Rank Two Tensors of a Diagonal Nature

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Abstract

The article explores the transformation of rank two tensors to bring out the fact that tensor transformations should be linear .Tensors of a diagonal nature have been used to this end.

Keywords: Rank two tensor, Contravariant tensor,Transformations.

Introduction

The article derives by considering the transformation of rank two tensors that these transformations are linear in nature. This is not ordinarily visible. Nevertheless the issue surfaces when diagonal tensors are considered.

Derivation

We consider a rank two tensor[1] $A^{\alpha\beta}$such that $A^{\alpha\beta}=0$ for $\alpha\neq\beta$. Off diagonal elements in the new frame of reference may not be zero.

Transformation of the rank two contravariant tensor:

$$\bar{A}^{\mu\nu}=\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial\bar{x}^{\nu}}{\partial x^{\beta}}A^{\alpha\beta}\ (1)$$

Inverse transformation

$$A^{\alpha\beta}=\frac{\partial x^{\alpha}}{\partial\bar{x}^{\mu}}\frac{\partial x^{\beta}}{\partial\bar{x}^{\nu}}\bar{A}^{\mu\nu}\ (2)$$

Diagonal elements

$$A^{\alpha\alpha}=\frac{\partial x^{\alpha}}{\partial\bar{x}^{\mu}}\frac{\partial x^{\alpha}}{\partial\bar{x}^{\nu}}\bar{A}^{\mu\nu}\ (3)$$

Considering specifically the diagonal components of $\bar{A}$,

$$\bar{A}^{\mu\mu}=\left(\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\right)^{2}A^{\alpha\alpha}\ (4)$$

$$\Longrightarrow \bar{A}^{\mu\mu} = \left(\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}\right)^2 \frac{\partial x^{\alpha}}{\partial \bar{x}^{\rho}} \frac{\partial x^{\alpha}}{\partial \bar{x}^{\sigma}} \bar{A}^{\rho\sigma} \ (5)$$

$$\Longrightarrow \left(\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}\right)^2 \frac{\partial x^{\alpha}}{\partial \bar{x}^{\rho}} \frac{\partial x^{\alpha}}{\partial \bar{x}^{\sigma}} = \delta_{\mu\rho}\delta_{\mu\sigma} (6)$$

In equation (5) $\rho$and σ are dummy indices but in equation (6) they are not dummies.

$$\Longrightarrow \frac{\partial \bar{x}^{\rho}}{\partial x^{\beta}} \frac{\partial \bar{x}^{\sigma}}{\partial x^{\gamma}} \frac{\partial x^{\alpha}}{\partial \bar{x}^{\sigma}} \left(\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}\right)^2 \frac{\partial x^{\alpha}}{\partial \bar{x}^{\rho}} \frac{\partial x^{\alpha}}{\partial \bar{x}^{\sigma}} = \frac{\partial \bar{x}^{\rho}}{\partial x^{\beta}} \frac{\partial \bar{x}^{\sigma}}{\partial x^{\gamma}} \delta_{\mu\rho}\delta_{\mu\sigma}$$

$$\Longrightarrow \left(\frac{\partial \bar{x}^{\rho}}{\partial x^{\beta}} \frac{\partial x^{\alpha}}{\partial \bar{x}^{\rho}}\right) \left(\frac{\partial \bar{x}^{\sigma}}{\partial x^{\gamma}} \frac{\partial x^{\alpha}}{\partial \bar{x}^{\sigma}}\right) \left(\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}\right)^2 = \frac{\partial \bar{x}^{\mu}}{\partial x^{\beta}} \frac{\partial \bar{x}^{\sigma}}{\partial x^{\gamma}} \ (7)$$

In equation (7) $\rho$and σ are dummy indices

$$\delta_{\alpha\beta}\delta_{\alpha\gamma} \left(\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}\right)^2 = \frac{\partial \bar{x}^{\mu}}{\partial x^{\beta}} \frac{\partial \bar{x}^{\sigma}}{\partial x^{\gamma}} \ (8)$$

If $\beta \neq \gamma$

$$\frac{\partial \bar{x}^{\mu}}{\partial x^{\beta}} \frac{\partial \bar{x}^{\sigma}}{\partial x^{\gamma}} = 0 \ \Longrightarrow \frac{\partial \bar{x}^{\mu}}{\partial x^{\beta}} = 0 \ or \frac{\partial \bar{x}^{\sigma}}{\partial x^{\gamma}} = 0(9)$$

Equation (9) indicates that the transformations have to be linear

Off diagonal elements in the new frame of reference

$$\bar{A}^{\mu\nu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \bar{x}^{\nu}}{\partial x^{\alpha}} A^{\alpha\alpha} \ (10)$$

$$A^{\alpha\alpha} = \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} \frac{\partial x^{\alpha}}{\partial \bar{x}^{\nu}} \bar{A}^{\mu\nu}$$

$$\bar{A}^{\mu\nu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \bar{x}^{\nu}}{\partial x^{\alpha}} \frac{\partial x^{\alpha}}{\partial \bar{x}^{\rho}} \frac{\partial x^{\alpha}}{\partial \bar{x}^{\sigma}} \bar{A}^{\rho\sigma} \ (11)$$

$$\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \bar{x}^{\nu}}{\partial x^{\alpha}} \frac{\partial x^{\alpha}}{\partial \bar{x}^{\rho}} \frac{\partial x^{\alpha}}{\partial \bar{x}^{\sigma}} = \delta_{\mu\rho}\delta_{\nu\sigma} \ (12)$$

