New trigonometric integrals with Barnes function

Denis GALLET
Rectorat de DIJON
2 G, rue Général Delaborde 21000 DIJON (France)
densg71@gmail.com

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Abstract
This time, I talk about some integrals in the continuity of my precedent paper Corrections about V. S. Adamchik’s papers (1). Now we can deduce from the three integrals, six new integrals and I give the general formulas in terms of Barnes function.

1 Definition
The Barnes function is defined as the following Weierstrass product:

\[ G(1 + z) = (2\pi)^\frac{z}{2} e^{-\frac{z(1 + z)}{2}} \sum_{k=1}^{\infty} \left(1 + \frac{z}{k}\right)^k e^{-z + \frac{k^2}{4}} \]  

(2)

where gamma is the Euler-Mascheroni constant.

The following properties of G are well-known.

2 Properties
\[ G(1) = 1 \]  

(3)

\[ G(1 + z) = G(z) \Gamma(z) \]  

(4)

\[ \log (G(1 + z)) = \frac{z \log (2\pi)}{2} - \frac{z(1 + z)}{2} + z \log (\Gamma(1 + z)) - \int_0^z \log (\Gamma(t + 1)) \, dt \]  

(5)

\[ \int_0^z \log (\Gamma(t + 1)) \, dt = \frac{z \log (2\pi)}{2} - \frac{z(1 + z)}{2} + z \log (\Gamma(1 + z)) - \log (G(z)) - \log (\Gamma(z)) \]  

(6)
3 Identity and rules

We know that 
\[
\frac{1}{e^{\pi x} - 1} = \frac{1}{(e^{\pi x} - 1)(e^{\pi x} + 1)} = \frac{1/2}{e^{\pi x} - 1} - \frac{1}{2} e^{\pi x}
\]

And in general in terms of a:
\[
\frac{1}{e^{a \pi x} - 1} = \frac{1}{e^{a \pi x} - 1} - \frac{2}{2} e^{a \pi x}
\]

So we need this identity for the general formula.

And now several rules for calculation:
\[
\zeta(1, -2, \frac{1}{2}) = -\frac{3\zeta(1, -2)}{4}
\]
\[
\sum_{k=1}^{k-1} \zeta(1, s, \frac{k}{2}) = (k^s - 1) \zeta(1, s) + k^s \zeta(s) \log(k) \text{ with } s=-2
\]
\[
\zeta(1, -2, 1 + t) = t^2 \log(t) + \zeta(1, -2, t)
\]

4 The first integral

\[
\int_0^\infty \frac{x^2}{e^{a \pi x} + 1} \arctan \left( \frac{x}{z} \right) dx
\]

Let A be the Glaisher–Kinkelin’s constant (7).

a and z are both positiv number.

I use the identity and the general formula of \( \int_0^\infty \frac{x^2}{e^{a \pi x} - 1} \arctan \left( \frac{x}{z} \right) dx \)

So I deduce the general formula of \( \int_0^\infty \frac{x^2}{e^{a \pi x} + 1} \arctan \left( \frac{x}{z} \right) dx \)

\[
- \frac{z^2 \log(G(a z))}{a^2} + \frac{11 z^2}{36} + \frac{z^2}{6} \log \left( \Gamma \left( \frac{a z}{2} \right) \right) - \frac{\log(\Gamma(a z)) z^2}{a^2} - 4 \frac{\log(\Gamma(\frac{a z}{2})) z}{a^2} - 4 \frac{\log(G(\frac{a z}{2})) z}{a^2} - z \ \frac{\log(\Gamma(\frac{a z}{2}))}{a^2} + \frac{z^2 \log(A)}{a^2} + 2 \frac{z \log(\Gamma(a z))}{a^2} \left( \frac{1-z}{2} \right) - 2 \frac{z \log(G(a z))}{a^2} - 2 \frac{z \log(G(\frac{a z}{2}))}{a^2} + 4 \frac{\zeta(1, -2, \frac{a z}{2})}{a^2}
\]

5 The second integral

\[
\int_0^\infty \frac{x^2 + z^2}{e^{a \pi x} + 1} \arctan \left( \frac{x}{z} \right) \cos \left( 2 \arctan \left( \frac{x}{z} \right) \right) dx
\]
a and z are both positive numbers.

I use the identity and the general formula of \( \int_0^\infty \frac{x^2 + z^2}{e^{\pi x} + 1} \arctan \left( \frac{x}{z} \right) \cos \left( 2 \arctan \left( \frac{x}{z} \right) \right) \, dx \)

So I deduce the general formula of \( \int_0^\infty \frac{x^2 + z^2}{e^{\pi x} + 1} \arctan \left( \frac{x}{z} \right) \cos \left( 2 \arctan \left( \frac{x}{z} \right) \right) \, dx \)

\[
\frac{2z^3 \log(2za) - 29z^3}{a^2} - \frac{z}{a^2} - \frac{z^2 \log(2)}{3a^2} + 4 \frac{z \log(G(az))}{a^2} + 4 \frac{z \log(G'(az))}{a^2} + \frac{\zeta(1, -2, za)}{a^3} - \frac{\zeta(1, -2, za)}{a^3}
\]

6 The third integral

\[
\int_0^\infty \frac{1}{e^{a \pi x} + 1} \log \left( x^2 + z^2 \right) \sin \left( 2 \arctan \left( \frac{x}{z} \right) \right) \left( x^2 + z^2 \right) \, dx
\]

a and z are both positive numbers.

