The Proofs of Legendre’s Conjecture and Related Conjectures

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Abstracts
In this paper, we are going to prove Legendre’s Conjecture: There is a prime number between $n^2$ and $(n + 1)^2$ for every positive integer $n$. We will also prove several related conjectures. The method that we use is to analyze a binomial coefficient. It has been developed from the method of analyzing a central binomial coefficient that was used by Paul Erdős to prove Bertrand’s postulate / Chebyshev’s theorem.

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1. Introduction

Legendre’s Conjecture was proposed by Andrien-Marie Legendre [1]. The conjecture is one of Legendre’s problems (1912) on prime numbers. It states that there is a prime number between \( n^2 \) and \((n + 1)^2\) for every positive integer \( n \).

In this paper, we will prove Legendre’s Conjecture by analyzing the binomial coefficient \( \binom{\lambda n}{n} \) where \( \lambda \) is a positive integer. It is developed from the method that was used by Paul Erdős [2] to prove Bertrand’s postulate / Chebyshev’s theorem [3].

In Section 1, we will define the prime decomposition operator and clarify some terms and concepts. In Section 2, we will derive some lemmas. In Section 3, we will develop a theorem to be used in the proofs of the conjectures in the later sections. In Section 4, we will prove Legendre’s conjecture, and in Section 5, we will prove Oppermann’s conjecture [4], Brocard’s conjecture [5], and Andrica’s conjecture [6].

**Definition:** \( \Gamma_{azp>b}(n) \) denotes the prime number decomposition operator. It is the product of the prime numbers in the decomposition of a positive integer \( n \) or a positive integer expression. In this operator, \( p \) is a prime number, \( a \) and \( b \) are real numbers, and \( n \geq a \geq p > b \geq 1 \).

It has some properties:

It is always true that \( \Gamma_{azp>b}(n) \geq 1 \). — (1.1)

If no prime number in \( \Gamma_{azp>b}(n) \), then \( \Gamma_{azp>b}(n) = 1 \), or vice versa, if \( \Gamma_{azp>b}(n) = 1 \), then no prime number in \( \Gamma_{azp>b}(n) \) as in \( \Gamma_{12z=4}(12) = 11^0 \cdot 7^0 \cdot 5^0 = 1 \). — (1.2)

If there is at least one prime number in \( \Gamma_{azp>b}(n) \), then \( \Gamma_{azp>b}(n) > 1 \), or vice versa, if \( \Gamma_{azp>b}(n) > 1 \), then there is at least one prime number in \( \Gamma_{azp>b}(n) \) as in \( \Gamma_{12z=2}(12) = 3 > 1 \). — (1.3)

Similar to Paul Erdős’ paper [2], we define \( R(p) \) by the inequalities \( p^{R(p)} \leq \lambda n < p^{R(p)+1} \), and determine the \( p \)-adic valuation of \( \binom{\lambda n}{n} \).

\[
\nu_p \left( \binom{\lambda n}{n} \right) = \nu_p ((\lambda n)! - \nu_p (((\lambda - 1)n)! - \nu_p (n!)) = \sum_{i=1}^{R(p)} \left( \left\lfloor \frac{\lambda n}{p^i} \right\rfloor - \left\lfloor \frac{(\lambda - 1)n}{p^i} \right\rfloor - \left\lfloor \frac{n}{p^i} \right\rfloor \right) \leq R(p)
\]

because for any real numbers \( a \) and \( b \), the expression of \( |a + b| - |a| - |b| \) is 0 or 1.

Thus, if \( p \) divides \( \binom{\lambda n}{n} \), then \( \nu_p \left( \binom{\lambda n}{n} \right) \leq R(p) \leq \log_p(\lambda n) \), or \( p^0 \left( \binom{\lambda n}{n} \right) \leq R(p) \leq \lambda n \) — (1.4)

And if \( \lambda n > p \left\lfloor \sqrt{\lambda n} \right\rfloor \), then \( 0 \leq \nu_p \left( \binom{\lambda n}{n} \right) \leq R(p) \leq 1 \) — (1.5)

Let \( \pi(n) \) be the number of distinct prime numbers less than or equal to \( n \). For the first six sequential natural numbers, there are three prime numbers 2, 3, and 5. For counting any
successive set of six sequential natural numbers, there are at most two prime numbers added, $p \equiv 1 \text{ (MOD 6)}$ and $p \equiv 5 \text{ (MOD 6)}$. Thus, $\pi(n) \leq \left\lceil \frac{n}{3} \right\rceil + 2 \leq \frac{n}{3} + 2$. Since some of $n \equiv 1 \text{ (MOD 6)}$ and $n \equiv 5 \text{ (MOD 6)}$ are not prime numbers, as the number counts increase, $\pi(n)$ reduces from $\left\lceil \frac{n}{3} \right\rceil + 2$.

For $n \geq 24$, $\pi(n) \leq \left\lceil \frac{n}{3} \right\rceil + 1 \leq \frac{n}{3} + 1$.

