The Dynamics of D-branes with Dirac-Born-Infeld and Chern-Simons/Wess-Zumino Actions

Boris Stoyanov

DARK MODULI INSTITUTE, Membrane Theory Research Department,
18 King William Street, London, EC4N 7BP, United Kingdom

E-mail: stoyanov@darkmodulinstitute.org

BRANE HEPLAB, Theoretical High Energy Physics Department,
East Road, Cambridge, CB1 1BH, United Kingdom

E-mail: stoyanov@braneheplab.org

SUGRA INSTITUTE, 125 Cambridge Park Drive, Suite 301,
Cambridge, 02140, Massachusetts, United States

E-mail: stoyanov@sugrainstitute.org

Abstract

We have explained, and shown by feature stringy examples, why a D-brane in superstring theory, when treated as a fundamental dynamical object, can be described by a map φ from an Azumaya/matrix manifold X^{Az} with a fundamental module with a connection (E, ∇) to the target spacetime Y. In this sequel, we construct a non-Abelian Dirac-Born-Infeld action functional $S_{\mathrm{DBI}}^{(\Phi,g,B)}(\varphi, \nabla)$ for such pairs (φ, ∇) when the target spacetime Y is equipped with a background (dilaton, metric, B)-field (Φ, g, B) from closed strings. We next develop a technical tool needed to study variations of this action and apply it to derive the first variation $\delta S_{\mathrm{DBI}}^{(\Phi,g,B)}/\delta(\varphi, \nabla)$ of $S_{\mathrm{DBI}}^{(\Phi,g,B)}$ with respect to (φ, ∇). The equations of motion that govern the dynamics of D-branes then follow. We introduce a new action $S_{standard}^{(\rho,h;\Phi,g,B,C)}$ for D-branes that is to D-branes as the Polyakov action is to fundamental strings. This 'standard action' is abstractly a non-Abelian gauged sigma model based on maps $\varphi : (X^{Az}, E; \nabla) \to Y$ from an Azumaya/matrix manifold X^{Az} with a fundamental module E with a connection ∇ to Y enhanced by the dilaton term, the gauge-theory term, and the Chern-Simons/Wess-Zumino term that couples (φ, ∇) to Ramond-Ramond field. In a special situation, this new theory merges the theory of harmonic maps and a gauge theory, with a nilpotent type fuzzy extension. A complete action for a D-brane world-volume must include also the Chern-Simons/Wess-Zumino term $S_{CS/WZ}^{(C)}(\varphi, \nabla)$ that governs how the D-brane world-volume couples with the Ramond-Ramond fields C on Y. The current notes lay down a foundation toward the dynamics of D-branes along the line of this research project.

1 Introduction

We have explained, and shown by feature stringy examples, why a D-brane in superstring theory, when treated as a fundamental dynamical object, can be described by a map φ from an Azumaya/matrix manifold $X^{\!Az}$, served as the D-brane world-volume, with a fundamental module with a connection (E,∇), served as the Chan-Paton bundle, to the target space-time Y. In this sequel, we construct a non-Abelian Dirac-Born-Infeld action functional $S_{\mathrm{DBI}}^{(\Phi,g,B)}(\varphi,\nabla)$ for such pairs (φ,∇) when the target spacetime Y is equipped with a background (dilaton, metric, B)-field (Φ,g,B) from closed strings. We next develop a technical tool needed to study variations of this action and apply it to derive the first variation $\delta S_{\mathrm{DBI}}^{(\Phi,g,B)}/\delta(\varphi,\nabla)$ of $S_{\mathrm{DBI}}^{(\Phi,g,B)}$ with respect to (φ,∇). The equations of motion that govern the dynamics of D-branes then follow. A complete action for a D-brane world-volume must include also the Chern-Simons/Wess-Zumino term $S_{CS/WZ}^{(C)}(\varphi,\nabla)$ that governs how the D-brane world-volume couples with the Ramond-Ramond fields C on Y. In the current notes, a version $S_{CS/WZ}^{(C,B)}(\varphi,\nabla)$ of non-Abelian Chern-Simons/Wess-Zumino action functional for (φ,∇) that follows the same guide with which we construct $S_{\mathrm{DBI}}^{(\Phi,g,B)}(\varphi,\nabla)$ is constructed for lower-dimensional D-branes (i.e. D(-1)-, D0-, D1-, D2-branes). Its first variation $\delta S_{CS/WZ}^{(C,B)}(\varphi,\nabla)/\delta(\varphi,\nabla)$ is derived and its contribution to the equations of motion for (φ,∇) follows. For D-branes of dimension ≥ 3, an anomaly issue needs to be understood in the current context. The current notes lay down a foundation toward the dynamics of D-branes along the line of this D-project. Some highlights of the history of how the Born-Infeld action and the Dirac-Born-Infeld action arise from open string theory and a list of issues one needs to resolve to convert such an action to that for coincident D-branes are given in the research article. They serve as a guide for the steps of our exceptional discussion.

