Relativist Gravitational Equations

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Abstract

The emphasis is not on precisely specifying the physical meaning of the scheme of the proposed gravitational equations, but on the calculation process from which they derive and whether they can be consistent and provide a calculation alternative that allows greater simplicity in obtaining acceptably satisfactory results to those already verified by general relativity.

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1 Introduction

From the field equations of General Relativity [1], [2], through a slight modification to the Schwarzschild metric[3], [4], two metrics are proposed, from which was obtained the equation of the trajectory of the planetary orbits of the solar system [5], exemplifying in particular for two bodies, the central massive body, the Sun, and the orbiting body, the planet Mercury.

$$ds^2 = \left(1 - \frac{2m}{r}\right)(cdt)^2 - \left(1 - \frac{\pi m}{l}\right)^{-1} dr^2 - r^2\left(d\theta^2 + sen^2\theta d\phi^2\right)$$

Or also:

$$ds^2 = \left(1 - \frac{2m}{r}\right)(cdt)^2 - dr^2 - \left(1 - \frac{2\pi m}{l}\right) r^2 \left(d\theta^2 + sen^2\theta d\phi^2\right)$$

Both equations of the trajectory derived from the exposed metrics admit the same solution of the equation of the trajectory given by the following expression:

$$u = F\left(1 - \frac{2\pi m}{l}\right)^{-1} - Fesen\phi\sqrt{1 - \frac{2\pi m}{l}}$$

Where u = 1 / r (r the radius of the orbit), ϕ is the angular coordinate, $F = \frac{mc^2}{h^2}$, $l = \frac{h^2}{mc^2}$, $m = \frac{GM}{c^2}$ is the Schwarzschild radius, e is the eccentricity, and h = vr = cte is the angular momentum per unit mass.

1.1 Relativist Equations of Binet

Solution that also fits the following equation of the trajectory or Binet's equation in special relativity:

$$\frac{d^2u}{d\phi^2} + u = F + \frac{2\pi m}{l} u$$

Expressing these equations in terms of r and time t:

$$\frac{d^2r}{dt^2} + r\omega^2 = -\frac{GM}{r^2} - \frac{2\pi K_\Omega M^2}{r^3}$$

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$$con\ K_\Omega = \frac{G^2}{c^2} = 4,9489\ 10^{-38}\ \left[m^4 \cdot kg^{-2} \cdot s^{-2}\right]$$

$$r = \frac{h^2}{mc^2}\left(1 - e\ sen\left(\phi\sqrt{1 - \frac{2\pi m}{l}}\right)\right)^{-1}$$

It is a solution to the equation of the trajectory, and verifies with the experimental data, such the case the advance of the perihelion of the planet Mercury orbiting around a massive body like the Sun. This advance for each orbit is given by:

$$\Delta\phi = 2\pi\left(1 - \sqrt{1 - \frac{2\pi m}{l}}\right)$$

With $m = \frac{GM}{c^2} = 1470 \quad m$ and $l = 5,62\ 10^{10} \quad m$

We obtain $\Delta\phi = 5,16\ 10^{-7} \quad [rad]$ per turn or orbit.

An extremely small value but taking into account that in 100 years it makes 414 laps around the Sun and going to seconds of arc: $5,16\ 10^{-7}\ \ 414\ \ 2,06265\ 10^5 = 44$" de arco for every 100 years.

What the experimental data verifies.

We note that the differential equation presents the first three terms that allow the classical calculation of planetary orbits to which the fourth relativistic term is added.

The solution r verifies the experimental data, such as the 44-arc perihelion advance of the planet Mercury orbiting a massive body like the Sun, exposed in the first part of this exposition.

1.2 Gravitational Force

Taking up the equation of the trajectory we can establish, taking m as the mass of the orbiting planet, the equation of motion:

$$m\frac{d^2r}{dt^2} + mr\omega^2 = -m\frac{GM}{r^2} - m\frac{2\pi K_\Omega M^2}{r^3}$$

With what the gravitational force:

$$F = m\left(\frac{GM}{r^2} + \frac{2\pi K_\Omega M^2}{r^3}\right)$$

Or also

$$F = m\left(\frac{GM}{r^2} + \frac{2\pi K_\Omega M^2}{r^3}\frac{v}{v}\right) = m\left(\frac{GM}{r^2} + v \times \frac{2\pi K_\Omega M^2}{hr^2}\right)$$

We observe that this last expression presents a great analogy with the Lorentz force with which we can establish two fields.

