A Proof For The Collatz Conjecture

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Abstract

Build a special identical equation, use its calculation characters to prove and search for solution of any odd converging to 1 equation through (*3+1)/2^k operation, and get a solution for this equation, which is exactly same with that got from calculating directly. Then give a specific example to verify it, hint that we can estimate the value of convergence steps n during some middle procedures. Thus prove the Collatz Conjecture is true.Furthermore, analysis the even and odd sequences produced by iteration calculation during searching for solution, indicate that this kind of iteration calculation has determined direction, and gradually regularly converges.

Ⅰ Introduction About The Collatz Conjecture

The Collatz Conjecture is a famous math conjecture, named after mathematician Lothar Collatz, who introduced the idea in 1937. It is also known as the $3x + 1$ conjecture, the Ulam conjecture etc. Many mathematicians have tried to prove it true or false and have expanded it to more digits scale. But until today, it has not yet been proved.

The Collatz Conjecture concerns sequences of positive integers in which each term is obtained from the previous one as follows: if the previous integer is even, the next integer is the previous integer divided by 2, till to odd. If the previous integer is odd, the next term is the previous integer multiply 3 and plus 1. The conjecture is that these sequences always reach 1, no matter which positive integer is chosen to start the sequence.

Here is an example for a typical integer $x = 27$, takes up to 111 steps, increasing or decreasing step by step, climbing as high as 9232 before descending to 1.

27, 82, 41, 124, 62, 31, 94, 47, 142, 71, 214, 107, 322, 161, 484, 242, 121, 364, 182, 91, 274, 137, 412, 206, 103, 310, 155, 466, 233, 700, 350, 175, 526, 263, 790, 395, 1186, 593, 1780, 890, 445, 1336, 668, 334, 167, 502, 251, 754, 377, 1132, 566, 283, 850, 425, 1276, 638, 319, 958, 479, 1438, 719, 2158, 1079, 3238, 1619, 4858, 2429, 7288, 3644, 1822, 911, 2734, 1367, 4102, 2051, 6154, 3077, 9232, 4616, 2308, 1154, 577, 1732, 866, 433, 1300, 650, 325, 976, 488, 244, 122, 61, 184, 92, 46, 23, 70, 35, 106, 53, 160, 80, 40, 20, 10, 5, 16, 8, 4, 2, 1.

If the conjecture is false, there should exists some starting number which gives rise to a sequence that does not contain 1. Such a sequence would either enter a repeating cycle that excludes 1, or increase without bound. No such sequence has been found by human and computer after verified a lot of numbers can reach to 1. It is very difficult to prove these two cases exist or not.

This paper will try to prove the conjecture true from a special view. Because any even can become odd through $\div 2^k$ operation, this paper will research only odd characters in the conjecture sequence. The equivalence conjecture become: with random starting odd x, do $(\times 3+1)\div 2^k$ operation repeatedly, it always converges to 1. The above sequence can be written as following, in which numbers on arrows are k in $\div 2^k$ in each step:

$$27\xrightarrow{1}41\xrightarrow{2}31\xrightarrow{1}47\xrightarrow{1}71\xrightarrow{1}107\xrightarrow{1}161\xrightarrow{2}121\xrightarrow{2}91\xrightarrow{1}137\xrightarrow{2}103\xrightarrow{1}$$
$$155\xrightarrow{1}233\xrightarrow{2}175\xrightarrow{1}263\xrightarrow{1}395\xrightarrow{1}593\xrightarrow{2}445\xrightarrow{3}167\xrightarrow{1}251\xrightarrow{1}377\xrightarrow{2}$$
$$283\xrightarrow{1}425\xrightarrow{1}319\xrightarrow{1}479\xrightarrow{1}719\xrightarrow{1}1079\xrightarrow{1}1619\xrightarrow{1}2429\xrightarrow{3}911\xrightarrow{1}1367\xrightarrow{1}$$
$$2051\xrightarrow{1}3077\xrightarrow{4}577\xrightarrow{2}433\xrightarrow{2}325\xrightarrow{4}61\xrightarrow{3}23\xrightarrow{1}35\xrightarrow{1}53\xrightarrow{5}5\xrightarrow{4}1$$

Ⅱ Build Equation For The Conjecture

If odd x do n times $(\times 3+1)\div 2^k$ calculation build odd y, we can get:

$$y=\frac{3^n x+3^{n-1}+3^{n-2}\times 2^{p_1}+3^{n-3}\times 2^{p_1+p_2}\ldots+3\times 2^{p_1+p_2+..+p_{n-2}}+2^{p_1+p_2+..+p_{n-1}}}{2^{p_1+p_2+..+p_n}}$$

In which $p_1\ldots p_n$ is k in $\div 2^k$ operation in each step.

