p_\alpha) : TX_\infty \overset{\mathcal{D}}{\to} TX_\alpha\}_M = \varprojlim\{TX_\alpha, T(p_{\alpha\beta})\}_M$$

i.e. $T : \mathcal{C} \to \mathcal{D}$ is continuous for inverse systems. ■

3.12 Remark: As we have seen, the definition of K-continuity of $T : \mathcal{C} \to \mathcal{D}$ assumes the existence of a unique morphism g, but does not specify the conditions for this morphism to exist. However, in the result just proved, the existence and uniqueness of g stated in §3.8 follows as a consequence of existing the inverse limit of $\{TX_\alpha, T(p_{\alpha\beta})\}_M$.

4 Hofmann's definition of continuity

Here we analyze the concept of continuity developed by Hofmann. The concept is stablished relative to a functor previously fixed and employs the standard definition of limit.

First we introduce the concept of constant functor induced by an object.

4.1 Def.: Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $D \in \mathrm{Obj}_{\mathcal{D}}$. We define a constant functor $D_{\mathcal{C}} : \mathcal{C} \to \mathcal{D}$ as follows

$D_{\mathcal{C}ob} : \mathrm{Obj}_{\mathcal{C}} \to \mathrm{Obj}_{\mathcal{D}}$

$C \to D_{\mathcal{C}ob}(C) := D$

$D_{\mathcal{C}mo} : \mathrm{Morf}_{\mathcal{C}} \to \mathrm{Morf}_{\mathcal{D}}$

$h : C \overset{\mathcal{C}}{\to} C' \to D_{\mathcal{C}mo}(h) := 1_D$ ■

Given a morphism it induces a natural transformation as follows:

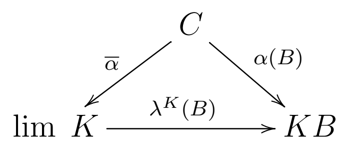
4.2 Def.: Given categories $\mathcal{C}$ and $\mathcal{D}$ and a morphism $F : D \overset{\mathcal{D}}{\to} D'$ we define a natural transformation

$F_{\mathcal{C}} : D_{\mathcal{C}} \dot{\to} D'_{\mathcal{C}}$ as
$F_{\mathcal{C}} : \text{Obj}_{\,\mathcal{C}} \to \text{Morf}_{\,\mathcal{D}}$
$\quad C \to F_{\mathcal{C}}(C) := F$ ■

Now, we recall the standard definition of the limit of a functor:

4.3 Def.: Limit of a functor
Let $K : \mathcal{B} \to \mathcal{C}$ be a functor. The limit of K consists of a pair $(\lim\ K, \lambda^K)$ with $\lim\ K \in \text{Obj}_{\,\mathcal{C}}$ and $\lambda^K : (\lim\ K)_{\mathcal{B}} \dot{\to} K$ such that $\forall C \in \text{Obj}_{\,\mathcal{C}}$, $\forall \alpha : C_{\mathcal{B}} \dot{\to} K$, $\exists! \overline{\alpha} : C \overset{\mathcal{C}}{\to} \lim\ K$ such that $\lambda^K\, \overline{\alpha}_{\mathcal{B}} = \alpha$. ■

Condition $\lambda^K\, \overline{\alpha}_{\mathcal{B}} = \alpha$ may be expressed in terms of the commutative diagram below ($\forall B \in \text{Obj}_{\,\mathcal{B}}$):



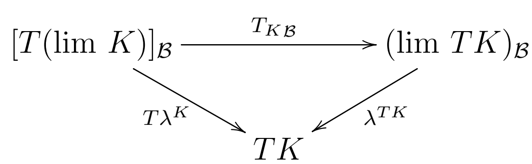
We say that λ^K is the *limit morphism* and $\lim\ K$ is the *limit object*. As an abuse of notation we write $\lim\ K$ as a shorthand for the pair $(\lim\ K, \lambda^K)$.

