Division Algebra Covariant Derivative

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Abstract

The classical definition for the gradient, divergence and curl utilizing the limit as the volume approaches zero, of the ratio of the integral over the enclosing surface divided by the integral over the enclosed volume can be consolidated to form an Ensemble Derivative. This form is constructed initially as a diffeomorphism between two compatible coordinate sets, one representing the intrinsic division algebra basis; u(v) and the other some basis v(u) within which differentiation is defined. The result is the structure for general covariance for division algebra analysis, where covariant differential equations are constructed by full applications of the Ensemble Derivative.

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There are alternate definitions for the familiar 3D gradient, divergence and curl that involve a limiting process on the ratio of the integral over the enclosing surface, to the integral over the enclosed volume as that volume approaches zero, while maintaining the point of application as an interior point.

The differential surface normal vector **dS** multiplies the scalar function for gradient.

$$\nabla \rho = \lim_{\int dV \to 0} \frac{\int \rho \, \mathbf{dS}}{\int dV}$$

For divergence, the scalar product of differential surface normal and function vector is used.

$$\nabla \bullet \mathbf{A} = \lim_{\int dV \to 0} \frac{\int \mathbf{A} \bullet \mathbf{dS}}{\int dV}$$

For curl, the cross product of differential surface normal and function vector is used.

$$\nabla \times \mathbf{A} = \lim_{\int dV \to 0} \frac{\int \mathbf{dS} \times \mathbf{A}}{\int dV}$$

The numerators in these three forms are examples of scalar - vector multiplication, scalar result vector - vector multiplication, and vector result vector - vector multiplication. All of these are present in the single product of two Quaternions.

A natural extension of the integral definitions for gradient, divergence and curl mentioned above would be to consolidate all three into one *ensemble* by replacing the three individual products with a single full algebraic product between a functional Quaternion algebraic element **A** and a 4D Quaternion differential surface normal, and replacing the 3D differential volume with the corresponding 4D differential volume. This would give us a holistic expression for the Quaternion Calculus operation of differentiation that is an ensemble of all three classical forms shown above.

Define the *Left Ensemble Derivative* and *Right Ensemble Derivative* as (see also reference [1])

$$\mathbf{E}(\mathbf{A}) = \lim_{\int dV \to 0} \frac{\int \mathbf{dS} * \mathbf{A}}{\int dV} \qquad (\mathbf{A})\mathbf{E} = \lim_{\int dV \to 0} \frac{\int \mathbf{A} * \mathbf{dS}}{\int dV}$$

We must of course define the derivative application from both sides if we are dealing with a non-commutative algebra. I take these integral representations to be the fundamental definition for n dimensional multivariate differentiation over a prescribed algebra, not just having utility as select alternate 3D forms. Thus, the definition will not be restricted to the Quaternions, it extends directly to the Octonions or *generally for any n dimensional algebra over an n dimensional vector space*. For Octonion Algebra, we just need to substitute the 8D algebraic element definitions for the differential surface normal and function **A**, the Octonion product * and 8D differential volume.

The integral definition for differentiation has the ability to easily move to a description using an alternate independent variable set. If a functional relationship exists between the two variable sets, Jacobian formalism can be used to cast the differential surface normal and differential volume element in terms of the new set of variables, Jacobians, Jacobian matrices and their co-factor matrices. This allows us to *define* differentiation more fundamentally in the form of a proper diffeomorphism between the intrinsic algebraic system and an alternate system existing within the rules of the same basic algebra. The transformation properties for differential equations then become intrinsic to the proper fundamental definition of differentiation itself, not an add-on, afterthought or something modified from a more simplistic (e.g. rectilinear) definition.

When partial differential equations are constructed free-form (by hand) within one coordinate system, they may not be appropriate in some other coordinate system even though there is a smooth map between systems and all partial derivatives are continuous. These differential equations are described as not being *covariant*. Problems can occur when the description of one coordinate system has dependencies on multiple independent variables spanning the alternate set. Since differentiation is a measure of functional variability within the neighborhood of the single point of application, this variability within the coordinate system itself is also in play. A proper, which is to say covariant, definition for differentiation must account for the variability of the coordinate system fundamentally. As defined, the Ensemble Derivative form does just that. The covariance comes from the consolidation of all subforms that were taken to be individually fundamental in the late 1800's. Any covariant expression must likewise be cast with full ensemble forms. To this end, *proper covariant differential equations describing physical effects must be constructed with whole applications of the Ensemble Derivative. The Ensemble Derivative becomes the fundamental building block for general covariance.*

Let's move on within an Octonion Algebra framework. For a given native Octonion space spanned by our familiar intrinsic basis elements, assign an algebraic element for rectilinear position as

$$\mathbf{u} = u_0\, e_0 + u_1\, e_1 + u_2\, e_2 + u_3\, e_3 + u_4\, e_4 + u_5\, e_5 + u_6\, e_6 + u_7\, e_7$$

The coefficients defining positional coordinates here are real valued, without bound and without granularity, allowing coordinate neighborhoods about any specific point to be smoothly and continuously defined. We may then define intrinsic and continuous Octonion functional elements as algebraic elements with coefficients that are real valued functions of position **u** in the form of

$$\mathbf{A}(\mathbf{u}) = A_0(\mathbf{u})\, e_0 + A_1(\mathbf{u})\, e_1 + A_2(u)\, e_2 + A_3(\mathbf{u})\, e_3 + A_4(\mathbf{u})\, e_4 + A_5(\mathbf{u})\, e_5 + A_6(\mathbf{u})\, e_6 + A_7(\mathbf{u})\, e_7$$

