On the Helicity Sign

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Massless angle and square spinors are described together using an index corresponding to their helicity sign. This can be applied to massive spinors as well. Relations between spinors can be written more compactly and some derivations are simplified. Massless polarisation vectors of different helicities can be treated together with this index. Three point and some higher point amplitudes are investigated within this ansatz and recursion with massless spinors is considered.

1. Introduction

The spinor helicity formalism, see for example the reviews in [1-4], is widely used for the calculation of amplitudes in particle physics. Massless states have only two helicities, positive and negative and exactly they are employed for amplitudes avoiding a lot of redundancy. Originally only massless states (or massive states in the high energy limit) could be described. Later massive spinor helicity variables were introduced in [5], [6] and [7]. For massless particles the little group is $U(1)$, while for massive particles the little group is $SU(2)$. Massive particles are described as $\lambda_\alpha^I, \tilde{\lambda}_{\dot\alpha}^I$ where $\alpha, \dot\alpha$ denote the $SL(2,\mathbb{C})$ indices and I, J the $SU(2)$ spin indices. Several authors have investigated amplitudes within this new formalism see for example [8-14].

Amplitudes are usually investigated in a certain helicity configuration and other helicity configurations are obtained by parity, cyclicity or other symmetries. Here we describe negative and positive helicity spinors together as $\left| \mathrm{i} \right)_\sigma = \{\left| \mathrm{i} \right\rangle, \left| \mathrm{i} \right]\}$ where σ is an index, which is connected to the helicity sign of the spinors. This can also be done for massive spinors $\left| \mathrm{i}^{\mathrm{I}} \right)_\sigma$ in a similar manner, where the index denotes the helicity category of the spinors. One advantage of this notation is that it allows one to write relations between spinors in more compact form. Polarisations with various helicity signs can be described together and their contraction with spinors is investigated. We also investigate massless and massive three point and some higher point amplitudes within this description and find that the index plays an important role there. Finally we consider recursion relations using the index notation.

2. Massless Spinor Helicity

We first discuss massless spinors and introduce the following notation describing angle and square spinors together

$$\left| \mathrm{i} \right)_\sigma = \left| \mathrm{i}_\sigma \right) = \begin{Bmatrix} \left| \mathrm{i} \right\rangle_\alpha & \sigma = - \\ \left| \mathrm{i} \right]^{\dot\alpha} & \sigma = + \end{Bmatrix} \; , \; \left(\mathrm{i} \right|_\sigma = \left(\mathrm{i}_\sigma \right| = \begin{Bmatrix} \left\langle \mathrm{i} \right|^\alpha & \sigma = - \\ \left[\mathrm{i} \right|_{\dot\alpha} & \sigma = + \end{Bmatrix} \tag{1}$$

The little group scaling is $\left| \mathrm{i} \right)_\sigma \to \mathrm{t}_\mathrm{i}^{-\sigma} \left| \mathrm{i} \right)_\sigma$ (writing $\sigma = \pm 1$ in terms) in agreement with $\left| \mathrm{i} \right\rangle \to \mathrm{t}_\mathrm{i} \left| \mathrm{i} \right\rangle$ and $\left| \mathrm{i} \right] \to \mathrm{t}_\mathrm{i}^{-1} \left| \mathrm{i} \right]$. Since angle (square) spinors have negative (positive) helicity, the index σ (memo σ=sign) is identical to the helicity sign of the spinor. Clearly one has to distinguish the index σ of spinors from the helicity sign of a particle $\sigma_\mathrm{p} = \mathrm{sign}(\mathrm{h}_\mathrm{p})$ in amplitudes, but because both are signs there always exists a relation $\sigma = \pm\sigma_\mathrm{p}$. As an example consider the three point amplitude for gluons $\left\langle 1\ 2 \right\rangle^3 / \left\langle 2\ 3 \right\rangle \left\langle 3\ 1 \right\rangle$, where all spinors have negative helicity sign $\sigma_\mathrm{i} = -1$ while the helicity signs of the particles are $\sigma_{\mathrm{p}1} = \sigma_{\mathrm{p}2} = -\sigma_{\mathrm{p}3} = -1$. In general particles in amplitudes are represented by spinors with both values of σ. The contraction between two spinors only makes sense if the spinors have the same sign σ corresponding to the contraction of undotted or dotted indices.

$$\left(i\ \ j \right)_\sigma = \left(i \right|_\sigma \cdot \left| j \right)_\sigma = \begin{Bmatrix} \langle i\ \ j \rangle & \sigma = - \\ [\, i\ j \,] & \sigma = + \end{Bmatrix} \tag{2}$$

Note that $\left(i\ \ j \right)_\sigma = -\left(j\ \ i \right)_\sigma$ and $\left(i\ \ i \right)_\sigma = 0$. The momentum of a particle is $p_i = \left| i \right)_\sigma \left(i \right|_{-\sigma}$ or equivalently

$$p_i = \left| i \right)_{-\sigma} \left(i \right|_\sigma = \begin{Bmatrix} \left| i \right\rangle \left[i \right| = p_{\alpha\dot\alpha}\,, \ \sigma = + \\ \left| i \right] \left\langle i \right| = \overline{p}^{\dot\alpha\alpha}\,, \ \sigma = - \end{Bmatrix} \tag{3}$$

One does not need to distinguish between $p = p_{\alpha\dot\alpha}$ and $\overline{p} = \overline{p}^{\dot\alpha\alpha}$ anymore because both cases are included. Momentum conservation (all outgoing particles) and the two Schouten identities now can be written in the compact form:

$$\sum_i p_i = \sum_i \left| i \right)_{-\sigma} \left(i \right|_\sigma = 0\ \ ,\ \left| i \right)_\sigma \left(j\ \ k \right)_\sigma + \left| j \right)_\sigma \left(k\ i \right)_\sigma + \left| k \right)_\sigma \left(i\ \ j \right)_\sigma = 0 \tag{4}$$

The product of two momenta for $\sigma = \pm$ is $s_{ij} = 2 p_i \cdot p_j = \mathrm{Tr}\left\{ \left| i \right)_\sigma \left(i \right|_{-\sigma} \cdot \left| j \right)_{-\sigma} \left(j \right|_\sigma \right\} = \left(i\ \ j \right)_{-\sigma} \left(j\ \ i \right)_\sigma = \left\langle i\ \ j \right\rangle \left[j\ \ i \right]$.
Contraction of a momentum with two other spinors of opposite signs evaluates as

$$\left(i_\sigma \left| p_k \right| j_{-\sigma} \right) = \left(i\, k\, j \right) = \left(i\ \ k \right)_\sigma \left(k\ \ j \right)_{-\sigma} = \begin{Bmatrix} \left[i\ \ k \right] \left\langle k\ \ j \right\rangle, \ \sigma = + \\ \left\langle i\ \ k \right\rangle \left[k\ \ j \right], \ \sigma = - \end{Bmatrix}$$

The massless Weyl equations are $p_i \left| i \right)_\sigma = \left| i \right)_{-\sigma} \left(i \right|_\sigma \cdot \left| i \right)_\sigma = 0$, $\left(i \right|_\sigma p_i = \left(i \right|_\sigma \cdot \left| i \right)_\sigma \left(i \right|_{-\sigma} = 0$.
A massless polarisation vector, where σ denotes both helicity signs, with reference spinor r is given by

$$\varepsilon_{i\sigma} = \varepsilon_i^\sigma = \sigma\sqrt{2}\, \frac{\left| i \right)_\sigma \left(r_i \right|_{-\sigma}}{\left(r_i\ \ i \right)_{-\sigma}} \tag{5}$$

The little group scaling of the polarisation is $\varepsilon_{i\sigma} \to t_i^{-2\sigma} \varepsilon_{i\sigma}$. Distinguishing between $\varepsilon = \varepsilon_{\alpha\dot\alpha}$ and $\overline{\varepsilon} = \varepsilon^{\dot\alpha\alpha}$ is not necessary any more, but one has to be careful in contracting only spinors with the same sign. We calculate the products $\varepsilon_{i\sigma} \cdot \varepsilon_{j\sigma'}$ for $\sigma' = \pm\sigma$ and the products with momenta $\varepsilon_{i\sigma} \cdot p_k$, comprising all cases with angle and square spinors after putting $\sigma = \pm$.

$$\varepsilon_{i\sigma} \cdot \varepsilon_{j\sigma'} = \frac{1}{2} \mathrm{Tr}\left\{ \sigma\sqrt{2}\, \frac{\left| i \right)_\sigma \left(r_i \right|_{-\sigma}}{\left(r_i\ \ i \right)_{-\sigma}} \cdot \sigma'\sqrt{2}\, \frac{\left| j \right)_{\sigma'} \left(r_j \right|_{-\sigma'}}{\left(r_j\ \ j \right)_{-\sigma'}} \right\} = \delta_{\sigma',\sigma} \frac{\left(j\ \ i \right)_\sigma \left(r_i\ \ r_j \right)_{-\sigma}}{\left(r_i\ \ i \right)_{-\sigma} \left(r_j\ \ j \right)_{-\sigma}} - \delta_{\sigma',-\sigma} \frac{\left(r_j\ \ i \right)_\sigma \left(r_i\, j \right)_{-\sigma}}{\left(r_i\ \ i \right)_{-\sigma} \left(r_j\ \ j \right)_\sigma} \tag{6}$$

$$\varepsilon_{i\sigma} \cdot p_k = \frac{1}{2} \mathrm{Tr}\left\{ \sigma\sqrt{2}\, \frac{\left| i \right)_\sigma \left(r_i \right|_{-\sigma}}{\left(r_i\ \ i \right)_{-\sigma}} \cdot \left| k \right)_{-\sigma} \left(k \right|_\sigma \right\} = \frac{\sigma}{\sqrt{2}} \frac{\left(k\ \ i \right)_\sigma \left(r_i\ \ k \right)_{-\sigma}}{\left(r_i\ \ i \right)_{-\sigma}}$$

