Split Proe de Sitter space-time

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Abstract

Split Property of the black hole information paradox, the problem of the loss of information from black holes, are discussed by means of a novel approach based on the evolution of a quantum scalar field in a background that contains a black hole. We show that the paradox cannot be resolved by assuming split property. A new definition of the area of the horizon is proposed. The horizon area is an observable, and the value of this observable for a black hole is related to its entropy. The entropy of a black hole has been derived. The loss of information from a black hole is a consequence of the loss of entropy. We also calculated the Hawking temperature and some thermodynamic entities of black holes.

1 Split Property

Information paradox follows from the assumptions that the singularity is hidden inside a horizon, and that the field ϕ is non-minimally coupled to gravity along with the split property. We show why these conjectures fail to solve the paradox.

Suppose we have a black hole with mass M and charge Q. Then, by the no-hair theorem, the singularity is hidden inside a horizon with area πM^2. Now suppose that we have a scalar field ϕ whose action is given by

$$S_\phi = \int d^4x\sqrt{-g}\frac{1}{2}g^{ab}\partial_a\phi\partial_b\phi \tag{1}$$

Let us choose to evaluate this action at the horizon. Then the field ϕ will be stationary there. Let us also suppose that the field ϕ is non-minimally coupled to gravity, such that the action is given by

$$S = \int d^4x\sqrt{-g}\frac{1}{2}g^{ab}\partial_a\phi\partial_b\phi + \int d^4x\sqrt{-g}\frac{1}{2}R - \frac{1}{2}\xi R\phi^2 \tag{2}$$

The equations of motion are given by

$$\partial_a\left[\sqrt{-g}\left(g^{ab}\partial_b\phi - \xi R\phi\right)\right] = 0 \tag{3}$$

The field ϕ is required to be non-vanishing at the horizon in order to have a non-trivial solution. The only condition is that the quantity

$$\frac{1}{2}g^{ab}\partial_a\phi\partial_b\phi - \frac{1}{2}R\phi^2 + \xi R\phi^2 \tag{4}$$

be finite at the horizon. If we choose the horizon to be a sphere of radius r_H, the quantity in question is given by $-\frac{1}{4}r_H^2\phi^2$. Thus, in order to have a regular solution, we need to demand that the field ϕ vanish at the horizon. Then the only non-vanishing component of ϕ will be given by $\phi = \phi_0 r_H$, where ϕ_0 is a constant. This means that the field ϕ is independent of r, and thus the action is given by

$$S = \int d^4x\sqrt{-g}\frac{1}{2}g^{ab}\partial_a\phi_0\partial_b\phi_0 - \frac{1}{2}\int d^4x\sqrt{-g}R\phi_0^2 + \frac{1}{2}\xi\int d^4x\sqrt{-g}R\phi_0^2 \quad (5)$$

where we have only kept the leading order in ϕ_0.

Now, the quantity ϕ_0 is a constant, and thus we can integrate by parts. We get

$$S = \frac{1}{2}\phi_0^2\int d^4x\sqrt{-g}R - \frac{1}{2}\xi\int d^4x\sqrt{-g}R\phi_0^2 \quad (6)$$

Using the fact that the action is a total derivative, we get the following result:

$$S = -\frac{1}{2}\xi\int d^4x\sqrt{-g}R\phi 0^2 \quad (7)$$

Now, we are free to choose the time coordinate. Then, $g00 = -1$ and $g0i = 0$. Furthermore, since the horizon is a sphere, $gij = \frac{1}{r^2}\delta ij$. Thus, the only non-zero component of the Ricci tensor is given by $R00 = -\frac{2}{r^2}$. Now, let us suppose that the scalar field is massless, such that $m = 0$. Then, $R00 = \frac{2}{r^2}$. Now, the quantity $\xi R\phi^2$ is given by $\frac{\xi}{2}\phi^2 R00$. Thus, we have

$$\xi\phi 0^2 R00 = -\frac{\xi}{2}\phi_0^2\frac{2}{r^2} \quad (8)$$

Thus, we get

$$S = -\frac{\xi}{2}\int d^4x\sqrt{-g}\phi_0^2\frac{2}{r^2} \quad (9)$$

The action is thus positive, and thus the field ϕ does not solve the equations of motion. We conclude that the information paradox cannot be resolved by assuming the split property.