Next define an alternate positional representation **v** for the space defined by **u**. The objective here will be to cast the definition of differentiation in terms of this alternate coordinate system and its transformation back to the native **u** coordinates. We will require a functional relationship between the intrinsic position **u** and the new system **v** such that there is a smooth, singular mapping of each unique position in **u** singularly to each unique position in **v**. We will also require a smooth mapping of each position in **v** singularly back to the same original unique position in **u**. Since we will not make any assumptions about the specific form the transformation will take, we will express the mappings as the unspecified functions

$u_j \rightarrow u_j(\mathbf{v})$
$v_k \rightarrow v_k(\mathbf{u})$

It will be convenient from now on to write expressions using summation notation and explicitly call out any exceptions, as implied in this representation for **u** where repeated indexes in a product term are summed over their range if not singularly stated on the other side of =.

$$u_0(\mathbf{v})\, e_0 + u_1(\mathbf{v})\, e_1 + u_2(\mathbf{v})\, e_2 + u_3(\mathbf{v})\, e_3 + u_4(\mathbf{v})\, e_4 + u_5(\mathbf{v})\, e_5 + u_6(\mathbf{v})\, e_6 + u_7(\mathbf{v})\, e_7 = u_j(\mathbf{v})\, e_j$$

Assume the following partial differentials exist and are continuous, and differentiate each sub-element coefficient of **u** with index j by partial ∂v_i to yield the tangent coefficient matrix **T**

$$T_{ij} = \partial u_j / \partial v_i$$

The determinant of $\mathbf{T} = |\, \mathbf{T}\, |$ is the Jacobian of the transformation $\mathbf{v} \rightarrow \mathbf{u}$. Define this as

$$J = |\, \mathbf{T}\, |$$

J must of course be non-zero for both transformations $\mathbf{u} \leftrightarrow \mathbf{v}$ to exist.

We can define the coefficients of hypercomplex system differential position displacement du_j as

$$du_j = T_{ij}\, dv_i$$

We may define an Octonion algebraic element basis set for the general curvilinear space **v** as **z**:

$$z_i = T_{ij}\, e_j$$

If we look at the magnitude (norm) of each z_i they will not in general be unity. While being so is not a problem mandating any adjustment, it sometimes is beneficial to represent using a unit magnitude basis element set for system **v**. Define the unity magnitude basis **w** then as

$$w_i = z_i / N(z_i) = T_{ij} / (T_{ik}\, T_{ik})^{1/2}\, e_j \qquad \text{fixed i and the sum over k is performed before the square root}$$

From this we may define an Octonion algebraic element equivalent to $\mathbf{A}(\mathbf{u})$ as

$$\mathbf{F}(\mathbf{v}) = F_i(\mathbf{v})\, z_i(\mathbf{v}) = F_i\, T_{ij}\, e_j$$

Here we emphasize both the coefficients and the new system basis vectors are functions of **v**. This is very important to remember when the calculus of the system is explored. If we let position **u'** correspond to position **v'** we must have

$$\mathbf{A}(\mathbf{u}') = \mathbf{F}(\mathbf{v}')$$

In this equality of course, the coefficients attached to like **e** basis elements on both sides match one by one, nothing more. This brings up one very important thing to keep in mind. For any diffeomorphism to or from an alternate hypercomplex coordinate system **v**, we never lose the fundamental basis elements

e_j. They are always present, and always fundamentally define the operation of multiplication the same way independent of the choice of **v**. This implies Octonion Algebraic Invariance is coordinate system invariant, being entirely determined by the underlying algebra common to all **v** systems.

The task now is to define the Ensemble Derivative form at a single point of application within the coordinate space for our **v** coordinate system using our fundamental limiting process above.

Define **C** as the matrix of co-factors of **T**, where C_{ij} is the co-factor for **T** element T_{ij}.

We may then define the inverse of **T** to be $\mathbf{T}^{-1} = (1/J)\, C^T$ where $C^T_{jk} = J\, \partial v_k / \partial u_j$ is the transpose of **C**.

The last equality comes about by $T_{ij}\, T_{jk}^{-1} = \delta_{ik}$ (1 if i=k, 0 if i≠k) and $(\partial u_j / \partial v_i \;\; \partial v_k / \partial u_j) = \delta_{ik}$

In the hypercomplex **v** system, the differential volume element is scaled by the Jacobian and can be expressed in the **v** system as

$$dV = J\, dv_0\, dv_1\, dv_2\, dv_3\, dv_4\, dv_5\, dv_6\, dv_7$$

Within the limit process above, the point of application is always an interior point of the volume we are taking in the limit tending towards zero. Since this limit definition process allows the enclosing surface to get arbitrarily close to this interior point, we may take the Jacobian within the differential volume element in the limit definition denominator outside of our volume integral, using instead its mean value defined as the Jacobian evaluated at the point of application.

The differential surface normal d**S** in the **v** system may be expressed in terms of co-factor expansion as the Octonion algebraic element

$$\mathbf{dS_i} = C_{ij}\, e_j\, dv_0\, dv_1\, dv_2\, dv_3\, dv_4\, dv_5\, dv_6\, dv_7 / dv_i$$
$$= J\, dv_i/du_j\, e_j\, dv_0\, dv_1\, dv_2\, dv_3\, dv_4\, dv_5\, dv_6\, dv_7 / dv_i$$

Define the unit surface normal algebraic elements as $\mathbf{n_i}$. We may then define

$$C_{ij}\, e_j\, dv_0\, dv_1\, dv_2\, dv_3\, dv_4\, dv_5\, dv_6\, dv_7 / dv_i = \mathbf{n_i}\, dS_i.$$

Unlike the Jacobian in the volume integral, the position the surface integral Jacobian is evaluated at is never at the point of application for the Ensemble Derivative. The limit process allows the surface to get arbitrarily close to the point of application, but it never gets there. Thus, the limit definition numerator Jacobian variation within the coordinate neighborhood of the point of application is very much in play, and as such it is not extricable from the limiting process on the surface integral.

