THE SPANNING METHOD AND THE LEHMER TOTIENT PROBLEM

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Abstract. In this paper we introduce and develop the notion of spanning of integers along functions $f : \mathbb{N} \longrightarrow \mathbb{R}$. We apply this method to a class of problems requiring to determine if the equations of the form $tf(n) = n - k$ has a solution $n \in \mathbb{N}$ for a fixed $k \in \mathbb{N}$ and some $t \in \mathbb{N}$. In particular, we show that

$$\#\{n \leq s \mid t\varphi(n) + 1 = n, \text{ **for some** } t \in \mathbb{N}\} \geq \frac{s}{2\log s}\prod_{p \mid \lfloor s \rfloor}(1 - \frac{1}{p})^{-1} - \frac{3}{2}e^{\gamma}$$

for $s \geq s_o$, where φ is the Euler totient function and $\gamma = 0.5772\cdots$ is the Euler-Macheroni constant.

1. **Introduction and problem statement**

The Euler totient function, denoted by $\varphi : \mathbb{N} \longrightarrow \mathbb{N}$, maps a natural number s to the count of integers $n \leq s$ that are coprime with s. For prime arguments, $\varphi(s)$ represents a unit left shift of the primes; specifically, $\varphi(p) = p - 1$, evident in the fact that $\varphi(p)$ divides $p - 1$. This function is multiplicative, exhibiting a property where for coprime natural numbers u and v, their product $n = u \cdot v$ satisfies $\varphi(n) = \varphi(u)\varphi(v)$.

Prompted by the Euler totient function's behavior, mathematician D.H. Lehmer posed the intriguing inquiry known as the Lehmer totient problem:

Question 1.1. Can the totient function of a composite number n divide $n - 1$?

This problem, akin in complexity to the elusive odd perfect number problem, has garnered considerable attention from mathematicians. D.H. Lehmer's initial contributions laid foundational progress by establishing that any such composite number n must be odd, square-free, and possess at least **seven** distinct prime factors. Subsequent advancements by Hagis and Cohen in 1980 refined this understanding, setting a lower bound of $n \geq 10^{20}$ and requiring **fourteen** distinct prime factors for a valid solution. Hagis further enhanced these bounds by proving that if 3 divides n, then $n \geq 10^{1937042}$ with a minimum of 298848 distinct prime factors. Notably, Luca's work [3] demonstrates that the count of Lehmer totient problem solutions less than or equal to x obeys the upper bound:

$$\leq \frac{\sqrt{x}}{(\log x)^{\frac{1}{2}+o(1)}}$$

where $o(1)$ is defined as a function that tends to zero as x tends to infinity.

Date: June 26, 2024.

Key words and phrases. fractional totient invariant function; span; measure; variation.

Here, $o(1)$ characterizes a function diminishing to zero as x tends to infinity, encapsulating the intricate behavior of solutions to this captivating mathematical conundrum.

In this paper we study the Lehmer totient problem using the lower bound

Lemma 1.2. *The lower bound holds*

$$\#\{n \leq s \mid t\varphi(n)+1=n, \ \textbf{for some } t \in \mathbb{N}\} \geq \frac{s}{2\log s}\prod_{p|\lfloor s\rfloor}(1-\frac{1}{p})^{-1}-\frac{3}{2}e^{\gamma}$$

for all $s \geq s_o$, where φ is the Euler totient function and $\gamma = 0.5772\cdots$ is the Euler-Macheroni constant.

In this paper, we denote $a|b$ to mean a divides b. Also when we write $f(n) = o(1)$ for an arithmetic function $f : \mathbb{N} \longrightarrow \mathbb{N}$, we mean $\lim\limits_{n\longrightarrow\infty} f(n) = 0$. Similarly when we write $f(n) = O(g(n))$, we mean there exists some fixed constant $c > 0$ such that for all sufficiently large values of n then $f(n) \leq c|g(n)|$. The notation $f(n) \ll g(n)$ is also alternatively used to convey the same meaning, where there is the flexibility to write the converse of the inequality as $f(n) \geq c|g(n)|$ for some fixed constant $c > 0$ such that for all sufficiently large values of n. In this case, we will write simply as $f(n) \gg g(n)$. We also write $f(n) \sim g(n)$ if and only if

$$\lim_{n\longrightarrow\infty}\frac{f(n)}{g(n)} = 1.$$

2. **Preliminary results**

In this paper, we find the following elementary inequalities useful. We will employ them in the course of establishing the main result of this paper.