The classical gravitational field:

$$g = \frac{GM}{r^2}$$

And a relativistic field to define:

$$K = \frac{2\pi K_\Omega M^2}{hr^2}$$

Finally we can express the force:

$$F = m\left(g + v \times K\right)$$

2 Potentials

In principle, it is not possible to establish with certainty how the field K affects or interacts with the gravitational field g, so to establish the potentials we will adopt the expression of the electromagnetic force as a reference.

2.1 Gravitational Potentials

Following the usual steps we can say that this force derives from a potential. Expressing the force as a function of the potentials:

$$F = m\left(-\nabla\Phi - \frac{\partial A}{\partial t} + v \times (\nabla \times A)\right)$$

$$v \times (\nabla \times A) = \nabla (v \cdot A) - \frac{dA}{dt} + \frac{\partial A}{\partial t}$$

$$F = m\left[-\nabla\Phi - \frac{dA}{dt} + \nabla (v \cdot A)\right]$$

With what we can provisionally anticipate:

$$g = -\nabla\Phi - \frac{\partial A}{\partial t}$$

$$K = \nabla \times A$$

Where Φ is the classical gravitational potential

$$\Phi = \frac{GM}{r}$$

And A is the relativistic potential:

$$A = \ 2\pi K_{\Omega}\frac{M^2}{hr}$$

Later we will indicate other expressions for the vector potential A.

2.2 Potential Energy U

With which we can establish an expression for the potential energy U (in some texts it is called as potential U, but this can be confused with the correctly called potential Φ).

The expressions of the force given above are derived from that of the expression of the potential energy U given by:

$$U = m(\Phi - v \cdot A)$$

$$U = m\left(\frac{GM}{r} - v\frac{2\pi K_{\Omega} M^2}{hr}\right)$$

With which, by deriving with respect to r, we can obtain the expression of the force previously exposed.
It is also possible to establish the following expression for the potential energy so that it does not depend on the speed v:

$$U = m\left(\frac{GM}{r} + \frac{\pi K_{\Omega} M^2}{r^2}\right)$$

With which we can establish the Lagrangian expression:

$$L = T - U$$

$$\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - m\frac{GM}{r} - m\frac{\pi K_{\Omega} M^2}{r^2} = L$$

For a body of small mass m moving in the equatorial plane $\theta = \pi/2$.

3 Duality of Angular Moment per Unit of Mass

The expression of the angular momentum per unit mass h is:

$$con\ h = vr = cte$$

In other words, h is a constant made up of two variables and the derivative of a constant is zero, because the differential of a constant is zero.

But if we calculate the derivative based on its variable components we obtain:
$\frac{dh}{dr} = \frac{a}{v}r + v$ and $\frac{dh}{dv} = r + \frac{v^2}{a}$
Where a is the acceleration. t If h is constant, it must be true that
$\frac{a}{v}r + v = 0$ and/or $r + \frac{v^2}{a} = 0$
Through an exhaustive analysis of the phenomenon in question, it can be calculated that one of the terms of the derivative is of the same numerical value and different sign and thus the derivative is zero.

But when the angular momentum per unit mass h is part of more complex terms, we can use the angular momentum h as such, or replace it with its components, and after performing the calculation procedures, it is not possible to say with certainty that the expressions resulting in both cases are equivalent.

However, the angular momentum per unit of mass h that does not present possible ambiguities as in the previous case and that we will call $\acute{h}$:

$$\acute{h} = mc$$

Where m is the Schwarzschild radius, since in this case its factors are constant.

Also it can be expressed as:

$$\acute{h} = \frac{GM}{c}$$

$$GM = \acute{h}c$$

In part III of this exposition, by analyzing the previous Lagrangian, it is possible to notice possible singularities of the angular momentum per unit of mass h.

4 Faraday Tensor

To determine the components of the tensor, we start from the four-potential A:

$$A^\alpha = \left(\frac{\Phi}{c}, \mathrm{A}\right)$$

Where Φ and A are the gravitational potential and the vector potential respectively defined above.

Assuming how we anticipate the definition of the gravitational field g and K:

$$g = -\nabla\Phi - \frac{\partial A}{\partial t}$$

$$K = \nabla \times \mathrm{A}$$

The law applies:

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$$
$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$$

Taking into account that the indices correspond to the coordinates: 0 corresponds to t, 1 to x, 2 to y and 3 to z.