The limit expression for **E**(**F**) then becomes

$$\mathbf{E}(\mathbf{F}) = \frac{1}{J} \lim_{\int dV \to 0} \frac{\int (J\;\, dv_i/du_j\;\, e_j\;\, dv_0\, dv_1\, dv_2\, dv_3\, dv_4\, dv_5\, dv_6\, dv_7 / dv_i) * \mathbf{F}}{\int dv_0\, dv_1\, dv_2\, dv_3\, dv_4\, dv_5\, dv_6\, dv_7}$$

The limit process will yield the following for the left application Ensemble Derivative **E** on the Octonion algebraic element **F**(**v**)

$$\mathbf{E}(\mathbf{F}(\mathbf{v})) = 1/J\; \partial/\partial v_i\, [\, J\, \partial v_i/\partial u_j\, e_j * \mathbf{F}(\mathbf{v})\,]$$

Since we have $C_{ij} = J\, \partial v_i/\partial u_j$ this may also be written as

$\mathbf{E}(\mathbf{F}(\mathbf{v})) = 1/J\ \partial/\partial v_i\ [\ C_{ij}\ e_j * \mathbf{F}(\mathbf{v})\ ]$

It is critical to emphasize the point that **F**(**v**) here is a regular Octonion algebraic element. It is not a vector of **v** system basis coefficients, it is the sum of all scalar functionals dependent on **v** that happen to be attached to the same intrinsic basis element set member, each in turn. Likewise, the calculated result for **E**(**F**(**v**)) is a regular Octonion algebraic element. If one desires the result to be represented by the set of coefficients attached to the **v** system basis elements **z**, additional manipulation is required. More on this shortly.

We have, rewriting from above using $\mathbf{F}(\mathbf{v}) = F_k\, T_{kl}\, e_l$

$\mathbf{E}(\mathbf{F}(\mathbf{v})) = 1/J\ \partial/\partial v_i\ [\ C_{ij}\ \ T_{kl}\ \ F_k\ ]\ \ e_j * e_l$

Here, the intrinsic basis elements are constants to the partial differentiation so can be moved out of the differentiation target. The F_k here are now the coefficients that scale the **v** system basis element z_k.

For both of these representations, the application of **E** from the right-side amounts to simply transposing the basis element products.

$(\mathbf{F}(\mathbf{v}))\mathbf{E} = 1/J\ \ \partial/\partial v_i\ [\ C_{ij}\ \ T_{kl}\ \ F_k\ ]\ e_l * e_j$

We can trivialize the transformation to determine the Ensemble Derivative in the **u** system by equating **u** and **v**. Then the Jacobian J = 1, and $\partial v_i/\partial u_j = \delta_{ij}$ and $\partial u_l/\partial v_k = \delta_{kl}$. The Ensemble Derivative on **A**(**u**) then becomes

$\mathbf{E}(\mathbf{A}(\mathbf{u})) = \partial/\partial u_i\ [\ \delta_{ij}\ \delta_{kl}\ A_k\ ]\ \ e_j * e_l$
$= \partial/\partial u_j\ A_l(\mathbf{u})\ e_j * e_l$

We can thus legitimately define *a posteriori* the "del" algebraic element operator as $\nabla_j = \partial/\partial u_j\ e_j$ and write

$\mathbf{E}(\mathbf{A}(\mathbf{u})) = \nabla * \mathbf{A}$

This differential operator multiplies like any other Octonion algebraic element under the rules of the selected algebra. The partial differentiation is applied as a scalar operation on all of the operand coefficients separate from algebraic multiplication of the basis elements which of course are constants not participating in these partials.

From above we had

$\mathbf{E}(\mathbf{F}(\mathbf{v})) = 1/J\ \ \partial/\partial v_i\ [\ J\ \partial v_i/\partial u_j\ e_j * \mathbf{F}(\mathbf{v})\ ]$

When differentiating with respect to equivalent intrinsic basis elements on coefficients that in one case are functions dependent on the variable **u**, and in the other case functions dependent on the variable **v**, we could infer an equivalence map

$\partial/\partial u_j\ [\ \ldots.\ \rightarrow\ 1/J\ \ \partial/\partial v_i\ [\ J\ \partial v_i/\partial u_j\ \ldots.$

This is different from the common chain rule, which states

$\partial/\partial u_j\ [\ \ldots.\ \rightarrow\ \ \partial v_i/\partial u_j\ \ \partial/\partial v_i\ [\ \ldots.$

The two will only be identical for the specific case when J $\partial v_i/\partial u_j$ is not a function of **v**. A fair conclusion on this would be that the chain rule is not as general as many assume it is. But it is the foundation for tensor transformations, defining the required tensor form invariance over transformation of variables. Within tensor analysis, the definition of differentiation must *repair* this problem somehow, for it is indeed broken. Enter the Christoffel symbols to the rescue. Their addition to the tensor analysis covariant derivative is a repair job on the unfortunate, but truthfully somewhat utilitarian invariant form.

The Ensemble Derivative on the other hand, correctly deals with the variability of the coordinate system coordinates themselves, and thus is *intrinsically and generally covariant*. A differential equation formed with *whole applications* of the Ensemble Derivative will be intrinsically covariant. This is an additional reason why I have given it the moniker *Ensemble* Derivative. No slicing, no dicing, no cleaver uses of vector identities glued together by hand required or even allowed to insure covariant results.