Note the contraction of ε_k with two spinors, which must have opposite signs σ for possible contractions

$$\left(i \right|_\sigma \varepsilon_{k\sigma'} \left| j \right)_{-\sigma} = \left(i \right|_\sigma \sigma'\sqrt{2}\, \frac{\left| k \right)_{\sigma'} \left(r_k \right|_{-\sigma'}}{\left(r_k\ \ k \right)_{-\sigma'}} \left| j \right)_{-\sigma} = \sigma\sqrt{2} \left(\delta_{\sigma',\sigma} \frac{\left(i\ \ k \right)_\sigma \left(r_k\ \ j \right)_{-\sigma}}{\left(r_k\ \ k \right)_{-\sigma}} - \delta_{\sigma',-\sigma} \frac{\left(j\ \ k \right)_{-\sigma} \left(r_k\ \ i \right)_\sigma}{\left(r_k\ \ k \right)_\sigma} \right)$$

For massless three point amplitudes one can multiply the first term by $\left(j\ \ i \right)_\sigma / \left(j\ \ i \right)_\sigma$, use momentum conservation $\left(r_k\ \ j \right)_{-\sigma} \left(j\ \ i \right)_\sigma = -\left(r_k\ \ k \right)_{-\sigma} \left(k\ \ i \right)_\sigma$ and similarly for the second term, to obtain an expression for all helicity signs

$$\left(i \right|_\sigma \varepsilon_{k\sigma'} \left| j \right)_{-\sigma} = \sigma\sqrt{2} \left(\delta_{\sigma',\sigma} \frac{\left(k\ i \right)_\sigma^2}{\left(j\ i \right)_\sigma} + \delta_{\sigma',-\sigma} \frac{\left(k\ \ j \right)_{-\sigma}^2}{\left(j\ i \right)_{-\sigma}} \right) \tag{7}$$

The three point amplitude $\mathcal{A}_3(f^{\sigma}\bar{f}^{-\sigma}\gamma^{\sigma'})$ is obtained by multiplying (7) with a coupling e and putting i, j, k = 1, 2, 3. Momentum conservation for a massless three point amplitude with complex momenta and the three legs i, j, k reads

$$\big|\,i\,\big)_{\sigma}\big(\,i\,\big|_{-\sigma}+\big|\,j\,\big)_{\sigma}\big(\,j\,\big|_{-\sigma}+\big|\,k\,\big)_{\sigma}\big(\,k\,\big|_{-\sigma}=0 \tag{8}$$

Multiplying from left with $\big(\,i\,\big|_{\sigma}$ and from right with $\big|\,k\,\big)_{-\sigma}$ one obtains $\big(\,i\;\;j\,\big)_{\sigma}\big(\,j\;\;k\,\big)_{-\sigma}=0$. By this one concludes that either the first or the second factor must vanish. If we take for example $\big(\,j\;\;k\,\big)_{-\sigma}=0$ then the bare three point amplitude $\bar{\mathcal{A}}_3$ (without couplings or color factors) must be given by a product of spinor contractions with the same σ. Then we get for the bare amplitude $\bar{\mathcal{A}}_3=\big(\,i\;\;j\,\big)_{\sigma}^{x_k}\big(\,j\;\;k\,\big)_{\sigma}^{x_i}\big(\,k\;\;i\,\big)_{\sigma}^{x_j}$. The little group scaling of a spinor is $\big|\,i\,\big)_{\sigma}\to t_i^{-\sigma}\big|\,i\,\big)_{\sigma}$ and therefore the amplitude scales for particle i as $\bar{\mathcal{A}}_3\to t_i^{-\sigma x_k}t_i^{-\sigma x_j}\bar{\mathcal{A}}_3\overset{!}{=}t_i^{-2h_i}\bar{\mathcal{A}}_3$. From this one obtains the equation $x_k+x_j=\sigma\cdot 2h_i$ plus two cyclic permutations. Adding two of these equations and subtracting the third one gives $x_k=\sigma(h-2h_k)$ and similar equations for x_j and x_i, where $h=h_i+h_j+h_k$. Substituting this into $\bar{\mathcal{A}}_3$ gives a compact formula for the bare three point amplitude where $\prod_{(ijk)}$ denotes the product over cyclic permutations of i, j, k.

$$\bar{\mathcal{A}}_3=\prod_{(ijk)}\big(\,i\;\;j\,\big)_{\sigma}^{\sigma(h-2h_k)} \tag{9}$$

From locality one knows [4] that the mass dimension of the three point amplitude must greater or equal than zero, i.e. $\big[\bar{\mathcal{A}}_3\big]=\sigma\big(3h-2\big(h_i+h_j+h_k\big)\big)=\sigma\cdot h\geq 0$. Therefore if $h>0$ then $\sigma=+$, if $h<0$ then $\sigma=-$, if $h=0$ then $\sigma=\pm$ is possible. One can easily check that formula (9) comprises all possible massless three point amplitudes of any helicity sign. One first calculates the total helicity h of the particles and from there the sign σ, determining whether one has to use angle or square brackets. This is of course the same formula as described in literature [1-4], but here given as one compact expression valid for all possible helicities. The amplitude given above must be multiplied by a coupling constant (and eventually appropriate color or symmetry factors not discussed here). Since the mass dimension of the three-particle amplitude must be 1, one obtains for the coupling constant

$$\mathcal{A}_3=\tilde{g}\cdot\bar{\mathcal{A}}_3,\quad \tilde{g}=\left\{\begin{array}{c} m,\ \text{if } h=0,\ [m]=1\\ g,\ \text{if } |h|=1,\ [g]=0\\ M_{Pl}^{-1},\ \text{if } |h|=2,\ [M_{Pl}]=1,\text{grav.}\\ \Lambda^{1-|h|},\ \text{if } |h|\geq 2,\ [\Lambda]=1,\text{nongrav.}\end{array}\right\} \tag{10}$$

We discuss here some simple examples showing how to obtain amplitudes for different helicities at once. Consider first the three gluon color ordered amplitude without coupling or color factors in Gervais-Neveu gauge as in [1] $\bar{\mathcal{A}}_3(1^{\sigma}\,2^{\sigma}\,3^{-\sigma})=-\sqrt{2}\big[\big(\varepsilon_1^{\sigma}\varepsilon_2^{\sigma}\big)\big(\varepsilon_3^{-\sigma}p_1\big)+\big(\varepsilon_2^{\sigma}\varepsilon_3^{-\sigma}\big)\big(\varepsilon_1^{\sigma}p_2\big)+\big(\varepsilon_3^{-\sigma}\varepsilon_1^{\sigma}\big)\big(\varepsilon_2^{\sigma}p_3\big)\big]$. With ε_i^{σ} given by (5) one obtains after using three point kinematics, momentum conservation and Schouten identities in agreement with [2]:

$$\bar{\mathcal{A}}_3(1^{\sigma}\,2^{\sigma}\,3^{-\sigma})=\sigma\frac{\big(\,1\;\;2\,\big)_{\sigma}^{3}}{\big(\,2\;\;3\,\big)_{\sigma}\big(\,3\;\;1\,\big)_{\sigma}} \tag{11}$$

As another example take the process $\bar{f}f\to\gamma\gamma$ [1] with momenta labelled as 1,2,3,4. Writing the amplitude the Feynman way one obtains with $\varepsilon_i^{\sigma_i}$ given by (5)

$$\mathcal{A}_4(\bar{f}^{-\sigma}f^{\sigma}\gamma^{\sigma_3}\gamma^{\sigma_4})=-e^2\left\{\big(\,2\,\big|_{\sigma}\varepsilon_4^{\sigma_4}\frac{p_1+p_3}{s_{13}}\varepsilon_3^{\sigma_3}\big|\,1\,\big)_{-\sigma}+\big(\,2\,\big|_{\sigma}\varepsilon_3^{\sigma_3}\frac{p_1+p_4}{s_{14}}\varepsilon_4^{\sigma_4}\big|\,1\,\big)_{-\sigma}\right\}$$

where we have left open the helicity signs of the photons. First observe, that we are forced to choose opposite helicity signs for the spinors 1,2 because there is an odd number of polarisations and propagators of the form $\big|\,a\,\big)_{\sigma'}\big(\,b\,\big|_{-\sigma'}$ between them. Furthermore, if ε_3 and ε_4 had the same helicity sign, then there is always one $\big|\,r\big)$ to be contracted with

