EXACT RAYTRACING FORMULAE FOR PARABOLIC AXIAL GRIN LENSES

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ABSTRACT

A novel raytracing method through parabolic axial gradient index lenses is presented. The study is based on developing the exact solution of the differential ray equations with initial conditions set by refraction at the first surface. The analysis offers potential benefits in reducing the computation effort and increasing the raytracing accuracy.

Subject terms: gradient-index lens, ray-tracing, axial gradient refractive index, optical design.

1. INTRODUCTION

Raytracing through gradient index lenses is usually carried out by using approximate numerical methods of solving the differential ray equations [1, 2]. To this end, Runge-Kutta algorithms are ranked among the most popular. They are computationally intensive and often require iterative adjustments of the step size to produce acceptable accuracy [3, 4].
There is interest in using axial GRIN materials for a variety of design applications due to the recent progress in the fabrication process[6]. In general, the GRIN technology offers a number of known benefits in replacing aspheric surfaces and enhancing the performance of conventional microoptics systems [5, 6].
This work reports on a closed form solution of the ray equations for axial GRIN elements displaying a parabolic distribution of the refractive index. The study is based on the theory of elliptic functions which has found extensive application in the dynamics of nonlinear oscillators[7,8].The exact solution of ray trajectories may increase the speed and computation accuracy for applications involving the analysis of the optical path difference (OPD) or spot diagrams. Power expanding the analytic form taken by the OPD in the exit pupil provides the basis for a correct description of the aberration polynomial.

2. RAYTRACING SETUP

Fig. 1 shows, for simplicity in only two dimensions, the image formation using a single element GRIN lens and an arbitrary oblique pencil.A skew ray with direction cosines (L,M,N) is launched from an object point placed in air towards the front spherical surface of the lens. After refraction at P_{-1} , the ray follows a curved path and reaches the back spherical surface at P_1. Refraction at P_1 directs the emerging ray to the image plane located in air at the distance "s_0" from the lens front surface. The z-axis coincides with

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the optical axis and the front/back curvatures are c_{-1} and c_1 , respectively. The lens thickness is "z_0"with the origin located at the vertex of the front surface.
We assume that the setup excludes any total internal reflection and the ray does not reach the outside lens surface during propagation from P_{-1} to P_1. This amounts to the condition:

$$x^2+y^2 < R^2 \tag{1}$$

in which x, y are arbitrary transverse ray coordinates and R is the lens outer radius.The axial GRIN is specified by the following refractive index distribution:

$$n(z) = n_{00} + n_{01} z + n_{02} z^2 \tag{2}$$

Let (L', M', N') denote the direction cosines of the refracted ray at P_{-1}, I and I' the incidence and refraction angles at P_{-1}. The refraction equations are[9]:

$$\begin{aligned} L' &= (L - K x_{-1}) / n_{-1} \\ M' &= (M - K y_{-1}) / n_{-1} \\ N' &= (N - K z_{-1} + n_{-1} \cos I' - \cos I) / n_{-1} \end{aligned} \tag{3}$$

where:

$$\begin{aligned} K &= c_{-1} (n_{-1} \cos I' - \cos I) \\ \cos I' &= [n_{-1}^2 - (1 - \cos^2 I)]^{1/2} / n_{-1} \\ \cos I &= L Q_{-1x} + M Q_{-1y} + N Q_{-1z} \end{aligned} \tag{4}$$

with $\mathbf{Q}_{-1}$ (Q_{-1x} , Q_{-1y} , Q_{-1z}) representing the normal unit vector at P_{-1} and :

$$n_{-1} = n(z_{-1}) \tag{5}$$

The direction cosines (3) and the position vector at P_{-1} given by $\mathbf{r}_{-1} = (x_{-1}, y_{-1}, z_{-1})$ define the raytracing initial conditions.