I use the identity and the general formula of \( \int_0^\infty \frac{\log(x^2 + z^2)}{e^{\pi x} + 1} \sin \left( 2 \arctan \left( \frac{x}{z} \right) \right) \left( x^2 + z^2 \right) \, dx \)

So I deduce the general formula of \( \int_0^\infty \frac{\log(x^2 + z^2)}{e^{\pi x} + 1} \sin \left( 2 \arctan \left( \frac{x}{z} \right) \right) \left( x^2 + z^2 \right) \, dx \)

\[
-\frac{z^3 \log(2za) + \frac{3z^3}{2} + \frac{z}{a^2} + \frac{2 \log(2)}{3a^2} - \frac{\log(a)z}{3a^2} - \frac{8 \log(G'(ax))z}{a^2}}{8} - \frac{\zeta(1, -2, za)}{a^3} - \frac{\zeta(1, -2, za)}{a^3}
\]

7 Integrals in terms of hyperbolic sine

We know the identity:

\[
\frac{1}{\sinh(\pi x)} = \frac{e^{\pi x} - 1}{e^{\pi x} + 1}
\]

Now in terms of a:

\[
\frac{1}{\sinh(a \pi x)} = \frac{e^{a \pi x} - 1}{e^{a \pi x} + 1}
\]

I use the same principle and we have three general formulas.
8 The fourth integral

\[\int_0^\infty \frac{x^2}{\sinh(a\pi x)} \arctan \left( \frac{x}{z} \right) \, dx\]

a and z are both positiv number.

The general formula is:

\[-\frac{z^3 \log(2)}{3} + \frac{2}{a} \frac{\log(\Gamma(\frac{2a}{a}))z^2}{2} - \frac{\log(\Gamma(az))z^2}{a} + \frac{z^2 \log(2\pi)}{2a} - 8 \frac{\log(\Gamma(\frac{az}{a}))z}{a^2} - 8 \frac{\log(\Gamma(\frac{az}{a}))z}{a^2} - 8 \frac{\log(\Gamma(\frac{az}{a}))z}{a^2} - 8 \frac{\log(\Gamma(\frac{az}{a}))z}{a^2}\]

9 The fifth integral

\[\int_0^\infty \frac{x^2 + z^2}{\sinh(a\pi x)} \arctan \left( \frac{x}{z} \right) \cos \left( 2 \arctan \left( \frac{x}{z} \right) \right) \, dx\]

a and z are both positiv number.

The general formula is:

\[\frac{4 z^3 \log(2)}{a} - \frac{z^2 \log(\tau)}{a} - 2 \frac{z^2 \log(2)}{a^2} + \frac{z^2 \log(az)}{a^2} + 8 \frac{\log(\Gamma(\frac{az}{a}))z}{a^2} + 8 \frac{\log(\Gamma(\frac{az}{a}))z}{a^2} - \frac{z \log(\Gamma(\frac{az}{a}))z}{a^2}\]

10 The sixth integral

\[\int_0^\infty \frac{\log \left( x^2 + z^2 \right) \left( x^2 + z^2 \right)}{\sinh(a\pi x)} \sin \left( 2 \arctan \left( \frac{x}{z} \right) \right) \, dx\]

a and z are both positiv number.

The general formula is:

\[-16 \frac{\log(\Gamma(\frac{az}{a}))z}{a^3} - 16 \frac{\log(\Gamma(\frac{az}{a}))z}{a^3} + 4 \frac{\log(\Gamma(az))z}{a^3} + 4 \frac{\log(\Gamma(az))z}{a^3} + 2 \frac{\log(\Gamma(\frac{az}{a}))z}{a^3} - 2 \frac{\log(\Gamma(\frac{az}{a}))z}{a^3}\]

\[12 \frac{z \log(2)}{a^2} + \log(2) \left( -2 z^3 + 2 \frac{z^2}{a} + \frac{4z}{3a} \right) + 2 \frac{z^2 \log(\tau)}{a} - \frac{\log(\tau)z}{a^3}\]
11 Applications

First example

Consider and calculate the closed form
\[ \int_0^\infty x^2 \arctan(2x) \frac{1}{e^{4\pi x} + 1} \, dx \]

So we see \( a = 3 \) and \( z = 1/2 \)