We need a preliminary result:

4.4 Res.: Let $K : \mathcal{B} \to \mathcal{C}$ and $T : \mathcal{C} \to \mathcal{D}$ be functors and let us assume that $\exists \lim\ K$. Then, $T \circ (\lim\ K)_{\mathcal{B}} = [T(\lim\ K)]_{\mathcal{B}}$.
Proof: It follows directly from definition 4.1. ■

The next result garantees the existence of a unique funtor T_K provided it exists the limits of K and TK.

4.5 Res.: Let $K : \mathcal{B} \to \mathcal{C}$ and $T : \mathcal{C} \to \mathcal{D}$ be functors. If $\exists \lim\ K$, $\exists \lim TK$ then $\exists! T_K : T(\lim\ K) \overset{\mathcal{D}}{\to} \lim TK$ such that $\lambda^{TK} T_{K\mathcal{B}} = T\lambda^K$, i.e. the diagram below is commutative



Proof: Since it exists $\lim\ K$ it follows there is a natural transformation $\lambda^K : (\lim\ K)_{\mathcal{B}} \dot{\to} K$. Using §4.4 we consider the natural transformation $T\lambda^K : [T(\lim\ K)]_{\mathcal{B}} \dot{\to} TK$. Since it also exists $\lim\ TK$ there is a natural transformation $\lambda^{TK} : (\lim\ TK)_{\mathcal{B}} \dot{\to} TK$ satisfying:

$$\forall \beta : D_{\mathcal{B}} \dot{\to} TK, \exists! \overline{\beta} : D \overset{\mathcal{D}}{\to} \lim\ TK \ \text{ such that } \lambda^{TK} \overline{\beta}_{\mathcal{B}} = \beta\ . \tag{10}$$

Identifying in (10): $D \equiv T(\lim\ K)$ and $\beta \equiv T\lambda^K$ we have that $\exists! \overline{\beta} : T(\lim\ K) \overset{\mathcal{D}}{\to} \lim\ TK$ such that $\lambda^{TK} \overline{\beta}_{\mathcal{B}} = T\lambda^K$. We identify the morphism T_K with $\overline{\beta}$ and this ends our proof. ■

We are now equipped to define $K-$continuity.

4.6 Def.: $K-$**continuity (according to Hofmann)**

Let $K:\mathcal{B}\to\mathcal{C}$ be a functor. The functor $T:\mathcal{C}\to\mathcal{D}$ is K-continuous iff
i. $\exists \lim K \Rightarrow \exists \lim TK$
ii. $T_K : T(\lim K) \to \lim TK$ is an isomorphism ■

Before we examine if this definition of K-continuity contemplate the continuity for inverse systems we need a preliminary result.

4.7 Res.: Let $K:\mathcal{B}\to\mathcal{C}$ be a projection such that $\exists \lim K$. For every inverse system on $\mathcal{C}$, $\{X_\alpha, p_{\alpha\beta}\}_M$, it exists $\{p_\alpha : \lim K \overset{\mathcal{C}}{\to} X_\alpha\}_M = \varprojlim\{X_\alpha, p_{\alpha\beta}\}_M$.

Proof: Let $\{X_\alpha, p_{\alpha\beta}\}_M$ be an inverse system in $\mathcal{C}$. If K is a projection then it exists $q_{\alpha\beta} : X_\beta \overset{\mathcal{B}}{\to} X_\alpha$ with $K(q_{\alpha\beta}) = p_{\alpha\beta}$. If it exists $\lim K$ we have verified the following conditions i, ii, iii:
i. There is a natural transformation $\lambda^K : (\lim K)_{\mathcal{B}} \dot{\to} K$ such that for the $q_{\alpha\beta} : X_\beta \overset{\mathcal{B}}{\to} X_\alpha$ we have $\lambda^K(X_\alpha) = K(q_{\alpha\beta})\lambda^K(X_\beta)$, i.e.

