A Method to Prove a Prime Number between $3N$ and $4N$

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Abstract

In this paper, we will prove that when an integer $n > 1$, there exists a prime number between $3n$ and $4n$. This is another step in the expansion of the Bertrand’s postulate - Chebyshev’s theorem after the proof of a prime number between $2n$ and $3n$.

Introduction

The Bertrand’s postulate - Chebyshev’s theorem states that for any positive integer $n$, there is always a prime number $p$ such that $n < p < 2n$. It was proved by Pafnuty Chebyshev in 1850 [1]. In 2006, M. El Bachraoui [2] expanded the theorem by proving that for any positive integer $n$, there is a prime number $p$ such that $2n < p < 3n$. In 2011, Andy Loo [3] expanded the theorem to that when $n \geq 2$, there exists a prime number in the interval $(3n, 4n)$. Recently, the author used a different method [4] to prove that a prime number exists between $2n$ and $3n$ by analyzing the binomial coefficient $\binom{3n}{n}$. In this paper, we will use the similar way to prove that a prime number exists between $3n$ and $4n$ by analyzing the binomial coefficient $\binom{4n}{n}$.

Definition: \( \Gamma_{a \geq p > b}\left(\frac{4n}{n}\right) \) denotes the prime factorization operator of $\binom{4n}{n}$. It is the product of the prime numbers in the decomposition of $\binom{4n}{n}$ in the range of $a \geq p > b$. In this operator, $p$ is a prime number, $a$ and $b$ are real numbers, and $4n \geq a \geq p > b \geq 1$.

It has some properties:

It is always true that $\Gamma_{a \geq p > b}\left(\frac{4n}{n}\right) \geq 1$ — (1)

If there is no prime number in $\Gamma_{a \geq p > b}\left(\frac{4n}{n}\right)$, then $\Gamma_{a \geq p > b}\left(\frac{4n}{n}\right) = 1$, or vice versa, if $\Gamma_{a \geq p > b}\left(\frac{4n}{n}\right) = 1$, then there is no prime number in $\Gamma_{a \geq p > b}\left(\frac{4n}{n}\right)$ — (2)

For example, $\Gamma_{12 \geq p > 8}\left(\frac{16}{4}\right) = 11^0 = 1$. No prime number is in $\binom{16}{4}$ in the range of $12 \geq p > 8$.

If there is at least one prime number in $\Gamma_{a \geq p > b}\left(\frac{4n}{n}\right)$, then $\Gamma_{a \geq p > b}\left(\frac{4n}{n}\right) > 1$, or vice versa, if $\Gamma_{a \geq p > b}\left(\frac{4n}{n}\right) > 1$, then there is at least one prime number in $\Gamma_{a \geq p > b}\left(\frac{4n}{n}\right)$ — (3)

For example, $\Gamma_{8 \geq p > 4}\left(\frac{16}{4}\right) = 5 > 1$. Prime number 5 is in $\binom{16}{4}$ in the range of $8 \geq p > 4$.

Let $v_p(n)$ be the $p$-adic valuation of $n$, the exponent of the highest power of $p$ that divides $n$. We define $R(p)$ by the inequalities $p^{R(p)} \leq 4n < p^{R(p)+1}$, and determine the $p$-adic valuation of $\binom{4n}{n}$.
For every integer \( n > 1 \), there exists at least a prime number \( p \) such that \( 3n < p \leq 4n \).

**Proof:**

By induction on \( n \), for \( n = 2 \), \( \binom{4n}{n} = \binom{8}{2} = 28 > \frac{4^{n-3}}{n \cdot 3^{n-3}} = \frac{512}{27} \approx 18.96 \)

If \( \binom{4n}{n} > \frac{4^{n-3}}{n \cdot 3^{n-3}} \) for \( n \) stands, then for \( n + 1 \),

\[
\binom{4(n+1)}{n+1} = \frac{4(n+4)(4n+3)(4n+2)(4n+1)}{(n+1)(3n+3)(3n+2)(3n+1)} \cdot \binom{4n}{n}
\]

\[
> \frac{4(n+4)(4n+3)(4n+2)(4n+1)}{(n+1)(3n+3)(3n+2)(3n+1)} \cdot \frac{4^{n-3}}{n \cdot 3^{n-3}} = \frac{4^{n+3}}{3n+3} \cdot \frac{4^{n+3}}{3n+3} \cdot \frac{4}{3n+3} \cdot \frac{4}{3n+3} \cdot \frac{4}{3n+3} \cdot \frac{4}{3n+3} = \frac{4^{n+3}}{(n+1) \cdot 3^{n+1}}
\]