We obtain
\[ \frac{35}{864} + \frac{\log(\Gamma(\frac{1}{4}))}{12} - \frac{\log(\pi)}{24} - \frac{2 \log(2)}{27} - \frac{\log(3)}{48} - \frac{K}{18\pi} - \frac{\log(A)}{36} + 4 \frac{\zeta(1,-2,1/4)}{27} \]

Where \( K \) is the Catalan’s constant \( (8) \)

Second example

Consider and calculate the closed form
\[ \int_0^\infty \arctan(3x) \cos(2 \arctan(3x)) \frac{1}{e^{4\pi x} + 1} \left( x^2 + \frac{1}{9} \right) \, dx \]

So we see \( z = 1/3 \) and \( a = 4 \)

We obtain
\[ -\frac{223}{7776} + \frac{\log(\Gamma(\frac{1}{4}))}{19} - \frac{\log(\pi)}{64} + \frac{25 \log(2)}{216} - \frac{11 \log(3)}{1296} - \frac{\pi \sqrt{3}}{432} + \frac{\psi(1,-2,\frac{1}{3})}{288\pi} \]

Or if you prefer,
\[ -\frac{223}{7776} + \frac{\log(\Gamma(\frac{1}{4}))}{19} - \frac{\log(\pi)}{64} + \frac{25 \log(2)}{216} - \frac{11 \log(3)}{1296} - \frac{\pi \sqrt{3}}{432} + \frac{\psi(1,-2,\frac{1}{3})}{288\pi} \]

Where \( \zeta(3) \) is the Apery’s constant \( (9) \) and I use the relation \( \zeta(1, -2) = -\frac{\zeta(3)}{4\pi^2} \)
And $\Psi \left( 1, \frac{1}{3} \right)$ is the trigamma function at $1/3$.  \hspace{1cm} (10)

**Third example**

Consider and calculate the closed form

$$\int_0^\infty \frac{1}{e^{\pi x} + 1} \log (x^2 + 25) \sin (2 \arctan (x/5)) \ (x^2 + 25) \, dx$$

So we see $z=5$ and $a=1/2$

We obtain

$$\frac{\frac{1135}{6} - 200 \log (\Gamma (1/4)) + 100 \log (\pi) + \frac{310 \log (2)}{3} + 80 \log (3) - 125 \log (5) + 40 \frac{\pi}{2} - 20 \log (A)}{8}$$

**Fourth example**

Consider and calculate the closed form

$$\int_0^\infty \frac{x^2}{\sinh (2 \pi x)} \arctan \left( \frac{x}{3} \right) \, dx$$

So we see $z=3$ and $a=2$

We obtain

$$\frac{3}{8} + \frac{9 \log (\pi)}{4} - \frac{19 \log (2)}{4} + \frac{9 \log (3)}{8} + \frac{\log (5)}{8} - \frac{9 \log (A)}{2} - \frac{7 \zeta (1, -2)}{8}$$

Or if you prefer

$$\frac{3}{8} + \frac{9 \log (\pi)}{4} - \frac{19 \log (2)}{4} + \frac{9 \log (3)}{8} + \frac{\log (5)}{8} - \frac{9 \log (A)}{2} + \frac{7 \zeta (3)}{32 \pi^2}$$

**Fifth example**

Consider and calculate the closed form

$$\int_0^\infty \frac{x^2 + 4}{\sinh (3 \pi x)} \arctan \left( \frac{x}{2} \right) \cos (2 \arctan (x/2)) \, dx$$

So we see $z=2$ and $a=3$

We obtain
\[-\frac{1}{9} - \frac{4 \log(\pi)}{3} + \frac{182 \log(2)}{27} - \log(3) - \frac{37 \log(5)}{27} + \frac{4 \log(A)}{3} + \frac{7 \zeta(1,-2)}{27}\]

Or if you prefer

\[-\frac{1}{9} - \frac{4 \log(\pi)}{3} + \frac{182 \log(2)}{27} - \log(3) - \frac{37 \log(5)}{27} + \frac{4 \log(A)}{3} - \frac{7 \zeta(3)}{108 \pi^2}\]

**Sixth example**

Consider and calculate the closed form

\[
\int_0^\infty \frac{1}{\sinh \left( \frac{x}{2} \right)} \log \left( x^2 + 9 \right) \sin \left( 2 \arctan \left( \frac{x}{3} \right) \right) \left( x^2 + 9 \right) dx
\]

So we see \( z=3 \) and \( a=1/2 \)

We obtain

\[144 \log \left( \Gamma \left( \frac{1}{4} \right) \right) - 72 \log(\pi) - 108 \log(2) - 48 \frac{K}{\pi}\]

## 12 References

(1): Denis Gallet, Corrections about V. S. Adamchik’s papers (2022)


(3),(4),(5) and (6): https://dlmf.nist.gov/5.17


(9): https://mathworld.wolfram.com/AperysConstant.html