$$\lambda^K(X_\alpha) = p_{\alpha\beta}\lambda^K(X_\beta)\ . \tag{11}$$

ii. For any $\beta : W_{\mathcal{B}} \dot{\to} K$ we also have $\beta(X_\alpha) : W \overset{\mathcal{C}}{\to} X_\alpha$ satisfies

$$\beta(X_\alpha) = p_{\alpha\beta}\beta(X_\beta)\ . \tag{12}$$

iii. There is a unique $\overline{\beta} : W \overset{\mathcal{C}}{\to} \lim K$ such that

$$\lambda^K(X_\alpha)\overline{\beta} = \beta(X_\alpha)\ . \tag{13}$$

Then, if we identify $p_\alpha = \lambda^K(X_\alpha)$ and $u_\alpha = \beta(X_\alpha)$ we notice that relations (11), (12), (13) correspond to the conditions (2) and (3). Then $\{p_\alpha : \lim K \overset{\mathcal{C}}{\to} X_\alpha\}_M = \varprojlim\{X_\alpha, p_{\alpha\beta}\}_M$. ■

We will now examine how K-continuity implies continuity relative to inverse systems.

4.8 Res.: Let $K:\mathcal{B}\to\mathcal{C}$ be a projection. If $\exists \lim K$ and $T:\mathcal{C}\to\mathcal{D}$ is K-continuous then T is continuous for inverse systems.

Proof: Let $K:\mathcal{B}\to\mathcal{C}$ be a projection and let us assume it exists $\lim K$. Let $T:\mathcal{C}\to\mathcal{D}$ be $K-$continuous. By definition $\exists \lim TK$, $\exists T_K : T(\lim K) \overset{\mathcal{D}}{\to} \lim TK$, which is an isomorphism. Let $\{X_\alpha, p_{\alpha\beta}\}_M$ be an inverse system on $\mathcal{C}$ and let us assume there is defined the inverse limit.

Since $K:\mathcal{B}\to\mathcal{C}$ is a projection then for $p_{\alpha\beta} : X_\beta \overset{\mathcal{C}}{\to} X_\alpha$ we have $q_{\alpha\beta} : X_\beta \overset{\mathcal{B}}{\to} X_\alpha$ such that $K(q_{\alpha\beta}) = p_{\alpha\beta}$.

If it exists $\lim K$ we have defined a natural transformation $\lambda^K : (\lim K)_{\mathcal{B}} \dot{\to} K$ such that $\lambda^K(X_\alpha) : \lim K \overset{\mathcal{C}}{\to} X_\alpha$ satisfies $\lambda^K(X_\alpha) = p_{\alpha\beta}\lambda^K(X_\beta)$. From 4.7 we identify $\{\lambda^K(X_\alpha) : \lim K \overset{\mathcal{C}}{\to} X_\alpha\}_M = \varprojlim\{X_\alpha, p_{\alpha\beta}\}_M$.

The functor T determines an inverse system $\{TX_\alpha, T(p_{\alpha\beta})\}_M$ in $\mathcal{D}$ and $T(\lambda^K(X_\alpha)) : T\lim K \overset{\mathcal{D}}{\to} TX_\alpha$ satisfies

$$T(\lambda^K(X_\alpha)) = T(p_{\alpha\beta})T(\lambda^K(X_\beta))\ . \tag{14}$$

If it exists $\lim\, TK$ we have defined a natural transformation $\lambda^{TK} : (\lim\, TK)_{\mathcal{B}} \dot{\to} TK$ such that $\lambda^{TK}(X_\alpha) : \lim\, TK \overset{\mathcal{D}}{\to} TX_\alpha$. Given $\beta : W_{\mathcal{B}} \dot{\to} TK$ we consider $\beta(X_\alpha) : W \overset{\mathcal{D}}{\to} TX_\alpha$, which satisfies

$$\beta(X_\alpha) = T(p_{\alpha\beta})\beta(X_\beta)\ . \tag{15}$$

Then $\exists!\overline{\beta} : W \overset{\mathcal{D}}{\to} \lim\, TK$ with $\lambda^{TK}(X_\alpha)\overline{\beta} = \beta(X_\alpha)$.