Lemma 2.1. *Let $S(x)$ denotes the sum of all prime number $\leq x$. Then the inequality holds*

$$S(x) > \frac{x^2}{2\log x}+\frac{x^2}{4\log^2 x}+\frac{x^2}{4\log^3 x}+\frac{1.2x^2}{8\log^4 x}.$$

for all $x \geq 905238547$.

Proof. For a proof see for instance [5]. □

Lemma 2.2 (The prime number theorem)**.** *Let $\pi(x)$ denotes the number of primes $\leq x$. Then*

$$\pi(x) \sim \frac{x}{\log x}.$$

Lemma 2.3 (Merten's formula)**.** *The asymptotic holds*

$$\prod_{p\leq s}(1-\frac{1}{p}) \sim \frac{e^{-\gamma}}{\log s}$$

as $s \longrightarrow \infty$, where $\gamma = 0.5772\cdots$ is the Euler-Macheroni constant.

Lemma 2.4 (Stieltjes-Lebesgue integral)**.** *Let $g : [a, b] \longrightarrow \mathbb{R}$ and $h : [a, b] \longrightarrow \mathbb{R}$ be right continuous and of bounded variation on $[a, b]$ and both having left limits. Then we have*

$$f(b)g(b) - f(a)g(a) = \int\limits_{(a,b]} f(t^-)dg(t) + \int\limits_{(a,b]} g(t^-)df(t) + \sum_{t\in(a,b]} \Delta f_t \Delta g_t$$

where $\Delta f_t = f(t) - f(t^-)$.

Lemma 2.5. *Let $\pi(x)$ denotes the number of primes $\leq x$. For all real numbers $x \geq 2$, we have*

$$\pi(x) < \frac{3}{2}\frac{x}{\log x}.$$

3. **The method of spanning along a function**

In this section we introduce and study the notion of **spanning** of integers along a function. We study this notion together with associated statistics and explore some applications.

Definition 3.1. Let $f : \mathbb{N} \longrightarrow \mathbb{R}$. Then we say $n \in \mathbb{N}$ is $k \in \mathbb{N}$ - step **spanned** along the function with multiplicity t if

$$tf(n) + k = n.$$

We call the set of all $n \in \mathbb{N}$ such that n is k - step spanned the k^{th} - step spanning set along f and denote by $\mathbb{S}_k(f)$. We call the set of all truncated k-step spanning set $\mathbb{S}_k(f) \cap \mathbb{N}_s := \mathbb{S}_k(f, s)$ the s^{th} scale spanned along f. We write the length of this spanned set as

$$|\mathbb{S}_k(f, s)| := \#\{n \leq s \mid tf(n) + k = n, \text{ for some } t \in \mathbb{N}\}.$$

It is easy to see that $|\mathbb{S}_k(f, s)| < s$.

3.1. **The s-level measure of spanned set.**

In this section we introduce the notion of the measure of the span set. We launch and examine the following languages.

Definition 3.2. By the s^{th} level measure of the span set $\mathbb{S}_k(f)$, denoted $\mathbb{M}_f(s, k)$, we mean the partial sum

$$\mathbb{M}_f(s, k) := \sum_{\substack{2\leq n\leq s\\ n\in\mathbb{S}_k(f)}} f(n).$$

Let us suppose that f is a right-continuous function and of bounded variation on $[j - 1, j)$ for all $j \geq 3$ with $j \in \mathbb{N}$ and with a left limit, then by applying the Stieltjes-Lebesgue integration by parts, we can write the s^{th} level measure of the span set in the form

$$\begin{aligned}\mathbb{M}_f(s,k) :&= \sum_{2\leq j\leq s}\sum_{\substack{j-1<n\leq j\\ n\in\mathbb{S}_k(f)}} f(n)\\ &= \sum_{2\leq j\leq s}\int\limits_{(j-1)}^{j} f(t)d|\mathbb{S}_k(f,t)|\\ &< \sum_{2\leq j\leq s}\bigg(f(j)|\mathbb{S}_k(f,j)|-f(j-1)|\mathbb{S}_k(f,j-1)|\bigg)\\ &= f(s)|\mathbb{S}_k(f,s)|-f(1)|\mathbb{S}_k(f,1)|.\end{aligned}$$

The following inequality is a simple consequence of the above analysis.

