Geometrical optics as U(1) local gauge theory in flat space-time

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We treat the geometrical optics as the classical limit of quantum electrodynamics i.e. an Abelian $U(1)$ local gauge theory in flat space-time. We formulate the eikonal equation in a (1+1)-dimensional Minkowskian (flat) space-time and we found that the refractive index as a function of the $U(1)$ gauge potential.

Keywords: *geometrical optics, quantum electrodynamics, Abelian $U(1)$ local gauge theory, eikonal equation, refractive index, gauge potential, Minkowskian (flat) space-time.*

The geometrical optics corresponds to the limiting case of a very small wavelength of light, $\lambda \to 0$[1,2], in comparison with the characteristic dimension of the problem[3] or in other words to each of the other scales present, so that the electromagnetic waves can be regarded *locally* as *plane waves* propagating through space-time[4]. In the other hand, the representation of the electromagnetic field by means of potentials specified *at every point in space* is essentially a description by means of *a continuous set of field variables*[5]. In classical electrodynamics, the change to the description by means of *a discrete set of field variables* is brought about by considering *the electromagnetic field in a large but finite volume*, so that the electromagnetic field in a finite volume can be expanded in terms of *travelling plane waves*[5]. We consider that by treating the electromagnetic field in a finite volume but large, it has a consequence that the electromagnetic wavelength (the wavelength of light, light is the electromagnetic waves) is *finite, discrete* and *small* respectively.

In case of *a steady (constant or unchanging in time*[6], *stationary*[7]*) monochromatic wave*, the frequency[8] is constant and the time dependence of *the eikonal*, ψ, *a function of space-time*, is given by a term $-f_\theta t$ (or we can write $\partial\psi/\partial t = -f_\theta$) where f_θ denotes (angular) frequency[3]. ψ is a large quantity (so the wavelength is very small) which is "almost linear" in the space coordinates and the time[3]. The relation between ψ and the electromagnetic field is mediated, as we will see implicitly, by the gauge potential. We interpret that f_θ is discrete due to λ is discrete, in turn ψ is also discrete. The discreteness of λ, f_θ, ψ is the consequence of our consideration that the electromagnetic field takes place in a finite volume. The discreteness is useful for treating the electromagnetic field as a quantum object.

Let us introduce ψ_1 which is also called *eikonal*[3]. The relation between ψ_1 and ψ can be expressed as[3]

$$\psi_1 = \frac{c}{f_\theta}\psi + ct \tag{1}$$

where the eikonal, ψ_1, is *a function of space coordinates only*[3], *"a length"*, *a real scalar function*[9] and c is the speed of light in vacuum. We consider that we need to replace ψ to ψ_1 because here we concern with a steady monochromatic wave only.

In a 1-dimensional space, the equation of ray[10] propagation in a transparent medium[11] can be written as[3,12–14]

$$|\vec{\nabla}\psi_1(x)| = |\vec{n}(x)| = n(x), \quad x \in \Omega \subset \mathbb{R}^1 \tag{2}$$

subject to $\psi_1(x)|_{\partial\Omega} = 0$ (the solution, $\psi_1(x)$, at the boundary, $\partial\Omega$, is equal to zero), Ω is *an open set*[13], *bounded*[15]*, with suitably smooth (well-behaved) boundary*[13] *in a 1-dimensional Euclidean (flat) space*, $\mathbb{R}^1$, $|.|$ denotes the Euclidean norm, a distance function[14], in 1-dimensional Euclidean space, $\vec{\nabla}$ denotes the gradient, $n(x)$ is *the refractive index*, *a real scalar function with positive values*, *the slowness (speed^{-1})* at x where x lies inside Ω[13]. We consider, *in a flat space, $n(x)$ is a constant function.* The function $n(x)$ is typically supplied as *known input, given*, and we seek *the solution*, $\psi_1(x)$, *the shortest time* needed to travel from x to *the boundary*, $\partial\Omega$[13]. Because ψ_1 is a function of coordinates only, then the refractive index is also a function of coordinates only (*i.e. a smooth continuous function of the position*[17]). Eq.(2) is called *the eikonal equation*[3,12], i.e. *a type of the first order non-linear partial differential equation*[13,18,19].

The eikonal equation is an approximated version of the wave equation[20], a typical example of steady-state Hamilton-Jacobi equations[21,22]. The eikonal equation can be derived from the Fermat's principle[23], the Euler-Lagrange equation[23] and Maxwell equations[12,13,24]. The Hamilton-Jacobi equations are a type of non-linear *hyperbolic* partial differential equations[25] and Maxwell equations can be formulated as a *hyperbolic* system of partial differential equations[26]. So, we consider *the eikonal equation as the (first order non-linear) hyperbolic partial differential equation.* The analysis of a partial differential equation for a steady state is very important, e.g. in the Atiyah-Singer index theorem (an effort for finding *the existence and uniqueness of solutions* to linear partial differential equations of *elliptic type*[27] on closed manifold[28,29]). *Why is the eikonal equation (2) a non-linear equation?* We consider the eikonal equation (2) as a non-linear[30] equation because there exists the Euclidean norm, $|.|$, in the eq.(2). The Euclidean norm has a non-linear property, $|\vec{v}+\vec{w}| \leq |\vec{v}| + |\vec{w}|$[31], where $\vec{v}$ and $\vec{w}$ are vectors.

