On the non-linear refractive index-curvature relation

Miftachul Hadi[1, 2]

[1)] *Physics Research Centre, Badan Riset dan Inovasi Nasional (BRIN), Puspiptek, Serpong, Tangerang Selatan 15314, Banten, Indonesia.*

[2)] *Institute of Mathematical Sciences, Kb Kopi, Jalan Nuri I, No.68, Pengasinan, Gn Sindur 16340, Bogor, Jawa Barat, Indonesia. E-mail: instmathsci.id@gmail.com*

The refractive index-curvature relation is formulated using the second rank tensor of Ricci curvature as a consequence of a scalar refractive index. A scalar refractive index describes (an isotropic) linear optics. In (an isotropic) non-linear optics, this scalar refractive index is decomposed into a contravariant fourth rank tensor of non-linear refractive index and a covariant fourth rank tensor of susceptibility. In topological space, both a contravariant fourth rank tensor of non-linear refractive index and a covariant fourth rank tensor of susceptibility are related to the Euler-Poincare characteristic, a topological invariant.

Keywords: *geometrical optics, Abelian gauge theory, refractive index, Ricci curvature, Riemann-Christoffel curvature, curvature form, curvature matrix, connection matrix, Gauss-Bonnet-Chern theorem, Euler-Poincare characteristic, topological invariant.*

In the geometrical optics, the refractive index-curvature relation which describes ray propagation in a steady (time-independent) state can be derived from the Fermat's principle[1–4]. The refractive index-curvature relation can be written as

$$\frac{1}{R} = \hat{N} \ . \ \vec{\nabla} \ \ln \ n(r) \tag{1}$$

where $1/R$ is a 1-dimensional space curvature, R is a radius of curvature, $\hat{N}$ is an unit vector along the principal normal or has the same direction with $\vec{\nabla}$ ln $n(r)$ and $n(r)$ is a 1-dimensional space refractive index. Eq.(1) tells us that the rays are therefore bent in the direction of increasing refractive index[1].

The dimension of the curvature in eq.(1) can be extended to any arbitrary number of dimensions[5]. In a $(3 + 1)$-dimensional space-time, eq.(1) can be written as

$$R_{\mu\nu} = g \ N_\mu \ \partial_\nu \ln n \tag{2}$$

where $R_{\mu\nu}$ is the second rank tensor of Ricci curvature[5,6], a function of the metric tensor $g_{\mu\nu}$, $g = |(\det \ g_{\mu\nu})|$, is a scalar, a real number. Why do we need to formulate the curvature in eq.(2) as the second rank tensor of Ricci curvature? It is because of the related refractive index in eq.(2) is the zeroth rank tensor, a scalar i.e. a real number.

The zeroth rank tensor (a scalar) of the refractive index describes *an isotropic linear optics*[7]. But, the refractive index can be not simply a scalar[8]. The refractive index can also be *a second rank tensor* which describes that the electric field component along one axis may be affected by the electric field component along another axis[8]. The second rank tensor of the refractive index describes *an anisotropic linear optics*[7]. Eq.(2) implies that the zeroth rank tensor of the refractive index related to the Ricci curvature describes naturally (an isotropic) linear optics.

How about the form of a refractive index related to non-linear optics? In optics, *non-linear properties of materials are usually described by non-linear susceptibilities*[9]. Mathematically, the optical response (as a result of interaction between light and optical transparent medium) can be expressed as a relationship between the polarization density[10,11], $\vec{P}$, and the electric field, $\vec{E}$.

In linear optics, a relationship between the polarization density and the electric field is simply expressed as[12,13]

$$\vec{P} = \varepsilon_0 \ \chi^{(1)} \ \vec{E} \tag{3}$$

where ε_0 is the permittivity of vacuum space, $\chi^{(1)}$ is *the first order susceptibility*[14] or *linear susceptibility, a scalar or a zeroth rank tensor*, whereas the polarization and the electric field are vectors.

In non-linear optics[15–17], the polarization density can be modelled as a power series of the electric field as below[12,13,18]

$$\begin{aligned} \vec{P} &= \varepsilon_0 \left[\chi^{(1)} \ \vec{E}^1 + \chi^{(2)} \ \vec{E}^2 + \chi^{(3)} \ \vec{E}^3 + ... \right] \\ &= \vec{P}^1 + \vec{P}^2 + \vec{P}^3 + ... \end{aligned} \tag{4}$$