3. INTEGRATION OF THE RAYTRACING MODEL

Let $\mathbf{r} = (x, y, z)$ represent the position vector of an arbitrary point inside the lens and "s" the arc length measured along the ray trajectory. The differential raytracing equations in a GRIN lens of a continuously varying refractive index are [4, 10]:

$$\mathbf{r}_{uu} = n(\mathbf{r}) \nabla [n(\mathbf{r})] \tag{6}$$

where $\mathbf{r}_{uu}$ denotes the second order derivative and "u" stands for the reduced coordinate defined via:

$$du = ds / n \tag{7}$$

with the choice $u = 0$ at P_{-1}.
Substituting (2) in (6) leads to:

$$\begin{aligned} x_{uu} &= 0 \\ y_{uu} &= 0 \\ z_{uu} &= N_0 + N_1 z + N_2 z^2 + N_3 z^3 \end{aligned} \tag{8}$$

in which:

$$\begin{aligned} N_0 &= n_{00}\, n_{01} \\ N_1 &= n_{01}{}^2 + 2\, n_{00}\, n_{02} \\ N_2 &= 3\, n_{01} n_{02} \\ N_3 &= 2\, n_{02}{}^2 \end{aligned} \tag{9}$$

Trivial integration of the first two equations in (8) yields:

$$\begin{aligned} x &= a_x + b_x u \\ y &= a_y + b_y u \end{aligned} \tag{10}$$

The integration constants a_x , a_y , b_x , b_y are determined from the initial conditions, that is:

$$a_x = x_{-1}\ ,\ a_y = y_{-1}\ ,\ b_x = n_{-1} L'\ ,\ b_y = n_{-1} M' \tag{11}$$

Processing the third equation in (8) requires use of the elliptic functions. Multiplying it by z_u and integrating gives:

$$z_u{}^2 = a + 2\, N_0\, z + N_1\, z^2 + (2/3)\, N_2\, z^3 + (1/2)\, N_3\, z^4 \tag{12}$$

The constant "a" is constrained by the ray direction at P_{-1} which requires:

$$(z_u)_{-1} = n_{-1} N' \tag{13}$$

and thus:

$$a = (n_{-1} N')^2 - [2 N_0 z_{-1} + N_1 z_{-1}^2 + (2/3) N_2 z_{-1}^3 + (1/2) N_3 z_{-1}^4] \qquad (14)$$

Let "α", "β", "γ" and "δ" be the distinct roots of the quartic equation:

$$z_u^2 = 0 \qquad (15)$$

and let "w" represent a new axial variable determined by:

$$w^2 (z) = (\beta - \delta) (z - \alpha) / [(\alpha - \delta) (z - \beta)] \qquad (16)$$

We now introduce the following parameters:

$$k^2 = (\beta - \gamma) (\alpha - \delta) / [(\alpha - \gamma) (\beta - \delta)]$$

$$G^2 = (\beta - \delta) (\alpha - \gamma)/4 \qquad (17)$$

$$h^2 = z_u^2 / (z - \alpha) (z - \beta) (z - \gamma) (z - \delta)$$

The standard solution for (12) assumes the form[11]:

$$w (u) = \mathrm{sn} (h G u , k) + w_{-1} \qquad (18)$$

in which the constant w_{-1} depends on the initial conditions via:

$$w_{-1}^2 = (\beta - \delta) (z_{-1} - \alpha) / [(\alpha - \delta) (z_{-1} - \beta)] \qquad (19)$$

Extracting the z-variable from (16) and combining it with (10) and (11) yields the ray trajectory in parametric format:

$$x (u) = x_{-1} + n_{-1} L' u$$

$$y (u) = y_{-1} + n_{-1} M' u \qquad (20)$$

$$z (u) = \{[\beta (\alpha - \delta) w^2(u) / (\beta - \delta)] - \alpha\} / \{[(\alpha - \delta) w^2(u)/(\beta - \delta)] - \alpha\}$$

4. RAY INTERSECTION WITH THE BACK SURFACE

The intersection point of (20) with the back spherical surface can be recursively computed with one of the existing numerical routines for root finding. We follow here Newton's algorithm as detailed in ref. 4 for the case of a meridional raytrace:

- compute the distance Δ from P_{-1} to the surface along the tangent line (fig 2).