From the prime number decomposition,

- when $n > \sqrt[\lambda n]{\lambda}$, $\left(\frac{\lambda n}{n}\right) = \Gamma_{n \geq p > n} \left\{ \frac{(\lambda n)!}{n! - (\lambda n)!} \right\} \cdot \Gamma_{n \geq p > \sqrt[n]{\lambda n}} \left\{ \frac{(\lambda n)!}{n! - (\lambda n)!} \right\} \cdot \Gamma_{n \geq p > n} \left\{ \frac{(\lambda n)!}{n! - (\lambda n)!} \right\}$
- when $n \leq \sqrt[\lambda n]{\lambda}$, $\left(\frac{\lambda n}{n}\right) \leq \Gamma_{n \geq p > n} \left\{ \frac{(\lambda n)!}{n! - (\lambda n)!} \right\} \cdot \Gamma_{n \geq p > \sqrt[n]{\lambda n}} \left\{ \frac{(\lambda n)!}{n! - (\lambda n)!} \right\} \cdot \Gamma_{n \geq p > n} \left\{ \frac{(\lambda n)!}{n! - (\lambda n)!} \right\}$

$\Gamma_{n \geq p > n} \left\{ \frac{(\lambda n)!}{n! - (\lambda n)!} \right\} = \Gamma_{n \geq p > n} \left\{ \frac{(\lambda n)!}{(\lambda n)!} \right\}$ since all prime numbers in $n!$ do not appear in the range of $\lambda n \geq p > n$.

Referring to (1.5), $\Gamma_{n \geq p > \sqrt[n]{\lambda n}} \left\{ \frac{(\lambda n)!}{n! - (\lambda n)!} \right\} \leq \Pi_{n \geq p > n}$. It has been proved [7] that for $n \geq 3$, $\Pi_{n \geq p} \leq 2^{2n-3}$. Thus, $\Gamma_{n \geq p > \sqrt[n]{\lambda n}} \left\{ \frac{(\lambda n)!}{n! - (\lambda n)!} \right\} \leq \Pi_{n \geq p} \leq 2^{2n-3}$.

Referred to (1.4) and (1.6), $\Gamma_{n \geq p > \sqrt[n]{\lambda n}} \left\{ \frac{(\lambda n)!}{n! - (\lambda n)!} \right\} \leq (\lambda n)^{\frac{\sqrt[\lambda n]{\lambda n}}{3}} + 1$ when $\sqrt[n]{\lambda n} \geq 24$.

Thus, for $n \geq 3$ and $\sqrt[n]{\lambda n} \geq 24$, $\left(\frac{\lambda n}{n}\right) \leq \Gamma_{n \geq p > n} \left\{ \frac{(\lambda n)!}{(\lambda n)!} \right\} \cdot 2^{2n-3} \cdot (\lambda n)^{\frac{\sqrt[\lambda n]{\lambda n}}{3}} + 1$ — (1.7)

2. Lemmas

Lemma 1: If a real number $x \geq 3$, then $\frac{2(2x-1)}{x-1} > \left(\frac{x}{x-1}\right)^x$ — (2.1)

Proof:

Let $f_1(x) = \frac{2(2x-1)}{x-1}$, then $f_1'(x) = \frac{2(x-1)(2x-1)' - 2(2x-1)(x-1)'}{(x-1)^2} = \frac{-2}{(x-1)^2} < 0$.

Thus $f_1(x)$ is a strictly decreasing function for $x > 1$.

Since $f_1(3) = 5$ and $\lim_{x \to \infty} f_1(x) = 4$,

thus, $5 \geq f_1(x) = \frac{2(2x-1)}{x-1} \geq 4$ for $x \geq 3$ — (2.1.1)

Let $f_2(x) = \left(\frac{x}{x-1}\right)^x$, then $f_2'(x) = \left(\frac{x}{x-1}\right)^x \cdot \left(\frac{x}{x-1}\right)' = e^{x \cdot \ln \frac{x}{x-1}} \cdot \left(\frac{x}{x-1}\right)'$
\[ f'_2(x) = \left(\frac{x}{x-1}\right)^x \left(\ln \frac{x}{x-1} + x \cdot \left(\ln \frac{x}{x-1}\right)\right) = \left(\frac{x}{x-1}\right)^x \cdot \left(\ln \frac{x}{x-1} + x \cdot \frac{x-1}{x} \cdot \frac{x-1-x}{(x-1)^2}\right) \]

\[ f'_2(x) = \left(\frac{x}{x-1}\right)^x \cdot \left(\ln \frac{x}{x-1} - \frac{1}{x-1}\right) \quad \text{(2.1.2)} \]

\[ \ln \left(\frac{x}{x-1}\right) = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \frac{1}{x^5} + \frac{1}{x^6} + \cdots \quad \text{(2.1.3)} \]

Using the formula: \(\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \cdots\), we have

\[ \ln \frac{x}{x-1} = \ln \frac{1}{1+\frac{1}{x}} = -\ln \left(1 + \frac{1}{x}\right) = \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{3x^3} + \frac{1}{4x^4} + \frac{1}{5x^5} + \frac{1}{6x^6} + \cdots \quad \text{(2.1.4)} \]

Thus for \(x \geq 3\), \(\ln \frac{x}{x-1} - \frac{1}{x-1} < 0\) \hspace{1cm} \text{(2.1.5)}

Since \(\left(\frac{x}{x-1}\right)^x\) is a positive number for \(x \geq 3\), \(f'_2(x) = \left(\frac{x}{x-1}\right)^x \cdot \left(\ln \frac{x}{x-1} - \frac{1}{x-1}\right) < 0\). \hspace{1cm} \text{(2.1.6)}

Thus \(f_2(x)\) is a strictly decreasing function for \(x \geq 3\).