where $\vec{E}^1 = \vec{E}$, $\vec{E}^2 = \vec{E} \ \vec{E}$, $\vec{E}^3 = \vec{E} \ \vec{E} \ \vec{E}$, etc. $\vec{P}^1$ is called the linear polarization, while $\vec{P}^2$, $\vec{P}^3$ *are the second and the third non-linear polarizations, respectively.* Thus, *the polarization is composed by linear and non-linear components*[13]. The first susceptibility term, $\chi^{(1)}$, corresponds to the linear (dimensionless) susceptibility. $\chi^{(a)}$ (meter/volt)$^{a-1}$ are *the subsequent non-linear susceptibilities*, where $a > 1$[14,19]. The quantities $\chi^{(2)}$ and $\chi^{(3)}$ are known as the second and the third order susceptibilities, respectively. *These first, second and third order susceptibilities, $\chi^{(1)}$, $\chi^{(2)}$, $\chi^{(3)}$, are the second, the third and the fourth rank tensors, respectively*[12]. In optical Kerr effect, the third order susceptibility, $\chi^{(3)}$, related to the non-linear refractive index[8].

Now, we have a question: if the non-linear refractive index is related to the third order susceptibility, $\chi^{(3)}$, and the third order susceptibility is the fourth rank tensor[12] then *how to define the non-linear refractive index related to the fourth rank tensor of the susceptibility?*

For a linearly polarized monochromatic light in *an isotropic medium* or a cubic crystal, *the non-linear refractive index*, n_2, can be expressed by[18]

$$n_0 = 12\pi \ (n_2)^{-1} \ \text{Re} \ \chi^{(3)} \tag{5}$$

where n_0 is *a linear refractive index*[20] and Re $\chi^{(3)}$ is a real part[21] of *the third order non-linear susceptibility.*

If the third order susceptibility, $\chi^{(3)}$, is the fourth rank tensor, χ_{pqrs}, and *an isotropic medium related to the linear refractive index, n_0, as a scalar or a zeroth rank tensor denoted by n* then from eq.(5) the non-linear refractive index, n_2, should be the fourth rank tensor, n_{pqrs}. So eq.(5) can be written as

$$n = 12\pi \; n^{pqrs} \; \chi_{pqrs} \tag{6}$$

where $n^{pqrs} = (n_{pqrs})^{-1}$.

Substituting (6) into (2), we obtain

$$R_{\mu\nu} = g \; N_\mu \; \partial_\nu \ln \left(12\pi \; n^{pqrs} \; \chi_{pqrs}\right) \tag{7}$$

Eq.(7) shows that in a non-linear optics, the Ricci curvature tensor is related to a (contravariant) fourth rank tensor of the refractive index (actually, the non-linear refractive index is a covariant fourth rank tensor).

We will formulate a curvature in a fibre bundle and we treat the geometical optics as a gauge theory[4]. *Is there a relationship between a fibre bundle and a gauge theory? Why do we need to formulate a curvature in a fibre bundle?* Originally, the fibre bundle and the gauge theory are developed independently. Until it was realized that *the curvature (in the fibre bundle) and the field strength (in Yang-Mills theory) are identical*[22]. *Simply speaking, the curvature in the fibre bundle is the field strength in the gauge theory.*

Because the geometrical optics can be treated as the Abelian $U(1)$ gauge theory[4], so we need to formulate the curvature in the refractive index-curvature relation as *an Abelian curvature form* in a fibre bundle. Probably, this is another reason why we really need to formulate a curvature in a curvature form instead of the Riemann-Christoffel curvature tensor. *A curvature form in a fibre bundle can be an Abelian (or a non-Abelian) which the Riemann-Christoffel curvature tensor can not be an Abelian*[23].

The curvature form, $\Omega_{\rho\sigma}$, can be written as[24,25]

$$\Omega_{\rho\sigma} = \sum R_{\rho\sigma\mu\nu} \; du^\mu \wedge du^\nu \tag{8}$$

where $R_{\mu\nu\rho\sigma}$ is the fourth rank tensor of Riemann-Christoffel curvature, u^μ, u^ν are local coordinates and $\wedge$ is a notation of the exterior (wedge) product (it satisfies the distributive, anti-commutative and associative laws)[24,25]. *The curvature form, $\Omega_{\rho\sigma}$, is an anti-symmetric matrix of 2-forms*[26,27]. The relation between the Ricci curvature tensor and the Riemann-Christoffel curvature tensor, we call the Ricci-Riemann relation, is $R_{\mu\nu} = g^{\rho\sigma} R_{\rho\sigma\mu\nu}$.

If we reformulate eq.(8) using eq.(2) and the Ricci-Riemann relation, we obtain

$$\Omega_{\rho\sigma} = \sum g \; g_{\rho\sigma} \; N_\mu \; \partial_\nu \ln n \; du^\mu \wedge du^\nu \tag{9}$$

Eq.(9) shows the relationship between the scalar refractive index and the curvature form in a (3+1)-dimensional space-time. Here, the scalar refractive index is a function of coordinates only (a smooth continuous function of the position[28]) which "lives" in a $(3+1)$-dimensional space-time[4].