$\left(2\right|_{\sigma}$ or $\left|1\right)_{-\sigma}$ and the amplitude can be made to vanish by proper choice of $\left|r\right)$. Starting from the left with the first summand in the above amplitude, one could contract the spinor $\left(2\right|_{\sigma}$ with $\left|4\right)_{\sigma_4}$ or $\left(r_4\right|_{-\sigma_4}$ giving two possibilities $\sigma_4=\sigma$ and therefore $\sigma_3=-\sigma$ or $\sigma_4=-\sigma$ and $\sigma_3=\sigma$. For the first option the second summand of the amplitude with $\left|r_4\right)$ pointing outwards can be made to vanish and similarly the first summand for the second option. Now we can choose $\left|r_3\right)=\left|r_4\right)=\left|1\right)$ leaving us after simplifying with the following amplitude:

$$\mathcal{A}_4=-2e^2\left\{\delta_{\sigma_4,\sigma}\frac{\left(2\;4\right)_{\sigma}\left(1\;3\right)_{-\sigma}}{\left(4\;1\right)_{\sigma}\left(3\;1\right)_{\sigma}}+\delta_{\sigma_4,-\sigma}\frac{\left(2\;3\right)_{\sigma}\left(1\;4\right)_{-\sigma}}{\left(3\;1\right)_{\sigma}\left(4\;1\right)_{\sigma}}\right\}$$

After multiplying with $\left(3\;2\right)_{\sigma}/\left(3\;2\right)_{\sigma}$ or $3\leftrightarrow 4$ and using momentum conservation one obtains finally

$$\mathcal{A}_4(\bar{f}^{-\sigma}f^{\sigma}\gamma^{\sigma_3}\gamma^{\sigma_4})=-2e^2\left\{\delta_{\sigma_4,\sigma}\frac{\left(2\;4\right)_{\sigma}^2}{\left(1\;3\right)_{\sigma}\left(2\;3\right)_{\sigma}}+\delta_{\sigma_4,-\sigma}\frac{\left(2\;3\right)_{\sigma}^2}{\left(1\;4\right)_{\sigma}\left(2\;4\right)_{\sigma}}\right\} \tag{12}$$

The calculation is analogous to the standard one, but gives one compact expression valid for all helicities.

3. Massive Spinor Helicity

We now discuss massive spinors introduced in [7] and adopt the notation of [12], [14]. Massive spinors are described by a pair of massless spinors $\lambda_{\alpha}^{I}=\left|i^{I}\right\rangle$ and $\tilde{\lambda}^{\dot{\alpha}I}=\left|i^{I}\right]$ (I=1,2) and we denote them together as $\left|i^{I}\right)_{\sigma}$.

$$\left|\mathbf{i}\right)_{\sigma}=\left|i^{I}\right)_{\sigma}=\left|i_{\sigma}^{I}\right)=\begin{Bmatrix}\left|i_{\alpha}^{I}\right\rangle & \sigma=-\\ \left|i^{\dot{\alpha},I}\right] & \sigma=+\end{Bmatrix},\;\left(\mathbf{i}\right|_{\sigma}=\left(i^{I}\right|_{\sigma}=\left(i_{\sigma}^{I}\right|=\begin{Bmatrix}\left\langle i^{\alpha,I}\right| & \sigma=-\\ \left[i_{\dot{\alpha}}^{I}\right| & \sigma=+\end{Bmatrix} \tag{13}$$

Since helicity is not a well defined quantum number for massive spinors one should better speak of helicity category [13]. The sign σ corresponds here to the helicity sign of the massless spinor remaining in the high energy limit. Contractions are only possible between spinors with the same sign σ

$$\left(\mathbf{i}\;\mathbf{j}\right)_{\sigma}=\begin{Bmatrix}\left\langle i^{I}\;\;j^{J}\right\rangle & \sigma=-\\ \left[i^{I}\;\;j^{J}\right] & \sigma=+\end{Bmatrix} \tag{14}$$

The momentum of a massive particle with momentum $\mathbf{p_i}$ is now given as

$$\mathbf{p}_i=\sigma\left|i^{I}\right)_{-\sigma}\left(i_I\right|_{\sigma}=\begin{Bmatrix}+\left|i^{I}\right\rangle\left[i_I\right|=\mathbf{p}_{i\alpha\dot{\alpha}}, & \sigma=+\\ -\left|i^{I}\right]\left\langle i_I\right|=\bar{\mathbf{p}}_i^{\dot{\alpha}\alpha}, & \sigma=-\end{Bmatrix} \tag{15}$$

Of course on can always switch $\sigma\to-\sigma$ and write $\mathbf{p_i}=-\sigma\left|i^{I}\right)_{\sigma}\left(i_I\right|_{-\sigma}$ equivalent to (15). Relations between massive spinors given in [12] can now be written in this compact form. In Appendix A we provide an explicit representation of massive spinors using the two vector notation of [15]. Another interesting representation given in [16] is also displayed there. So one can show that massive spinors fulfil the following relations: (where $a,b=\alpha,\beta$ or $\dot{\beta},\dot{\alpha}$ depending on σ)

$$\begin{aligned}
&\left(i^{J}\;\;i^{K}\right)_{\sigma}=\sigma m_i\,\epsilon^{JK},\;\left(i_J\;\;i_K\right)_{\sigma}=-\sigma m_i\,\epsilon_{JK},\;\left(i^{J}\;\;i_K\right)_{\sigma}=-\sigma m_i\,\delta_K^J,\;\left(i_J\;\;i^{K}\right)_{\sigma}=\sigma m_i\,\delta_J^K\\
&\left(i^{J}\;\;i_J\right)_{\sigma}=-\left(i_J\;\;i^{J}\right)_{\sigma}=-\sigma 2m_i,\;\left|i^{J}\right)_{\sigma}\left(i_J\right|_{\sigma}=-\left|i_J\right)_{\sigma}\left(i^{J}\right|_{\sigma}=\sigma m_i\,\delta_a^b\\
&\mathbf{p}_i=-\sigma\left|i^{J}\right)_{\sigma}\left(i_J\right|_{-\sigma}=\sigma\left|i_J\right)_{\sigma}\left(i^{J}\right|_{-\sigma}\\
&\mathbf{p}_i\left|i^{I}\right)_{-\sigma}=m_i\left|i^{I}\right)_{\sigma},\;\left(i^{I}\right|_{-\sigma}\mathbf{p}_i=-m_i\left(i^{I}\right|_{\sigma}
\end{aligned} \tag{16}$$

The spinors $\left|i^{I}\right)_{\sigma}$ are tailor made for Dirac spinors which are defined as (compare with [12]):

$$u^I_{i,\sigma} = \big|\, i^I\big)_\sigma = \begin{pmatrix} \big| i^I \big\rangle \\ \big| i^I \big] \end{pmatrix}, \quad v^I_{i,\sigma} = -\sigma \big|\, i^I\big)_\sigma = \begin{pmatrix} \big| i^I \big\rangle \\ -\big| i^I \big] \end{pmatrix} \tag{17}$$
$$\bar u_{iI,\sigma} = \sigma\big(\, i_I \big|_\sigma = \big(-\big\langle i_I \big| \quad \big[i_I \big|\big), \quad \bar v_{iI,\sigma} = \big(\, i_I \big|_\sigma = \big(\big\langle i_I \big| \quad \big[i_I \big|\big)$$

Orthogonality relations and spin sums can easily be checked with (16), (17) and using $\sigma^2 = 1$, $\sum_\sigma \sigma^2 = 2$, $\sum_\sigma \sigma = 0$ as the following examples show:

$$\sum_\sigma \bar u_{iI,\sigma} u^J_{i,\sigma} = \sum_\sigma \sigma\big(\, i_I \big|_\sigma \big| i^J\big)_\sigma = \sum_\sigma \sigma^2\, m_i \delta^J_I = 2 m_i \delta^J_I, \quad \sum_\sigma \bar v_{iI,\sigma} u^J_{i,\sigma} = \sum_\sigma \big(\, i_I \big|_\sigma \cdot \big| i^J\big)_\sigma = m_i \delta^J_I \cdot \sum_\sigma \sigma = 0$$
$$u^I_{i,\sigma} \bar u_{iI,\sigma'} = \big| i^I\big)_\sigma \sigma'\big(\, i_I \big|_{\sigma'} = \delta_{\sigma',\sigma}\, \sigma \big| i^I\big)_\sigma \big(\, i_I \big|_\sigma + \delta_{\sigma',-\sigma} \cdot -\sigma \big| i^I\big)_\sigma \big(\, i_I \big|_{-\sigma} = \begin{pmatrix} m_i & p_i \\ \bar p_i & m_i \end{pmatrix}$$

In summary we obtain:

$$\sum_\sigma \bar u_{iI,\sigma} u^J_{i,\sigma} = 2 m_i \delta^J_I, \quad \sum_\sigma \bar v_{iI,\sigma} v^J_{i,\sigma} = -2 m_i \delta^J_I, \quad \sum_\sigma \bar u_{iI,\sigma} v^J_{i,\sigma} = 0, \quad \sum_\sigma \bar v_{iI,\sigma} u^J_{i,\sigma} = 0$$
$$u^I_{i,\sigma} \bar u_{iI,\sigma'} = \begin{pmatrix} m_i & p_i \\ \bar p_i & m_i \end{pmatrix} = \not p + m_i, \quad v^I_{i,\sigma} \bar v_{iI,\sigma'} = \begin{pmatrix} -m_i & p_i \\ \bar p_i & -m_i \end{pmatrix} = \not p - m_i \tag{18}$$

Massive polarisations are defined as follows, where the round upper bracket means symmetrisation in I, J.

$$\boldsymbol{\varepsilon}_i = \frac{\sqrt{2}}{m_i} \big| \mathbf{i}\, \big)_\sigma \big(\, \mathbf{i} \big|_{-\sigma} = \varepsilon^{IJ}_i = \frac{\sqrt{2}}{m_i} \big| i^{(I}\big)_\sigma \big(\, i^{J)} \big|_{-\sigma} \tag{19}$$

In the high energy limit one obtains for I=J=1 $\varepsilon^{11}_i = \varepsilon^+_i = \sqrt{2}\,\big|\,\zeta\big\rangle\big[\, i\,\big| / \big\langle i\ \zeta\big\rangle$, for I=J=2 $\varepsilon^{22}_i = \varepsilon^-_i = \sqrt{2}\,\big|\, i\big\rangle\big[\,\zeta\big| / \big[\, i\ \ \zeta\,\big]$ and finally for I=1 J=2 or vice versa $\varepsilon^{12}_i = \varepsilon^{21}_i = \varepsilon^L_i = p_i / \sqrt{2} m_i$ i.e. a pure gauge.