$$\frac{\partial \bar{x}^{\rho}}{\partial x^{\beta}} \frac{\partial \bar{x}^{\sigma}}{\partial x^{\alpha}} \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \bar{x}^{\nu}}{\partial x^{\alpha}} \frac{\partial x^{\alpha}}{\partial \bar{x}^{\rho}} \frac{\partial x^{\alpha}}{\partial \bar{x}^{\sigma}} = \frac{\partial \bar{x}^{\rho}}{\partial x^{\beta}} \frac{\partial \bar{x}^{\sigma}}{\partial x^{\alpha}} \delta_{\mu\rho}\delta_{\nu\sigma} \ (13)$$

$$\left(\frac{\partial \bar{x}^{\rho}}{\partial x^{\beta}} \frac{\partial x^{\alpha}}{\partial \bar{x}^{\rho}}\right) \left(\frac{\partial \bar{x}^{\sigma}}{\partial x^{\gamma}} \frac{\partial x^{\alpha}}{\partial \bar{x}^{\sigma}}\right) \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \bar{x}^{\nu}}{\partial x^{\alpha}} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\beta}} \frac{\partial \bar{x}^{\nu}}{\partial x^{\alpha}}$$

$$\delta_{\alpha\beta}\delta_{\alpha\gamma}\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial\bar{x}^{\nu}}{\partial x^{\alpha}}=\frac{\partial\bar{x}^{\mu}}{\partial x^{\beta}}\frac{\partial\bar{x}^{\nu}}{\partial x^{\alpha}}\ (14)$$

If $\beta\neq\gamma$

$$\frac{\partial\bar{x}^{\mu}}{\partial x^{\beta}}\frac{\partial\bar{x}^{\nu}}{\partial x^{\alpha}}=0\Longrightarrow\frac{\partial\bar{x}^{\nu}}{\partial x^{\alpha}}=0\ or\ \frac{\partial\bar{x}^{\mu}}{\partial x^{\beta}}=0(15)$$

Again equation (15) indicates that the transformations have to be linear.

Next we consider the contrast case:

From (1) and (2) we write the following for $A^{\alpha\beta}$hich I not diagonal

$$\bar{A}^{\mu\nu}=\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial\bar{x}^{\nu}}{\partial x^{\beta}}\frac{\partial x^{\alpha}}{\partial\bar{x}^{\rho}}\frac{\partial x^{\beta}}{\partial\bar{x}^{\sigma}}\bar{A}^{\rho\sigma}$$

$$\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial\bar{x}^{\nu}}{\partial x^{\beta}}\frac{\partial x^{\alpha}}{\partial\bar{x}^{\rho}}\frac{\partial x^{\beta}}{\partial\bar{x}^{\sigma}}=\delta^{\mu}{}_{\rho}\delta^{\nu}{}_{\sigma}$$

$$\frac{\partial\bar{x}^{\rho}}{\partial x^{\gamma}}\frac{\partial\bar{x}^{\sigma}}{\partial x^{\delta}}\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial\bar{x}^{\nu}}{\partial x^{\beta}}\frac{\partial x^{\alpha}}{\partial\bar{x}^{\rho}}\frac{\partial x^{\beta}}{\partial\bar{x}^{\sigma}}=\frac{\partial\bar{x}^{\rho}}{\partial x^{\gamma}}\frac{\partial\bar{x}^{\sigma}}{\partial x^{\delta}}\delta^{\mu}{}_{\rho}\delta^{\nu}{}_{\sigma}$$

$$\left(\frac{\partial\bar{x}^{\rho}}{\partial x^{\gamma}}\frac{\partial x^{\alpha}}{\partial\bar{x}^{\rho}}\right)\left(\frac{\partial\bar{x}^{\sigma}}{\partial x^{\delta}}\frac{\partial x^{\beta}}{\partial\bar{x}^{\sigma}}\right)\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial\bar{x}^{\nu}}{\partial x^{\beta}}=\frac{\partial\bar{x}^{\mu}}{\partial x^{\gamma}}\frac{\partial\bar{x}^{\nu}}{\partial x^{\delta}}$$

$$\delta^{\alpha}{}_{\gamma}\delta^{\beta}{}_{\delta}\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial\bar{x}^{\nu}}{\partial x^{\beta}}=\frac{\partial\bar{x}^{\mu}}{\partial x^{\gamma}}\frac{\partial\bar{x}^{\nu}}{\partial x^{\delta}}$$

$$\frac{\partial\bar{x}^{\mu}}{\partial x^{\gamma}}\frac{\partial\bar{x}^{\nu}}{\partial x^{\delta}}=\frac{\partial\bar{x}^{\mu}}{\partial x^{\gamma}}\frac{\partial\bar{x}^{\nu}}{\partial x^{\delta}}$$

There is no discrepancy since we have $\delta^{\alpha}{}_{\gamma}\delta^{\beta}{}_{\delta}$and not $\delta^{\alpha}{}_{\gamma}\delta^{\alpha}{}_{\delta}$

The issue of linearity of transformations is not discernible now though its presence is not being opposed to in these calculations.

Conclusions

As claimed we have considered tensors of a diagonal nature to bring out the fact that ternsor transformations have to be linear in their nature

References

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