Different Metrics with Different Cosmological Constants

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Abstract

In this article, the author demonstrates, that different metrics and therefore different gravitational systems have different cosmological constants and thus different total energy densities. Each gravitational system has its own individual metric, which differs from that one of another gravitational system. Coordinate transformations are only possible within one and the same metric, which means within one and the same gravitational system. As a result, our universe is traversed by a myriad of different metrics, from which one must select the appropriate one, that describes the gravitational system to be considered.

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1. Introduction

A coordinate transformation is always possible if the metric ds^2 is equal to or identical with the metric $d\tilde{s}^2$, so that [1, 2]

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \tilde{g}_{\kappa\lambda} d\tilde{x}^\kappa d\tilde{x}^\lambda = d\tilde{s}^2, \quad (1)$$

where

$$\tilde{g}_{\kappa\lambda} = J^\mu_\kappa J^\nu_\lambda g_{\mu\nu},$$

$g_{\mu\nu}$ as well as $\tilde{g}_{\kappa\lambda}$ are metric tensors, and

$$J^\mu_\kappa = \frac{\partial x^\mu}{\partial \tilde{x}^\kappa}$$

is the Jacobi matrix. Any covariant tensor of second rank is transformed in the same manner [1, 2],

$$\tilde{F}_{\kappa\lambda} = J^\mu_\kappa J^\nu_\lambda F_{\mu\nu},$$

while a scalar remains invariant under a coordinate transformation, $\tilde{S} = S$.

Einstein's field equations with the cosmological constant in their mixed-tensor representation show the fundamental conservation law of total energy, momentum, and stress [3]. This is why the cosmological term cannot be neglected, otherwise one would violate this fundamental conservation law. The cosmological term in Einstein's field equations represents the total energy-momentum density tensor [3]. Einstein's field equations with the cosmological term are valid in any frame of reference and must be fulfilled in order to describe gravity.

The cosmological constant is not a universal constant, but a constant parameter [3, 4]. This is not surprising, because different gravitational systems and therefore different metrics ds^2 have different total energy densities and thus different cosmological constants. In empty spacetime, the total energy density with respect to the metric ds^2 (gravitational system) equals the energy density of the gravitational field with respect to the same metric (gravitational system) [3],

$$\kappa^{-1} \Lambda \delta^0_0 = -\kappa^{-1} G^0_0.$$

2. Theory

The following findings are also valid within matter, however for simplicity it is sufficient to consider empty space-times, where $T_{\mu\nu} = 0$ and $\tilde{T}_{\kappa\lambda} = 0$.

In empty space-times, Einstein's field equations with different cosmological constants $\Lambda \neq \tilde{\Lambda}$ read

$$G_{\mu\nu} = -\Lambda g_{\mu\nu}, \quad (2)$$
$$\tilde{G}_{\kappa\lambda} = -\tilde{\Lambda} \tilde{g}_{\kappa\lambda}. \quad (3)$$

The respective Einstein tensors are functions of the respective metric tensors,

$$G_{\mu\nu} = G_{\mu\nu}(g_{\mu\nu}), \quad \tilde{G}_{\kappa\lambda} = \tilde{G}_{\kappa\lambda}(\tilde{g}_{\kappa\lambda}),$$

and the respective metric tensors depend on the respective cosmological constants,

$$g_{\mu\nu} = g_{\mu\nu}(\Lambda), \quad \tilde{g}_{\kappa\lambda} = \tilde{g}_{\kappa\lambda}(\tilde{\Lambda}).$$

It is of great importance to recognize, that Einstein's field equations (3) cannot be yielded by performing a coordinate transformation from Einsteins field equations (2) in case $\Lambda \neq \tilde{\Lambda}$. This is only possible if $\Lambda = \tilde{\Lambda}$. Consequently in case $\Lambda \neq \tilde{\Lambda}$, the metrics ds^2 and $d\tilde{s}^2$ are different from each other, and hence Eq. (1) is not applicable, because

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \neq \tilde{g}_{\kappa\lambda} d\tilde{x}^\kappa d\tilde{x}^\lambda = d\tilde{s}^2. \quad (4)$$

3. Discussion

As a simple example, the two metric tensors could be those ones of two different gravitational systems wherein $\Lambda \neq \widetilde{\Lambda}$, which describe the respective empty space-times of two different non-rotating stars with masses M and $\widetilde{M}$, respectively,

$$g_{\mu\nu} = diag[-B(r), A(r), r^2, r^2 \sin^2\theta],$$
$$\widetilde{g}_{\kappa\lambda} = diag[-\widetilde{B}(\tilde{r}), \widetilde{A}(\tilde{r}), \tilde{r}^2, \tilde{r}^2 \sin^2\tilde{\theta}],$$

where the metric coefficients are given by

$$B(r) = 1 - \frac{2GM}{c^2 r} - \frac{\Lambda r^2}{3}, \quad A(r) = B^{-1}(r),$$
$$\widetilde{B}(\tilde{r}) = 1 - \frac{2G\widetilde{M}}{c^2 \tilde{r}} - \frac{\widetilde{\Lambda} \tilde{r}^2}{3}, \quad \widetilde{A}(\tilde{r}) = \widetilde{B}^{-1}(\tilde{r}).$$