Proposition 3.1 (Spanning inequality)**.** *Let f be a right-continuous function and of bounded variation on $[x,x+1)$ for $x\geq 1$ with $x\in\mathbb{N}$ and have left limits. Then the inequality holds*

$$|\mathbb{S}_k(f,s)| \geq \frac{1}{f(s)}\sum_{\substack{2\leq n\leq s\\ n\in\mathbb{S}_k(f)}} f(n)+\frac{f(1)|\mathbb{S}_k(f,1)|}{f(s)}.$$

It is important to note that this inequality does not hold in general. As it is informed by the spanning method, it only holds for functions that are right continuous and of bounded variation on intervals of the form $[x,x+1)$ for $x\geq 1$ with $x\in\mathbb{N}$ and additionally have left limits, generally known as cadlag functions. Indeed the challenge of approaching the Lehmer totient problem using the spanning method is to construct an appropriate cadlag function for the Euler totient function. The next section studies an extension of the Euler totient function.

4. The fractional Euler totient invariant function

In this section we introduce and study a new function defined on the real line. We launch the following languages.

Definition 4.1. By the fractional Euler totient invariant function, we mean the function $\tilde{\varphi}:[1,\infty)\longrightarrow\mathbb{R}$ such that

$$\tilde{\varphi}(a)=\varphi(\lfloor a\rfloor)+\{a\}$$

where φ is the Euler totient function and $\lfloor\cdot\rfloor$ and $\{\cdot\}$ is the floor and the fractional part of a real number, respectively.

The fractional Euler totient invariant function turns out to be an interesting function that in some way extends the Euler totient function to the reals. Even though the notion of co-primality in not well-defined on the entire real line, it captures the intrinsic property of the Euler totient function defined on the positive integers. In essence, the Euler totient function and the fractional totient invariant function coincides on the set of positive integers. Next, we examine some elementary properties of the fractional Euler totient invariant function in the following sequel.

Proposition 4.1. *The following properties of the fractional totient invariant function holds*

(i) *If a is a positive integer, then* $\tilde{\varphi}(a) = \varphi(a)$.
(ii) $\tilde{\varphi}(a) < a$ *for all* $a > 1$.

Remark 4.2. We now state an analytic property of the fractional totient invariant function. In fact, the fractional totient invariant function can be seen as a slightly continuous analogue of the Euler totient function on subsets of the reals.

Proposition 4.2. *The function* $\tilde{\varphi} : [1, \infty) \longrightarrow \mathbb{R}$ *with*

$$\tilde{\varphi}(a) = \varphi(\lfloor a \rfloor) + \{a\}$$

is right-continuous and of bounded variation on $[x, x+1)$ *for* $x \geq 1$ *with* $x \in \mathbb{N}$ *and have left limits.*

5. **Main result**

Lemma 5.1. *The lower bound holds*

$$\#\{n \leq s \mid t\varphi(n) + 1 = n, \ \textbf{for some } t \in \mathbb{N}\} \geq \frac{s}{2\log s} \prod_{p \mid \lfloor s \rfloor} (1 - \frac{1}{p})^{-1} - \frac{3}{2} e^{\gamma}$$

for all $s \geq s_o$, *where* φ *is the Euler totient function and* $\gamma = 0.5772\cdots$ *is the Euler-Macheroni constant.*

Proof. By appealing to Proposition 3.1, we obtain the lower bound

$$\#\{2 \leq n \leq s \mid t\tilde{\varphi}(n) + 1 = n, \ \textbf{for some } t \in \mathbb{N}\} \geq \frac{1}{\tilde{\varphi}(s)} \sum_{\substack{2 \leq n \leq s \\ n \in \mathbb{S}_1(\tilde{\varphi})}} \tilde{\varphi}(n) + \frac{1}{\tilde{\varphi}(s)}$$

$$\geq \frac{1}{\tilde{\varphi}(s)} \sum_{\substack{2 \leq n \leq s \\ n \in \mathbb{S}_1(\tilde{\varphi})}} \tilde{\varphi}(n). \tag{5.1}$$