- define the starting recurrence variable as:

$$u_0 = \Delta / n_{-1} \tag{21}$$

- define Newton's function as:

$$F(u) = 2\ [z(u)/c_1] - [x^2(u) + y^2(u) + z^2(u)] \tag{22}$$

- iterate the recurrence variable with:

$$u_{n+1} = u_n - F(u_n)/F_u(u_n) \tag{23}$$

- find the intersection point P_0 of the back surface with a parallel to the optical axis passing through the n-th iterate P_n (fig 2).

- evaluate the separation between the axial coordinate of the n-th iterate (z_n) and the axial coordinate of P_0:

$$\delta_n = z_n - z_0 = c_1 F(u_n) / [2\,(1 - c_1 z_n)] \tag{24}$$

where:

$$z_n = z(u_n) \tag{25}$$

The stopping criterion is met when δ_n falls below a threshold value which sets the endpoint of the recurrence variable (u_l). The position vector locating P_l is then given by: $\mathbf{r}_l = (x(u_l), y(u_l), z(u_l))$.

5. FINAL REFRACTION AND RAYTRACE

Once the intersection point of the ray trajectory with the back surface has been found, one needs to compute the final refraction followed by the ray transfer to the image plane.
The direction cosines of the ray incident at P_l are determined from the local slopes of (20), that is:

$$\begin{aligned} (x_u)_l / n_{-1} &= L' \\ (y_u)_l / n_{-1} &= M' \\ (z_u)_l / n_l &= N'' \end{aligned} \tag{26}$$

in which:

$$n_l = n(z_l) \tag{27}$$

The direction cosines of the ray incident at P_1 are obtained from (7) and (26) as follows:

$$(x_s)_1 = (x_u)_1 / n_1 = (n_{-1}/ n_1)\, L'$$

$$(y_s)_1 = (y_u)_1 / n_1 = (n_{-1}/ n_1)\, M' \qquad (28)$$

$$(z_s)_1 = (z_u)_1 / n_1 = N''$$

and comply with the constraint:

$$(x_s)_1^2 + (y_s)_1^2 + (z_s)_1^2 = 1 \qquad (29)$$

Let $\mathbf{Q_1}$ (Q_{1x} , Q_{1y} , Q_{1z}) designate the normal unit vector at P_1 (fig.1). The refraction equations are:

$$L_1 = n_1 (x_s)_1 - K' x_1$$

$$M_1 = n_1 (y_s)_1 - K' y_1 \qquad (30)$$

$$N_1 = (n_1 N'' - K' z_1 + \cos I_1' - n_1 \cos I_1)$$

in which:

$$K' = c_1 (\cos I_1' - n_1 \cos I_1)$$

$$\cos I_1' = [1 - n_1^2 (1 - \cos^2 I)]^{1/2} \qquad (31)$$

$$\cos I_1 = (x_s)_1 Q_{1x} + (y_s)_1 Q_{1y} + N'' Q_{1z}$$

The intercept coordinates of the ray at the image plane are given by:

$$x_i = (s_0 - z_1) (L_1/ N_1) + x_1$$

$$y_i = (s_0 - z_1) (M_1/ N_1) + y_1 \qquad (32)$$

$$z_i = s_0$$

6. SUMMARY

We have described a new raytracing method in axial GRIN singlets with a parabolic distribution of the refractive index. Direct integration of the ray equations allows exact computation of light paths for a given set of initial conditions. This approach eliminates the need for numerical routines, except for the back surface refraction where ray intercepts are derived using an iterative Newton algorithm.