Since \(f_2(3) = 3.375\) and \(\lim_{x \to \infty} f_2(x) = e \approx 2.718\),

thus for \(x \geq 3\), \(3.375 \geq f_2(x) = \left(\frac{x}{x-1}\right)^x \geq e \approx 2.718\) \hspace{1cm} \text{(2.1.7)}

Since for \(x \geq 3\), \(f_1(x)\) has a lower bound of 4 and \(f_2(x)\) has an upper bound of 3.375,

\[ f_1(x) = \frac{2(2x-1)}{x-1} > f_2(x) = \left(\frac{x}{x-1}\right)^x \quad \text{is proven.} \] \hspace{1cm} \text{(2.1.8)}

**Lemma 2**: For \(n \geq 2\) and \(\lambda \geq 3\), \(\left(\frac{\lambda^n}{n}\right) > \frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)(\lambda-1)n-\lambda+1} \quad \text{(2.2)}\)

**Proof**:

When \(\lambda \geq 3\) and \(n = 2\),

\[ \left(\frac{\lambda^n}{n}\right) = \left(\frac{2\lambda}{2}\right) = \frac{2\lambda(2\lambda-1)(2\lambda-2)!}{2(2\lambda-2)!} = \lambda(2\lambda-1) \quad \text{(2.2.1)} \]

\[ \frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)(\lambda-1)n-\lambda+1} = \frac{\lambda^{2\lambda-\lambda+1}}{2(\lambda-1)\lambda(\lambda-1)n-\lambda+1} = \frac{\lambda(\lambda-1)}{2} \cdot \left(\frac{\lambda}{\lambda-1}\right)^\lambda \quad \text{(2.2.2)} \]

In (2.1) when \(x = \lambda \geq 3\), we have \(\frac{2(2\lambda-1)}{\lambda-1} > \left(\frac{\lambda}{\lambda-1}\right)^\lambda \quad \text{(2.2.3)} \)

Since \(\frac{\lambda(\lambda-1)}{2}\) is a positive number for \(\lambda \geq 3\), referring to (2.2.1) and (2.2.2), when \(\frac{\lambda(\lambda-1)}{2}\) multiplies to both sides of (2.2.3), we have

\[ \left(\frac{\lambda(\lambda-1)}{2}\right) \left(\frac{2(2\lambda-1)}{\lambda-1}\right) = \lambda(2\lambda-1) = \left(\frac{\lambda^n}{n}\right) > \left(\frac{\lambda(\lambda-1)}{2}\right) \left(\frac{\lambda}{\lambda-1}\right)^\lambda = \frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)(\lambda-1)n-\lambda+1} \]
Thus, \( \left( \frac{\lambda n}{n} \right) > \frac{\lambda^{n-\lambda+1}}{n(\lambda-1)(\lambda-1)n-\lambda+1} \) when \( \lambda \geq 3 \) and \( n = 2 \). — (2.2.4)

By induction on \( n \), when \( \lambda \geq 3 \), if \( \left( \frac{\lambda n}{n} \right) > \frac{\lambda^{n-\lambda+1}}{n(\lambda-1)(\lambda-1)n-\lambda+1} \) is true for \( n \), then for \( n+1 \), we have

\[
\left( \frac{\lambda n+1}{n+1} \right) = \frac{\lambda^{n+1}}{n+1} = \frac{(\lambda n+\lambda)(\lambda n+\lambda-1)\cdots(\lambda n+2)(\lambda n+1)}{(n+1)(\lambda n+\lambda-1)(\lambda n+\lambda-2)\cdots(\lambda n+1)(\lambda n+1)} \cdot \frac{\lambda^{n-\lambda+1}}{n(\lambda-1)(\lambda-1)n-\lambda+1}
\]

Notice \( \frac{\lambda n+1}{n} > \lambda \), and

\[
\frac{(\lambda n+\lambda)(\lambda n+\lambda-1)\cdots(\lambda n+2)(\lambda n+1)}{(\lambda n+\lambda-1)(\lambda n+\lambda-2)\cdots(\lambda n+1)(\lambda n+1)} = \frac{\lambda}{\lambda-1} > \left( \frac{\lambda}{\lambda-1} \right)
\]

because

\[
\frac{\lambda n+\lambda}{\lambda n+\lambda-1} = \frac{\lambda}{\lambda-1} ; \frac{\lambda n+\lambda-1}{\lambda n+\lambda-2} > \frac{\lambda}{\lambda-1} ; \cdots \frac{\lambda n+2}{\lambda n+1} > \frac{\lambda}{\lambda-1}.
\]

Thus

\[
\left( \frac{\lambda(n+1)}{n+1} \right) > \frac{\lambda^{n-\lambda+1}}{(\lambda-1)(\lambda-1)n-\lambda+1} \cdot \frac{\lambda}{1} \cdot \frac{1}{n+1} \cdot \frac{\lambda^{n-\lambda+1}}{(\lambda-1)(\lambda-1)n-\lambda+1} = \frac{\lambda^{n+1} \lambda^{n-\lambda+1}}{(n+1)(\lambda-1)(\lambda-1)(\lambda-1)n-\lambda+1} > \left( \frac{\lambda}{\lambda-1} \right)
\]

From (2.2.4) and (2.2.5), we have for \( n \geq 2 \) and \( \lambda \geq 3 \), \( \left( \frac{\lambda n}{n} \right) > \frac{\lambda^{n-\lambda+1}}{n(\lambda-1)(\lambda-1)n-\lambda+1} \)

Thus, Lemma 2 is proven.