Let us introduce the general form of *the curvature matrix*, Ω, which is *a matrix of exterior two-forms* as below[24]

$$\Omega = d\omega - \omega \wedge \omega \tag{10}$$

where ω is *the connection matrix.* We see that eq.(10) is a non-Abelian, a non-linear equation.

Can the curvature matrix, Ω, in eq.(10) be an Abelian, a linear equation? An Abelian curvature matrix means that the second term in the right hand side of eq.(10), $\omega \wedge \omega$, vanish. It can be done if *the isometry group*, $G = U(1)$, then the Killing vector fields, $\xi_i \in u(1)$ (the Lie algebra of $U(1)$)[4]. So in case of $G = U(1)$[29], we have

$$\Omega = d\omega \tag{11}$$

We see that eq.(11) is an Abelian, a linear equation.

Is there a relationship between the curvature matrix, Ω (10), and the curvature form, $\Omega_{\rho\sigma}$ (8)? Yes (there is)[30]. If $\Omega_{\rho\sigma}$ and $\omega_{\rho\sigma}$ denote *the components* of curvature and connection matrices, Ω and ω, respectively then we can write[24]

$$\Omega = (\Omega_{\rho\sigma}), \quad \omega = (\omega_{\rho\sigma}) \tag{12}$$

So, the curvature matrix (10) can be written using the curvature form[25] as below

$$\Omega_{\rho\sigma} = d\omega_{\rho\sigma} - \omega_\rho{}^\tau \wedge \omega_{\tau\sigma} \tag{13}$$

In case of the Killing vector fields, $\xi_i \in u(1)$[4], the curvature form (13) becomes

$$\Omega_{\rho\sigma} = d\omega_{\rho\sigma} \tag{14}$$

Eq.(9) is *the equation of an Abelian curvature form.*

By substituting eq.(9) into eq.(4), we obtain

$$d\omega_{\rho\sigma} = \sum g \; g_{\rho\sigma} \; N_\mu \; \partial_\nu \ln n \; du^\mu \wedge du^\nu \tag{15}$$

We call eq.(15) *the Abelian curvature form-scalar refractive index relation.*

Let us define *the pfaffian of the curvature matrix* Ω as below[24,31]

$$\text{pf } \Omega \equiv \sum \; \epsilon_{\rho_1\sigma_1 \ldots \rho_{2q}\sigma_{2q}} \; \Omega_{\rho_1\sigma_1} \wedge \ldots \wedge \Omega_{\rho_{2q}\sigma_{2q}} \tag{16}$$

where Ω is *any even-size complex* $2q \times 2q$ *anti-symmetric matrix* (if Ω is *an odd size* complex anti-symmetric matrix, the corresponding pfaffian is defined to be *zero*), $\epsilon_{\rho_1\sigma_1 \ldots \rho_{2q}\sigma_{2q}}$ is the $2q$-th rank Levi-Civita tensor which has value +1 or -1 according as its indices form an even or odd permutation of $1, \ldots, 2q$, and its otherwise zero, and the sum is extended over all indices from 1 to $2q$. Here, $\rho_1 < \sigma_1, \ \ldots \ , \ \rho_{2q} < \sigma_{2q}$ and $\rho_1 < \rho_2 < \ldots < \rho_{2q}$[24,31]. Shortly, the pfaffian of Ω (16) can be rewritten shortly as

$$\text{pf } \Omega = \sum \; \epsilon_{\rho\sigma} \; \Omega_{\rho\sigma} \tag{17}$$

By substituting eq.(9) into eq.(17), we obtain

$$\text{pf}\ \Omega = \sum \epsilon_{\rho\sigma}\ \sum g\ g_{\rho\sigma}\ N_\mu\ \partial_\nu \ln n\ du^\mu \wedge du^\nu \quad (18)$$

Using the pfaffian of Ω, *the Gauss-Bonnet-Chern theorem*[32–34] says that[24,33]

$$(-1)^q \frac{1}{2^{2q}\pi^q q!} \int_{M^{2q}} \text{pf}\ \Omega = \chi(M^{2q}) \quad (19)$$

where q is *a natural number*, $\chi(M^{2q})$ is *the Euler-Poincare characteristic*[35,36] *of the even dimensional oriented compact Riemannian manifold, M^{2q}*. The Euler-Poincare characteristic is *a topological invariant*[24].