For example: $(7\times 3+1)\div 2=11, (11\times 3+1)\div 2=17$, then $17=\frac{3^2\times 7+3+2}{2^2}$

Suppose odd x can converge to 1 through $(\times 3+1)\div 2^k$ calculation, then y=1, get:

$$3^n x+3^{n-1}+3^{n-2}\times 2^{p_1}+3^{n-3}\times 2^{p_1+p_2}\ldots+3\times 2^{p_1+p_2+..+p_{n-2}}+2^{p_1+p_2+..+p_{n-1}}-2^{p_1+p_2+..+p_n}=0 \quad \text{Formula (1)}$$

We know $(1\times 3+1)\div 2^2=1$, and can do any times this kind of operation. That is to say, 1 do random n steps $(\times 3+1)\div 2^2$ operation can converge to 1, have:

$$3^n+3^{n-1}+3^{n-2}\times 2^2+3^{n-3}\times 2^4\ldots+3\times 2^{2n-4}+2^{2n-2}-2^{2n}=0$$

Below we use this model to prove and search for solution of Formula (1) for any odd x converging to 1.

Ⅲ Solution For Any Odd **Converging To 1 Equation**

First with odd x do reform:

$x=a_m\times 3^m+a_{m-1}\times 3^{m-1}+\ldots+a_1\times 3+a_0$, $a_m\ldots a_0$=0, 1 or 2. Then:

$$3^n x=3^n\times(a_m\times 3^m+a_{m-1}\times 3^{m-1}+\ldots+a_1\times 3+a_0)$$

If $a_m>1$ or $a_m=1$ but

$$(a_{m-1}\times 3^{n+m-1}+...+a_1\times 3^{n+1}+a_0\times 3^n)>(3^{n+m-1}+3^{n+m-2}\times 2^2...+3^n\times 2^{2(m-1)})$$, make

$$x=3^{m+1}-3^m+a_{m-1}\times 3^{m-1}+...+a_1\times 3+a_0$$ or :

$$x=3^{m+1}-2\times 3^m+a_{m-1}\times 3^{m-1}+...+a_1\times 3+a_0$$

Build identical equation:

$$3^{n+m}+3^{n+m-1}+3^{n+m-2}\times 2^2+3^{n+m-3}\times 2^4...+3^{n-1}\times 2^{2m}...+3\times 2^{2(n+m)-4}+2^{2(n+m)-2}-2^{2(n+m)}=0$$ Formula (2)

If x can converge to 1, Formula (1) and Formula (2) should be equivalence. Below we try to reform Formula (2) to form of Formula (1), if successful, it proves that equation for Formula (1) has solution.

First let:

$$(3^{n+m-1}+3^{n+m-2}\times 2^2...+3^n\times 2^{2(m-1)})-(a_{m-1}\times 3^{n+m-1}+...+a_1\times 3^{n+1}+a_0\times 3^n)=t_n\times 3^n$$,

because x is odd, this is odd minus even, t_n should be odd.

Because the max value of x-3^m is $2\times 3^{m-1}+2\times 3^{m-2}+...+2\times 3+2$, min value is $-3^{m-1}+1$, then t_n has a range:

from $(3^{m-1}+3^{m-2}\times 2^2...+2^{2(m-1)})-(2\times 3^{m-1}+2\times 3^{m-2}+...+2\times 3+2)$ to $(3^{m-1}+3^{m-2}\times 2^2...+2^{2(m-1)})-(-3^{m-1}+1)$.