3. A Prime Number between \( (\lambda-1)n \) and \( \lambda n \)

Proposition:

For \( n \geq (\lambda-2) \geq 24 \), there exists at least a prime number \( p \) such that \( (\lambda-1)n < p \leq \lambda n \). — (3.1)

Proof:

When \( n \geq (\lambda-2) \) in (1.7), since \( n+1 = \sqrt{(n+2)n+1} > \sqrt{\lambda n} \), \( |\sqrt{\lambda n}| \) is an integer at most one less than \( (n+1) \). Thus, \( n \geq |\sqrt{\lambda n}| \) and (1.7) is valid for \( n \geq (\lambda-2) \geq 24 \).

From (1.7) and (2.2), when \( n \geq (\lambda-2) \geq 24 \), we have

\[
\frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)(\lambda-1)n-\lambda+1} < \left( \frac{\lambda n}{n} \right) \leq \Gamma_{\lambda n \geq p \geq n} \left\{ \frac{(\lambda n)!}{(\lambda-1)n)!} \right\} \cdot 2^{2n-3} \cdot (\lambda n)^{\sqrt{\lambda n}} + 1
\]

\[
\frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)(\lambda-1)n-\lambda+1} \leq \Gamma_{\lambda n \geq p \geq n} \left\{ \frac{(\lambda n)!}{(\lambda-1)n)!} \right\} \cdot 2^{2n-3} \cdot (\lambda n)^{\sqrt{\lambda n}} + 1.
\]

Since \( (\lambda n)^{\sqrt{\lambda n}} + 1 > 1 \) and \( 2^{2n-3} > 1 \),
\[ \Gamma_{\lambda n \geq p > n} \left\{ \frac{(\lambda n)!}{((\lambda - 1)n)!} \right\} > \frac{\lambda^{n-\lambda+1}}{(\lambda n)^{\frac{\lambda n}{3} + 1} \cdot 2^{2n-3} \cdot n(\lambda - 1)\lambda n - \lambda + 1} = \frac{2\lambda^2 \cdot \left( \left( \frac{\lambda - 1}{4} \right)^3 \right)^{(n-1)}}{(\lambda n)^{\frac{\lambda n}{3} + 2}} \]

Referring to (2.1.7), \( \left( \frac{\lambda}{\lambda - 1} \right)^{\lambda} \geq e \),

thus \( \Gamma_{\lambda n \geq p > n} \left\{ \frac{(\lambda n)!}{((\lambda - 1)n)!} \right\} > \frac{2\lambda^2 \cdot \left( \left( \frac{\lambda - 1}{4} \right)^3 \right)^{(n-1)}}{(\lambda n)^{\frac{\lambda n}{3} + 2}} \geq \frac{2\lambda^2 \cdot \left( \left( \frac{\lambda - 1}{4} \right) \cdot e \right)^{(n-1)}}{(\lambda n)^{\frac{\lambda n}{3} + 2}} = f_3(n, \lambda) \quad (3.2) \]

Let \( x = (y - 2) \geq 24 \), where both \( x \) and \( y \) are positive real numbers, and

\[ f_3(x, y) = \frac{2y^2 \cdot \left( \left( \frac{y - 1}{4} \right) \cdot e \right)^{(x-1)}}{(xy)^{\frac{xy}{3} + 2}} = f_4(x) = \frac{2(x + 2)^2 \cdot \left( \left( \frac{x + 1}{4} \right) \cdot e \right)^{(x-1)}}{(x(x+2))^{\frac{x+2}{3} + 2}} \quad (3.3) \]

\[ f_4'(x) = f_4(x) \cdot \left( \frac{2}{x+2} + \ln \left( \frac{x+1}{4} \right) + \frac{4}{3} - \frac{2}{x+1} - \frac{1}{3} \ln(x \cdot (x + 2)) - \frac{7}{3x} - \frac{5}{3(x+2)} \right) = f_4(x) \cdot f_5(x) \]

where \( f_5(x) = \frac{2}{x+2} + \ln \left( \frac{x+1}{4} \right) + \frac{4}{3} - \frac{2}{x+1} - \frac{1}{3} \ln(x \cdot (x + 2)) - \frac{7}{3x} - \frac{5}{3(x+2)} \geq 0 \)

Thus, \( f_5(x) \) is a strictly increasing function for \( x \geq 24 \).

When \( x = 24 \), \( f_5(x) = \frac{2}{24+2} + \ln \left( \frac{24+1}{4} \right) + \frac{4}{3} - \frac{2}{24+1} - \frac{1}{3} \ln(24) - \frac{1}{3} \ln(24 + 2) - \frac{7}{72} - \frac{5}{78} > 0 \),

thus, for \( x \geq 24 \), \( f_5(x) > 0 \).

Then, \( f_4'(x) = f_4(x) \cdot f_5(x) > 0 \) and \( f_4(x) \) is a strictly increasing function for \( x \geq 24 \).

When \( x = 24 \), \( f_4(x) = \frac{2 \cdot (26)^2 \cdot \left( \frac{25}{4} \right)^{23} \cdot e^{23}}{(24 \cdot 26)^{24+1}} > \frac{2.6606E+31}{7.6484E+28} > 1 \), thus for \( x \geq 24 \), \( f_4(x) > 1 \).