By substituting eq.(18) into eq.(19), we obtain the Gauss-Bonnet-Chern theorem as below

$$\chi(M^{2q}) = (-1)^q \frac{1}{2^{2q}\pi^q q!} \int_{M^{2q}} \sum \epsilon_{\rho\sigma} \sum g\ g_{\rho\sigma}\ N_\mu\ \partial_\nu \ln n\ du^\mu \wedge du^\nu \quad (20)$$

We see from eq.(20), the scalar refractive index is related to the Euler-Poincare characteristic. *Because the Euler-Poincare characteristic is a topological invariant*[37,38] *then the scalar refractive index is also a topological invariant.*

In non-linear optics, by substituting eq.(6) into eq.(20), we obtain the Gauss-Bonnet-Chern theorem as below

$$\chi(M^{2q}) = (-1)^q \frac{1}{2^{2q}\pi^q q!} \int_{M^{2q}} \sum \epsilon_{\rho\sigma} \sum g\ g_{\rho\sigma}\ N_\mu\ \partial_\nu \ln\left(12\pi\ n^{pqrs}\ \chi_{pqrs}\right) du^\mu \wedge du^\nu \quad (21)$$

We see from eqs.(6), (21),*the scalar refractive index is decomposed into the contravariant fourth rank tensor of the non-linear refractive index and the fourth rank tensor of susceptibilty. Does it mean that, as a consequence of the decomposed scalar refractive index, the related Euler-Poincare characteristic is also decomposed? Because the Euler-Poincare characteristic is a topological invariant*[37,38] *then does it mean that the decomposed scalar refractive index i.e. the contravariant fourth rank tensor of the non-linear refractive index and the fourth rank tensor of susceptibilty are topological invariants?*

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[11] Light is an electromagnetic wave, and the electric field of this wave oscillates perpendicularly to the direction of light propagation. If the direction of the electric field of light is well defined, it is polarized light. The most common source of polarized light is a laser[10].

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[14] The susceptibility of a material or substance describes its response to an applied field (Wikipedia, *Susceptibility*). The magnetic susceptibility is *a measure* of how much a material will become magnetized in an applied magnetic field (Wikipedia, *Magnetic susceptibility*). The electric susceptibility is *a dimensionless proportionality constant* that indicates the degree of polarization of a dielectric material in response to an applied electric field (Wkipedia, *Electric susceptibility*).

[15] Wikipedia, *Nonlinear system.*

[16] Wikipedia, *Nonlinear optics.*

[17] A non-linear system is a system in which the change of the output is not proportional to the change of the input[15]. In optics, the non-linearity is typically observed only at very high intensities (field strength) of light such as those provided by lasers[16].

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[19] Francois Cardarelli, *Materials Handbook: A Concise Desktop Reference*, Springer, 2017.

[20] Some values of the linear refractive index, n_0, and the non-linear refractive index, n_2, of some oxides are shown in Table I of Dimitrov-Sakka[18]. Figure 1 of Dimitrov-Sakka shows the "exponential relation" between the linear refractive index, n_0, and the non-linear refractive index, n_2[18]. It means that the non-linear refractive index increases exponentially with the increasing of the linear refractive index.

[21] In general, susceptibility is a complex quantity. The real part is related to the refraction, while the imaginary part is related to the absorption.

[22] Chen Ning Yang, *Topology and Gauge Theory in Physics*, International Journal of Modern Physics A, Vol. 27, No. 30 (2012) 1230035.

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[26] WolframMathWorld, *Antisymmetric Matrix*, https://mathworld.wolfram.com/AntisymmetricMatrix.html

[27] An antisymmetric matrix is a square matrix that satisfies the identity $A = -A^T$ where A^T is the matrix transpose. *All $n \times n$ antisymmetric matrices of odd size (i.e. if n is odd) are singular (determinant of matrix is equal to zero)*. Antisymmetric matrices are commonly called "skew symmetric matrices" by mathematicians[26].

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[34] *Gauss-Bonnet formula expresses the global invariant, $\chi(M)$, as the integral of a local invariant*, which is perhaps the most desirable relationship between local and global properties[32].
For *even-dimensional* oriented compact Riemannian manifold, M^{2n}, the Gauss-Bonnet-Chern theorem is a special case of the Atiyah-Singer index theorem[33].

[35] Milosav M. Marjanovic, *Euler-Poincare Characteristic - A Case of Topological Convincing*, The Teaching of Mathematics, 2014, Vol. XVII, 1, pp. 21–33.

[36] The Euler-Poincare characteristic starts from *Euler's polyhedron formula (a number)* which appeared first in a note submitted by Euler to the Proceedings of the Petersburg Academy of 1752/53. Henri Poincare who defined *an integer* to be *a topological property* of all other geometric objects. *The Euler-Poincare characteristic is a stable topological property*[35].

[37] Topological Invariant. Encyclopedia of Mathematics. https://encyclopediaofmath.org/wiki/Topological_invariant.

[38] Topological invariant is any property of a topological space that is *invariant* under *homeomorphisms*[37]. Homeomorphisms are, roughly speaking, the mappings that preserve all the topological properties of a given space.