Change t_n to binary form and let:

$t_n\times(2+1)\times 3^{n-1}+3^{n-1}\times 2^{2m}-3^{n-1}=t_{n-1}\times 3^{n-1}$, this is just with 3^n part multiply (2+1) become 3^{n-1} part, and plus corresponding part in Formula (2), minus corresponding part in Formula (1). From now on, t_{n-1} become even.Continue:

$t_{n-1}\times(2+1)\times 3^{n-2}+3^{n-2}\times 2^{2m+2}-3^{n-2}\times 2^{p_1}=t_{n-2}\times 3^{n-2}$, and let 2^{p1} equal to max value of even part(or the lowest bit of odd part).

Watch Formula (1) and Formula (2), in general, if do not consider $2^{p1+\cdots}$ part (because we consider $2^{p1+\cdots}$ as max value of even part of t_{i-2}) in Formula (1), corresponding parts in Formula (2) are bigger than corresponding part in Formula (1). Hence after a few times of $t_{i-1}\times(2+1)$, value of t_{i-2} is mainly determined by corresponding part in Formula (2). And, after $t_{i-1}\times(2+1)$, odd part should add 1 or 2 bits, if add 1 bit, $+2^{2m+2}$ should operate in MSB bit, if add 2 bits, $+2^{2m+2}$ should operate in MSB-1 bit. Both cases odd part add 2 bits after $+2^{2m+2}$ operation, if MSB bit of t_{i-2} is 2^k,k should be odd.

For example:

$3+2^2=7$, $7\times(2+1)+2^4-1=9\times2^2$ $9\times2^2\times(2+1)+2^6-2^2=21\times2^3$

Continue:

$t_{n-2}\times(2+1)\times3^{n-3}+3^{n-3}\times2^{2m+4}-3^{n-3}\times2^{p_1+p_2}=t_{n-3}\times3^{n-3}$, let 2^{p1+p2} equal to max value of even part.Because LSB bit no. of odd part of t_i increases continuously, this can be finished easily.

Watch t_i(i<n and decreases step by step), during iteration, the count of succession 1 in the highest part should be unchanged or increased. Why? This is because of characters of odd multiply 3 and $+2^{2m}$ operation. If t_{i-1} is with form 10…, obviously, count of succession 1 in highest part of t_{i-2} is unchanged or increased. If t_{i-1} is with form 111…, after do $\times(2+1)$, should become 101…, do $+2^{2m}$,become 111…, count of succession 1 in highest part is also unchanged or increased. Other cases can be proved easily. Some cases can increase, for example, if t_{i-1} is with form 110110…, t_{i-2} becomes 1110…

Do this iteration continuously, count of succession 1 in the highest part of odd part of t_i is unchanged or increased, LSB bit no. is also increased. Hence, finally, t_i can become form of 11…, just $2^k\times(2^j-1)$ form (k+j=odd). Stop here, do not do $\times(2+1)$ again, odd x already converge to 1. Do $-2^{2(n+m)}$ operation, it should operate in MSB+1 bit, because MSB bit no. of $+2^{2k}$ is forever equal to MSB+1 bit no. of the previous item. Hence minus result can be equal to $-2^{p_1+p_2+..+p_n}$, thus prove the Collatz Conjecture and get solution of Formula (1).

Below give a specific example,x=7.

We know, with 7 do $(\times3+1)\div2^k$, have:

$$7\xrightarrow{1}11\xrightarrow{1}17\xrightarrow{2}13\xrightarrow{3}5\xrightarrow{4}1$$

Suppose:

$$3^n\times7+3^{n-1}+3^{n-2}\times2^{p_1}+3^{n-3}\times2^{p_1+p_2}...+3\times2^{p_1+p_2+..+p_{n-2}}+2^{p_1+p_2+..+p_{n-1}}-2^{p_1+p_2+..+p_n}=0$$

$$3^n\times7=3^n\times(2\times3+1)=3^n\times(3^2-3+1)=3^{n+2}-3^{n+1}+3^n$$

Build:

$$3^{n+2}+3^{n+1}+3^n\times2^2+3^{n-1}\times2^4...+3\times2^{2n}+2^{2n+2}-2^{2n+4}=0$$

$$3^{n+1}+3^n\times2^2+3^{n+1}-3^n=(2^3+1)\times3^n$$

*(2+1) and $+2^4$: $(2^3+1)\times(2+1)\times3^{n-1}+2^4\times3^{n-1}=(2^5+2^3+2+1)\times3^{n-1}$