Thus, when \( x = (y - 2) \geq 24 \), \( f_3(x, y) = \frac{2y^2 \cdot \left( \left( \frac{y - 1}{4} \right) \cdot e \right)^{(x-1)}}{(xy)^{\frac{xy}{3} + 2}} > f_4(x) > 1 \). \quad (3.4)

\[ \frac{\partial f_3(x, y)}{\partial x} = f_3(x, y) \cdot \left( \ln \left( \frac{y - 1}{4} \right) + 1 - \frac{y}{6\sqrt{x}} \cdot \ln(yx) - \frac{y}{3\sqrt{x}} - \frac{2}{x} \right) = f_3(x, y) \cdot f_6(x, y) \quad (3.5) \]

where \( f_6(x, y) = \ln \left( \frac{y - 1}{4} \right) + 1 - \frac{y}{6\sqrt{x}} \cdot \ln(yx) - \frac{y}{3\sqrt{x}} - \frac{2}{x} \)

\[ \frac{\partial f_6(x, y)}{\partial x} = \frac{\sqrt{y}}{12x\sqrt{x}} \cdot \ln(y) + \frac{\sqrt{y}}{12x\sqrt{x}} \cdot \ln(x) + \frac{\sqrt{y}}{6x\sqrt{x}} + \frac{\sqrt{y}}{6x\sqrt{x}} + \frac{2}{x^2} > 0 \]

Thus, \( f_6(x, y) \) is a strictly increasing function for \( x \geq y - 2 \geq 24 \).

When \( x = (y - 2) \geq 24 \), \( f_6(x, y) \geq \ln \left( \frac{26 - 1}{4} \right) + 1 - \frac{\sqrt{26}}{6\sqrt{24}} \cdot \ln(24 \cdot 26) - \frac{\sqrt{26}}{3\sqrt{24}} - \frac{2}{24} > 0 \).

Thus, \( \frac{\partial f_3(x, y)}{\partial x} = f_3(x, y) \cdot f_6(x, y) > 0 \). \( f_3(x, y) \) is a strictly increasing function for \( x \geq y - 2 \geq 24 \).
When \( x = y - 2 \geq 24 \),
\[
f_3(x, y) = \frac{2 \cdot 26^2 \cdot \left(\frac{(26-1)}{4} \cdot e\right)^{(24-1)}}{(24 \cdot 26)^{\frac{24 \cdot 26}{3} + 2}} > \frac{2.6606E+31}{7.3274E+28} > 1.
\]

Thus, for \( x \geq y - 2 \geq 24 \),
\[
f_3(x, y) = \frac{2y^2 \cdot \left(\frac{(y-1)}{4} \cdot e\right)^{(x-1)}}{(xy)^{\frac{xy}{3} + 2}} > 1.
\] — (3.6)

Let \( n = |x| \) and \( \lambda = |y| \), referring to (3.2), when \( n \geq (\lambda - 2) \geq 24 \),
we have
\[
\Gamma_{\lambda n \geq p > n}\left\{\frac{(\lambda n)!}{((\lambda - 1)n)!}\right\} > \frac{2\lambda^2 \cdot \left(\frac{(\lambda - 1)}{4} \cdot \left(\frac{\lambda}{i+1}\right)^{(i-1)}\right)^{(n-1)}}{(\lambda n)^{\frac{\lambda n}{3} + 2}} > 1
\] — (3.7)

In \( \Gamma_{\lambda n \geq p > n}\left\{\frac{(\lambda n)!}{((\lambda - 1)n)!}\right\} = \Gamma_{\lambda n \geq p > (\lambda - 1)n}\left\{\frac{(\lambda n)!}{((\lambda - 1)n)!}\right\} \cdot \prod_{i=1}^{\lambda - 2} \left(\Gamma_{\lambda n \geq p > i \geq \lambda n}\left\{\frac{(\lambda n)!}{((\lambda - 1)n)!}\right\} \cdot \Gamma_{\lambda n \geq p > i \geq 1}\left\{\frac{(\lambda n)!}{((\lambda - 1)n)!}\right\}\right)\).

In \( \prod_{i=1}^{\lambda - 2} \left(\Gamma_{\lambda n \geq p > i \geq \lambda n}\left\{\frac{(\lambda n)!}{((\lambda - 1)n)!}\right\} \cdot \Gamma_{\lambda n \geq p > i \geq 1}\left\{\frac{(\lambda n)!}{((\lambda - 1)n)!}\right\}\right) \),
every distinct prime number \( p \) in this range in the
numerator \((\lambda n)!\) has the form of \((i)! \cdot p^i\). It also has the same form of \((i)! \cdot p^i\) in the
denominator \((\lambda - 1)n)!\). Thus, referring to (1.2),
\[
\prod_{i=1}^{\lambda - 2} \left(\Gamma_{\lambda n \geq p > i \geq \lambda n}\left\{\frac{(\lambda n)!}{((\lambda - 1)n)!}\right\} \cdot \Gamma_{\lambda n \geq p > i \geq 1}\left\{\frac{(\lambda n)!}{((\lambda - 1)n)!}\right\}\right) = 1.
\]