Thus for \( n \geq 2 \), \( \binom{4n}{n} > \frac{4^{n-3}}{n \cdot 3^{n-3}} \)

Applying (7) into (6):

For \( n \geq 3 \),

\[
\frac{4^{n-3}}{n \cdot 3^{n-3}} < \binom{4n}{n} < \Gamma_{\geq 2 \sqrt{n}} \frac{(4n)!}{n! \cdot (3n)!} \cdot 2^{2n-3} \cdot \Gamma_{\geq 2 \sqrt{n}} \frac{(4n)!}{n! \cdot (3n)!}
\]

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**Proposition**

Thus, if \( p \) divides \( \binom{4n}{n} \), then \( \binom{4n}{n} \leq R(p) \leq \log_p(4n) \), or \( p^{\nu_p(\binom{4n}{n})} \leq p^{R(p)} \leq 4n \)  

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**Referring to (5),** \( \Gamma_{\geq 2 \sqrt{n}} \frac{(4n)!}{n! \cdot (3n)!} \leq \prod_{n \geq p} p \).

It has been proved [5] that \( \prod_{n \geq p} p < 2^{2n-3} \) when \( n \geq 3 \).

Thus for \( n \geq 3 \),

\[
\binom{4n}{n} < \Gamma_{\geq 2 \sqrt{n}} \frac{(4n)!}{n! \cdot (3n)!} \cdot 2^{2n-3} \cdot \Gamma_{\geq 2 \sqrt{n}} \frac{(4n)!}{n! \cdot (3n)!}
\]  

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Let \( \pi(n) \) be the number of distinct prime numbers less than or equal to \( n \). Among the first six consecutive natural numbers are three prime numbers 2, 3 and 5. Then, for each additional six consecutive natural numbers, at most one can add two prime numbers, \( p \equiv 1 \pmod{6} \) and \( p \equiv 5 \pmod{6} \). Thus, \( \pi(n) \leq \left\lfloor \frac{n}{3} \right\rfloor + 2 \leq \frac{n}{3} + 2 \). — (9)

Referring to (4) and (9),

\[
\Gamma_{[2\sqrt{n}]} \geq p \left\{ \left( \frac{4n!}{n!(3n)!} \right) \right\} = \Gamma_{[2\sqrt{n}]} \geq p \left\{ \left( \frac{4n}{n} \right) \right\} \leq (4n)^{\pi(2\sqrt{n})} \leq (4n)^{\frac{2\sqrt{n}+2}{3}}
\]

Applying (10) into (8): \[
\frac{4^{4n^3}}{n(3n-3)^{2n^3}} < \Gamma_{[2\sqrt{n}]} \geq p \left\{ \left( \frac{4n!}{n!(3n)!} \right) \right\} \cdot 2^{2n^3} \cdot (4n)^{\frac{2\sqrt{n}+2}{3}}
\]

Since for \( n \geq 3 \), both \( 2^{2n^3} > 0 \) and \( (4n)^{\frac{2\sqrt{n}+2}{3}} > 0 \)

\[
\Gamma_{[2\sqrt{n}]} \geq p \left\{ \left( \frac{4n!}{n!(3n)!} \right) \right\} > \frac{4^{4n^3}}{n(3n-3)^{2n^3}} \frac{2^{2n^3}}{(4n)^{\frac{2\sqrt{n}+2}{3}}} = \frac{27}{2} \left( \frac{4}{3} \right)^{3n} \frac{4}{3} \frac{3^n}{(4n)^{\frac{2\sqrt{n}+2}{3}}}
\]

Let \( f(x) = \frac{w}{u} \) where \( x, u, w \) are real numbers and \( x \geq 42 \), \( u = \frac{27}{2} \left( \frac{4}{3} \right)^{3x} \), \( w = (4x)^{\frac{2\sqrt{x}+9}{3}} \)

\[
\frac{du}{dx} = \left( \frac{27}{2} \cdot \left( \frac{4}{3} \right)^{3x} \right) = \frac{27}{2} \left( \frac{4}{3} \right)^{3x} \cdot 3 \cdot \ln \left( \frac{4}{3} \right) = u \cdot 3 \cdot \ln \left( \frac{4}{3} \right)
\]

\[
\frac{dw}{dx} = \left( (4x)^{\frac{2\sqrt{x}+9}{3}} \right) = \left( (4x)^{\frac{2\sqrt{x}+9}{3}} \right) \left( \frac{\ln(4x)}{3\sqrt{x}} + \frac{2\sqrt{x}+9}{3x} \right) = w \left( \frac{\ln(x)+\ln(4)+2}{3\sqrt{x}} + \frac{3}{x} \right)
\]

\[
f'(x) = \left( \frac{w}{u} \right)' = \frac{w(u)' - u(w)'}{u^2} = \frac{w}{u} \left( 3 \cdot \ln \left( \frac{4}{3} \right) - \frac{\ln(x)+\ln(4)+2}{3\sqrt{x}} - \frac{3}{x} \right)
\]