However, just as we combined the separate limit on integration ratio representations for divergence, gradient and curl to define the Ensemble Derivative, we will be able to algebraically separate portions of its full description into their individual representations in the **v** basis system. The "how to do this" comes from understanding the notion of what each of these three forms are. What descriptively separates them is their algebraic description embodied by their intrinsic basis element products ever present for any **v** system definition. The Ensemble Derivative intrinsic basis element products are sums over all indexes j and l on the form $e_j * e_l$ where these indexes also are present in the **T** and **C** portions. If we were to restrict indexes j and l to j = l and sum both over the range 1 to 3 in our Quaternion representation, the result after also summing over all i and k without restriction will be the negative of the divergence as represented in the **v** system. We will do this later to verify the Ensemble Derivative form reproduces the correct known results for spherical-polar coordinates.

Let us next take a look at the volume integral of the Ensemble derivative. We have

$$\int \mathbf{E}(\mathbf{F}(\mathbf{v}))\, dV = \int\ 1/J\ \partial/\partial v_i\ [\ C_{ij}\ \ T_{kl}\ \ F_k\ ]\ \ e_j * e_l\ \ \ J\, dv_0\, dv_1\, dv_2\, dv_3\, dv_4\, dv_5\, dv_6\, dv_7$$

Integrating over each index i individually using The Fundamental Law of Calculus we have

$$\int \mathbf{E}(\mathbf{F}(\mathbf{v}))\, dV = \int [\ C_{ij}\ \ T_{kl}\ \ F_k\ ]\ \ e_j * e_l\ \ dv_0\, dv_1\, dv_2\, dv_3\, dv_4\, dv_5\, dv_6\, dv_7 / dv_i$$

From above this is equivalent to, taking the n-volume integration over the enclosing (n–1)-surface S with surface normal differential set $\mathbf{dS_i}$

$$\int \mathbf{E}(\mathbf{F}(\mathbf{v}))\, dV = \int \mathbf{dS_i} * \mathbf{F}(\mathbf{v}) \quad \text{summed over index i}$$

This is the generalized Stokes theorem for the Ensemble Derivative, or what should be called The Fundamental Law of Multivariate Calculus over any n-dimensional space with an n-dimensional algebra defining the operation *. Just as the Ensemble Derivative unifies the classical operations of curl, divergence and gradient, this expression unifies Stokes' Theorem, the Gauss Divergence Theorem and Green's Theorem. The equivalent representation for application of the Ensemble Derivative from the right is easily shown as

$$\int (\mathbf{F}(\mathbf{v}))\mathbf{E}\, dV = \int \mathbf{F}(\mathbf{v}) * \mathbf{dS_i} \quad \text{summed over index i}$$

For many applications of differentiation, it is optimal to have the results expressed in terms of the coefficients that scale each of the **v** system basis elements. We touched on this above when it was pointed out that the **v** system function being differentiated is placed within the Ensemble Derivative as a regular algebraic element, and the result that pops out is likewise a regular algebraic element. The reason they must be is the operation * defined by the applied algebra must be performed within the definition of the Ensemble derivative and is independent of any **v** system definition. The specific summed quasi **v** system coefficients attached to each of the two intrinsic basis elements multiplied are scalar multiplied to set the coefficient sum attached to the product result, which becomes a portion of the final result post partial differentiation.

Two different ways to look at **F**(**v**) or result **R**(**v**) = **E**(**F**(**v**)) generally use a common form **M**(**v**)

$\mathbf{M}(\mathbf{v}) = M_k\, T_{kl}\, e_l = M_k\, z_k$ k not summed but separately maintained, index l summed

$\mathbf{M}(\mathbf{v}) = M_k\, T_{kl}\, e_l = M'_l\, e_l$ l not summed but separately maintained, index k summed

What we put into the Ensemble Derivative and what it gives back is the latter form. Each of the M'_l are known sums of coefficients going in, and coming out post calculation. The M_k are not necessarily readily apparent in these sums due to the **z** structure. If they are of interest, we need a way to map between M'_l and M_k.

If we have all M_k in hand, forming the algebraic element **M**(**v**) from them is easy to build since **T** is known. Therefore, the map $M_k \rightarrow M'_l$ is simply $M'_l = M_k\, T_{kl}$.

If we want to retrieve each M_k from known M'_l, we must do a bit more work to formulate their proper extraction. We have $T_{kl} = \partial u_l/\partial v_k$ so

$M'_l = M_k\, T_{kl} = M_k\, \partial u_l/\partial v_k$ so multiplying both sides by $\partial v_m/\partial u_l$ then summing over l we get

$M'_l\, \partial v_m/\partial u_l = M_k\, \partial u_l/\partial v_k\, \partial v_m/\partial u_l$ but $(\partial u_l/\partial v_k\, \partial v_m/\partial u_l)$ sum l = δ_{km} so

$M'_l\, \partial v_m/\partial u_l = M_k\, \delta_{km} = M_m$ and we have $\partial v_m/\partial u_l = 1/J\, C_{ml}$ so the map $M'_l \rightarrow M_m$ is

$M_m = 1/J\, C_{ml}\, M'_l$

Another nicety is to represent **F**(**v**) and the result of its Ensemble derivative in terms of a unity magnitude **v** system basis $w_k = z_k/|z_k|$. This is neither here nor there in developing **T** and generating its cofactor matrix **C**. These must remain direct results of the relationship defining **u**(**v**) and its partial derivatives forming **T** leading to **C** independent of the use or not of a unit norm **v** basis. We can however represent **F**(**v**) in terms of the **v** system unit norm basis and expect its Ensemble Derivative to likewise represented. We have