We introduce a new action $S_{standard}^{(\rho,h;\Phi,g,B,C)}$ for D-branes that is to D-branes as the (Brink-Di Vecchia-Howe/Deser-Zumino/) Polyakov action is to fundamental superstrings. This action depends both on the (dilaton field ρ, metric h) on the underlying topology X of the D-brane world-volume and on the background (dilaton field Φ, metric g, B-field B, Ramond-Ramond field C) on the target space-time Y; and is naturally a non-Abelian gauged sigma model — based on maps $\varphi:(X^{\!Az},E;\nabla)\to Y$ from an Azumaya/matrix manifold $X^{\!Az}$ with a fundamental module E with a connection ∇ to Y — enhanced by the dilaton term that couples (φ,∇) to (ρ,Φ), the B-coupled gauge-theory term that couples ∇ to B, and the Chern-Simons/Wess-Zumino term that couples (φ,∇) to (B,C) in our standard action $S_{standard}^{(\rho,h;\Phi,g,B,C)}$. Before one can do so, one needs to resolve the built-in obstruction of pull-push of covariant tensors under a map from a noncommutative manifold to a commutative manifold. Such issue already appeared in the construction of the non-Abelian Dirac-Born-Infeld action. In this note, we give a hierarchy of various admissible conditions on the pairs (φ,∇) that are enough to resolve the issue while being open-string compatible. This improves our understanding of admissible conditions beyond. With the noncommutative analysis, we develop further in this note some covariant differential calculus for such maps and use it to define the standard action for D-branes. After promoting the setting to a family version, we work out the first variation and hence the corresponding equations of motion for D-branes of the standard action and the second variation of the kinetic term for maps and the dilaton term in this action. Compared with the non-Abelian Dirac-Born-Infeld action constructed in the research article along the same line, the current standard action is clearly much more manageable. Classically and mathematically and in the special case where the background (Φ,B,C) on Y is set to vanish, this new theory is a merging of the theory of harmonic maps and a gauge theory (free to choose either a Yang-Mills theory or other kinds of applicable gauge theory) with a nilpotent type fuzzy extension. The current bosonic setting is the first step toward fermionic D-branes and their quantization as fundamental dynamical objects, in parallel to what happened for fundamental superstrings with inclusion of exceptional type extremal brane systems.

2 The first variation of the Dirac-Born-Infeld action

Given an admissible Lorentzian map,

$$\varphi \;:\; (X^{Az}, E; \nabla) \;\longrightarrow\; (Y, g, B, \Phi)\,,$$

let $T := (-\varepsilon, \varepsilon) \subset R^1$ and $\varphi_t : (X^{Az}, E; \nabla^t) \to (Y, g, B, \Phi)$, $t \in T$, be a differentiable T-family of admissible Lorentzian maps that deforms $\varphi =: \varphi_0$. In this subsection we derive in steps the first variation

$$\left.\frac{d}{dt}\right|_{t=0} S_{\mathrm{DBI}}^{(\Phi,g,B)}(\varphi_t, \nabla^t)$$

of the Dirac-Born-Infeld action. The derivation for the other two situations: (Y, g) Lorentzian and φ_t spacelike, and (Y, g) Riemannian and φ_t Riemannian, are completely the same.