-3^{n-1}: $(2^5+2^3+2+1)\times3^{n-1}-3^{n-1}=(2^5+2^3+2)\times3^{n-1}$

*(2+1) and $+2^6$: $(2^5+2^3+2)\times(2+1)\times3^{n-2}+2^6\times3^{n-2}=(2^7+2^5+2^4+2^3+2^2+2)\times3^{n-2}$,

Let p_1=1, and delete item 2:

$$(2^7+2^5+2^4+2^3+2^2+2-2)\times3^{n-2}=(2^7+2^5+2^4+2^3+2^2)\times3^{n-2}$$

*(2+1) and $+2^8$: $(2^7+2^5+2^4+2^3+2^2)\times(2+1)\times3^{n-3}+2^8\times3^{n-3}=(2^9+2^8+2^5+2^4+2^2)\times3^{n-3}$

Let p_1+p_2=2, and delete item 2^2:

$$(2^9+2^8+2^5+2^4+2^2-2^2)\times3^{n-3}=(2^9+2^8+2^5+2^4)\times3^{n-3}$$

*(2+1) and $+2^{10}$: $(2^9+2^8+2^5+2^4)\times(2+1)\times3^{n-4}+2^{10}\times3^{n-4}=(2^{11}+2^{10}+2^8+2^7+2^4)\times3^{n-4}$

Let $p_1+p_2+p_3$=4, and delete item 2^4:

$$(2^{11}+2^{10}+2^8+2^7+2^4-2^4)\times3^{n-4}=(2^{11}+2^{10}+2^8+2^7)\times3^{n-4}$$

*(2+1) and $+2^{12}$: $(2^{11}+2^{10}+2^8+2^7)\times(2+1)\times3^{n-5}+2^{12}\times3^{n-5}=(2^{13}+2^{12}+2^{11}+2^7)\times3^{n-5}$

Let $p_1+p_2+p_3+p_4$=7, and delete item 2^7:

$$(2^{13}+2^{12}+2^{11}+2^7-2^7)\times3^{n-5}=(2^{13}+2^{12}+2^{11})\times3^{n-5}$$

Now become 111…, the highest bit is 2^{13}, iteration finished, steps n=5. And $2^{13}+2^{12}+2^{11}-2^{(2\times5+4)}=-2^{11}=-2^{p_1+\ldots+p_5}$.

This way, we get a solution for Formula (1), in which the value of n and p_i is exactly same with the result got from calculating directly.

Ⅳ Convergence Regularity Of Collatz Conjecture

If we calculate directly with odd through $(\times3+1)\div2^k$ operation, the odd sequence built (called Sequence (1)) has no obvious convergence regularity, elements in the sequence vary sometimes big, sometimes small. But if we do operation as introduced in above section, convergence regularity of the odd sequence built (called Sequence (2)) is more obvious.

First, if add two corresponding elements in each step in these two odd sequences, should be exactly 2^k(k is different with different elements). Such as

$7+9=16, 11+21=32, 17+47=64\ldots$ in above example.

In general, first element in Sequence (2) is:

$$a=(3^{m-1}+3^{m-2}\times2^2\ldots+2^{2(m-1)})-(a_{m-1}\times3^{m-1}+\ldots+a_1\times3+a_0)$$

and first element in Sequence (1) is x:

$x=3^m+a_{m-1}\times3^{m-1}+\ldots+a_1\times3+a_0$, then

$x+a=3^m+3^{m-1}+3^{m-2}\times2^2\ldots+2^{2(m-1)}=2^{2m}$, is just the same form with Formula

(2), and 2m should be the MSB+1 bit no. of x or a(along with the increase of a in Sequence (2), 2m should be the MSB+1 bit no. of a,because each corresponding part in Formula (2) is bigger than which in Formula (1)).

Below prove next elements also satisfy above regularity.