7. REFERENCES

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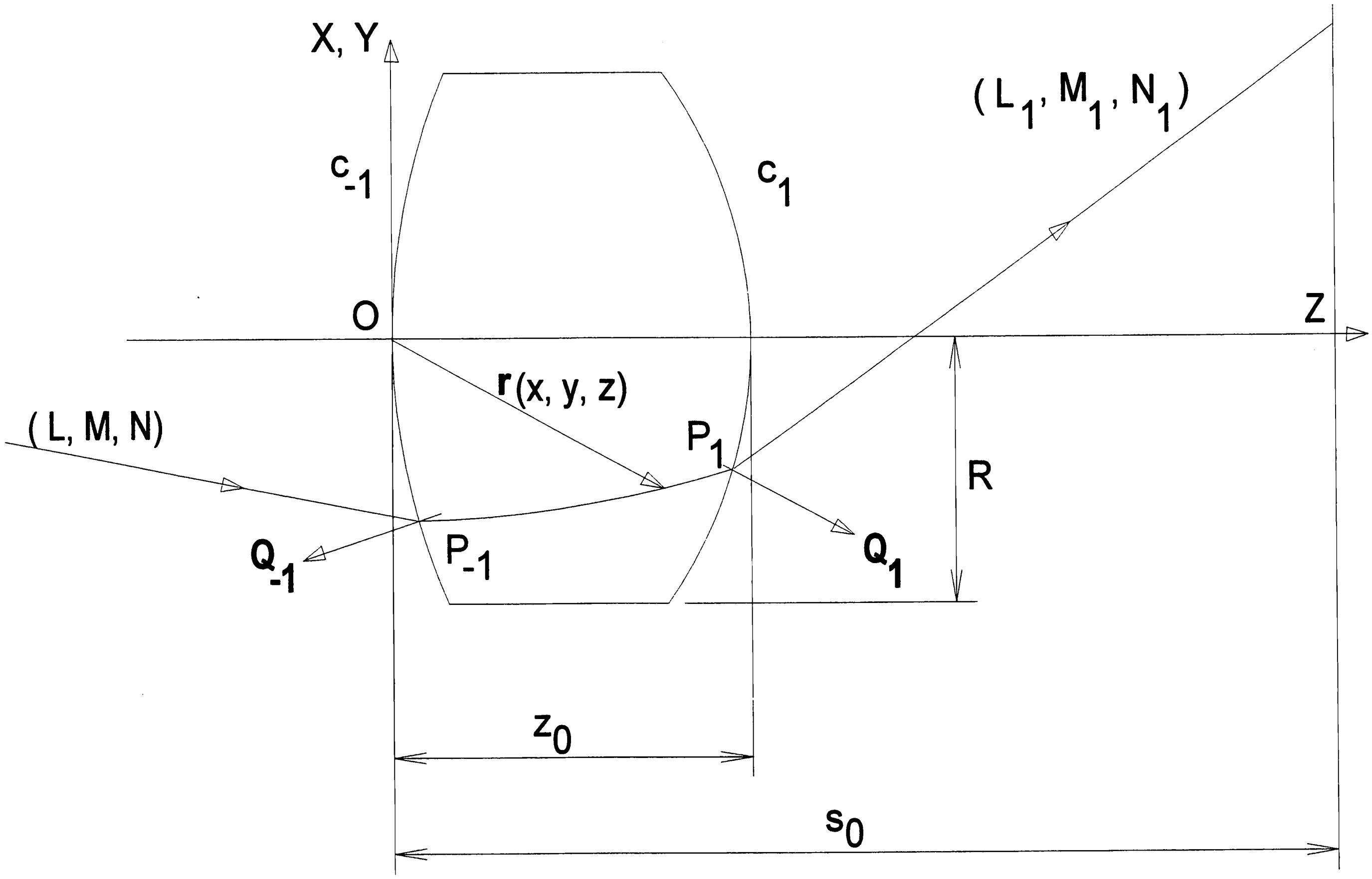