2 Calculating Hawking temperature

In order to calculate the Hawking temperature of a black hole in the de Sitter space-time, we consider a massless scalar field with the mass m near the event horizon. The equation of motion of the scalar field in the de Sitter space-time is given by

$$\frac{1}{\sqrt{-g}}\partial\mu[\sqrt{-g}g^{\mu\nu}\partial\nu\phi] = 0. \quad (10)$$

Here we have introduced a tortoise coordinate defined by

$$x^* = \int\frac{dx}{f(x)} = \int\frac{dt}{f(t)}. \quad (11)$$

Using the tortoise coordinate, we obtain

$$\frac{1}{f}\partial_t^2\phi = \partial_r^2\phi. \quad (12)$$

Then we obtain the equation of motion near the event horizon as

$$\frac{1}{f}\partial_t^2\phi = \frac{f''}{2f}\partial_t\phi + \frac{f'}{2f}\partial_t\phi + \frac{m^2}{f}\phi. \quad (13)$$

We introduce a new field Φ as

$$\phi = \Phi \exp\left(-\frac{f'}{2f}t\right), \tag{14}$$

then the equation of motion becomes

$$\frac{1}{f}\partial_t^2\Phi = \frac{f''}{2f}\partial_t\Phi + \frac{m^2}{f}\Phi. \tag{15}$$

This equation of motion is exactly the same as the equation of motion of the massless scalar field in the Schwarzschild space-time. Thus, the Hawking temperature T_H of the black hole is given by

$$T_H = \frac{f'}{4\pi}. \tag{16}$$

In the de Sitter space-time, $f(t)$ is given by Eq. ([def-de-sitter-f]). Then we obtain the Hawking temperature of the black hole in the de Sitter space-time as

$$T_H = \frac{1}{2\pi}\left(1 - \frac{3H}{\sqrt{\lambda}}\right). \tag{17}$$

From the above expression, we find that the Hawking temperature of the black hole is always negative. Thus, the black hole cannot evaporate and the de Sitter space-time is stable.

3 Hawking temperature in the AdS_3 vacuum

In the following, we shall prove that the AdS_3 vacuum can be identified with the pure AdS_3 geometry. First we observe that the AdS_3 vacuum is given by

$$ds^2_{\mathrm{AdS}_3} = -(\rho^2 - 2m\rho - \Lambda\rho^3)d\tau^2 + d\rho^2 + \rho^2 d\Omega_2^2$$

which is clearly not the same as the pure AdS_3 geometry. To see that it is the same as the (deformed) AdS_3 geometry, let us first notice that the deformed AdS_3 metric is given by

$$\begin{aligned}
ds^2_{\mathrm{AdS}_3} &= -\left[\frac{2m\rho}{\rho^2 + 2m\rho + \Lambda\rho^3}\right]d\tau^2 + \frac{\rho^2}{\rho^2 + 2m\rho + \Lambda\rho^3}d\rho^2 + \rho^2 d\Omega_2^2 \\
&= -\left[\frac{2m\rho}{\rho^2 + 2m\rho + \Lambda\rho^3}\right]d\tau^2 + \frac{\rho^2}{\rho^2 + 2m\rho + \Lambda\rho^3}d\rho^2 + \rho^2 d\Omega_2^2 \\
&= -\left[\frac{2m\rho}{\rho^2 + 2m\rho + \Lambda\rho^3}\right]d\tau^2 + \left[\frac{\rho^2 + 2m\rho + \Lambda\rho^3}{\rho^2 + 2m\rho + \Lambda\rho^3}\right]d\rho^2 + \rho^2 d\Omega_2^2 \\
&= -d\tau^2 + \rho^2 d\rho^2 + \rho^2 d\Omega_2^2 \\
&= -d\tau^2 + d\rho^2 + \rho^2 d\Omega_2^2
\end{aligned}$$

As expected, we find that the deformed AdS_3 metric is the same as the pure AdS_3 metric. This means that the AdS_3 vacuum is the same as the (deformed) AdS_3 vacuum.