From 4.5 T_K satisfies $\lambda^{TK}(X_\alpha)T_K = T(\lambda^K(X_\alpha))$ and since T_K is isomorphism we obtain $\lambda^{TK}(X_\alpha) = T(\lambda^K(X_\alpha))T_K^{-1} \therefore \lambda^{TK}(X_\alpha)\overline{\beta} = T(\lambda^K(X_\alpha))T_K^{-1}\overline{\beta}$, which becomes

$$\beta(X_\alpha) = T(\lambda^K(X_\alpha))T_K^{-1}\overline{\beta}\ . \tag{16}$$

Then considering (14), (15) and (16) we have shown that $\{T(\lambda^K(X_\alpha)) : T\lim\, K \overset{\mathcal{D}}{\to} TX_\alpha\}_M$ satisfies the properties (2) and (3) relative to the inverse system $\{TX_\alpha, T(p_{\alpha\beta})\}_M$, therefore

$$\{T(\lambda^K(X_\alpha)) : T\lim\, K \overset{\mathcal{D}}{\to} TX_\alpha\}_M = \varprojlim\{TX_\alpha, T(p_{\alpha\beta})\}_M$$

i.e. T is continuous for inverse systems. ■

We now propose a modification of Hofmann's definition in order to relate it to the Bacon-Cordier-Porter's definition.

4.9 Def.: Let $K : \mathcal{B} \to \mathcal{C}$ be a functor. The functor $T : \mathcal{C} \to \mathcal{D}$ is $K-$continuous iff
i. $\exists \lim\, K \Rightarrow \exists \lim\, TK$
ii. $T_K : T(\lim\, K) \to \lim\, TK$ is an isomorphism
iii. $\forall \alpha : C_{\mathcal{B}} \dot{\to} K$, $\forall \beta : D_{\mathcal{B}} \dot{\to} TK$, $\exists!\chi : D \overset{\mathcal{D}}{\to} TC$ such that $T_K^{-1}\overline{\beta} = T(\overline{\alpha})\chi$
i.e. the diagram below is commutative

$$\begin{array}{ccc} D & \xrightarrow{\ \overline{\beta}\ } & \lim\, TK \\ {\scriptstyle \chi}\downarrow & & \downarrow{\scriptstyle T_K^{-1}} \\ TC & \xrightarrow{\ T(\overline{\alpha})\ } & T(\lim\, K) \end{array} \qquad ■$$

4.10 Remark: In condition iii, the morphisms $\overline{\alpha}$ and $\overline{\beta}$ are uniquely obtained from the natural transformations α and β due to the existence of $\lim\, K$ and $\lim\, TK$.

4.11 Res.: Let $K : \mathcal{B} \to \mathcal{C}$ be a functor such that $\exists \lim\, K$. If $T : \mathcal{C} \to \mathcal{D}$ is $K-$continuous according to definition 4.9 then T is $K-$continuous according to the Bacon-Porter-Cordier definition.

Proof: Let $C \in \mathrm{Obj}_{\mathcal{C}}$, $D \in \mathrm{Obj}_{\mathcal{D}}$ and consider a map

$$V^* : \cup_{B\in \mathrm{Obj}_{\mathcal{B}}} \mathrm{Morf}_{\mathcal{C}}(C, KB) \to \cup_{B\in \mathrm{Obj}_{\mathcal{B}}} \mathrm{Morf}_{\mathcal{D}}(D, TKB)$$

satisfying (4). In order to show that T is $K-$continuous we must verify (5). Then let us assume $f : C \overset{\mathcal{C}}{\to} KB$, $f' : C \overset{\mathcal{C}}{\to} KB'$ and $h : B \overset{\mathcal{B}}{\to} B'$ satisfying $f' = K(h)f$ and $V^*(f') = TK(h)V^*(f)$.