Let \( f_1(x) = 3 \cdot \ln \left( \frac{4}{3} \right) - \frac{\ln(x)+\ln(4)+2}{3\sqrt{x}} - \frac{3}{x} \)

Since \( f_1'(x) = \frac{\ln(x)+\ln(4)+2}{6x\sqrt{x}} + \frac{3}{x^2} > 0 \), when \( x > 1 \), \( f_1(x) \) is a strictly increasing function.

When \( x = 42 \), \( f_1(x) = 3 \cdot \ln \left( \frac{4}{3} \right) - \frac{\ln(x)+\ln(4)+2}{3\sqrt{x}} = 0.863 - 0.367 - 0.071 = 0.425 > 0 \).

Thus, when \( x \geq 42 \), \( f_1(x) > 0 \).

Since when \( x \geq 42 \), \( u, w, f_1(x) \) are greater than zero, \( f'(x) = \frac{w}{u} \cdot f_1(x) > 0 \).

Thus \( f(x) \) is a strictly increasing function for \( x \geq 42 \). Then when \( x \geq 42 \), \( f(x+1) > f(x) \).

Let \( x = n \geq 42 \), then \( f(n+1) > f(n) = \frac{27}{2} \cdot \left( \frac{4}{3} \right)^{3n} \frac{3^n}{(4n)^{\frac{2\sqrt{n}+2}{3}}} \)

Since for \( n = 42 \), \( f(n) = \frac{27}{2} \cdot \left( \frac{4}{3} \right)^{126} \frac{3^{126}}{168} \approx 7.457E+16 \cdot 1.952E+16 > 1 \), and since

\[
f(n+1) > f(n), \text{ by induction on } n, \text{ when } n \geq 42, f(n) = \frac{27}{2} \cdot \left( \frac{4}{3} \right)^{3n} \frac{3^n}{(4n)^{\frac{2\sqrt{n}+2}{3}}} > 1. \quad (12)
\]
Applying (12) to (11): When \( n \geq 42 \), \( \Gamma_{4n \geq p > n} \left\{ \frac{(4n)!}{(3n)!} \right\} > \frac{27}{2} \cdot \frac{\left( \frac{4}{3} \right)^{3n}}{2^{\sqrt{4n+9}} \cdot (4n)} > 1. \)

Thus when \( n \geq 42 \),
\[
\Gamma_{4n \geq p > n} \left\{ \frac{(4n)!}{(3n)!} \right\} \\
= \Gamma_{4n \geq p > 3n} \left\{ \frac{(4n)!}{(3n)!} \right\} \cdot \Gamma_{3n \geq p > 2n} \left\{ \frac{(4n)!}{(3n)!} \right\} \cdot \Gamma_{2n \geq p > 3n} \left\{ \frac{(4n)!}{(3n)!} \right\} \cdot \Gamma_{n \geq p > 4n} \left\{ \frac{(4n)!}{(3n)!} \right\} \cdot \Gamma_{4n \geq p > n} \left\{ \frac{(4n)!}{(3n)!} \right\} > 1.
\]

If there is any prime number \( p \) such that \( 3n \geq p > 2n \), then \( (4n)! \) has a factor of \( p \) in this range, and \( (3n)! \) also has the same factor of \( p \). Thus, they cancel to each other in \( \frac{(4n)!}{(3n)!} \) with no prime number in this range. Referring to (2), \( \Gamma_{3n \geq p > 2n} \left\{ \frac{(4n)!}{(3n)!} \right\} = 1. \)

If there is any prime number \( p \) such that \( \frac{3n}{2} \geq p > \frac{4n}{3} \), then \( (4n)! \) has the product of \( p \cdot 2p \), and \( (3n)! \) also has the same product of \( p \cdot 2p \). Thus, they cancel to each other in \( \frac{(4n)!}{(3n)!} \) with no prime number in this range. Referring to (2), \( \Gamma_{\frac{3n}{2} \geq p > \frac{4n}{3}} \left\{ \frac{(4n)!}{(3n)!} \right\} = 1. \)