$T_{kl}\, e_l = z_k$ and therefore $|z_k|$ = the norm of the algebraic element formed by the kth row of T_{kl} call N_k where we have

$N_k = (T_{kl}\, T_{kl})^{1/2}$ of course summing over l before taking the square root

Replacing z_k in in the equality $M_k\, z_k = M'_j\, e_j$ with $z_k/|z_k|$ leads to the modification of the algebraic element to orthonormal **v** system basis coefficients map

$M_k = N_k/J\, C_{kj}\, M'_j$

We must modify for the unity norm **v** system representation in algebraic elements to

$\mathbf{F}(\mathbf{v}) = F_k\, z_k/|z_k| = F_k\, w_k = F_k\, T_{kl}/N_k\, e_l$

These maps in both directions, whether or not unity scaled by choice, between algebraic element coefficients and **v** system basis coefficients are not limited to functions to form the Ensemble derivative on or its result. They are the general transformations between the intrinsic basis and the **v** basis systems.

Since the Ensemble Derivative is applicable to Quaternions, and they are a quite a bit simpler than Octonions, it will be instructive to work out the case where the **v** system is spherical-polar coordinates with the addition of a time variable in the Quaternion scalar position, since we know or at least can readily look up what the correct answers should be. Define the **v** system as independent variables t, r, θ and φ with the corresponding mapping to the rectilinear **u** system having coefficients

$u_0 = c\ t$
$u_1 = r \sin(\theta) \cos(\varphi)$
$u_2 = r \sin(\theta) \sin(\varphi)$
$u_3 = r \cos(\theta)$

We are taking t to represent the scalar time component of algebraic time here, since we are limiting the discussion to $\mathbb{H}$. With c having the dimensions of speed: distance/time, all **u** system coefficients have dimension of length as we must require for the intrinsic rectilinear system. On the other hand, we have complete freedom for the dimensionality of the **v** system, as diverse as shown. We have for **T** then

$T_{00} = c$ $\quad T_{0j} = T_{j0} = 0$ for $j \neq 0$

$T_{11} = \sin(\theta) \cos(\varphi)$	$T_{12} = \sin(\theta) \sin(\varphi)$	$T_{13} = \cos(\theta)$
$T_{21} = r \cos(\theta) \cos(\varphi)$	$T_{22} = r \cos(\theta) \sin(\varphi)$	$T_{23} = -r \sin(\theta)$
$T_{31} = -r \sin(\theta) \sin(\varphi)$	$T_{32} = r \sin(\theta) \cos(\varphi)$	$T_{33} = 0$

The Jacobian for this transformation is $|\ \mathbf{T}\ | = c\ r^2 \sin(\theta)$. Now for the co-factor matrix **C** we have

$C_{00} = r^2 \sin(\theta)$ $\quad C_{0j} = C_{j0} = 0$ for $j \neq 0$

$C_{11} = c\ r^2 \sin^2(\theta) \cos(\varphi)$	$C_{12} = c\ r^2 \sin^2(\theta) \sin(\varphi)$	$C_{13} = c\ r^2 \sin(\theta) \cos(\theta)$
$C_{21} = c\ r \sin(\theta) \cos(\theta) \cos(\varphi)$	$C_{22} = c\ r \sin(\theta) \cos(\theta) \sin(\varphi)$	$C_{23} = -c\ r \sin^2(\theta)$
$C_{31} = -c\ r \sin(\varphi)$	$C_{32} = c\ r \cos(\varphi)$	$C_{33} = 0$

Typical literature for spherical-polar coordinates shows the functional coefficients as unity basis scaling factors. Since we will be looking for a direct comparison and such scaling impacts the calculus we are doing, we will do the same here. The norms for the **z** basis $z_i = T_{ij}\ e_j$ for the **v** system can be seen to be

$N(z_0) = N_0 = c$ $\quad N(z_1) = N_1 = 1$ $\quad N(z_2) = N_2 = r$ $\quad N(z_3) = N_3 = r \sin(\theta)$

Before we do the differentiations $\partial/\partial v_i$ we must replace the coefficients z_k with the orthonormal basis $w_k = z_k / N_k$ and make $\mathbf{F}(\mathbf{v})$ appropriate for the **w** basis: $\mathbf{F}(\mathbf{v}) \rightarrow F_k\ T_{kl} / N_k\ e_l$.

The full Ensemble Derivative is the sum of results for $\mathbf{E}(\mathbf{F}(\mathbf{v})) = 1/J\ \ \partial/\partial v_i\ [\ C_{ij}\ \ T_{kl}\ \ F_k / N_k\ ]\ \ e_j * e_l$ for each basis element product combination. Since we will first be looking for comparisons with the results for gradient, divergence and curl, we must individually sort them out from the full ensemble. To do this we must understand the rectilinear equivalence is provided by the algebraic product $e_j * e_l$. We need only pick out the easily determined product pairs for each of the separate differential forms we seek.