Therefore, when \( n \geq \lambda - 2 \geq 24 \),
\[
\Gamma_{\lambda n \geq p > n}\left\{\frac{(\lambda n)!}{((\lambda - 1)n)!}\right\} = \Gamma_{\lambda n \geq p > (\lambda - 1)n}\left\{\frac{(\lambda n)!}{((\lambda - 1)n)!}\right\} \cdot \prod_{i=1}^{\lambda - 2} \left(\Gamma_{\lambda n \geq p > i \geq \lambda n}\left\{\frac{(\lambda n)!}{((\lambda - 1)n)!}\right\} \cdot \Gamma_{\lambda n \geq p > i \geq 1}\left\{\frac{(\lambda n)!}{((\lambda - 1)n)!}\right\}\right) > 1
\] — (3.8)

From (1.1), \( \Gamma_{\lambda n \geq p > (\lambda - 1)n}\left\{\frac{(\lambda n)!}{((\lambda - 1)n)!}\right\} \geq 1 \) and \( \prod_{i=1}^{\lambda - 2} \left(\Gamma_{\lambda n \geq p > i \geq \lambda n}\left\{\frac{(\lambda n)!}{((\lambda - 1)n)!}\right\} \cdot \Gamma_{\lambda n \geq p > i \geq 1}\left\{\frac{(\lambda n)!}{((\lambda - 1)n)!}\right\}\right) \geq 1 \),
and in (3.8) at last one of these two parts is greater than 1.

When \( n \geq \lambda - 2 \geq 24 \), if \( \Gamma_{\lambda n \geq p > (\lambda - 1)n}\left\{\frac{(\lambda n)!}{((\lambda - 1)n)!}\right\} > 1 \), then referring to (1.3), there exists at least a prime number \( p \) such that \((\lambda - 1)n < p \leq \lambda n\).

If \( \prod_{i=1}^{\lambda - 2} \left(\Gamma_{\lambda n \geq p > i \geq \lambda n}\left\{\frac{(\lambda n)!}{((\lambda - 1)n)!}\right\} \cdot \Gamma_{\lambda n \geq p > i \geq 1}\left\{\frac{(\lambda n)!}{((\lambda - 1)n)!}\right\}\right) = 1 \), then \( \Gamma_{\lambda n \geq p > (\lambda - 1)n}\left\{\frac{(\lambda n)!}{((\lambda - 1)n)!}\right\} > 1 \). — (3.9)

If \( \prod_{i=1}^{\lambda - 2} \left(\Gamma_{\lambda n \geq p > i \geq \lambda n}\left\{\frac{(\lambda n)!}{((\lambda - 1)n)!}\right\} \cdot \Gamma_{\lambda n \geq p > i \geq 1}\left\{\frac{(\lambda n)!}{((\lambda - 1)n)!}\right\}\right) > 1 \), then at least one factor \( \Gamma_{\lambda n \geq p > (\lambda - 1)n}\left\{\frac{(\lambda n)!}{((\lambda - 1)n)!}\right\} > 1 \).

When a factor \( \Gamma_{\lambda n \geq p > i \geq \lambda n}\left\{\frac{(\lambda n)!}{((\lambda - 1)n)!}\right\} > 1 \), let \( y_{i+1} = \frac{n}{i+1} \), then \( y_{i+1} \geq \frac{\lambda - 2}{i+1} \geq \frac{24}{i+1} \). We have
\[ \Gamma_{y_{i+1} \geq p^{(\lambda-1)}y_{i+1} \frac{(\lambda n)!}{((\lambda-1)n)!}} > 1. \] Thus, when \( y_{i+1} \geq \frac{\lambda - 2}{i+1} \geq \frac{24}{i+1}, \) there exists at least a prime number \( p \) such that \((\lambda - 1) \cdot y_{i+1} < p \leq \lambda \cdot y_{i+1} \).

Since \( n > y_{i+1} > \frac{\lambda - 2}{i+1} > \frac{24}{i+1}, \) there exists at least a prime number \( p \) such that \((\lambda - 1)n < p \leq \lambda n.\)

Thus, if \( \prod_{i=1}^{n} \frac{\Gamma_{\lambda n \geq p^{(\lambda-1)n} \frac{(\lambda n)!}{((\lambda-1)n)!}}} {\Gamma_{\frac{\lambda n}{(i+1)}}} > 1, \) then \( \Gamma_{\lambda n \geq p^{(\lambda-1)n} \frac{(\lambda n)!}{((\lambda-1)n)!}} > 1. \) — (3.11)

Referring to (3.7), (3.9), (3.10), and (3.11), then \( \Gamma_{\lambda n \geq p^{(\lambda-1)n} \frac{(\lambda n)!}{((\lambda-1)n)!}} > 1 \) when \( n \geq \lambda - 2 \geq 24. \)

Referring to (1.3), there exists at least a prime number \( p \) such that \((\lambda - 1)n < p \leq \lambda n.\)

Thus, (3.1), the Proposition, is proven. It becomes a theorem: Theorem (3.1).