One can obtain massive three point amplitudes by bolding the massless amplitude in (9). The first question arising immediately is: what should we do with the term $\sigma(h - 2h_k)$ in the exponent. A massive particle of spin s has $(2s+1)$ states with $s_z = \{-s..s\}$ in integer steps. In the limit $m \to 0$ only the two extreme values {-s, s} remain corresponding to the two helicity values. The connection between spin and helicity is therefore $s = \sigma \cdot h$ and for every single particle $s_k = \sigma_k \cdot h_k$, but for three point amplitudes with only massless particles one has $\sigma_k = \sigma$.
But there is another difference compared to the massless three point amplitude. Massive amplitudes can have brackets with both signs σ and the coupling constant may depend on the sign σ of the brackets. This leads to the following expression for the amplitude also given in [13], where a sum over various structures is implied.

$$\mathcal{A}_3 = \tilde g_{ijk} \prod_{(ijk)} \big(\, \mathbf{i}\ \mathbf{j}\, \big)^{(s-2s_k)}_{\sigma_{ij}} \tag{20}$$

Clearly the exponents must be positive, for further constraints see [13]. Consider an amplitude with two massive particles (of equal masses $m_1 = m_2 = m$ and spins $s_1 = s_2 = 1/2$) and one massive particle (of mass m_3 and spin $s_3 = 1$) yielding $s = s_1 + s_2 + s_3 = 2$. For renormalisable interactions one has $\sigma = \sigma_{23} = -\sigma_{31}$. We factor out m_3 for the correct mass dimension and absorb a factor $\sqrt{2}$ in the coupling to obtain from (20) and (19) the amplitude
$\mathcal{A}_3 = \tilde g \big(\, \mathbf{1 2}\, \big)^{(s-2s_3)}_{\sigma_{12}} \big(\, \mathbf{2\ 3}\, \big)^{(s-2s_1)}_{\sigma_{23}} \big(\, \mathbf{3\ 1}\, \big)^{(s-2s_2)}_{\sigma_{31}} = \frac{\tilde g}{m_3} \big(\, \mathbf{2\ 3}\, \big)_{\sigma_{23}} \big(\, \mathbf{3\ 1}\, \big)_{\sigma_{31}} = \tilde g \big(\, \mathbf{2} \big|_\sigma \boldsymbol{\varepsilon}_3 \big| \mathbf{1}\, \big)_{-\sigma} \xrightarrow[m_3 \to 0]{} g \big(\, \mathbf{2} \big|_\sigma \varepsilon^{\sigma'}_3 \big| \mathbf{1}\, \big)_{-\sigma}$. This form was shown in [12] to be equivalent to the standard form $\mathcal{A}_3 = g\, x^\sigma \big(\, \mathbf{1}\ \ \mathbf{2}\, \big)_{-\sigma}$ given in [7]. To see this we write the Schouten identities as $\big|\, i\,\big)_{-\sigma} \big(\, j\ \ k\,\big)_{-\sigma} + \big|\, j\,\big)_{-\sigma} \big(\, k\ i\,\big)_{-\sigma} + \big|\, k\,\big)_{-\sigma} \big(\, i\ \ j\,\big)_{-\sigma} = 0$ and focus on the case with minimal coupling.

Together with $|\,i\,)=|\,\mathbf{1}^{I}),\ |\,j\,)=|\,\mathbf{2}^{J}),\ |\,k)=|\,\varsigma)$ one obtains after multiplying by $(\,3|_{\sigma}\,\mathbf{p_1}$, using momentum conservation $\mathbf{p_1}=-\mathbf{p}_2-p_3$ and equations of motion $\mathbf{p}_1|\,\mathbf{1}^{I})_{-\sigma}=m|\,\mathbf{1}^{I})_{\sigma}$, $\mathbf{p_2}|\,\mathbf{2}^{J})_{-\sigma}=m|\,\mathbf{2}^{J})_{\sigma}$, the relation

$$(\,\mathbf{1}^{I}\ \ \mathbf{2}^{J}\,)_{-\sigma}\frac{(\varsigma_{-\sigma}\,\mathbf{1}\,3_{\sigma})}{m(\,\varsigma\ 3\,)_{-\sigma}}=\frac{(\,3\ \mathbf{2}^{J}\,)_{\sigma}(\,\varsigma\ \mathbf{1}^{I}\,)_{-\sigma}+(\,3\ \mathbf{1}^{I}\,)_{\sigma}(\,\varsigma\ \mathbf{2}^{J}\,)_{-\sigma}}{(\,\varsigma\ 3\,)_{-\sigma}}=\frac{1}{\sqrt{2}}\sum_{\sigma'}\left(\mathbf{1}^{I}_{\sigma'}\,\varepsilon_{3,\sigma}\,\mathbf{2}^{J}_{-\sigma'}\right). \tag{21}$$

The second equality is obtained using the massless polarisation vector $\varepsilon_{3,\sigma}=\sqrt{2}\left(|\,3\,)_{\sigma}(\,\varsigma|_{-\sigma}\right)/(\,3\ \varsigma\,)_{-\sigma}$ [12], [14] and the sum runs over $\sigma'=-\sigma,+\sigma$. With the x-factor introduced in [7]

$$x^{\sigma}=\frac{(\varsigma_{-\sigma}\,\mathbf{1}\,3_{\sigma})}{m(\,\varsigma\ 3\,)_{-\sigma}}=\sqrt{2}\,\varepsilon_{3,\sigma}\cdot\mathbf{p_1} \tag{22}$$

the 3-pt amplitude for the minimal coupling interaction of fermion, antifermion, massless spin 1 boson now can be written in the compact form valid for all helicity signs:

$$\mathcal{A}_3\left(\mathbf{1}_{\Psi^c}\,\mathbf{2}_{\Psi}\,3^{\sigma}\right)=g(\,\mathbf{1}^{I}\ \ \mathbf{2}^{J}\,)_{-\sigma}\,x^{\sigma}=g\frac{1}{\sqrt{2}}\sum_{\sigma'}\left(\mathbf{1}^{I}_{\sigma'}\,\varepsilon_{3,\sigma}\,\mathbf{2}^{J}_{-\sigma'}\right)\ , \tag{23}$$

By dividing (23) by $(\,\mathbf{1}^{I}\ \ \mathbf{2}^{J}\,)_{-\sigma}$ for fixed I, J one obtains for the x-factor the expression

$$x^{\sigma}=\frac{\sum_{\sigma'}\left(\mathbf{1}^{I}_{\sigma'}\,\varepsilon_{3,\sigma}\,\mathbf{2}^{J}_{-\sigma'}\right)}{\sqrt{2}(\,\mathbf{1}^{I}\ \ \mathbf{2}^{J}\,)} \tag{24}$$

The x-factor can be understood as arising from the contraction of two spinors (opposite helicities) with massless boson polarisation and therefore contains an auxiliary spinor as the boson polarisation does.

What about gravitons? The graviton polarisation is given as a product by $\varepsilon_3^{\mu\nu}=\varepsilon_3^{\mu}\,\varepsilon_3^{\nu}$ [1] or in bispinor notation as $\varepsilon_{3,2\sigma}=\varepsilon_{3,\sigma}\,\varepsilon_{3,\sigma}$. The contraction of two massless spin 1 polarisations with same helicity sign would yield zero according to (6) by choosing $r_i=r_j$. To obtain the amplitude one should contract each of those spin 1 polarisations with the spinors **1**, **2** with opposite but arbitrary helicity signs. The arising possibilities are $\left(\mathbf{1}_{\sigma'}\,\varepsilon_{3,\sigma}\,\mathbf{2}_{-\sigma'}\right)\left(\mathbf{1}_{\sigma''}\,\varepsilon_{3,\sigma}\,\mathbf{2}_{-\sigma''}\right)\rightarrow$ $\left(\mathbf{1}_{-\sigma}\,\varepsilon_{3,\sigma}\,\mathbf{2}_{\sigma}\right)\left(\mathbf{1}_{-\sigma}\,\varepsilon_{3,\sigma}\,\mathbf{2}_{\sigma}\right),\left(\mathbf{1}_{-\sigma}\,\varepsilon_{3,\sigma}\,\mathbf{2}_{\sigma}\right)\left(\mathbf{1}_{\sigma}\,\varepsilon_{3,\sigma}\,\mathbf{2}_{-\sigma}\right),\left(\mathbf{1}_{\sigma}\,\varepsilon_{3,\sigma}\,\mathbf{2}_{-\sigma}\right)\left(\mathbf{1}_{-\sigma}\,\varepsilon_{3,\sigma}\,\mathbf{2}_{\sigma}\right),\left(\mathbf{1}_{\sigma}\,\varepsilon_{3,\sigma}\,\mathbf{2}_{-\sigma}\right)\left(\mathbf{1}_{\sigma}\,\varepsilon_{3,\sigma}\,\mathbf{2}_{-\sigma}\right)$ and their sum equals $\left(\sum_{\sigma'}(\,\mathbf{1}_{\sigma'}\,|\,\varepsilon_{3,\sigma}\,|\,\mathbf{2}_{-\sigma'}\,)\right)^2$. The amplitude for minimal coupling of two equal mass fermions to a massless graviton, where $g_G=1/M_{pl}$, can be written as using (24)