For the gradient with time derivative, we must restrict the Ensemble Derivative result basis index range to products $e_j * e_0$ for j: 0 to 3 and i,k summed over their full range, we have the result algebraic element for $\mathbf{E}(\mathbf{F}(\mathbf{v})) = 1/J\ \ \partial/\partial v_i\ [\ C_{ij}\ T_{k0}\ \ (1/N(z_k))\ F_k\ ]\ \ e_j * e_0$

$\{ 1/c\ \partial/\partial t\ (F_0) \}\ e_0$

$\{ \sin(\theta) \cos(\varphi)\ \partial/\partial r\ (F_0) + \cos(\theta) \cos(\varphi)\ \partial/\partial\theta\ (F_0) / r - \sin(\varphi)\ \partial/\partial\varphi\ (F_0) / r \sin(\theta) \}\ e_1$

$\{ \sin(\theta) \sin(\varphi)\ \partial/\partial r\ (F_0) + \sin(\varphi) \cos(\theta)\ \partial/\partial\theta\ (F_0) / r + \cos(\varphi)\ \partial/\partial\varphi\ (F_0) / r \sin(\theta) \}\ e_2$

$\{ \cos(\theta)\ \partial/\partial r\ (F_0) - \sin(\theta)\ \partial/\partial\theta\ (F_0) / r \}\ e_3$

Mapping to the **v** system basis, its coefficients are

$\{ 1/c\ \partial/\partial t\ (F_0) \}\ w_t$

$\{ \partial/\partial r\ (F_0) \}\ w_r$

$\{ \partial/\partial\theta\ (F_0) / r \}\ w_\theta$

$\{ \partial/\partial\varphi\ (F_0) / r \sin(\theta) \}\ w_\varphi$

This is the expected form for the spherical-polar gradient of F_0 + F_0 time derivative.

For $e_n * e_n$ for n: 1 to 3 we have the algebraic element form for the negated divergence

$\{- \partial/\partial\varphi\ (F_\varphi) / r \sin(\theta) - \partial/\partial r\ (F_r) - F_\theta \cos(\theta) / r \sin(\theta) - 2F_r / r - \partial/\partial\theta\ (F_\theta) / r \}\ e_0$

Mapping to the **v** system basis, its coefficients are the same

$\{- \partial/\partial\varphi\ (F_\varphi) / r \sin(\theta) - \partial/\partial r\ (F_r) - F_\theta \cos(\theta) / r \sin(\theta) - 2F_r / r - \partial/\partial\theta\ (F_\theta) / r \}\ w_t$

This is the expected form for the negated divergence in spherical-polar coordinates.

Next the components of the curl. The curl is the evaluation of

$$
\begin{aligned}
&= 1/J\ \{ \partial/\partial v_i [C_{i2} T_{j3}\ (1/N(z_j)) F_j]\ e_2 * e_3 + \partial/\partial v_k [C_{k3} T_{l2}\ (1/N(z_l)) F_l]\ e_3 * e_2 \} \\
&+ 1/J\ \{ \partial/\partial v_i [C_{i3} T_{j1}\ (1/N(z_j)) F_j]\ e_3 * e_1 + \partial/\partial v_k [C_{k1} T_{l3}\ (1/N(z_l)) F_l]\ e_1 * e_3 \} \\
&+ 1/J\ \{ \partial/\partial v_i [C_{i1} T_{j2}\ (1/N(z_j)) F_j]\ e_1 * e_2 + \partial/\partial v_k [C_{k2} T_{l1}\ (1/N(z_l)) F_l]\ e_2 * e_1 \}
\end{aligned}
$$

The algebraic element result is

$\{ \cos(\theta) \cos(\varphi)\ \partial/\partial\varphi\ (F_r) / r \sin(\theta) - \cos(\varphi)\ \partial/\partial\varphi\ (F_\theta) / r - F_\theta \sin(\varphi) / r - \cos(\theta) \cos(\varphi)\ \partial/\partial r\ (F_\varphi)$
$+ \sin(\theta) \cos(\varphi)\ \partial/\partial\theta\ (F_\varphi) / r - \sin(\varphi)\ \partial/\partial r\ (F_\theta) + \sin(\varphi)\ \partial/\partial\theta\ (F_r) / r \}\ e_1$

$\{- \sin(\varphi) \cos(\theta)\ \partial/\partial r\ (F_\varphi) + \sin(\theta) \sin(\varphi)\ \partial/\partial\theta\ (F_\varphi) / r + \sin(\varphi) \cos(\theta)\ \partial/\partial\varphi\ (F_r) / r \sin(\theta)$
$- \sin(\varphi)\ \partial/\partial\varphi\ (F_\theta) / r + F_\theta \cos(\varphi) / r + \cos(\varphi)\ \partial/\partial r\ (F_\theta) - \cos(\varphi)\ \partial/\partial\theta\ (F_r) / r \}\ e_2$

$\{ \sin(\theta)\ \partial/\partial r\ (F_\varphi) + \cos(\theta)\ \partial/\partial\theta\ (F_\varphi) / r - \partial/\partial\varphi\ (F_r) / r - \cos(\theta)\ \partial/\partial\varphi\ (F_\theta) / r \sin(\theta) + F_\varphi / r \sin(\theta) \}\ e_3$

Mapping to the **v** system basis, its coefficients are

$\{ \partial/\partial\theta\ (F_\varphi) / r + F_\varphi \cos(\theta) / r \sin(\theta) - \partial/\partial\varphi\ (F_\theta) / r \sin(\theta) \}\ w_r$

$\{ \partial/\partial\varphi\ (F_r) / r \sin(\theta) - \partial/\partial r\ (F_\varphi) - F_\varphi / r \}\ w_\theta$

$\{ F_\theta / r + \partial/\partial r (F_\theta) - \partial/\partial\theta (F_r) / r \} w_\varphi$

This can be recognized as the curl expressed in spherical-polar coordinates.