Each of these different two metrics is the de Sitter-Schwarzschild metric [5] of the respective non-rotating stars. However, a coordinate transformation between these two different de Sitter-Schwarzschild metrics is not possible, because Einstein's field equations (3) cannot be yielded by a coordinate transformation from Einstein's field equations (2) owing to the fact that $\Lambda \neq \widetilde{\Lambda}$.

In special relativity, the metric tensors $g_{\mu\nu} = \eta_{\mu\nu}$ and $\widetilde{g}_{\kappa\lambda} = \eta_{\kappa\lambda}$, where

$$\eta_{\alpha\beta} = diag(-1, \quad 1, \quad 1, \quad 1)$$

is the Minkowski-metric tensor. Hence, $\Lambda = \widetilde{\Lambda} = 0$, $ds^2 = d\tilde{s}^2$, and the Jacobi matrix is replaced by the transformation matrix of the Lorentz transformation [1, 2]. A vanishing cosmological constant $\Lambda = 0$ does not describe a gravitational system, because in this case the total energy density is zero. This scenario would be in case of a total empty universe without any matter [3] – the nothing – wherein $T_{\mu\nu} = 0$. Because there is no curved spacetime in this case, one can consider the Minkowski metric, so that $G_{\mu\nu} = 0$. The author denominates this kind of Minkowski metric a global Minkowski metric.

In contrast to a global Minkowski metric, there exist local inertial frames, which the author in this context also denominates local reference frames [6], inter alia, in Ref. [3]. These are reference frames, which describe the free fall. Therein, the Minkowski-metric tensor only exists in one single point, namely in the origin of the local inertial frame [7], where the laws of special relativity are valid. The Riemann tensor does not vanish in a local inertial frame, which demonstrates, that spacetime is curved. There arise tidal forces even very close to the origin of the local inertial frame. With respect to these local inertial frames (local reference frames), the cosmological constant never vanishes although there locally exists a Minkowski metric, the tensor of which does not contain a cosmological constant. However, one can always make a coordinate transformation to a local inertial frame, whereby the cosmological term in Einstein's field equations does not disappear,

$$\widetilde{G}_{\mu\nu} = -\Lambda g_{\mu\nu} \quad \rightarrow \quad G_{\rho\sigma} = -\Lambda \eta_{\rho\sigma}.$$

4. Conclusions

Each gravitational system has its own individual total energy density, that is proportional to its cosmological constant, the latter of which is not a universal constant, but a constant parameter. In empty spacetime, the total energy density with respect to the metric ds^2 (gravitational system) equals the energy density of the gravitational field with respect to the same metric (gravitational system) [3]. A coordinate transformation is not possible in case the cosmological constant Λ with respect to the metric ds^2 differs from the cosmological constant $\widetilde{\Lambda}$ with respect to the metric $d\tilde{s}^2$, see Eq. (4), because Einstein's field equations (3) cannot be yielded by performing a coordinate transformation from Einsteins field equations (2) in case $\Lambda \neq \widetilde{\Lambda}$. Thus, each gravitational system has its own individual metric, which differs from that one of another gravitational system. Coordinate transformations are only possible within one and the same metric, which means within one and the same gravitational system. As a result, our universe is traversed by a myriad of different metrics [5], from which one must select the appropriate one, that describes the gravitational system to be considered.

Because of their different cosmological constants $\Lambda \neq \widetilde{\Lambda}$, different metrics have different scalar curvatures,

$$R = 4\Lambda, \quad \widetilde{R} = 4\widetilde{\Lambda}.$$

For simplicity also here empty space-times are considered, but this finding is of course also valid within matter.

There exist myriads of different metrics independently from each other in the whole universe [5]. They in general also do not merge with each other. Moreover, it is impossible to continuously join all of these myriads of different metrics. Metrics with different cosmological constants cannot be merged. This would be inconsistent with Einstein's field equations with the cosmological term, because two or even more merged metrics with different cosmological constants cannot fulfill Einstein's field equations with the cosmological term at one and the same time. Different metrics describe different gravitational systems and possess different cosmological constants and thus different total energy densities. Merging metrics is only possible for metrics where $\Lambda = \widetilde{\Lambda}$, which is in case of inner and outer metrics of the same celestial objects.

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