$$\mathcal{A}_3\left(\mathbf{1}_{\Psi^c}\,\mathbf{2}_{\Psi}\,3_G^{2\sigma}\right)=g_G m(\,\mathbf{1}\ \ \mathbf{2}\,)_{-\sigma}\,x^{2\sigma}=g_G m(\,\mathbf{1}\ \ \mathbf{2}\,)_{-\sigma}\left(\frac{\sum_{\sigma'}(\,\mathbf{1}_{\sigma'}\,|\,\varepsilon_{3,\sigma}\,|\,\mathbf{2}_{-\sigma'}\,)}{\sqrt{2}(\,\mathbf{1}\ \ \mathbf{2}\,)_{-\sigma}}\right)^2 \tag{25}$$

Thus we also understand the graviton fermion amplitude as arising from the contraction of two massless spin 1 polarisations (building the graviton polarisation) with spinors of opposite helicities, in accordance with $\text{gravity}=\text{YM}^2$.

Finally we note that the Parke-Taylor formula [18] for both n-pt MHV and anti-MHV amplitudes for $n\geq 4$ gluons can be written as

$$\mathcal{A}_n\left(1^{\sigma}2^{\sigma}3^{-\sigma}..n^{-\sigma}\right)=\frac{(\,1\ 2\,)_{\sigma}^{4}}{(\,1\ 2\,)_{\sigma}..(\,n\ 1\,)_{\sigma}} \tag{26}$$

For $n\geq 5$ with $[\mathcal{A}_n]<0$ the index of the spinors equals minus the total helicity sign of the amplitude (which is $-\sigma$).

4. Recursion relations

We consider massless BCFW see [2], [3], [16] and adopt the notation of [2]. One can write the spinor shifts in a compact form, assuming for the first that the adjacent particles 1 and n have helicity signs σ and $-\sigma$ (in order to avoid many indices). Since there are always adjacent particles of different helicity signs in nontrivial amplitudes, this is possible in every case. The helicity sign of spinors is denoted as σ', which in general may be different from σ. The spinor shifts are given as: $\left|\hat{1}\right)_{\sigma'} = \left|1\right)_{\sigma'} - z\frac{1-\sigma'\sigma}{2}\left|n\right)_{\sigma'}$, $\left|\hat{n}\right)_{\sigma'} = \left|n\right)_{\sigma'} + z\frac{1+\sigma'\sigma}{2}\left|1\right)_{\sigma'}$. For $\sigma' = \pm\sigma$ one gets the following shifts using $\sigma^2 = 1$:

$$\left|\hat{1}\right)_{\sigma} = \left|1\right)_{\sigma}, \left|\hat{1}\right)_{-\sigma} = \left|1\right)_{-\sigma} - z\left|n\right)_{-\sigma}, \left|\hat{n}\right)_{\sigma} = \left|n\right)_{\sigma} + z\left|1\right)_{\sigma}, \left|\hat{n}\right)_{-\sigma} = \left|n\right)_{-\sigma} \tag{27}$$

The momenta are then $\hat{p}_1 = \left|1\right)_{-\sigma}\left(1\right|_{\sigma} - z\left|n\right)_{-\sigma}\left(1\right|_{\sigma}$, $\hat{p}_n = \left|n\right)_{-\sigma}\left(n\right|_{\sigma} + z\left|n\right)_{-\sigma}\left(1\right|_{\sigma}$ and the shift fulfils the relations $\left(\hat{1}\ \hat{n}\right)_{\sigma} = \left(1\ n\right)_{\sigma}$, $\left(\hat{1}\ k\right)_{\sigma} = \left(1\ k\right)_{\sigma}$, $\left(k\ \hat{n}\right)_{-\sigma} = \left(k\ n\right)_{-\sigma}$, $\hat{p}_1 + \hat{p}_n = p_1 + p_n$, $p_1 \cdot q = p_n \cdot q = q^2 = 0$, $q = \left|n\right)_{-\sigma}\left(1\right|_{\sigma}$.
For $\sigma = +$ one obtains the valid $(+-)$ shift in [2] $\left|\hat{1}\right] = \left|1\right]$, $\left|\hat{1}\right\rangle = \left|1\right\rangle - z\left|n\right\rangle$, $\left|\hat{n}\right] = \left|n\right] + z\left|1\right]$, $\left|\hat{n}\right\rangle = \left|n\right\rangle$
with momenta $\hat{p}_1 = \left|1\right\rangle\left[1\right| - z\left|n\right\rangle\left[1\right|$, $\hat{p}_n = \left|n\right\rangle\left[n\right| + z\left|n\right\rangle\left[1\right|$.
For $\sigma = -$ one gets the valid $(-+)$ shift in [3] (case I) $\left|\hat{1}\right\rangle = \left|1\right\rangle$, $\left|\hat{1}\right] = \left|1\right] - z\left|n\right]$, $\left|\hat{n}\right\rangle = \left|n\right\rangle + z\left|1\right\rangle$, $\left|\hat{n}\right] = \left|n\right]$
with momenta $\hat{p}_1 = \left|1\right\rangle\left[1\right| - z\left|1\right\rangle\left[n\right|$, $\hat{p}_n = \left|n\right\rangle\left[n\right| + z\left|1\right\rangle\left[n\right|$.
Finally in the case of equal helicity signs of particle 1 and n the shift can be used as well, since this gives always a valid shift with good asymptotic for $z \to \infty$.

The recursion relation gives the n-point amplitude as $\mathcal{A}_n = \sum_{i,s}\mathcal{A}_L^s(z_i)\frac{1}{P_i^2}\mathcal{A}_R^{-s}(z_i)$ with $z_i = P_i^2 / \left(n_{-\sigma}\ P_i\ 1_{\sigma}\right)$ [2]. During BCFW recursion with gluons one encounters the 3-point subamplitudes $\mathcal{A}_3^L\left(\hat{1}, 2, -\hat{P}\right)$ and $\mathcal{A}_3^R\left(\hat{P}, n-1, \hat{n}\right)$ where P denotes the internal gluon. The minus sign convention reads as $\left|-P\right)_{\sigma} = -\sigma\left|P\right)_{\sigma}$. The left subamplitude $\mathcal{A}_3^L$ is given by (using that the amplitude vanishes for three equal helicity signs):

$$\mathcal{A}_3^L\left(\hat{1}^{\sigma}, 2^{\sigma_2}, -\hat{P}_{12}^{\sigma_P}\right) = \delta_{\sigma_2,\sigma}\delta_{\sigma_P,\sigma}\cdot 0 + \delta_{\sigma_2,-\sigma}\delta_{\sigma_P,\sigma}\cdot - \frac{\left(\hat{P}_{12}\ \hat{1}\right)_{\sigma}^3}{\left(\hat{1}\ 2\right)_{\sigma}\left(2\ \hat{P}_{12}\right)_{\sigma}} +$$
$$\delta_{\sigma_2,\sigma}\delta_{\sigma_P,-\sigma}\cdot - \frac{\left(\hat{1}\ 2\right)_{\sigma}^3}{\left(2\ \hat{P}_{12}\right)_{\sigma}\left(\hat{P}_{12}\ \hat{1}\right)_{\sigma}} + \delta_{\sigma_2,-\sigma}\delta_{\sigma_P,-\sigma}\cdot - \frac{\left(2\ \hat{P}_{12}\right)_{-\sigma}^3}{\left(\hat{P}_{12}\ \hat{1}\right)_{-\sigma}\left(\hat{1}\ 2\right)_{-\sigma}}$$