Thus, when \( n \geq 42 \),
\[
\Gamma_{4n \geq p > n} \left\{ \frac{(4n)!}{(3n)!} \right\} = \Gamma_{4n \geq p > 3n} \left\{ \frac{(4n)!}{(3n)!} \right\} \cdot \Gamma_{2n \geq p > 3n} \left\{ \frac{(4n)!}{(3n)!} \right\} \cdot \Gamma_{n \geq p > 4n} \left\{ \frac{(4n)!}{(3n)!} \right\} > 1. \quad (13)
\]

Referring to (1), \( \Gamma_{4n \geq p > 3n} \left\{ \frac{(4n)!}{(3n)!} \right\} \geq 1, \Gamma_{2n \geq p > 3n} \left\{ \frac{(4n)!}{(3n)!} \right\} \geq 1, \) and \( \Gamma_{n \geq p > 4n} \left\{ \frac{(4n)!}{(3n)!} \right\} \geq 1. \)

If \( \Gamma_{\frac{2n}{2} \geq p > \frac{3n}{2}} \left\{ \frac{(4n)!}{(3n)!} \right\} = 1 \) or \( \Gamma_{\frac{4n}{3} \geq p > n} \left\{ \frac{(4n)!}{(3n)!} \right\} = 1, \) it will drop out from (13).

If \( n \geq 42 \) and \( \Gamma_{4n \geq p > 3n} \left\{ \frac{(4n)!}{(3n)!} \right\} > 1, \) then referring to (3), there exists at least a prime number \( p \) such that \( 3n < p \leq 4n. \)
\[
\Gamma_{\frac{2n}{2} \geq p > \frac{3n}{2}} \left\{ \frac{(4n)!}{(3n)!} \right\} = \Gamma_{4 \cdot \frac{n}{2} \geq p > 3 \cdot \frac{n}{2}} \left\{ \frac{(4n)!}{(3n)!} \right\}.
\]

If \( \frac{n}{2} \geq 21 \) and, \( \Gamma_{4 \cdot \frac{n}{2} \geq p > 3 \cdot \frac{n}{2}} \left\{ \frac{(4n)!}{(3n)!} \right\} > 1, \) let \( m_1 = \frac{n}{2} \), then when \( m_1 \geq 21, \) there exists at least a prime number \( p \) such that \( 3m_1 < p \leq 4m_1. \) Since \( n \geq 42 > m_1 \geq 21, \) the statement is also valid for \( n. \) Thus, when \( n \geq 42, \) if \( \Gamma_{4n \geq p > 3n} \left\{ \frac{(4n)!}{(3n)!} \right\} > 1, \) then \( \Gamma_{4n \geq p > 3n} \left\{ \frac{(4n)!}{(3n)!} \right\} > 1, \) there exists at least a prime number \( p \) such that \( 3n < p \leq 4n. \)
\[
\Gamma_{\frac{4n}{3} \geq p > n} \left\{ \frac{(4n)!}{(3n)!} \right\} = \Gamma_{4 \cdot \frac{n}{3} \geq p > 3 \cdot \frac{n}{3}} \left\{ \frac{(4n)!}{(3n)!} \right\}.
\]
If \( \frac{n}{3} \geq 14 \) and, \( \Gamma_{4n \geq p > 3n} \{ \frac{(4n)!}{(3n)!} \} > 1 \), let \( m_2 = \frac{n}{3} \), then when \( m_2 \geq 14 \), there exists at least a prime number \( p \) such that \( 3m_2 < p \leq 4m_2 \). Since \( n \geq 42 > m_2 \geq 14 \), the statement is also valid for \( n \). Thus, when \( n \geq 42 \), if \( \Gamma_{4n \geq p > 3n} \{ \frac{(4n)!}{(3n)!} \} > 1 \), then \( \Gamma_{4n \geq p > 3n} \{ \frac{(4n)!}{(3n)!} \} > 1 \), there exists at least a prime number \( p \) such that \( 3n < p \leq 4n \). — (16)

From the right side of (13), at least one of these 3 factors is greater than one when \( n \geq 42 \). From (14), (15), and (16), when \( n \geq 42 \) and any one of these 3 factors is greater than one, there exists at least a prime number \( p \) such that \( 3n < p \leq 4n \). — (17)

Table 1 shows that when \( 2 \leq n \leq 42 \), there is a prime number \( p \) such that \( 3n < p \leq 4n \). — (18)

Thus, the proposition is proven by combining (17) and (18): For every integer \( n > 1 \), there exists at least a prime number \( p \) such that \( 3n < p \leq 4n \). — (19)

Table 1: For \( 2 \leq n \leq 42 \), there is a prime number \( p \) such that \( 3n < p \leq 4n \).

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References