In a $(1+1)$-dimensional space-time, the gradient operator, $\vec{\nabla}$, in eq.(2) is replaced by the covariant fourgradient, ∂_μ. So, eq.(2) becomes

$$||\partial_\mu \psi_1(x)|| = n(x) \tag{3}$$

where μ runs from 1 to 1+1 by considering that the time derivative of ψ_1 is equal to zero. We consider that the eikonal equation (3) describes the propagation of wavefronts (field discontinuities) in a (1+1)-dimensional Minkowskian (pseudo-Euclidean[1], flat) space-time[32]. We see from eq.(3), *the zeroth rank tensor (a scalar)* of the refractive index describes *an isotropic linear optics*[33]. *It means that a flat space-time descibes an isotropic linear optics*[34]. But, the refractive index can also be *a second rank tensor* which describes that the electric field component along one axis may be affected by the electric field component along another axis[35]. The second rank tensor of the refractive index describes *an anisotropic linear optics*[33].

In a $(1+1)$-dimensional Minkowskian space-time and related to the gauge theory, *a four-vector potential* (a combination of *an electric scalar potential* and *a magnetic vector potential*[36,37]) of the geometrical optics is replaced by *a four-vector field*[43] or *the gauge potential*[4,39–43] (which makes *the related field tensor invariant under the gauge transformation*) as written below

$$\vec{B}_\mu = \vec{a}_\mu \; e^{i\psi} \tag{4}$$

where $\psi(x,t)$, as we mentioned, is *the eikonal* (*a real phase*[4]) and $\vec{a}_\mu(x,t)$ is *a complex amplitude*[4], a slowly varying function of space coordinate and time[3]. We see from eq.(4), $e^{i\psi}$ is *a scalar function* (more precisely, *a complex scalar function, dimensionless*), $\vec{B}_\mu$ is *a complex*[4,44] *quantity (a complex four-vector field)*. $\vec{B}_\mu$ as $\vec{a}_\mu$, can be interpreted as the oscillating variable[45], the displacement from *an equilibrium*[46], a position at infinity where the gauge potential is assumed equal to zero.

The treatment of the geometrical optics as an Abelian $U(1)$ *local gauge theory has a consequence that the gauge potential of the geometrical optics and the Maxwell's theory are the same, i.e. both are the Abelian* $U(1)$ *gauge potential,* $\vec{B}_\mu^{\,U(1)}$. In other words, *the related field strength of the geometrical optics and the Maxwell's theory are, in principle, the same.* So, we can rewrite eq.(4) as

$$\vec{B}_\mu^{\,U(1)} = \vec{a}_\mu \; e^{i\psi} \tag{5}$$

We can say that eq.(5) is the core of this article. The treatment of the gauge potential of geometrical optics as an Abelian $U(1)$ gauge potential has implication that we can formulate[65] the geometrical optics related to a fibre bundle (global geometry) theory where gauge potential is identical to connection[66]. Eq.(5) expresses the Abelian $U(1)$ gauge potential of the geometrical optics in a $(1+1)$-dimensional Minkowskian space-time. Eq.(5) can be written as[4]

$$\vec{B}_\mu^{U(1)} \; \underline{\vec{a}}^\mu = \vec{a}_\mu \; \underline{\vec{a}}^\mu \; e^{i\psi} = (\mathbf{a}\cdot\underline{\mathbf{a}}) \; e^{i\psi} = a^2 \; e^{i\psi} = e^{i\psi} \tag{6}$$

where $\underline{\vec{a}}^\mu$ is *a complex conjugate of* $\vec{a}_\mu$, and a is *a scalar amplitude*[4] which we can take its value as 1.

Using *Euler's formula*, eq.(6) can be written as

$$\cos\psi + i\sin\psi = \vec{B}_\mu^{U(1)} \; \underline{\vec{a}}^\mu \tag{7}$$

Eq.(7) shows us that $\vec{B}_\mu^{U(1)} \; \underline{\vec{a}}^\mu$ is *a complex scalar function.* To simplify the problem, we take the real part of (7) only, we obtain

$$\cos\psi = \text{Re}\;(\vec{B}_\mu^{U(1)} \; \underline{\vec{a}}^\mu) \tag{8}$$

where ψ in eq.(8), i.e. a real phase ("a gauge") is *an angle.* This angle has value

$$\psi = \arccos\left[\text{Re}\left(\vec{B}_\mu^{U(1)} \; \underline{\vec{a}}^\mu\right)\right] \tag{9}$$

By substituting eq.(9) into eq.(1), we obtain

$$\psi_1 = \frac{c}{f_\theta}\arccos\left[\text{Re}\left(\vec{B}_\mu^{U(1)} \; \underline{\vec{a}}^\mu\right)\right] + ct \tag{10}$$

and by substituting eq.(10) into the eikonal equation (3), we obtain

$$\left|\left|\partial_\nu\left\{\frac{c}{f_\theta}\arccos\left[\text{Re}\left(\vec{B}_\mu^{U(1)} \; \underline{\vec{a}}^\mu\right)\right] + ct\right\}\right|\right| = n \tag{11}$$

where n is *a dimensionless quantity, a real scalar function of* 1-*coordinate* which "lives" in a $(1+1)$-dimensional Minkowskian space-time.

As we mentioned, the analysis of a partial differential equation for steady state is very important for finding the existence and uniqueness of solutions to partial differential equations (PDEs). *Related to the existence and uniqueness of solutions to PDEs, does eq.(11) have a solution? In general, what are the characteristics of a partial diferential equation which has a solution? What is a consequence if we treat the eikonal in eq.(11), as a complex scalar function? Roughly speaking, does a solution of a (complex) eikonal equation generate a non-trivial topological configurations*[9,64]?

If gauge potential is identical to connection, *what is a identical form of amplitude (gauge potential is related to amplitude as written in eq.(5)) in fibre bundle theory?* In quantum theory, roughly speaking, the commutation relation is formulated using canonical field variables which are amplitudes[5]. So, *what does the commutation relation look like if we are using the identical form of amplitude in fibre bundle?* We hope that these questions can be used as a guidance for next research.

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