The on-shell condition for the internal gluon is $\hat{P}_{12}^2 = \left(2\ \hat{1}\right)_{-\sigma}\left(\hat{1}\ 2\right)_{\sigma} = \left(\left(2\ 1\right)_{-\sigma} - z\left(2\ n\right)_{-\sigma}\right)\left(1\ 2\right)_{\sigma} = 0$ from which one sees immediately that $z = \left(1\ 2\right)_{-\sigma} / \left(n\ 2\right)_{-\sigma}$. Equation (27) together with Schouten gives
$\left|\hat{1}\right)_{\sigma} = \left|1\right)_{\sigma}, \left|\hat{1}\right)_{-\sigma} = \left|2\right)_{-\sigma}\left(n\ 1\right)_{-\sigma} / \left(n\ 2\right)_{-\sigma}, \hat{p}_1 = \left|2\right)_{-\sigma}\left(1\right|_{\sigma}\cdot\left(n\ 1\right)_{-\sigma} / \left(n\ 2\right)_{-\sigma}$,
$\hat{P}_{12} = \left|2\right)_{-\sigma}\left(\left(2\right|_{\sigma} + \frac{\left(n\ 1\right)_{-\sigma}}{\left(n\ 2\right)_{-\sigma}}\left(1\right|_{\sigma}\right) = \left|\hat{P}_{12}\right)_{-\sigma}\left(\hat{P}_{12}\right|_{\sigma}$. Thereby we get

$$\left(\hat{P}_{12}\ \hat{1}\right)_{\sigma} = \left(2\ 1\right)_{\sigma}, \left(\hat{1}\ 2\right)_{\sigma} = \left(1\ 2\right)_{\sigma}, \left(2\ \hat{P}_{12}\right)_{\sigma} = \frac{\left(n\ 1\right)_{-\sigma}}{\left(n\ 2\right)_{-\sigma}}\left(2\ 1\right)_{\sigma}$$
$$\left(2\ \hat{P}_{12}\right)_{-\sigma}^3 = 0^3, \left(\hat{P}_{12}\ \hat{1}\right)_{-\sigma} = 0, \left(\hat{1}\ 2\right)_{-\sigma} = 0 \tag{28}$$

Now $\mathcal{A}_3^L$ is obtained as

$$\mathcal{A}_3^L\left(\hat{1}^{\sigma},2^{\sigma_2},\hat{P}_{12}^{\sigma_P}\right)=\delta_{\sigma_2,\sigma}\delta_{\sigma_P,\sigma}\cdot 0+\delta_{\sigma_2,-\sigma}\delta_{\sigma_P,\sigma}\cdot\frac{\left(n\,2\right)_{-\sigma}\left(2\,1\right)_{\sigma}}{\left(n\,1\right)_{-\sigma}}+\delta_{\sigma_2,\sigma}\delta_{\sigma_P,-\sigma}\cdot\frac{\left(n\,2\right)_{-\sigma}\left(2\,1\right)_{\sigma}}{\left(n\,1\right)_{-\sigma}}+\delta_{\sigma_2,-\sigma}\delta_{\sigma_P,-\sigma}\cdot 0 \tag{29}$$

The right subamplitude of $\mathcal{A}_3^L\left(\hat{1}^{\sigma},2^{\sigma_2},-\hat{P}_{12}^{\sigma_P}\right)$ is $\mathcal{A}_{n-1}^R\left(\hat{P}_{12}^{-\sigma_P},3^{\sigma_3},..,(n-1)^{\sigma_{n-1}},\hat{n}^{-\sigma}\right)$. Using from (27) $\left|\hat{n}\right)_{-\sigma}=\left|n\right)_{-\sigma}$, $\left|\hat{n}\right)_{\sigma}=\left|n\right)_{\sigma}+\frac{\left(1\ 2\right)_{-\sigma}}{\left(n\ 2\right)_{-\sigma}}\left|1\right)_{\sigma}$, $\left|\hat{P}_{12}\right)_{-\sigma}=\left|2\right)_{-\sigma}$, $\left|\hat{P}_{12}\right)_{\sigma}=\left|2\right)_{\sigma}+\frac{\left(n\ 1\right)_{-\sigma}}{\left(n\ 2\right)_{-\sigma}}\left|1\right)_{\sigma}$ one derives for the adjacent particles

$$\begin{aligned}&\left(\hat{P}_{12}\ 3\right)_{-\sigma}=\left(2\ 3\right)_{-\sigma},\ \left(\hat{P}_{12}\ 3\right)_{\sigma}=\frac{\left(n_{-\sigma}\,P_{12}\,3_{\sigma}\right)}{\left(n\ 2\right)_{-\sigma}},\ \left(\hat{n}\ \hat{P}_{12}\right)_{-\sigma}=\left(n\ 2\right)_{-\sigma},\left(\hat{n}\ \hat{P}_{12}\right)_{\sigma}=\frac{P_{12n}^2}{\left(n\ 2\right)_{-\sigma}}\\&\left(n-1,\hat{n}\right)_{-\sigma}=\left(n-1,n\right)_{-\sigma},\ \left(n-1,\hat{n}\right)_{\sigma}=\frac{\left((n-1)_{\sigma}\ P_{1n}\ 2_{-\sigma}\right)}{\left(n\ 2\right)_{-\sigma}}.\end{aligned} \tag{30}$$

These relations are helpful in evaluating the right subamplitude $\mathcal{A}_{n-1}^R$.

Similarly one obtains for the right 3-point subamplitude

$$\begin{aligned}\mathcal{A}_3^R\left(\hat{P}_{n-1,n}^{-\sigma_P},(n-1)^{\sigma_{n-1}},\hat{n}^{-\sigma}\right)&=\delta_{\sigma_P,\sigma}\delta_{\sigma_{n-1},-\sigma}\cdot 0+\delta_{\sigma_P,\sigma}\delta_{\sigma_{n-1},\sigma}\frac{\left(\hat{n}\ \hat{P}_{n-1,n}\right)_{-\sigma}^3}{\left(\hat{P}_{n-1,n},n-1\right)_{-\sigma}\left(n-1,\hat{n}\right)_{-\sigma}}\\&+\delta_{\sigma_P,-\sigma}\delta_{\sigma_{n-1},-\sigma}\frac{\left(n-1,\hat{n}\right)_{-\sigma}^3}{\left(\hat{n},\hat{P}_{n-1,n}\right)_{-\sigma}\left(\hat{P}_{n-1,n},n-1\right)_{-\sigma}}+\delta_{\sigma_P,-\sigma}\delta_{\sigma_{n-1},\sigma}\frac{\left(\hat{P}_{n-1,n},n-1\right)_{\sigma}^3}{\left(n-1,\hat{n}\right)_{\sigma}\left(\hat{n}\ \hat{P}_{n-1,n}\right)_{\sigma}}.\end{aligned}$$

The on-shell condition is $\hat{P}_{n-1,n}^2=\left(n-1,\hat{n}\right)_{-\sigma}\left(\hat{n},n-1\right)_{\sigma}=\left(n-1,n\right)_{-\sigma}\left(\left(n,n-1\right)_{\sigma}+z\left(1,n-1\right)_{\sigma}\right)=0$ and gives $z=\left(n,n-1\right)_{\sigma}/\left(n-1,1\right)_{\sigma}$. From (27) and Schouten one derives $\left|\hat{n}\right)_{\sigma}=\left|n-1\right)_{\sigma}\left(n\ 1\right)_{\sigma}/\left(n-1,1\right)_{\sigma}$, $\left|\hat{n}\right)_{-\sigma}=\left|n\right)_{-\sigma}$, $\hat{P}_{n-1,n}=\left(\left|n-1\right)_{-\sigma}+\frac{\left(n\ 1\right)_{\sigma}}{\left(n-1,1\right)_{\sigma}}\left|n\right)_{-\sigma}\right)\left(n-1\right|_{\sigma}=\left|\hat{P}_{n-1,n}\right)_{-\sigma}\left(\hat{P}_{n-1,n}\right|_{\sigma}$ and from this

$$\begin{aligned}&\left(\hat{n}\ \hat{P}_{n-1,n}\right)_{-\sigma}=\left(n,n-1\right)_{-\sigma},\ \left(\hat{P}_{n-1,n},n-1\right)_{-\sigma}=\frac{\left(n\ 1\right)_{\sigma}}{\left(n-1,1\right)_{\sigma}}\left(n,n-1\right)_{-\sigma},\ \left(n-1,\hat{n}\right)_{-\sigma}=\left(n-1,n\right)_{-\sigma},\\&\left(\hat{P}_{n-1,n},n-1\right)_{\sigma}^3=0^3,\ \left(n-1,\hat{n}\right)_{\sigma}=0,\ \left(\hat{n}\ \hat{P}_{n-1,n}\right)_{\sigma}=0.\end{aligned} \tag{31}$$

By inserting them $\mathcal{A}_3^R$ is evaluated as

$$\begin{aligned}\mathcal{A}_3^R\left(\hat{P}_{n-1,n}^{-\sigma_P},(n-1)^{\sigma_{n-1}},\hat{n}^{-\sigma}\right)&=\delta_{\sigma_P,\sigma}\delta_{\sigma_{n-1},-\sigma}\cdot 0+\delta_{\sigma_P,\sigma}\delta_{\sigma_{n-1},\sigma}\cdot\frac{\left(n-1,n\right)_{-\sigma}\left(n-1,1\right)_{\sigma}}{\left(n\ 1\right)_{\sigma}}\\&+\delta_{\sigma_P,-\sigma}\delta_{\sigma_{n-1},-\sigma}\cdot\frac{\left(n-1,n\right)_{-\sigma}\left(n-1,1\right)_{\sigma}}{\left(n\ 1\right)_{\sigma}}+\delta_{\sigma_P,-\sigma}\delta_{\sigma_{n-1},\sigma}\cdot 0\end{aligned} \tag{32}$$

