u and Pi

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abstract

In this note we give some relations between u and Pi

Keywords: number u , number Pi, integrals, series.

Introduction

Recall that

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\cdots$$

$$\pi=\frac{15\sqrt{3}}{4}-6\int_0^1\sqrt{1-2\sin\left(\frac{1}{3}\sin^{-1}\left(1-\frac{81x^2}{128}\right)\right)}\,dx$$

$$\pi=\frac{3\sqrt{3}}{4}+\frac{16\sqrt{2}}{3}\int_0^{\sqrt{3}/2}\sqrt{1-\sin\left(3\sin^{-1}\left(\frac{1-x^2}{2}\right)\right)}\,dx$$

The number u is defined by

$$u=1+\sin(1+\sin(1+\sin(1+\cdots)))=1.9345632\ldots$$

In this note we give some relations between u and Pi.

u and Pi

Entry 1.

$$\pi=2u-2\sqrt{2}\sqrt{2-u}\left(1+\sum_{n=1}^{\infty}\binom{2n}{n}\frac{2^{-3n}(2-u)^n}{2n+1}\right)$$

Entry 2.

$$\pi = 2u - 2\sqrt{u(2-u)} \sum_{n=0}^{\infty} \left(1 - \frac{u}{2}\right)^n \sum_{k=0}^{[n/2]} \binom{2n-2k}{n-k} \binom{n-k}{k} \frac{(-1)^k}{2n-2k+1}$$

Entry 3.

$$\pi = 2u - u\sqrt{2(2-u)} \times$$

$$\sum_{n=0}^{\infty} \left(1 - \frac{u}{2}\right)^n 2^{-2n} \sum_{k=0}^{n} 2^{2k} \binom{2n-2k}{n-k} \sum_{m=0}^{[k/2]} \binom{2k-2m}{k-m} \binom{k-m}{m} \frac{(-1)^m}{2k-2m+1}$$

Entry 4.

$$u = 1 + \sin u$$

$$\pi = 2 + 2\sin u + 2\sin^{-1}(\cos u)$$

$$\pi = 2u - 2\cos^{-1}(\sin u) = 2u - 2\cos^{-1}(u-1)$$

$$\pi = 4u - 4\tan^{-1}\left(\frac{u - 1 + \sqrt{2u - u^2}}{u - 1 - \sqrt{2u - u^2}}\right)$$

$$\pi = \frac{4}{3}u + \frac{4}{3}\tan^{-1}\left(\frac{u - 1 - \sqrt{2u - u^2}}{u - 1 + \sqrt{2u - u^2}}\right)$$

$$\pi = 2u - 2\tan^{-1}\left(\frac{\sqrt{2u - u^2}}{u - 1}\right)$$

$$\pi = 2u - 2\sin^{-1}\sqrt{2u - u^2} = 2u - 4\sin^{-1}\sqrt{1 - \frac{u}{2}}$$

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