Now for something a bit more involved, the second order **v** system equivalent of $-\nabla^2$ **A**, where here we will take ∇^2 to be the full D'Alembertian. The Ensemble form once again is

$\mathbf{E}(\mathbf{F}(\mathbf{v})) = 1/J\ \partial/\partial v_i\ [\ C_{ij}\ \ T_{kl}\ \ F_k\ ]\ \ e_j * e_l$

The equivalent form for $-\nabla^2$ **A** is created by doing the complete Ensemble Derivative but with fixed index j, then a second application on its result once again with the same fixed j index, adding the results of each of these fixed j second order Ensemble Derivatives for all j values. The algebraic element result for our Quaternion spherical-polar **v** system is

$$\{ 1/c^2\ \partial^2/\partial t^2 (F_0) - \cos(\theta)\ \partial/\partial\theta (F_0) / r^2 \sin(\theta) - \sin^{-2}(\theta)\ \partial^2/\partial\varphi^2 (F_0) / r^2 \sin^2(\theta) - 2\ \partial/\partial r (F_0) / r$$
$$- \partial^2/\partial r^2 (F_0) - \partial^2/\partial\theta^2 (F_0) / r^2 \}\ e_0$$

$$\{ -2 \sin(\theta) \cos(\varphi)\ \partial/\partial r (F_r) / r - \sin(\theta) \cos(\varphi)\ \partial^2/\partial r^2 (F_r) - 2 \cos(\theta) \cos(\varphi)\ \partial/\partial r (F_\theta) / r$$
$$+ 2F_\theta \cos(\theta) \cos(\varphi) / r^2 - 3 \cos(\theta) \cos(\varphi)\ \partial/\partial\theta (F_r) / r^2 - \cos^2(\theta) \cos(\varphi)\ \partial/\partial\theta (F_\theta) / r^2 \sin(\theta)$$
$$- \cos(\theta) \cos(\varphi)\ \partial^2/\partial\theta^2 (F_\theta) / r^2 + \sin(\varphi) \cos(\theta)\ \partial/\partial\theta (F_\varphi) / r^2 \sin(\theta) + 2 \sin(\varphi)\ \partial/\partial\varphi (F_r) / r^2 \sin(\theta)$$
$$+ 2 \sin(\varphi) \cos(\theta)\ \partial/\partial\varphi (F_\theta) / r^2 \sin^2(\theta) + F_\theta \cos(\theta) \cos(\varphi) / r^2 \sin^2(\theta) + \sin(\varphi)\ \partial^2/\partial\varphi^2 (F_\varphi) / r^2 \sin^2(\theta)$$
$$- F_\varphi \sin(\varphi) / r^2 \sin^2(\theta) + 1/c^2 \sin(\theta) \cos(\varphi)\ \partial^2/\partial t^2 (F_r) + 1/c^2 \cos(\theta) \cos(\varphi)\ \partial^2/\partial t^2 (F_\theta)$$
$$- 1/c^2 \sin(\varphi)\ \partial^2/\partial t^2 (F_\varphi) - \cos(\theta) \cos(\varphi)\ \partial^2/\partial r^2 (F_\theta) + 2 F_r \sin(\theta) \cos(\varphi) / r^2$$
$$- \sin(\theta) \cos(\varphi)\ \partial^2/\partial\theta^2 (F_r) / r^2 + 2 \sin(\theta) \cos(\varphi)\ \partial/\partial\theta (F_\theta) / r^2 - \cos(\varphi)\ \partial^2/\partial\varphi^2 (F_r) / r^2 \sin(\theta)$$
$$- \cos(\theta) \cos(\varphi)\ \partial^2/\partial\varphi^2 (F_\theta) / r^2 \sin^2(\theta) + 2\cos(\varphi)\ \partial/\partial\varphi (F_\varphi) / r^2 \sin^2(\theta) + \sin(\varphi)\ \partial^2/\partial r^2 (F_\varphi)$$
$$+ 2 \sin(\varphi)\ \partial/\partial r (F_\varphi) / r + \sin(\varphi)\ \partial^2/\partial\theta^2 (F_\varphi) / r^2 \}\ e_1$$

$$\{ -2 \sin(\theta) \sin(\varphi)\ \partial/\partial r (F_r) / r - \sin(\theta) \sin(\varphi)\ \partial^2/\partial r^2 (F_r) - 2 \sin(\varphi) \cos(\theta)\ \partial/\partial r (F_\theta) / r$$
$$+ 2 F_\theta \sin(\varphi) \cos(\theta) / r^2 - 3 \sin(\varphi) \cos(\theta)\ \partial/\partial\theta (F_r) / r^2 - \sin(\varphi) \cos^2(\theta)\ \partial/\partial\theta (F_\theta) / r^2 \sin(\theta)$$
$$- \sin(\varphi) \cos(\theta)\ \partial^2/\partial\theta^2 (F_\theta) / r^2 - \cos(\theta) \cos(\varphi)\ \partial/\partial\theta (F_\varphi) / r^2 \sin(\theta) - 2 \cos(\varphi)\ \partial/\partial\varphi (F_r) / r^2 \sin(\theta)$$
$$- 2 \cos(\theta) \cos(\varphi)\ \partial/\partial\varphi (F_\theta) / r^2 \sin^2(\theta) + F_\theta \sin(\varphi) \cos(\theta) / r^2 \sin^2(\theta) - \cos(\varphi)\ \partial^2/\partial\varphi^2 (F_\varphi)/ r^2 \sin^2(\theta)$$
$$+ F_\varphi \cos(\varphi) / r^2 \sin^2(\theta) - \sin(\varphi) \cos(\theta)\ \partial^2/\partial r^2 (F_\theta) + 2 F_r \sin(\theta) \sin(\varphi) / r^2$$
$$- \sin(\theta) \sin(\varphi)\ \partial^2/\partial\theta^2 (F_r) / r^2 + 2 \sin(\theta) \sin(\varphi)\ \partial/\partial\theta (F_\theta) / r^2 + 1/c^2 \sin(\theta) \sin(\varphi)\ \partial^2/\partial t^2 (F_r)$$
$$+ 1/c^2 \sin(\varphi) \cos(\theta)\ \partial^2/\partial t^2 (F_\theta) + 1/c^2 \cos(\varphi)\ \partial^2/\partial t^2 (F_\varphi) - \sin(\varphi)\ \partial^2/\partial\varphi^2 (F_r) / r^2 \sin(\theta)$$
$$- \sin(\varphi) \cos(\theta)\ \partial^2/\partial\varphi^2 (F_\theta) / r^2 \sin^2(\theta) + 2 \sin(\varphi)\ \partial/\partial\varphi (F_\varphi) / r^2 \sin^2(\theta) - \cos(\varphi)\ \partial^2/\partial r^2 (F_\varphi)$$
$$- 2 \cos(\varphi)\ \partial/\partial r (F_\varphi) / r - \cos(\varphi)\ \partial^2/\partial\theta^2 (F_\varphi) / r^2 \}\ e_2$$