Next we estimate each term on the right-hand side of the inequality. Since $\varphi(p) = p - 1$ for any prime number $p \in \mathbb{P}$, we obtain the lower bound

$$\sum_{\substack{2 \leq n \leq s \\ n \in \mathbb{S}_1(\tilde{\varphi})}} \tilde{\varphi}(n) \geq \sum_{p \leq s} \varphi(p)$$

$$= \sum_{p \leq s} p - \pi(s).$$

By applying Lemma 2.1, we obtain the lower bound for sufficiently large values of s

$$\sum_{p \leq s} p - \pi(s) \geq \frac{s^2}{2\log s} - \pi(s)$$

so that by appealing to the decomposition

$$\varphi(\lfloor s \rfloor) = \lfloor s \rfloor \prod_{p \mid \lfloor s \rfloor} (1 - \frac{1}{p}) \sim s \prod_{p \mid \lfloor s \rfloor} (1 - \frac{1}{p})$$

with $\tilde{\varphi}(s) \sim \varphi(\lfloor s \rfloor)$ and Lemma 2.3, we obtain the lower bound

$$\frac{1}{\tilde{\varphi}(s)} \sum_{\substack{2 \leq n \leq s \\ n \in \mathbb{S}_1(\tilde{\varphi})}} \tilde{\varphi}(n) \geq \frac{s}{2\log s} \prod_{p \mid \lfloor s \rfloor} (1 - \frac{1}{p})^{-1} - \frac{1}{\tilde{\varphi}(s)} \pi(s) \tag{5.2}$$

for all $s \geq s_o$. By plugging the lower bound in (5.2) into (5.1) and applying Lemma 2.5, we obtain the lower bound

$$\begin{aligned}\#\{2 \leq n \leq s \mid t\tilde{\varphi}(n)+1=n,\ \textbf{for some}\ t \in \mathbb{N}\} &\geq \frac{s}{2\log s}\prod_{p|\lfloor s\rfloor}(1-\frac{1}{p})^{-1}-\frac{\pi(s)}{\varphi(\lfloor s\rfloor)}\\ &\geq \frac{s}{2\log s}\prod_{p|\lfloor s\rfloor}(1-\frac{1}{p})^{-1}-\frac{3}{2}e^{\gamma}\end{aligned}$$

for all $s \geq s_o$. ☐

Theorem 5.2. *There exists a composite $n \in \mathbb{N}$ such that $\varphi(n)|n-1$.*

Proof. Suppose on the contrary that there exists no composite $n \in \mathbb{N}$ such that $\varphi(n)|n-1$. Then for all $s \geq s_o$, we obtain the lower bound by appealing to Lemma 6.1

$$\pi(s) \geq \frac{s}{2\log s}\prod_{p|\lfloor s\rfloor}(1-\frac{1}{p})^{-1}-\frac{3}{2}e^{\gamma}$$

where $\pi(s)$ is the prime counting function. Now, we construct an infinite set of composites

$$\mathcal{C} := \{s \in \mathbb{N} \mid s := \prod_{\substack{p \leq p_o\\ p,p_o \in \mathbb{P}}} p\}.$$

It can be checked that for all sufficiently large composites s in the infinite set $\mathcal{C}$, we have

$$\prod_{p|s}(1-\frac{1}{p})^{-1} \geq 3 \tag{5.3}$$

so that

$$\pi(s) \geq \frac{3}{2}\frac{s}{\log s}-\frac{3}{2}e^{\gamma}$$

which contradicts the prime number theorem. ☐

6. Conclusion and further remarks

The present study adeptly navigates a significant impediment that might have otherwise hindered previous investigations in this domain. The inherent limitation of the Euler totient function, restricted to positive integers and lacking one-sided continuity over the real numbers, posed a formidable challenge, now elegantly surmounted within this paper. Introducing a refined variant of the Euler totient function tailored to specific subsets of real numbers, characterized by right continuity while upholding the essence of the original function, paves a seamless path beyond this anticipated obstacle.

Moreover, this proof leverages two seminal achievements of the twentieth century, rooted in the rich history of eighteenth and nineteenth-century mathematics: the prime number theorem and the Mertens formula. Employing the innovative spanning method, this work integrates these foundational results to establish the ensuing lower bound.

Lemma 6.1. *The lower bound holds*

$$\#\{n \leq s \mid t\varphi(n) + 1 = n, \ \textbf{for some } t \in \mathbb{N}\} \geq \frac{s}{2\log s} \prod_{p \mid \lfloor s \rfloor} (1 - \frac{1}{p})^{-1} - \frac{3}{2}e^{\gamma}$$

for all $s \geq s_o$, where φ is the Euler totient function.

This lower bound is used as the main toolbox to show existence of a certain composite (large) that satisfies the divisibility relation $n|\varphi(n)$. The spanning method and it's variant could in principle be used in careful manner to study related problems, which is not the main goal of this paper.

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