The left subamplitude of $\mathcal{A}_3^R\left(\hat{P}_{n-1,n}^{-\sigma_P},(n-1)^{\sigma_{n-1}},\hat{n}^{-\sigma}\right)$ is $\mathcal{A}_{n-1}^L\left(\hat{1}^{\sigma},2^{\sigma_2},..,(n-2)^{\sigma_{n-2}},-\hat{P}_{n-1,n}^{\sigma_P}\right)$. With $\left|\hat{1}\right)_{\sigma}=\left|1\right)_{\sigma}$ $\left|\hat{1}\right)_{-\sigma}=\left|1\right)_{-\sigma}+\frac{\left(n-1,n\right)_{\sigma}}{\left(n-1,1\right)_{\sigma}}\left|n\right)_{-\sigma}$, $\left|\hat{P}_{n-1,n}\right)_{\sigma}=\left|n-1\right)_{\sigma}$, $\left|\hat{P}_{n-1,n}\right)_{-\sigma}=\left|n-1\right)_{-\sigma}+\frac{\left(n\ 1\right)_{\sigma}}{\left(n-1,1\right)_{\sigma}}\left|n\right)_{-\sigma}$ one easily derives

$$\left(\hat{P}_{n-1,n}\ \hat{1}\right)_{\sigma}=\left(n-1,1\right)_{\sigma},\ \left(\hat{P}_{n-1,n}\ \hat{1}\right)_{-\sigma}=\frac{P^{2}_{n-1,n,1}}{\left(n-1,1\right)_{\sigma}},\ \left(\hat{1}\ 2\right)_{\sigma}=\left(1\ 2\right)_{\sigma},\ \left(\hat{1}\ 2\right)_{-\sigma}=\frac{\left((n-1)_{\sigma}\,P_{1n}\ 2_{-\sigma}\right)}{\left(n-1,1\right)_{\sigma}},$$

$$\left(n-2,\hat{P}_{n-1,n}\right)_{\sigma}=\left(n-2,n-1\right)_{\sigma},\ \left(n-2,\hat{P}_{n-1,n}\right)_{-\sigma}=\frac{\left((n-2)_{-\sigma}\,P_{n-1,n}\ 1_{\sigma}\right)}{\left(n-1,1\right)_{\sigma}} \tag{33}$$

which can be useful for evaluating $\mathcal{A}^{L}_{n-1}$.

In summary one sees that the nonzero left and right three point subamplitudes are

$$\mathcal{A}^{L}_{3}\left(\hat{1}^{\sigma},2^{-\sigma},\hat{P}^{\sigma}_{12}\right)=\mathcal{A}^{L}_{3}\left(\hat{1}^{\sigma},2^{\sigma},\hat{P}^{-\sigma}_{12}\right)=\frac{\left(n\ 2\right)_{-\sigma}\left(2\ 1\right)_{\sigma}}{\left(n\ 1\right)_{-\sigma}}$$

$$\mathcal{A}^{R}_{3}\left(\hat{P}^{-\sigma}_{n-1,n},(n-1)^{\sigma},\hat{n}^{-\sigma}\right)=\mathcal{A}^{R}_{3}\left(\hat{P}^{\sigma}_{n-1,n},(n-1)^{-\sigma},\hat{n}^{-\sigma}\right)=\frac{\left(n-1,n\right)_{-\sigma}\left(n-1,1\right)_{\sigma}}{\left(n\ 1\right)_{\sigma}}$$

while $\mathcal{A}^{L}_{3}\left(\hat{1}^{\sigma},2^{-\sigma},\hat{P}^{-\sigma}_{12}\right)=\mathcal{A}^{R}_{3}\left(\hat{P}^{\sigma}_{n-1,n},(n-1)^{\sigma},\hat{n}^{-\sigma}\right)=0$ together with amplitudes with the same helicity sign.

For subamplitudes with more than three particles both sides one can use the following equations for simplification. From $P_{a,b}=p_{a}+..+p_{b}$, $\hat{P}_{1,i-1}=P_{1,i-1}-z\left|n\right)_{-\sigma}\left(1\right|_{\sigma}$, $\hat{P}_{i,n}=P_{i,n}+z\left|n\right)_{-\sigma}\left(1\right|_{\sigma}$, $\left|\hat{1}\right)_{\sigma}=\left|1\right)_{\sigma}$, $\left|\hat{n}\right)_{-\sigma}=\left|n\right)_{-\sigma}$ one derives

$$\left(n\ \hat{P}_{i,n}\right)_{-\sigma}\left(\hat{P}_{i,n}\ k\right)_{\sigma}=\left(n_{-\sigma}\,\hat{P}_{i,n}\,k_{\sigma}\right)=\left(n_{-\sigma}\,P_{i,n}\,k_{\sigma}\right),\ \left(k\ \hat{P}_{1,i-1}\right)_{-\sigma}\left(\hat{P}_{1,i-1}\ 1\right)_{\sigma}=\left(k_{-\sigma}\,\hat{P}_{1,i-1}\,1_{\sigma}\right)=\left(k_{-\sigma}\,P_{1,i-1}\,1_{\sigma}\right) \tag{34}$$

which can be employed in many helicity configurations. With the helicity sign index and the equations above one can easily derive amplitudes in various helicity configurations.

Consider as an example the 6 point amplitudes with split helicities. From (29) we know that if $\sigma_{2}=\sigma$ then $\sigma_{P}=-\sigma$ and from (32) that if $\sigma_{5}=-\sigma$ then $\sigma_{P}=\sigma$.

$$\mathcal{A}_{6}\left(1^{\sigma},2^{\sigma},3^{\sigma},4^{-\sigma},5^{-\sigma},6^{-\sigma}\right)=\mathcal{A}^{L}_{3}\left(\hat{1}^{\sigma},2^{\sigma},\hat{P}^{-\sigma}\right)\frac{1}{P^{2}_{12}}\mathcal{A}^{R}_{5}\left(\hat{P}^{\sigma},3^{\sigma},4^{-\sigma},5^{-\sigma},\hat{6}^{-\sigma}\right)+\mathcal{A}^{L}_{5}\left(\hat{1}^{\sigma},2^{\sigma},3^{\sigma},4^{-\sigma},\hat{P}^{-\sigma}\right)\frac{1}{P^{2}_{56}}\mathcal{A}^{R}_{3}\left(\hat{P}^{\sigma},5^{-\sigma},\hat{6}^{-\sigma}\right)$$

For the left and right three point subamplitudes one can use (29) and (32) as well as (30) and (33) for the other subamplitudes to obtain the result of [2] with $\left\langle i\ j\right\rangle\rightarrow\left(i\ j\right)_{-\sigma},\left[i\ j\right]\rightarrow\left(i\ j\right)_{\sigma}$ valid for both signs σ.

A similar thing can be done for the alternating helicity configuration, where (34) should be used in addition to equations (28)-(33). If $\sigma_{2}=-\sigma$ from (29), then $\sigma_{P}=\sigma$ else the amplitude vanishes, the same is valid for $\sigma_{5}=\sigma$ then $\sigma_{P}=\sigma$ in equation (32).

As an example for $\sigma_{n}=\sigma$ one can take $\mathcal{A}_{6}\left(1^{\sigma},2^{-\sigma},3^{-\sigma},4^{-\sigma},5^{-\sigma},6^{\sigma}\right)=\mathcal{A}^{L}_{5}\left(\hat{1}^{\sigma},2^{-\sigma},3^{-\sigma},4^{-\sigma},-\hat{P}^{\sigma}\right)\frac{1}{P^{2}_{56}}\mathcal{A}^{R}_{3}\left(\hat{P}^{-\sigma},5^{-\sigma},\hat{6}^{\sigma}\right)$, where the left three point amplitude with the configuration $\mathcal{A}^{L}_{3}\left(\hat{1}^{\sigma},2^{-\sigma},\hat{P}^{-\sigma}_{12}\right)$ vanishes due to (29). With equations (33) plus equations above one obtains the Parke-Taylor formula for n=6.

As last example consider the amplitude involving quarks and gluons $\mathcal{A}_{4}\left(1^{\sigma}_{q},2^{-\sigma}_{q},3^{\sigma}_{g},4^{-\sigma}_{g}\right)$. There are several restrictions for possible shifts see [6] and we use the shift in (27) where n=4. The left subamplitude is a quark, anti-quark, gluon amplitude given by (7). Since it arises from the contraction of a polarisation with two fermions we know that they must have opposite helicity signs. The right subamplitude is the three gluon amplitude see (11) and also would vanish if the gluons had the same helicity sign. The left subamplitude vanishes for $\sigma_{P}=-\sigma$ according (28) and therefore one obtains together with (28) and (30) the result

$$\mathcal{A}_{4}\left(1^{\sigma}_{q},2^{-\sigma}_{q},3^{\sigma}_{g},4^{-\sigma}_{g}\right)=\frac{\left(\hat{P}_{12}\ \hat{1}\right)^{2}_{\sigma}}{\left(\hat{1}\ 2\right)_{\sigma}}\cdot\frac{1}{\left(1\ 2\right)_{-\sigma}\left(2\ 1\right)_{\sigma}}\cdot\frac{\left(\hat{4}\ \hat{P}_{12}\right)^{3}_{-\sigma}}{\left(\hat{P}_{12}\ 3\right)_{-\sigma}\left(3\ \hat{4}\right)_{-\sigma}}=-\frac{\left(4\ 2\right)^{3}_{-\sigma}\left(4\ 1\right)_{-\sigma}}{\left(1\ 2\right)_{-\sigma}\left(2\ 3\right)_{-\sigma}\left(3\ 4\right)_{-\sigma}\left(4\ 1\right)_{-\sigma}}$$

We shortly comment on how soft and collinear limits for gluon tree amplitudes see [1-3] or [19] could be written more compactly. With $\mathcal{A}_n(...a,s^\sigma,b...)\xrightarrow{s\to 0}\mathrm{Soft}(a,s^\sigma,b)\cdot\mathcal{A}_{n-1}(...a,b...)$ it is easy to write the soft function in a compact form as $\mathrm{Soft}(a,s^\sigma,b)=\sigma\dfrac{(\,a\ \ b\,)_{-\sigma}}{(\,a\ \ s\,)_{-\sigma}(\,s\ \ b\,)_{-\sigma}}$.