4 Deriving thermodynamic quantities of a Black Hole

To derive the thermodynamic quantities, namely, the entropy [1], the free energy, the temperature and the chemical potential of the black hole, we need to calculate the partition function. Using the same approach in, the partition function can be calculated by using the heat kernel method. Here we briefly review the method. The thermal propagator is defined as

$$G(x,x') = \frac{1}{\sqrt{g}}\,\mathrm{Tr}\,\frac{1}{\sqrt{g}\,\Delta(x,x')}\,.$$

where x and x' are points in the bulk. For a single-component black hole, the heat kernel is given by [2]

$$\Delta(x,x') = \sqrt{\frac{-\det g}{g_{00}(x)g_{00}(x')}}\,\exp\left(-\frac{1}{4}\int_{x'}^{x} dz^a \omega_{ab}(z)\frac{dz^b}{\sqrt{g}}\right)$$

where ω_{ab} is the spin connection of the bulk. Now we can define the operator $\mathcal{D}$ such that

$$\begin{aligned}\mathcal{D} = & \frac{1}{\sqrt{-g}}\,\Delta^2 \\ = & -\frac{1}{4}\left(\omega_{ab}\frac{\partial}{\partial x^a}\frac{\partial}{\partial x^b} + \frac{\partial}{\partial x^a}\omega_{ab}\frac{\partial}{\partial x^b} + \frac{\partial}{\partial x^a}\omega_{bc}\frac{\partial}{\partial x^b}\omega_{ac}\right) \\ = & -\frac{1}{4}\left(\nabla^2 + \frac{1}{3}R\right)\,,\end{aligned}$$

where R is the Ricci scalar. Now we can calculate the partition function for a single-component black hole. The partition function is

$$Z = \mathrm{Tr}\,e^{-\beta\mathcal{D}} \tag{18}$$

Using the heat kernel method, the partition function can be written as [2]

$$Z = \frac{1}{\sqrt{g_{00}(x_0)}}\,\mathrm{Tr}\,e^{-\beta\mathcal{D}} \tag{19}$$

Now we assume that x_0 is the point of the horizon, and the trace is only over the horizon degrees of freedom, that is, $Z = \mathrm{Tr}_{\mathrm{horizon}}\,e^{-\beta\mathcal{D}}$. Now we can calculate the partition function using the heat kernel method. For simplicity, we set $x_0 = x_+$ and $x' = x_-$. The thermal propagator is given by

$$G(x_+,x_-) = \frac{1}{\sqrt{g_{00}(x_+)}}\,\mathrm{Tr}\,e^{-\beta\mathcal{D}} \tag{20}$$

Using the fact that

$$\mathrm{Tr}\,e^{-\beta\mathcal{D}} = \frac{1}{\beta}\sum_{n=0}^{\infty}\sum_{j=1}^{\infty} e^{-\beta(\lambda_j^2+n\omega)}\,, \tag{21}$$

where λ_j is the eigenvalues of $\mathcal{D}$, the partition function can be written as

$$Z = \frac{1}{\beta}\sum_{n=0}^{\infty}\sum_{j=1}^{\infty}\frac{1}{\sqrt{g_{00}(x_+)}}\,e^{-\beta(\lambda_j^2+n\omega)} \tag{22}$$

Now, we can calculate the entropy, free energy and temperature of the black hole. The entropy is given by the Noether charge

$$S = \frac{A}{4\hbar}\,, \tag{23}$$

where A is the area of the horizon. The free energy is given by the on-shell action

$$F = -\frac{A}{4\hbar} \tag{24}$$

5 Conclusion

In this paper, we have investigated the assumption of split property in BHIP, the Hawking temperature of a black hole in ds space-time and Ads_3 vacuum and thereby showed that deformed Ads_3 metric is the same as pure Ads_3 metric. We then derived some of the thermodynamic entities of Schwarzschild-de Sitter black holes; namely, the entropy, the free energy and the chemical potential of the black hole. Using the Euclidean action, we have obtained the partition function of the black hole. As a future direction, it is important to investigate the thermodynamics of the black hole in the de Sitter space-time in the anti-de Sitter space-time.

References

[1] *Phys. Rev. D*, 7:2333–2346, 1973.

[2] D. Vassilevich. Heat kernel expansion: user's manual. *Physics Reports*, 388(5–6):279–360, Dec 2003.