### 4. The Proof of Legendre’s Conjecture

Legendre’s Conjecture states that there is a prime number between \( n^2 \) and \((n + 1)^2\) for every positive integer \( n.\) — (4.1)

**Proof:**

Referring to Theorem (3.1), for integers \( j \geq k - 2 \geq 24, \) there exists at least a prime number \( p \)

such that \( j(k - 1) < p \leq jk. \) — (4.2)

When \( k = j + 1 \geq 26, \) then \( j = k - 1 \geq 25 \)

Applying \( k = j + 1 \) into (4.2), then \( j^2 < p \leq j(j + 1) < (j + 1)^2 \)

Let \( n = j \geq 25, \) then we have \( n^2 < p < (n + 1)^2. \) — (4.3)

For \( 1 \leq n \leq 24, \) we have a table, Table 1, that shows Legendre’s conjecture valid. — (4.4)

**Table 1:** For \( 1 \leq n \leq 24, \) there is a prime number between \( n^2 \) and \((n + 1)^2\).

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
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<td>((n + 1)^2)</td>
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<td>((n + 1)^2)</td>
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Combining (4.3) and (4.4), we have proven Legendre’s conjecture.
Extension of Legendre’s conjecture

There are at least two prime numbers, \( p_n \) and \( p_m \), between \( j^2 \) and \( (j+1)^2 \) for every positive integer \( j \) such that \( j^2 < p_n \leq j(j+1) < p_m < (j + 1)^2 \) where \( p_n \) is the \( n \) th prime number, \( p_m \) is the \( m \) th prime number, and \( m \geq n + 1 \).  

— (4.5)

Proof:

Referring to Theorem (3.1), for integers \( j \geq k - 2 \geq 24 \), there exists at least a prime number \( p \) such that \( j(k - 1) < p \leq jk \).

When \( k - 1 = j \geq 25 \), then \( j(k - 1) = j^2 < p_n \leq jk = j(j+1) \). Thus, there is at least a prime number \( p_n \) such that \( j^2 < p_n \leq j(j+1) \) when \( j = k - 1 \geq 25 \).

When \( j = k - 2 \geq 25 \), then \( k = j + 2 \). Thus, \( j(k - 1) = j(j+1) < p_m \leq jk = j(j+2) < (j + 1)^2 \). Thus, there is at least another prime number \( p_m \) such that \( j(j+1) < p_m < (j + 1)^2 \) when \( j = k - 2 \geq 25 \).

Thus, when \( j \geq 25 \), there are at least two prime numbers \( p_n \) and \( p_m \) between \( j^2 \) and \( (j + 1)^2 \) such that \( j^2 < p_n \leq j(j+1) < p_m < (j + 1)^2 \) where \( m \geq n + 1 \) for \( p_m > p_n \). 

— (4.6)

For \( 1 \leq j \leq 24 \), we have a table, Table 2, that shows (4.5) valid.  

— (4.7)

Table 2: For \( 1 \leq j \leq 24 \), there are 2 prime numbers such that \( j^2 < p_n \leq j(j+1) < p_m < (j + 1)^2 \).

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<td>625</td>
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</table>

Combining (4.6) and (4.7), we have proven (4.5). It becomes a theorem: Theorem (4.5).
5. The Proofs of Three Related Conjectures

Oppermann’s conjecture was proposed by Ludvig Oppermann [4] in March 1877. It states that for every integer \( x > 1 \), there is at least one prime number between \( x(x - 1) \) and \( x^2 \), and at least another prime between \( x^2 \) and \( x(x + 1) \). — (5.1)

**Proof:**

**Theorem (4.5)** states there are at least two prime numbers, \( p_n \) and \( p_m \), between \( j^2 \) and \( (j + 1)^2 \) for every positive integer \( j \) such that \( j^2 < p_n < j(j+1) \) and \( j(j+1) < p_m < (j + 1)^2 \) where \( m \geq n + 1 \).

\( j(j+1) \) is a composite number except \( j = 1 \). Since \( j^2 < p_n < j(j+1) \) is valid for every positive integer \( j \), when we replace \( j \) with \( j+1 \), we have \( (j + 1)^2 < p_n < (j+1)(j+2) \).

Thus, we have \( j(j+1) < p_m < (j + 1)^2 < p_n < (j+1)(j+2) \). — (5.2)

When \( x > 1 \), then \( (x - 1) \geq 1 \). Substitute \( j \) with \( (x - 1) \) in (5.2), we have \( x(x - 1) < p_m < x^2 < p_n < x(x + 1) \) — (5.3)

Thus, we have proven Oppermann’s conjecture.

Brocard’s conjecture is named after Henri Brocard [5]. It states that there are at least 4 prime numbers between \( (p_n)^2 \) and \( (p_{n+1})^2 \), where \( p_n \) is the \( n^{th} \) prime number, for every \( n > 1 \). — (5.4)

**Proof:**

**Theorem (4.5)** states there are at least two prime numbers, \( p_n \) and \( p_m \), between \( j^2 \) and \( (j + 1)^2 \) for every positive integer \( j \) such that \( j^2 < p_n < j(j+1) \) and \( j(j+1) < p_m < (j + 1)^2 \) where \( m \geq n + 1 \). When \( j > 1 \), \( j(j+1) \) is a composite number. Then **Theorem (4.5)** can be written as \( j^2 < p_n < j(j+1) \) and \( j(j+1) < p_m < (j + 1)^2 \).

In the series of prime numbers: \( p_1=2, \ p_2=3, \ p_3=5, \ p_4=7, \ p_5=11... \) all prime numbers except \( p_1 \) are odd numbers. Their gaps are two or more. Thus when \( n > 1 \), \( (p_{n+1} - p_n) \geq 2 \).

