The Irrationality of Odd and Even Zeta Values

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Abstract

We show that using the denominators of the terms of $\zeta(n)-1 = z_n$ as decimal bases gives all rational numbers in (0,1) as single decimals. We also show the partial sums of z_n are not given by such single digits using the partial sum's terms. These two properties yield a proof that z_n is irrational. The proof shows neighborhoods exist around partial sums that have rational numbers with their reduced fraction's denominators greater than any given natural number. The union of such neighborhoods gives an epsilon neighborhood for z_n that has only such large denominators. This implies z_n is irrational for both odd and even n.

1 Introduction

Apery's $\zeta(3)$ is irrational proof [1] and its simplifications [3, 8] are the only proofs that a specific odd argument for $\zeta(n)$ is irrational. The irrationality of even arguments of zeta are a natural consequence of Euler's formula [2]:

$$\zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = (-1)^{n-1} \frac{2^{2n-1}}{(2n!)} B_{2n} \pi^{2n}. \tag{1}$$

Apery also showed $\zeta(2)$ is irrational, and Beukers, based on the work (tangentially) of Apery, simplified both proofs. He replaced Apery's mysterious recursive relationships with multiple integrals. See Poorten [10] for the history of Apery's proof; Havil [5] gives an overview of Apery's ideas and attempts to demystify them. Also of interest is Huylebrouck's [6] paper giving an historical context for the main technique used by Beukers.

Attempts to generalize the techniques of the one odd success seem to be hopelessly elusive. Apery's and other ideas can be seen in the work of Rivoal and Zudilin [11, 12]. Their results, that there are an infinite number of odd n such that $\zeta(n)$ is irrational and at least one of the cases 5,7,9, 11 likewise irrational do suggest a radically different approach is necessary.

Let

$$z_n = \zeta(n) - 1 = \sum_{j=2}^{\infty} \frac{1}{j^n} \text{ and } s_k^n = \sum_{j=2}^{k} \frac{1}{j^n}.$$

We show that every rational number in $(0,1)$ can be written as a single decimal using the denominators of a term in z_n as a number basis. But the partial sums can't be expressed with such a single decimal using the denominators of its terms as number bases. These two properties yield a proof that all z_n are irrational.

Properties of z_n

We define a decimal set.

Definition 1. *Let*

$$d_{j^n} = \{1/j^n, \ldots, (j^n - 1)/j^n\} = \{.1, \ldots, .(j^n - 1)\} \text{ base } j^n.$$

That is d_{j^n} consists of all single decimals greater than 0 and less than 1 *in base j^n. The decimal set for j^n is*

$$D_{j^n} = d_{j^n} \setminus \bigcup_{k=2}^{j-1} d_{k^n}.$$

The set subtraction removes duplicate values.

Definition 2.

$$\bigcup_{j=2}^{k} D_{j^n} = \Xi_k^n$$

The union of decimal sets gives all rational numbers in $(0,1)$.

Lemma 1.

$$\bigcup_{j=2}^{\infty} D_{j^n} = \mathbb{Q}(0,1)$$

Proof. Every rational $a/b \in (0,1)$ is included in a d_{b^n} and hence in some D_{r^n} with $r \leq b$. This follows as $ab^{n-1}/b^n = a/b$ and as $a < b$, per $a/b \in (0,1)$, $ab^{n-1} < b^n$ and so $a/b \in d_{b^n}$. $\square$

Next we show $s_k^n \notin \Xi_k^n$; that is: we show that partial sums of z_n can't be expressed as a single decimal using number bases given by the denominators of the partial's terms.

Lemma 2. *If $s_k^n = r/s$ with r/s a reduced fraction, then 2^n divides s.*

Proof. The set $\{2, 3, \ldots, k\}$ will have a greatest power of 2 in it, a; the set $\{2^n, 3^n, \ldots, k^n\}$ will have a greatest power of 2, na. Also $k!$ will have a powers of 2 divisor with exponent b; and $(k!)^n$ will have a greatest power of 2 exponent of nb. Consider

$$\frac{(k!)^n}{(k!)^n} \sum_{j=2}^{k} \frac{1}{j^n} = \frac{(k!)^n/2^n + (k!)^n/3^n + \cdots + (k!)^n/k^n}{(k!)^n}. \tag{2}$$

The term $(k!)^n/2^{na}$ will pull out the most 2 powers of any term, leaving a term with an exponent of $nb - na$ for 2. As all other terms but this term will have more than an exponent of 2^{nb-na} in their prime factorization, we have the numerator of (2) has the form

$$2^{nb-na}(2A + B),$$

where $2 \nmid B$ and A is some positive integer. This follows as all the terms in the factored numerator have powers of 2 in them except the factored term $(k!)^n/2^{na}$. The denominator, meanwhile, has the factored form

$$2^{nb}C,$$

where $2 \nmid C$. This leaves 2^{na} as a factor in the denominator with no powers of 2 in the numerator, as needed. $\square$

Lemma 3. *If $s_k^n = r/s$ with r/s a reduced fraction and p is a prime such that $k > p > k/2$, then p^n divides s.*

Proof. First note that $(k, p) = 1$. If $p|k$ then there would have to exist r such that $rp = k$, but by $k > p > k/2$, $2p > k$ making the existence of such a natural number $r > 1$ impossible.