And for the components of the four-vector: 0 for the scalar Φ, and 1, 2, 3, for the components in x, y, z, of the potential vector A.

Then as an example:

$$F^{01} = \partial^0 A^1 - \partial^1 A^0 = \frac{1}{c}\frac{\partial A_x}{\partial t} - \left(-\frac{\partial\left(\frac{\phi}{c}\right)}{\partial x}\right) = \frac{1}{c}\left(\frac{\partial A_x}{\partial t} + \frac{\partial \phi}{\partial x}\right) = -\frac{g_x}{c}$$

$$F^{12} = \partial^1 A^2 - \partial^2 A^1 = -\frac{\partial A_y}{\partial x} + \frac{\partial A_x}{\partial y} = -K_z$$

And so on, the 16 components of the resulting gravitational tensor are calculated:

$$F_{\alpha\beta} = \left[\begin{array}{cccc} 0 & g_x/c & g_y/c & g_z/c \\ g_x/c & 0 & K_z & -K_y \\ g_y/c & -K_z & 0 & K_x \\ g_z/c & K_y & -K_x & 0 \end{array}\right]$$

4.1 Gravitational Equations I

From this tensor the gravitational equations can be established:

$$\nabla_a F^{ab} = \left(\frac{4\pi}{c}\right) J^b$$

$$\nabla^a F^{bc} + \nabla^b F^{ca} + \nabla^c F^{ab} = 0$$

More explicitly:

$$F^{\alpha\beta}_{,\gamma} + F^{\beta\gamma}_{,\alpha} + F^{\gamma\alpha}_{,\beta} = \frac{\partial F^{\alpha\beta}}{\partial x^\gamma} + \frac{\partial F^{\beta\gamma}}{\partial x^\alpha} + \frac{\partial F^{\gamma\alpha}}{\partial x^\beta} = 0$$

$$F^{\alpha\beta}_{,\beta} = \frac{\partial F^{\alpha\beta}}{\partial x^\beta} = \frac{K_\Omega}{G} J^\alpha$$

$$J^\alpha = (c\rho \quad 2\pi(\; J_x \quad J_y \quad J_z))$$

Where J is the surface density of mass flux M, and ρ is the volumetric density of mass M/V.
Example:
For $\alpha = 0$, $\beta = 1$ and $\gamma = 2$

$$\frac{\partial F^{01}}{\partial x^2} + \frac{\partial F^{12}}{\partial x^0} + \frac{\partial F^{20}}{\partial x^1} = \frac{\partial g_x/c}{\partial y} + \frac{\partial K_z}{c\partial t} + \frac{\partial g_y/c}{\partial x} = 0$$

$$\frac{\partial g_x/c}{\partial y} + \frac{\partial g_y/c}{\partial x} = -\frac{\partial K_z}{c\partial t} \qquad \Rightarrow \quad \nabla \times g = -\frac{\partial K}{\partial t}$$

For $\alpha = 1$, $\beta = 2$ and $\gamma = 3$

$$\frac{\partial F^{12}}{\partial x^3} + \frac{\partial F^{23}}{\partial x^1} + \frac{\partial F^{31}}{\partial x^2} = \frac{\partial K_Z}{\partial z} + \frac{\partial K_x}{\partial x} + \frac{\partial K_y}{\partial y} = 0$$

$$\Rightarrow \quad \nabla \cdot \mathrm{K} = 0$$

$$\frac{\partial F^{\alpha\beta}}{\partial x^\beta} = \frac{\partial F^{01}}{\partial x^1} + \frac{\partial F^{02}}{\partial x^2} + \frac{\partial F^{03}}{\partial x^3} = \frac{\partial g_x/c}{\partial x} + \frac{\partial g_y/c}{\partial y} + \frac{\partial g_z/c}{\partial z} = \frac{K_\Omega}{G} J^\alpha = \frac{K_\Omega}{G} J^0 = \frac{K_\Omega}{G} c\rho$$

$$\Rightarrow \nabla \cdot g = G\rho$$

$$\frac{\partial F^{\alpha\beta}}{\partial x^\beta} = \frac{\partial F^{10}}{\partial x^0} + \frac{\partial F^{12}}{\partial x^2} + \frac{\partial F^{13}}{\partial x^3} = \frac{\partial g_x/c}{c\partial t} + \frac{\partial K_z}{\partial y} + \frac{\partial -K_y}{\partial z} = \frac{K_\Omega}{G} J^{\alpha} = \frac{K_\Omega}{G} 2\pi\; J^{1} = \frac{K_\Omega}{G} 2\pi J^x$$

$$\frac{\partial K_z}{\partial y} - \frac{\partial K_y}{\partial z} = \frac{K_\Omega}{G} 2\pi J^x - \frac{\partial g_x/c}{c\partial t}$$

$$\Rightarrow \nabla \times K = \frac{2\pi K_\Omega}{G} J + \frac{K_\Omega}{G^2}\frac{\partial g}{\partial t}$$