As the major part of the discussion is local and around $0 \in T$, we will assume that ε is small enough and set the computation over a small enough coordinate chart $U \subset X$ (with coordinate functions $\boldsymbol{x} = (x^1, \cdots, x^m)$ so that $E|_U$ is trivializable and trivialized, and $\varphi_t(U)$ is contained in a coordinate chart $V \subset Y$ (with coordinate functions $y = (y^1, \cdots, y^n)$). Recall from Sec. 3.2 that, over U,

$$\begin{aligned} S_{\mathrm{DBI}}^{(\Phi,g,B)}|_U(\varphi_t, \nabla^t) \;&=\; -T_{m-1}\int_U Re\left(Tr\left(e^{-\varphi_t^\diamond\Phi}\sqrt{-\,SymDet_U\left(\varphi_t^\diamond(g+B) + 2\pi\alpha' F_{\nabla^t}\right)}\right)\right)\\ &=\; -T_{m-1}\int_U Re\left(Tr\left(e^{-\varphi_t^\sharp(\Phi)}\sqrt{-SymDet\left(\sum_{i,j}\varphi_t^\sharp(E_{ij})D_\mu^t\varphi_t^\sharp(y^i)D_\nu^t\varphi_t^\sharp(y^j) \;+\; 2\pi\alpha'\,[\nabla_\mu^t, \nabla_\nu^t]\right)_{\mu\nu}}\right)\right) d^m\boldsymbol{x}\,. \end{aligned}$$

Here, we set the notation for the tensors and connections involved as follows:

- $g + B \;=\; \sum_{i,j}(g_{ij} + B_{ij})\, dy^i \otimes dy^j \;=:\; \sum_{i,j} E_{ij} dy^i \otimes dy^j$, with $g_{ij} = g_{ji}$, $B_{ij} = -B_{ji}$,
- $\nabla^t \;=\; d + A^t \;=\; \sum_\mu(\partial_\mu + A_\mu^t)\, dx^\mu$ is the connection on $E|_U$,
- $D^t \;=\; d + [A^t, \cdot\,] \;=\; \sum_\mu(\partial_\mu + [A_\mu^t, \cdot\,])\, dx^\mu$ is the ∇^t-induced connection on $End_C(E|_U)$,
- $d^m\boldsymbol{x} \;:=\; dx^1 \wedge \cdots \wedge dx^m$ is compatible with the orientation on U;

and, for later use,

$$\dot{\varphi}^\sharp(y^i) \;:=\; \left.\frac{d}{dt}\right|_{t=0}\left(\varphi_T^\sharp(y^i)\right)\,, \qquad {}^\bullet(\varphi^\sharp(\boldsymbol{y}^{\boldsymbol{d}})) := \left.\frac{d}{dt}\right|_{t=0}(\varphi_T^\sharp(\boldsymbol{y}^{\boldsymbol{d}}))\,, \qquad \dot{A}_\mu \;:=\; \left.\frac{d}{dt}\right|_{t=0} A_\mu^T\,.$$

We assume further that the local chart U and $\varphi > 0$ are small enough so that the construction over $U_T := U \times (-\varepsilon, \varepsilon)$ in Sec. 4.1, with $p \in U \times \{0\} \subset U_T$, applies simultaneously to $e^{-\Phi}$ and E_{ij}, $i, j = 1, \cdots, n$, to give the local expression of $\varphi_T^\sharp(\Phi)$ and $\varphi_T^\sharp(E_{ij})$, $i, j = 1 \ldots, n$, in terms of elements in the polynomial ring over $C^\infty(U_T)$

$$\varphi_T^\sharp(\Phi)\,,\; \varphi_T^\sharp(E_{ij}) \;\in\; \left(\oplus_{j=1}^s C^\infty(U_T)\cdot Id_{E_T^{(j)}}\right)[\,\varphi_T^\sharp(y^1), \cdots, \varphi_T^\sharp(y^n)\,]$$

of multi-degree $\le (r-1, \cdots, r-1)$. Associated to these settings and with the notation from Remark/Notataion 4.2.3.5, recall that

$$\begin{aligned} e^{-\varphi_T^\sharp(\Phi)} \;&=\; \varphi_T^\sharp(e^{-\Phi}) \;&&=\; R^{e^{-\Phi}}[0]|_{\boldsymbol{y}^{\boldsymbol{d}}\varphi_T^\sharp(\boldsymbol{y})^{\boldsymbol{d}}}\,,\\ \left.\frac{d}{dt}\right|_{t=0} e^{-\varphi_T^\sharp(\Phi)} \;&=\; \left.\frac{d}{dt}\right|_{t=0}\varphi_T^\sharp(e^{-\Phi}) \;&&=\; R^{e^{-\Phi}}[1]|_{\boldsymbol{y}^{\boldsymbol{d}}\,{}^\bullet(\varphi_T^\sharp(\boldsymbol{y})^{\boldsymbol{d}})}\,; \end{aligned}$$