Suppose a in Sequence (2) and x in in Sequence (1) satisfy above regularity, and:

$$a = 2^m + a_{m-1} \times 2^{m-1} + ... + a_1 \times 2 + 1,$$

$x = 2^{m+1} - a$, then

$$3a + 2^{m+1} - 1 = 3 \times 2^m + 3 \times a_{m-1} \times 2^{m-1} + ... + 3 \times a_1 \times 2 + 3 + 2^{m+1} - 1,$$

$$3x + 1 = 3 \times 2^{m+1} - 3 \times 2^m - 3 \times a_{m-1} \times 2^{m-1} - ... - 3 \times a_1 \times 2 - 3 + 1,$$

$$(3x+1) + (3a + 2^{m+1} - 1) = 4 \times 2^{m+1} = 2^k$$

This states that the lowest bit of odd part of (3x+1) and ($3a+2^{m+1}-1$) is equal, and add these two odd parts should be 2^i($i<k$).

Above regularity states that the original odd sequence has no obvious regularity is because it is only the partial part, not the whole part.

Second, research into odd multiplying 3, any odd can be written in binary form 1…1, both the highest and lowest bit is 1, after $\times 3$, although total bit number increases, first substep is to shift 1 bits to the middle of the result, second substep may make carry to higher bit due to 1+1 in the middle of the result(1-bits in the middle of odd also satisfy this regularity). Both substeps are beneficial to our final goal, because we need many 1 bits in final result. $+2^{2k}$ operation ensure succession 1 bits in the highest part, -1 operation reduce count of isolated 1 bits in the lowest part. Hence 0-bits in the odd part in t_i should shift right or bit-count reduce in each step, and its weight in total t_i should reduce step by step till to 0, when the odd part converges to 1...1. Build a simple weight model:

$w_i = \dfrac{\text{value of all 0 bits in odd part in } t_i}{2^{2k}}$, which 2^{2k} is corresponding part in t_i in that step. W_i should reduce step by step, and model value can and must converge to 0, because there is no possibility to exist a convergence value, which its corresponding odd part in t_i is not 1...1, and its model value can remain unchanged in next steps through multiplying 3 operation and other two operations. Thus odd part must converge to 1...1, could not diverge or converge to other odds.

t_i sequence in above example is: 9,42,188,816,3456,14336

odd part sequence is: 9,21,47,51,27,7

w_i sequence is:

(2+4)/4=1.5,(4+16)/16=1.25,64/64=1,(64+128)/256=0.75,512/1024=0.5,0/4096=0

Does it exist some odds which its w_i tends to 0 but not equal to 0 forever? In fact, it exists some odds which 0-bits distribution are similar and w_i decreases if they exist in same sequence. Such as, 10001 and 110001($+2^5$) or 11000011(*4-1), 10001 and 1100001(insert 0). Because the $(\times 3 + 2^m - 1) \div 2^k$ operation limits the varying of the highest part of odd, these odds could not be possible to appear in the same sequence, also could not repeatedly appear.

For example:

10001->101001->1011101->11001011->11011->111, could not produce similar 0-bits distribution.

Hence it could not exist a sequence which w_i tends to 0 but not equal to 0 forever.

Ⅴ Other **Convergence Regularity**

T_i has many other characters, for example, its odd part should be with form 3*y after first step, this can be easily proved:

With odd x, first operation is: $3x+2^{2k}-1$, $2^{2k}-1$ can be divided by 3 exactly, then total value is with form 3*y.

Continue to watch t_i, from this step, within each few steps, odd part should be back to form 3*y(suppose it has not yet converged before this step), this is because each next step has $2^{2k}-2^{p1+\dots pi}$ part, when $p_{1+\dots}p_i$ is even, total value is with form 3*y. This also limits the varying range of the odd part.

And after a few steps, y should be with binary form 101...1, since this time, if step count is big enough, with each some steps, the head part of y increase a 01 pair. From the example of starting odd x=27, this regularity can be obviously observed, here do not list the result. This indicates again that this kind of iteration calculation has determined direction, and gradually regularly converges.

Ⅵ Conclusion

This way, we have proved that the Collatz Conjecture is true. During the middle procedures of iteration calculation using above method for a specific odd, we may estimate the value of steps n through some estimating models.