4.2 Gravitational Equations II

Considering all the above, and exemplifying in some cases with the analyzed two-body system (Sol Mercury), and omitting some calculation procedures for brevity, so the following scheme of gravitational equations is exposed in a heuristic way, which is not intended be a rigorous exposition, but present them for your consideration.

Given the fields established in the Binet equation in special relativity:

$$\overrightarrow{g}_{(r)} = \frac{GM}{r^2}\left[\frac{m}{s^2}\right] \qquad \overrightarrow{K}_{(r)} = 2\pi K_\Omega \frac{M^2}{hr^2}\left[\frac{1}{s}\right] \qquad K_\Omega = \frac{G^2}{c^2} \qquad h = vr = cte$$

It is possible to set:

$$\oint g \cdot ds = GM \qquad \nabla \cdot g = G\rho$$

$$\oint K \cdot ds = 0 \qquad \nabla \cdot K = 0$$

$$\oint g \cdot dl = -\frac{\partial \Phi_K}{\partial t} \qquad \nabla \times g = -\frac{\partial K}{\partial t}$$

$$\oint K \cdot dl = \frac{2\pi K_\Omega}{G}\dot{M} + \frac{K_\Omega}{G^2}\frac{d\Phi_g}{dt} \qquad \nabla \times K = \frac{2\pi K_\Omega}{G}J + \frac{K_\Omega}{G^2}\frac{\partial g}{\partial t}$$

$$\dot{m} = \frac{GMm}{r^2 v_{merc}} \qquad \dot{M} = \frac{GMm}{r^2 v_{sol}}\left[\frac{kg}{s}\right] \qquad \dot{M} = R_M \dot{m} \qquad R_M = \frac{m_i}{m_j} = \frac{M}{m}$$

$$field\ H = \frac{\dot{M}}{r}\left[\frac{kg}{m\ s}\right]$$

$$\Phi_K = 2\pi\frac{G}{c^2}\dot{M}\oint\frac{ds}{r}$$

$$\frac{\mathrm{d}\Phi_K}{dt} = 2\pi\frac{G}{c^2}\ddot{M}r = 2\pi\frac{G}{c^2}\dot{M}v$$

$$\Phi_g = GM\oint\frac{ds}{r^2}$$

$$\frac{\mathrm{d}\Phi_g}{dt} = -\frac{GM}{r}\cdot v \ = G\dot{M}$$

$$Potential\ \Phi = -\frac{GM}{r}$$

$$Potential\ \mathrm{A} = -2\pi K_\Omega\frac{M^2}{h^2}r\dot{\phi} = -2\pi K_\Omega\frac{M^2}{h^2}v = -2\pi K_\Omega\frac{M^2}{h\ r} = -v\frac{2\pi}{c^2}\frac{GM}{r} = -\frac{2\pi G\dot{M}}{c^2}$$

$$g_{(r)} = \nabla\Phi$$

$$K_{(r)} = \nabla\times\mathrm{A}$$

$$F_r = -m\frac{GM}{r^2}\left(1 + v\frac{2\pi GM}{c^2 h}\right) = -m\left(\frac{GM}{r^2} + v\times 2\pi K_\Omega\frac{M^2}{hr^2}\right)$$

$$F_r = -m\left(\frac{GM}{r^2} + v\times\frac{2\pi G\dot{M}}{c^2 r}\right)$$

5 The GM Constant

For a circular orbit the magnitude of the velocity is given by $v^2 = \frac{GM}{r} = \frac{GM}{a} = cte$.

And for an elliptical orbit $v^2 = GM(\frac{2}{r} - \frac{1}{a})$

Where a is the semi-major axis of the ellipse.