and

$$\begin{aligned} \varphi_T^\sharp(E_{ij}) \;&=\; R^{E_{ij}}[0]|_{\boldsymbol{y}^{\boldsymbol{d}}\varphi_T^\sharp(\boldsymbol{y})^{\boldsymbol{d}}}\,,\\ \left.\frac{d}{dt}\right|_{t=0}\varphi_T^\sharp(E_{ij}) \;&=\; R^{E_{ij}}[1]|_{\boldsymbol{y}^{\boldsymbol{d}}\,{}^\bullet(\varphi_T^\sharp(\boldsymbol{y})^{\boldsymbol{d}})}\,, \end{aligned}$$

for $i, j = 1, \ldots, n$. For simplicity of notation, it is understood that $R^{E_{ij}}[1]$ is evaluated at $t=0$ in the expression $R^{E_{ij}}[1]|_{\boldsymbol{y}^{\boldsymbol{d}}\,\bullet\,(\varphi_T^\sharp(\boldsymbol{y})^{\boldsymbol{d}})}$; and similarly for induced expressions that follow this.

The first variation of each ingredient in the Dirac-Born-Infeld action

(a) The first variation of $e^{-\varphi^\sharp(\Phi)}$ *and* $\varphi^\sharp(E_{ij})$ Then, it follows from Proposition 4.2.3.1 that

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0}\left(e^{-\varphi_T^\sharp(\Phi)}\right) &= R^{e^{-\Phi}}[1]|_{\boldsymbol{y}^{\boldsymbol{d}}\,\bullet\,(\varphi^\sharp(\boldsymbol{y}^{\boldsymbol{d}}))} \\
= \sum_{i'=1}^{n}\sum_{d=0}^{\bullet}\sum_{\boldsymbol{d},\,|\boldsymbol{d}|=d}\;\sum_{\vec{\pi}\in\vec{Ptn}(1,d),\,i_{(\vec{\pi},\,\boldsymbol{d})}=i'} & ([\partial_{y^{i'}}^{\vec{\pi}}]R^{e^{-\Phi}}[1]_{(\boldsymbol{d})})^L(\varphi^\sharp(\boldsymbol{y}))\cdot\dot{\varphi}^\sharp(y^{i'})\cdot([\partial_{y^{i'}}^{\vec{\pi}}]R^{e^{-\Phi}}[1]_{(\boldsymbol{d})})^R(\varphi^\sharp(\boldsymbol{y})) \\
=: \sum_{i'=1}^{n}\;\sum_{d,\,\boldsymbol{d},\,\vec{\pi};\,|\boldsymbol{d}|=d,\,i_{(\vec{\pi},\boldsymbol{d})}=i'} & R^{e^{-\Phi}}[1]^{\,L}_{(\boldsymbol{d},\,\vec{\pi})}(\varphi^\sharp(\boldsymbol{y}))\cdot\dot{\varphi}^\sharp(y^{i'})\cdot R^{e^{-\Phi}}[1]^{\,R}_{(\boldsymbol{d},\,\vec{\pi})}(\varphi^\sharp(\boldsymbol{y}))
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0}\left(\varphi_T^\sharp(E_{ij})\right) &= R^{E_{ij}}[1]|_{\boldsymbol{y}^{\boldsymbol{d}}\,\bullet\,(\varphi^\sharp(\boldsymbol{y}^{\boldsymbol{d}}))} \\
= \sum_{i'=1}^{n}\sum_{d=0}^{\bullet}\sum_{\boldsymbol{d},\,|\boldsymbol{d}|=d}\;\sum_{\vec{\pi}\in\vec{Ptn}(1,d),\,i_{(\vec{\pi},\,\boldsymbol{d})}=i'} & ([\partial_{y^{i'}}^{\vec{\pi}}]R^{E_{ij}}[1]_{(\boldsymbol{d})})^L(\varphi^\sharp(\boldsymbol{y}))\cdot\dot{\varphi}^\sharp(y^{i'})\cdot([\partial_{y^{i'}}^{\vec{\pi}}]R^{E_{ij}}[1]_{(\boldsymbol{d})})^R(\varphi^\sharp(\boldsymbol{y})) \\
=: \sum_{i'=1}^{n}\;\sum_{d,\,\boldsymbol{d},\,\vec{\pi};\,|\boldsymbol{d}|=d,\,i_{(\vec{\pi},\boldsymbol{d})}=i'} & R^{E_{ij}}[1]^{\,L}_{(\boldsymbol{d},\,\vec{\pi})}(\varphi^\sharp(\boldsymbol{y}))\cdot\dot{\varphi}^\sharp(y^{i'})\cdot R^{E_{ij}}[1]^{\,R}_{(\boldsymbol{d},\,\vec{\pi})}(\varphi^\sharp(\boldsymbol{y}))\,.
\end{aligned}$$