$$\{ 1/c^2 \cos(\theta)\ \partial^2/\partial t^2 (F_r) - 1/c^2 \sin(\theta)\ \partial^2/\partial t^2 (F_\theta) - F_\theta \sin^3(\theta) / r^2 + 2 \sin(\theta)\ \partial/\partial r (F_\theta) / r$$
$$+ \sin(\theta)\ \partial^2/\partial r^2 (F_\theta) - 2 \cos(\theta)\ \partial/\partial r (F_r) / r + 2 F_r \cos(\theta) / r^2 - \cos^2(\theta)\ \partial/\partial\theta (F_r) / r^2 \sin(\theta)$$
$$- \cos(\theta)\ \partial^2/\partial\theta^2 (F_r) / r^2 + 3 \cos(\theta)\ \partial/\partial\theta (F_\theta) / r^2 - \cos(\theta)\ \partial^2/\partial\varphi^2 (F_r) / r^2 \sin^2(\theta)$$
$$+ \partial^2/\partial\varphi^2 (F_\theta) / r^2 \sin(\theta) - \cos(\theta)\ \partial^2/\partial r^2 (F_r) + 2 \sin(\theta)\ \partial/\partial\theta (F_r) / r^2 + \sin(\theta)\ \partial^2/\partial\theta^2 (F_\theta) / r^2$$
$$+ F_\theta \cos^4(\theta) / r^2 \sin(\theta) \}\ e_3$$

Mapping to the **v** system basis, its coefficients are

$$\{ 1/c^2\ \partial^2/\partial t^2 (F_0) - \cos(\theta)\ \partial/\partial\theta (F_0) / r^2 \sin(\theta) - \partial^2/\partial\varphi^2 (F_0) / r^2 \sin^2(\theta) - 2\ \partial/\partial r (F_0) / r$$
$$- \partial^2/\partial r^2 (F_0) - \partial^2/\partial\theta^2 (F_0) / r^2 \}\ w_t$$

$\{ -2\, \partial/\partial r\, (F_r) / r - \partial^2/\partial r^2\, (F_r) + 1/c^2\, \partial^2/\partial t^2\, (F_r) + 2F_r / r^2 - \partial^2/\partial\theta^2\, (F_r) / r^2$
$- \cos(\theta)\, \partial/\partial\theta\, (F_r) / r^2 \sin(\theta) + 2\, F_\theta \cos(\theta) / r^2 \sin(\theta) + 2\, \partial/\partial\varphi\, (F_\varphi) / r^2 \sin(\theta)$
$+ 2\, \partial/\partial\theta\, (F_\theta) / r^2 - \partial^2/\partial\varphi^2\, (F_r) / r^2 \sin^2(\theta) \}\, w_r$

$\{ -2\, \partial/\partial r\, (F_\theta) / r - \partial^2/\partial\theta^2\, (F_\theta) / r^2 + 1/c^2\, \partial^2/\partial t^2\, (F_\theta) - \partial^2/\partial r^2\, (F_\theta) - \partial^2/\partial\varphi^2\, (F_\theta) / r^2 \sin^2(\theta)$
$- \cos(\theta)\, \partial/\partial\theta\, (F_\theta) / r^2 \sin(\theta) + 2 \cos(\theta)\, \partial/\partial\varphi\, (F_\varphi) / r^2 \sin^2(\theta) - 2\, \partial/\partial\theta\, (F_r) / r^2 + F_\theta / r^2 \sin^2(\theta) \}\, w_\theta$

$\{ - \cos(\theta)\, \partial/\partial\theta\, (F_\varphi) / r^2 \sin(\theta) - 2\, \partial/\partial\varphi\, (F_r) / r^2 \sin(\theta) - 2 \cos(\theta)\, \partial/\partial\varphi\, (F_\theta) / r^2 \sin^2(\theta)$
$- \partial^2/\partial\varphi^2\, (F_\varphi) / r^2 \sin^2(\theta) + F_\varphi / r^2 \sin^2(\theta) + 1/c^2\, \partial^2/\partial t^2\, (F_\varphi) - \partial^2/\partial r^2\, (F_\varphi) -2\, \partial/\partial r\, (F_\varphi) / r$
$- \partial^2/\partial\theta^2\, (F_\varphi) / r^2 \}\, w_\varphi$

This can be verified to be the spherical-polar equivalent of $-\nabla^2 \mathbf{A}$. Each time we move to the unity norm **v** system basis, there is significant simplification.

The Quaternion Ensemble Derivative has thus faithfully reproduced the gradient, divergence and curl in spherical-polar coordinates, as well as the second order form for $-\nabla^2 \mathbf{A}$. This validates the Ensemble Derivative as the general form for Quaternion differentiation, and by extension to Octonion differentiation, or any n dimensional space where multiplication is defined by its intrinsic n dimensional algebra.

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