The reasoning is much the same as in Lemma 2; cf. Chapter 2, Problem 21 in [2], solution in [7]. Consider

$$\frac{(k!)^n}{(k!)^n}\sum_{j=2}^{k}\frac{1}{j^n} = \frac{(k!)^n/2^n + \cdots + (k!)^n/p^n + \cdots + (k!)^n/k^n}{(k!)^n}. \qquad (3)$$

As $(k, p) = 1$, only the term $(k!)^n/p^n$ will not have p in it. The sum of all such terms will not be divisible by p, otherwise p would divide $(k!)^n/p^n$. As $p < k$, p^n divides $(k!)^n$, the denominator of r/s, as needed. ☐

Lemma 4. *For any $k \geq 2$, there exists a prime p such that $k < p < 2k$.*

Proof. This is Bertrand's postulate [4]. ☐

Theorem 1. *If $s_k^n = \frac{r}{s}$, with r/s reduced, then $s > k^n$.*

Proof. Using Lemma 4, for even k, we are assured that there exists a prime p such that $k > p > k/2$. If k is odd, $k - 1$ is even and we are assured of the existence of prime p such that $k - 1 > p > (k - 1)/2$. As $k - 1$ is even, $p \neq k - 1$ and $p > (k - 1)/2$ assures us that $2p > k$, as $2p = k$ implies k is even, a contradiction.

For both odd and even k, using Lemma 4, we have assurance of the existence of a p that satisfies Lemma 3. Using Lemmas 2 and 3, we have $2^n p^n$ divides the denominator of r/s and as $2^n p^n > k^n$, the proof is completed. ☐

Corollary 1.

$$s_k^n \notin \Xi_k^n$$

Proof. This is a restatement of Theorem 1. ☐

z_n is irrational

Lemma 5. *There exists a neighborhood, an ϵ_k for s_k^n such that if $p/q \in N_{\epsilon_k}(s_k^n)$, p/q a reduced fraction, then $q > k^n$.*

Proof. This follows from Corollary (1). Let $\epsilon_k = \min(\{|x - s_k^n| : x \in \Xi_k^n\})$. By Lemma 1, Ξ_k^n consists of all rational numbers with reduced denominators less then k^n. ☐

One can visualize neighborhoods of s_k^n as ellipses around the x-axis of shrinking width and growing height, where the height indicates the size of all rational denominators in the x-axis interval. A given rational p/q can be understood as two parallel lines distance q above and below the x-axis. For any q the ever increasing height of the ellipses will eventually exclude fractions with q denominators from all neighborhoods from some s_k^n on. This is the idea of the proof.

Definition 3. *Let*

$$\mathcal{E}(q) = max\{\epsilon_k : p/q \notin N_{\epsilon_k}(s_k^n)\}$$

Theorem 2. z_n *is irrational.*

Proof. Suppose, to obtain a contradiction, that z_n is the reduced fraction p/q. Then, using convergence, for $\epsilon = \mathcal{E}(q)$ there exists a natural number N such that for all $k > N$, $z_n - s_k^n < \epsilon$; that is $z_n \in N_\epsilon(s_k^n)$. By Lemma 5, this implies $z_n \neq p/q$, a contradiction. □

Conclusion

Finally, this result surviving public scrutiny, there is the possibility of its relevance to the premier number theory open problem: the Riemann hypotheses. I have some hope that the equivalent of number bases (plural) in the complex number system might allow the same exclusions used here (irrational not rational) to carry over to a zero versus not a zero. There are Gaussian integers and Gaussian primes; might there be forms of number bases that inform us of the location of zeros for the ever wonderful and mysterious zeta function.

References

[1] Apéry, R. (1979). Irrationalité de $\zeta(2)$ et $\zeta(3)$. *Astérisque* 61: 11-13.

[2] Apostol, T. M. (1976). *Introduction to Analytic Number Theory.* New York: Springer.

[3] Beukers, F. (1979). A Note on the irrationality of $\zeta(2)$ and $\zeta(3)$, *Bull. London Math. Soc.* 11: 268–272.

[4] Hardy, G. H., Wright, E. M., Heath-Brown, R. , Silverman, J. , Wiles, A. (2008). *An Introduction to the Theory of Numbers*, 6th ed. London: Oxford Univ. Press.

[5] J. Havil (2012). *The Irrationals.* Princeton, NJ: Princeton Univ. Press.

[6] Huylebrouck, D. (2001). Similarities in irrationality proofs for π, $\ln 2$, $\zeta(2)$, and $\zeta(3)$, *Amer. Math. Monthly* 108(10): 222–231.

[7] Hurst, G. (2014). Solutions to Introduction to Analytic Number Theory by Tom M. Apostol. https://greghurst.files.wordpress.com/2014/02/apostol_intro_to_ant.pdf

[8] Nesterenko, Y. V. (1996). A few remarks on $\zeta(3)$, *Math. Zametki* 59(6): 865–880.

[9] Niven, I. (2006). Irrational Numbers. MAA.

[10] van der Poorten, A. (1978/9). A proof that Euler missed... Apery's proof of the irrationality of $\zeta(3)$, an informal report. *Math. Intelligencer* 1(4): 195–203.

[11] Rivoal, T. (2000). La fonction zeta de Riemann prend une infinit de valeurs irrationnelles aux entiers impairs, *Comptes Rendus de l'Acadmie des Sciences, Srie I. Mathmatique* 331: 267-270.

[12] Zudilin, W. W. (2001). One of the numbers $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$ is irrational. *Russian Mathematical Surveys* 56(4): 747–776.