(b) The first variation of $D_\mu\varphi^\sharp(y^i)$ *and* $F_{\mu\nu}$ By straightforward computation,

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0}\left(D_\mu^T\varphi_T^\sharp(y^i)\right) &= \frac{d}{dt}\Big|_{t=0}\left(\partial_\mu\varphi_T^\sharp(y^i)\,+\,[A_\mu^T,\varphi_T^\sharp(y^i)]\right) \\
&= D_\mu\dot{\varphi}^\sharp(y^i)\,-\,[\varphi^\sharp(y^i),\dot{A}_\mu]
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0}F_{\mu\nu}^T &= \frac{d}{dt}\Big|_{t=0}[\nabla_\mu^T,\nabla_\nu^T] \\
&= D_\mu\dot{A}_\nu\,-\,D_\nu\dot{A}_\mu\,.
\end{aligned}$$

The first variation of the Dirac-Born-Infeld action

With all the ingredients prepared, the computation of the first variation of $S_{\mathrm{DBI}}(\varphi,\nabla)$ is now straightforward, though some of the expressions may look complicated due to noncommutativity. We proceed in five steps.

Step (1) : *Input from all the pieces*

Let

$$\boldsymbol{M}_{\mu\nu}(t)\;:=\;\sum_{i,j}\varphi_t^\sharp(E_{ij})D_\mu^t\varphi_t^\sharp(y^i)D_\nu^t\varphi_t^\sharp(y^j)\,+\,2\pi\alpha'\,[\nabla_\mu^t,\nabla_\nu^t]\;\in\;C^\infty(End_C(E|_U))$$

and $\boldsymbol{M}(t) := [\boldsymbol{M}_{\mu\nu}(t)]_{\mu\nu}$ the $m \times m$ matrix with (μ,ν)-entry $\boldsymbol{M}_{\mu\nu}(t)$. Then,

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} S_{\mathrm{DBI}}(\varphi_t,\nabla^t) &= -T_{m-1}\left.\frac{d}{dt}\right|_{t=0}\int_U Re\left(Tr\left(e^{-\varphi_t^\sharp(\Phi)}\sqrt{-SymDet(\boldsymbol{M}(t))}\right)\right)d^m\boldsymbol{x}\\ &= -T_{m-1}\int_U Re\left(Tr\left.\frac{d}{dt}\right|_{t=0}\left(e^{-\varphi_t^\sharp(\Phi)}\sqrt{-SymDet(\boldsymbol{M}(t))}\right)\right)d^m\boldsymbol{x};\end{aligned}$$

$$\begin{aligned}&Tr\left.\frac{d}{dt}\right|_{t=0}\left(e^{-\varphi_t^\sharp(\Phi)}\sqrt{-SymDet(\boldsymbol{M}(t))}\right)\\ &\quad= Tr\left(\left(R^{e^{-\Phi}}[1]_{t=0}\right)|_{\boldsymbol{y}^I\,\boldsymbol{\cdot}(\varphi^\sharp(\boldsymbol{y}^I))}\cdot\sqrt{-SymDet(\boldsymbol{M}(0))}\right) + Tr\left(e^{-\varphi^\sharp(\Phi)}\left.\frac{d}{dt}\right|_{t=0}\sqrt{-SymDet(\boldsymbol{M}(t))}\right).\end{aligned}$$

Since (φ,∇) is admissible, $e^{-\varphi^\sharp(\Phi)}$ and $\sqrt{-SymDet(\boldsymbol{M}(t))}$ commute. Thus,

$$Tr\left(e^{-\varphi^\sharp(\Phi)}\left.\frac{d}{dt}\right|_{t=0}\sqrt{-SymDet(\boldsymbol{M}(t))}\right) = \frac{-1}{2}\,Tr\left(e^{-\varphi^\sharp(\Phi)}\sqrt{-SymDet(\boldsymbol{M}(0))}^{\,-1}\cdot\left.\frac{d}{dt}\right|_{t=0}SymDet(\boldsymbol{M}(t))\right).$$