We assume $\lim\, K$ exists then $\exists\lambda^K : (\lim\, K)_{\mathcal{B}} \to K$ such that for a natural transformation $\alpha : C_{\mathcal{B}} \dot{\to}$

K satisfying $\alpha(B) = f$, $\alpha(B') = f'$ we have that $\alpha(B') = K(h)\alpha(B)$ for the given $h : B \overset{\mathcal{B}}{\to} B'$, and also $\lambda^K(B)\overline{\alpha} = \alpha(B)$ for a unique $\overline{\alpha} : C \overset{\mathcal{C}}{\to} \lim K$.

Since T is $K-$continuous and it exists $\lim K$ by definition it also exists $\lim TK$. Then $\exists \lambda^{TK} : (\lim TK)_{\mathcal{B}} \dot{\to} TK$ such that for a natural transformation $\beta : D_{\mathcal{B}} \dot{\to} TK$ satisfying $\beta(B) = V^*(f)$, $\beta(B') = V^*(f')$ we have $\beta(B') = TK(h)\beta(B)$ for the $h : B \overset{\mathcal{B}}{\to} B'$ previously considered. In addition we have $\lambda^{TK}(B)\overline{\beta} = \beta(B)$ for a unique $\overline{\beta} : D \overset{\mathcal{D}}{\to} \lim TK$.

The result given in §4.5 provides us with an isomorphism $T_K : T(\lim K) \overset{\mathcal{D}}{\to} \lim TK$ satisfying $\lambda^{TK}(B) = T(\lambda^K(B))T_K^{-1}$, then $\lambda^{TK}(B)\overline{\beta} = T(\lambda^K(B))T_K^{-1}\overline{\beta}$ i.e.

$$\beta(B) = T(\lambda^K(B))T_K^{-1}\overline{\beta} \tag{17}$$

and from 4.9.iii $\exists! \chi : D \overset{\mathcal{D}}{\to} TC$ such that $\beta(B) = T(\lambda^K(B))T(\overline{\alpha})\chi = T(\alpha(B))\chi$ i.e. $\beta(B) = T(\alpha(B))\chi$, or using the identifications $\alpha(B) = f$, $V^*(f) = \beta(B)$ we obtain $V^*(f) = T(f)\chi$, which proves (5). This shows that T is $K-$continuous according to the Bacon-Cordier-Porter definition. ■

5 Conclusion

The concept of continuity of a functor is not consensually established and in our work we have focused our attention on three definitions. The first one, due to Holsztynski, deals with inverse systems. The second one, due to Bacon, Cordier and Porter, defines continuity relative to a fixed functor K and employs a framework that depends neither on inverse systems nor on the limit concept. The third definition, due to Hofmann, also establishes the concept of continuity relative to a fixed functor K and doesn't depend on inverse systems but employs the concept of limit of a functor. In its original form none of them includes simultaneously the other two as particular cases, however, we showed that the definitions given by Bacon-Cordier-Porter and Hofmann allow to include Holzstynski's definition of continuity if we take the fixed functor K as a projection. It is only with the addition of condition 4.9iii to Hofmann's original definition that we obtain a definition of continuity that becomes sufficiently general to include the two previous ones.

It is not clear what elements could be added to or even what modifications could be made in the Bacon-Cordier-Porter definition in order to make it the most general one. From the analysis of sections 3 and 4 we observe that for $f : C \overset{\mathcal{C}}{\to} KX_\alpha$ the key point is to harmonize equations $V^*(f) = T(f)g$ (see (5)) with $V^*(f) = T(\lambda^K(X_\alpha))T_K^{-1}\overline{\beta}$ (see (17)), which is the farther we can go in §4.10 without imposing condition 4.9iii. Such attempt, if possible, may demand some modifications on the definitions given in section 3, which consequently would affect the contents of the Shape theory formulated by Bacon-Cordier-Porter leading to a modified form. The analysis of the properties of this modified Shape theory in connection with the application to topological algebras as given by Hofmann deserves investigation.

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