For the collinear limit of two adjacent particles a, b with $\left|\,a\,\right)_\sigma=\sqrt{z}\left|\,P\,\right)_\sigma$, $\left|\,b\,\right)_\sigma=\sqrt{1-z}\left|\,P\,\right)_\sigma$, $P=k_a+k_b$ and $\mathcal{A}_n(...a^{\sigma_a},b^{\sigma_b}...)\xrightarrow{a\|b}\sum_{\sigma_P}\mathrm{Split}_{-\sigma_P}(z,a^{\sigma_a},b^{\sigma_b})\cdot\mathcal{A}_{n-1}(...P^{\sigma_P}...)$ one can write the non vanishing split functions in the following compact form: (where $\sigma'=\sigma_a\sigma_b\sigma_P$) $\mathrm{Split}_{-\sigma_P}(z,a^{\sigma_a},b^{\sigma_b})=\sigma'\dfrac{z^{1-\sigma_b\sigma_P}(1-z)^{1-\sigma_a\sigma_P}}{\sqrt{z(1-z)}(\,a\ \ b\,)_{-\sigma'}}$. One can check that this agrees with the split functions given in [2], [19].

5. Summary

In summary we have described angle and square spinors together using an index connected to their helicity sign. An advantage of this description is that it allows writing many relations between the spinors in a more compact form and simplifies some derivations. Polarisations of different helicity signs and their contraction with spinors can be treated together. Three particle amplitudes can be written compactly and some higher point amplitudes were considered. It is possible to extend this description to massive spinors as well. For the important case of three point amplitudes containing two equal mass fermions and a massless boson of spin one or two we eventually gained an additional understanding of the x-factor, used to write them down. We also considered recursion with massless particles and showed that the helicity sign index may be useful in the calculation of some amplitudes.

Of course there exist a couple of problems not discussed here, which might be interesting: investigate other higher point amplitudes with the helicity sign index, especially amplitudes with loops, amplitudes with massive particles or perhaps in connection with twistors.

Appendix A: Spinor representations

Here we provide an explicit representation for massive spinors used in [12][14] based on momentum

$$p^{\mu} = \begin{pmatrix} E & P\sin(\theta)\cos(\phi) & P\sin(\theta)\sin(\phi) & P\cos(\theta) \end{pmatrix} \tag{A1}$$

Using the Pauli matrices, the momentum can be written in bispinor form $p = p_{\alpha\dot{\alpha}} = p_{\mu}\sigma^{\mu}$ and $\overline{p} = p^{\dot{\alpha}\alpha} = p_{\mu}\overline{\sigma}^{\mu}$, with $c = \cos(\theta/2)$, $s = sin(\theta/2)exp(i \cdot \phi)$, $s^{*} = sin(\theta/2)exp(-i \cdot \phi)$.

$$p = p_{\alpha\dot{\alpha}} = \begin{pmatrix} E - P(cc - ss^{*}) & -2Pcs^{*} \\ -2Pcs & E + P(cc - ss^{*}) \end{pmatrix}, \ \overline{p} = p^{\dot{\alpha}\alpha} = \begin{pmatrix} E + P(cc - ss^{*}) & 2Pcs^{*} \\ 2Pcs & E - P(cc - ss^{*}) \end{pmatrix} \tag{A2}$$

We note the massive spinors in the 2-vector notation used in [15], which is better readable than enumerating all eight 2x2 matrices. Note that lowercase index spinors are obtained by $|\mathrm{i_I}\rangle = \epsilon_{\mathrm{IJ}} |\mathrm{i^J}\rangle$ and mirrored spinors by $|\ \rangle \rightarrow \langle\ |$, similarly for square brackets. Gluing together for example $|\mathrm{n_i}\rangle$ and $|\mathrm{i}\rangle$ leads to the 2x2 matrix notation.

$$|\mathrm{i^I}\rangle = \big(|\mathrm{n_i}\rangle \ \ |\mathrm{i}\rangle\big), \ \langle\mathrm{i^I}| = \big(\langle\mathrm{n_i}| \ \ \langle\mathrm{i}|\big), \ |\mathrm{i^I}] = \big(|\mathrm{i}] \ \ -|\mathrm{n_i}]\big), \ [\mathrm{i^I}| = \big([\mathrm{i}| \ \ -[\mathrm{n_i}|\big) \tag{A3}$$

$$|\mathrm{i_I}\rangle = \big(-|\mathrm{i}\rangle \ \ |\mathrm{n_i}\rangle\big), \ \langle\mathrm{i_I}| = \big(-\langle\mathrm{i}| \ \ \langle\mathrm{n_i}|\big), \ |\mathrm{i_I}] = \big(|\mathrm{n_i}] \ \ |\mathrm{i}]\big), \ [\mathrm{i_I}| = \big([\mathrm{n_i}| \ \ [\mathrm{i}|\big)$$

$$|\mathrm{i}\rangle = \sqrt{\mathrm{E_i + P_i}}\begin{pmatrix} -\mathrm{s_i^*} \\ \mathrm{c_i} \end{pmatrix}, \ |\mathrm{n_i}\rangle = \sqrt{\mathrm{E_i - P_i}}\begin{pmatrix} \mathrm{c_i} \\ \mathrm{s_i} \end{pmatrix}, \ \langle\mathrm{i}| = \sqrt{\mathrm{E_i + P_i}}\begin{pmatrix} \mathrm{c_i} \\ \mathrm{s_i^*} \end{pmatrix}, \ \langle\mathrm{n_i}| = \sqrt{\mathrm{E_i - P_i}}\begin{pmatrix} \mathrm{s_i} \\ -\mathrm{c_i} \end{pmatrix} \tag{A4}$$

$$|\mathrm{i}] = \sqrt{\mathrm{E_i + P_i}}\begin{pmatrix} \mathrm{c_i} \\ \mathrm{s_i} \end{pmatrix}, \ |\mathrm{n_i}] = \sqrt{\mathrm{E_i - P_i}}\begin{pmatrix} \mathrm{s_i^*} \\ -\mathrm{c_i} \end{pmatrix}, \ [\mathrm{i}| = \sqrt{\mathrm{E_i + P_i}}\begin{pmatrix} -\mathrm{s_i} \\ \mathrm{c_i} \end{pmatrix}, \ [\mathrm{n_i}| = \sqrt{\mathrm{E_i - P_i}}\begin{pmatrix} \mathrm{c_i} \\ \mathrm{s_i^*} \end{pmatrix}$$

The momentum is $\mathrm{p_i} = |\mathrm{i^I}\rangle[\mathrm{i_I}| = |\mathrm{i}\rangle[\mathrm{i}| + |\mathrm{n_i}\rangle[\mathrm{n_i}|$ or $\overline{\mathrm{p}}_{\mathrm{i}} = -|\mathrm{i^I}]\langle\mathrm{i_I}| = |\mathrm{i}]\langle\mathrm{i}| + |\mathrm{n_i}]\langle\mathrm{n_i}|$ and one can check that the following relations as well as the relations in (16) are satisfied.

$$\langle\mathrm{i\ n_i}\rangle = \mathrm{m_i}, \ [\mathrm{i\ n_i}] = -\mathrm{m_i} \tag{A5}$$

Another interesting representation was given in [16], see also [5], [6]. Writing momenta in bispinor form $p = p_{\alpha\dot{\alpha}} = p_{\mu}\sigma^{\mu}$ one can decompose a massive momentum obeying $p^2 = m^2$ in terms of two null momenta k, q:

$$p = k + \frac{m^2}{2k \cdot q} q \tag{A6}$$

The massive spinors then can be written as (raising and lowering SU(2) indices I,J goes with $\epsilon^{JK} = -\epsilon_{JK} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$).

$$|p^{\mathrm{I}}\rangle = \left(\frac{m}{\langle k\ q\rangle}|q\rangle \ \ |k\rangle\right), \ |p^{\mathrm{I}}] = \left(|k] \ \ \frac{m}{[k\ q]}|q]\right)$$

$$|p_{\mathrm{I}}\rangle = \left(-|k\rangle \ \ \frac{m}{\langle k\ q\rangle}|q\rangle\right), \ |p_{\mathrm{I}}] = \left(-\frac{m}{[k\ q]}|q] \ \ |k]\right) \tag{A7}$$

The spinors $\langle p^{\mathrm{I}}|$, $[p^{\mathrm{I}}|$ are obtained with $|k\rangle \rightarrow \langle k|$, $|q\rangle \rightarrow \langle q|$ and similarly for square brackets and lower indices. It is easy to show equations (16) with this representation and the connection with the representation above is given by:

$$|\mathrm{n}\rangle = \frac{m}{\langle k\ q\rangle}|q\rangle, \ |\mathrm{n}] = -\frac{m}{[k\ q]}|q] \tag{A8}$$

Then one obtains as in (A5) $\langle\mathrm{k\ n}\rangle = \mathrm{m}$, $[\mathrm{k\ n}] = -\mathrm{m}$.

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