So the v^2r product:

$v^2r = GM = v^2a = hv = cte$ Circular orbit

and

$v^2r = GM\left(2 - \frac{r}{a}\right) = \frac{GM}{a}(2a - r) = hv$ Elliptical orbit

5.1 Distance and Average Speed Options

If we apply the formula for the perimeter of the ellipse

$$Perim\ elipse = Cosh\left(\frac{a-b}{1.7420328\ a + 1.3636247\ b}\right)^2 \pi \left(\frac{a+b}{2} + \sqrt{\frac{a^2+b^2}{2}}\right)$$

Or very approximate:

$$Perim\ elipse = 4\frac{(a-b)^2 + \pi ab}{a+b}$$

Two options are presented to estimate the mean distance between two bodies:

1. To adopt the major radius or semi-major axis a of the ellipse as the mean radius or mean distance.
2. Mean distance: $distancia_{media} = 2a - b$

Average velocity is given by: Average speed: $v_{med} = \frac{Per\acute{i}m\ elipse}{Per\acute{i}odo\ \acute{O}rbita}$

In the case of only two bodies that orbit each other, as the case we have exemplified, the Sun and the planet Mercury, both describe symmetrical elliptical orbits around a common focus, where the planet Mercury describes a much larger orbit that encloses the small elliptical orbit of the Sun. Let us consider that the distance from Mercury to the common focus in the aphelion is 69.9 million kilometers, and at that same moment the aphelion of the Sun opposite that of Mercury reaches a distance from the common focus of only 11.6 kilometers, that is, an insignificant value compared to that of Mercury.

In general, the length of the semi-major axis of the elliptical orbit of the orbiting planet is the one that best adjusts to adopt it as the average distance between the two bodies, so the average speed is given by:

Average speed: $v_{med} = \frac{2\pi a}{Per\acute{i}odo\ \acute{O}rbita}$

5.2 The product Angular Moment By Speed I

The following table is compiled from data usually present in the literature.

Planeta		Distancia Kilómetros	Velocidad media orbital Km/s	Producto $v^2r=hv$
1	Mercurio	57.910.000	47.87	1,32714E+20
2	Venus	108.200.000	35.02	1,32697E+20
3	La Tierra	149.597.870	29.78	1,32671E+20
4	Marte	227.940.000	24.07	1,3206E+20
5	Ceres	413.700.000	17.88	1,32258E+20
6	Júpiter	778.330.000	13.05	1,32552E+20
7	Saturno	1.429.400.000	9.64	1,32834E+20
8	Urano	2.870.990.000	6.81	1,33145E+20
9	Neptuno	4.504.300.000	5.43	1,32809E+20
10	Plutón	5.913.520.000	4.72	1,31744E+20
11	Makemake	6.850.000.000	4.41	1,33219E+20

Note that despite the wide range of distances and speeds, the product $v^2{}_{media}\ r_{medio} = hv_{media}$ presents a value that significantly remains regularly constant around the value of the GM product.

$$GM = 6.67384\ 10^{-11}\ 1,989\ 10^{30} = 1,3274\ 10^{20}$$

5.3 The product Angular Moment By Speed II

However, if we adopt option 2 as the mean radius or mean distance, that is $distancia_{media} = 2a - b$, the product hv becomes more regularly constant close to GM.

And using: Average speed: $v_{med} = \frac{Perím\ elipse}{Período\ Órbita}$

Noting that:

$$GM \cong hv_{media} \cong v_{med}^2(2a - b)$$

Which approximates Kepler's law $GM = \frac{4\pi^2 a^3}{T^2}$

6 The K Field

The calculation process carried out previously allows us to notice a great symmetry with the development of the electromagnetic field, which would lead us to suppose that the K field is consistent from a theoretical point of view.

A great similarity has been established between the electrostatic field and the classical gravitational field, but the great difference between them is that the former has two poles, while the gravitational one does not, and there are no masses that repel each other, so in principle the analogous conformation between magnetic field and K field would not be the same either. In other words, the lines of force of the electric and magnetic fields have been established both individually and when they interact with each other, and those of the gravitational field also even when several masses interact.

The electric field has been determined independently by rubbing two bodies and observing how it attracts a material with opposite charge or repels a material with equal charge. The magnetic field has been observed independently from the magnetite stone, attracting or repelling different metals.

The gravitational field has been observed for a long time although its evidence was made formal through classical mechanics. The K field has not been established in any observation, or some anomaly or disturbance with which it is consistent to associate it even at current levels of observation and measurement.

But perhaps this virtual field can have some practical use in some situations, such as the one exposed in the first part of this exposition, when calculating the advance of the perihelion of the planet Mercury in a satisfactory way.

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