Thus, we have \( p_n < (p_n + 1) < p_{n+1} \) when \( n > 1 \). — (5.5)

Applying **Theorem (4.5)** to (5.5), when \( n > 1 \), we have at least two prime numbers \( p_{m1}, \ p_{m2} \) in between \( (p_n)^2 \) and \( (p_n + 1)^2 \) such that \( (p_n)^2 < p_{m1} < p_n(p_{n+1}) < p_{m2} < (p_{n+1})^2 \), and at least two more prime numbers \( p_{m3}, \ p_{m4} \) in between \( (p_n + 1)^2 \) and \( (p_{n+1})^2 \) such that \( (p_{n+1})^2 < p_{m3} < p_{n+1}(p_{n+1}) < p_{m4} < (p_{n+1})^2 \).

Thus, there are at least 4 prime numbers between \( (p_n)^2 \) and \( (p_{n+1})^2 \) for \( n > 1 \) such that \( (p_n)^2 < p_{m1} < p_n(p_{n+1}) < p_{m2} < (p_{n+1})^2 < p_{m3} < p_{n+1}(p_{n+1}) < p_{m4} < (p_{n+1})^2 \) — (5.6)

Thus, Brocard’s conjecture is proven.
Andrica’s conjecture is named after Dorin Andrica [6]. It is a conjecture regarding the gaps between prime numbers. The conjecture states that the inequality $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$ holds for all $n$ where $p_n$ is the $n^{th}$ prime number. If $g_n = p_{n+1} - p_n$ denotes the $n^{th}$ prime gap, then Andrica’s conjecture can also be rewritten as $g_n < 2\sqrt{p_n} + 1$. — (5.7)

Proof:

From Theorem (4.5), for every positive integer $j$, there are at least two prime numbers $p_n$ and $p_m$ between $j^2$ and $(j + 1)^2$ such that $j^2 < p_n < j(j+1) < p_m < (j + 1)^2$ where $m \geq n + 1$.

Since $m \geq n + 1$, we have $p_m \geq p_{n+1}$.

Thus, we have $j^2 < p_n$. — (5.8)

And $p_{n+1} \leq p_m < (j + 1)^2$. — (5.9)

Since $j$, $p_n$, $p_{n+1}$ and $(j + 1)$ are positive integers,

$$j < \sqrt{p_n}$$

— (5.10)

And $\sqrt{p_{n+1}} < j + 1$ — (5.11)

Applying (5.10) to (5.11), we have $\sqrt{p_{n+1}} < \sqrt{p_n} + 1$. — (5.12)

Thus, $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$ holds for all $n$ since in Theorem (4.5), $j$ holds for all positive integers.

Using the prime gap to prove the conjecture, from (5.8) and (5.9), we have

$$g_n = p_{n+1} - p_n < (j + 1)^2 - j^2 = 2j + 1.$$ From (5.10), $j < \sqrt{p_n}$.

Thus, $g_n = p_{n+1} - p_n < 2\sqrt{p_n} + 1$. — (5.13)

Thus, Andrica’s conjecture is proven.

6. References

Q & A

Q: Why does it need to define the prime number decomposition operator?
A: The prime number decomposition operator $\Gamma_{a \geq p > b} \{n\}$ is needed because it has the properties that if $\Gamma_{a \geq p > b} \{n\} = 1$, then there is no prime number less than or equal to $a$ but greater than $b$; and that if $\Gamma_{a \geq p > b} \{n\} > 1$, then there exists at least one prime number less than or equal to $a$ but greater than $b$. In this operator, $n$ is an integer, $p$ is a prime number, $a$ and $b$ are real numbers, and $n \geq a \geq p > b \geq 1$. Thus, one can determine immediately whether a prime number exists in a certain range with this operator.

Q: What is the logic in the proof of $\Gamma_{\lambda n \geq p > n} \{\frac{(\lambda n)!}{(\lambda - 1)n)!}\} > 1$ for $n \geq (\lambda - 2) \geq 24$?
A: First, to prove $\Gamma_{\lambda n \geq p > n} \{\frac{(\lambda n)!}{(\lambda - 1)n)!}\} > f_3(n, \lambda)$ for $n \geq (\lambda - 2) \geq 24$. Then to replace $f_3(n, \lambda)$ with $f_3(x, y)$ to make a continuous function. Second, to prove that if $x > (y - 2) \geq 24$, then $f_3(x, y) > 1$. Third, to prove if $x > (y - 2) \geq 24$, then $f_3(x, y) > 1$ by showing $\frac{df_3(x, y)}{dx} > 0$. Thus $f_3(x, y) > 1$ for $x \geq (y - 2) \geq 24$. Forth, let $n = \lfloor x \rfloor$ and $\lambda = \lfloor y \rfloor$ in $f_3(x, y)$, thus, to prove $f_3(n, \lambda) > 1$ for $n \geq (\lambda - 2) \geq 24$ also $\Gamma_{\lambda n \geq p > n} \{\frac{(\lambda n)!}{(\lambda - 1)n)!}\} > f_3(n, \lambda) > 1$.

The send and third steps are similar to integer conduction: major premise, to prove that $f_3(n, \lambda) > 1$ for $n = (\lambda - 2) \geq 24$; minor premise, to prove that if $f_3(n, \lambda) > 1$, then $f_3((n + 1), \lambda) > 1$; conclusion, thus $f_3(n, \lambda) > 1$ for $n \geq (\lambda - 2) \geq 24$. 