fig. 1

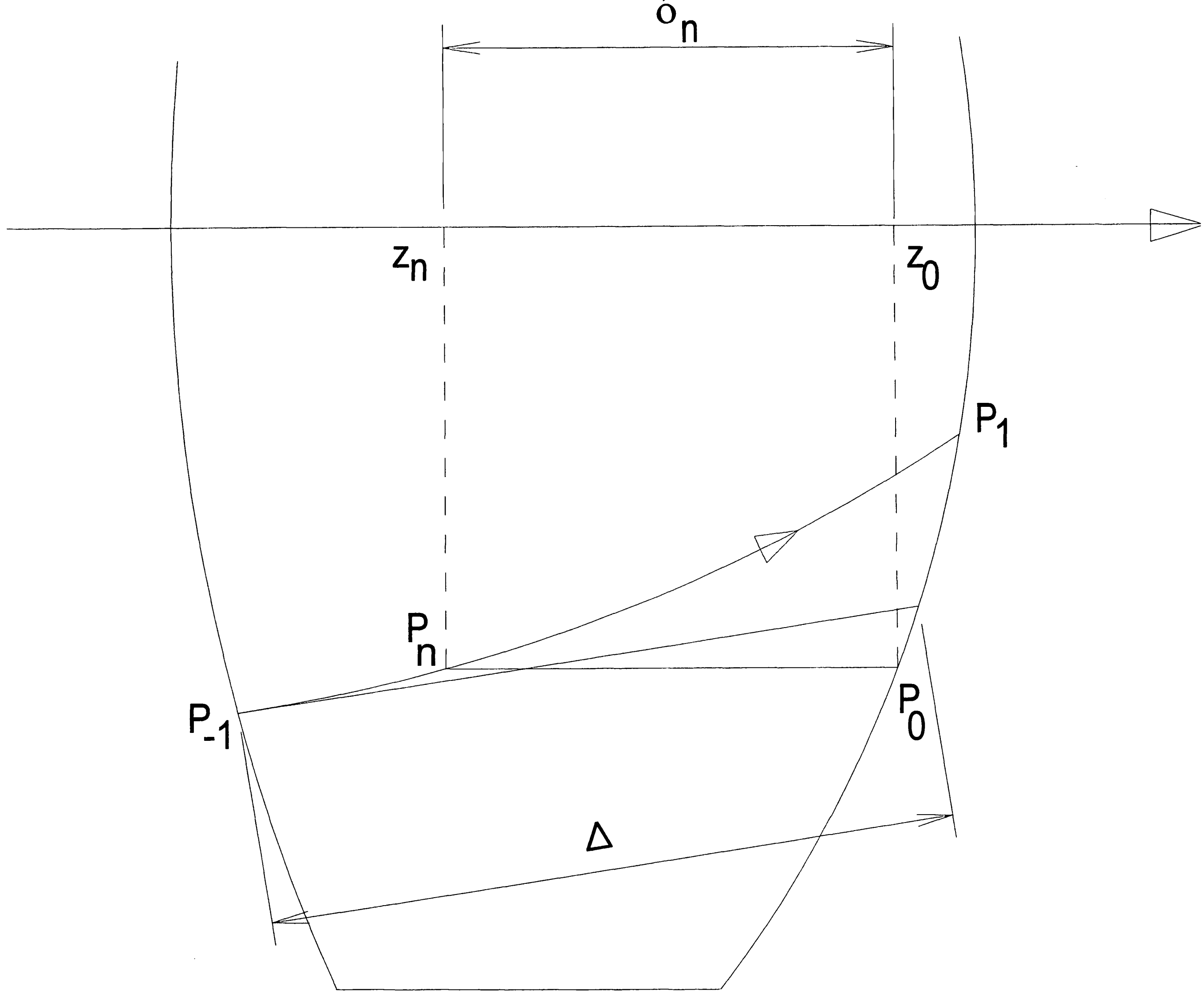

fig. 2