Denote by $[\,\cdot\,]^\top$ the transpose of the matrix $[\,\cdot\,]$ and let

$$\boldsymbol{M}(t) = \begin{bmatrix}\boldsymbol{M}_{(1)}(t)\\ \cdots\\ \boldsymbol{M}_{(m)}(t)\end{bmatrix} = \left[\boldsymbol{M}_{(1)}^\top,\,\cdots,\,\boldsymbol{M}_{(m)}^\top\right]^\top$$

be the presentation of $\boldsymbol{M}(t)$ in terms of its row vectors and denote $\left.\frac{d}{dt}\right|_{t=0}\boldsymbol{M}_{(\mu)}(t)$ by $\dot{\boldsymbol{M}}_{(\mu)}(0)$, for $\mu = 1,\,\ldots\,,m$. Then

$$\left.\frac{d}{dt}\right|_{t=0}SymDet(\boldsymbol{M}(t)) = \sum_{\mu=1}^m SymDet([\boldsymbol{M}_{(1)}(0)^\top,\,\cdots,\,\boldsymbol{M}_{(\mu-1)}(0)^\top,\,\dot{\boldsymbol{M}}_{(\mu)}(0)^\top,\,\boldsymbol{M}_{(\mu+1)}(0)^\top,\,\cdots,\,\boldsymbol{M}_{(m)}(0)^\top]^\top).$$

Denote $\left.\frac{d}{dt}\right|_{t=0}\boldsymbol{M}_{\mu\nu}(t)$ by $\dot{\boldsymbol{M}}_{\mu\nu}(0)$, for $\mu,\,\nu = 1,\,\ldots\,,m$. Then, the ν-th entry in $\dot{\boldsymbol{M}}_{(\mu)}(0)$ is given by

$$\begin{aligned}\dot{\boldsymbol{M}}_{\mu\nu}(0) &= \sum_{i,j}R^{E_{ij}}[1]|_{\boldsymbol{y}^{\boldsymbol{d}}\,\boldsymbol{\cdot}(\varphi^\sharp(\boldsymbol{y}^{\boldsymbol{d}}))}\cdot D_\mu\varphi^\sharp(y^i)D_\nu\varphi^\sharp(y^j)\\ &\quad+\sum_{i,j}\varphi^\sharp(E_{ij})\cdot(D_\mu\dot{\varphi}^\sharp(y^i) - [\varphi^\sharp(y^i),\dot{A}_\mu])\cdot D_\nu\varphi^\sharp(y^j)\\ &\quad+\sum_{i,j}\varphi^\sharp(E_{ij})D_\mu\varphi^\sharp(y^i)\cdot(D_\nu\dot{\varphi}^\sharp(y^j) - [\varphi^\sharp(y^j),\dot{A}_\nu]) + 2\pi\alpha'(D_\mu\dot{A}_\nu - D_\nu\dot{A}_\mu).\end{aligned}$$

With

$$\boldsymbol{M}_{\mu\nu}(0) = \sum_{i,j}\varphi^\sharp(E_{ij})D_\mu\varphi^\sharp(y^i)D_\nu\varphi^\sharp(y^j) + 2\pi\alpha'[\nabla_\mu,\nabla_\nu],$$

one has altogether:

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} S_{\mathrm{DBI}}(\varphi_t,\nabla^t) &= -T_{m-1}\left.\frac{d}{dt}\right|_{t=0}\int_U Re\left(Tr\left(e^{-\varphi_t^\sharp(\Phi)}\sqrt{-SymDet(\boldsymbol{M}(t))}\right)\right)d^m\boldsymbol{x}\\ &= -T_{m-1}\int_U Re\Big(Tr\Big(R^{e^{-\Phi}}[1]|_{\boldsymbol{y}^{\boldsymbol{d}}\,\boldsymbol{\cdot}(\varphi^\sharp(\boldsymbol{y}^{\boldsymbol{d}}))}\cdot\sqrt{-SymDet(\boldsymbol{M}(0))}\\ &\qquad-\frac{1}{2}e^{-\varphi^\sharp(\Phi)}\sqrt{-SymDet(\boldsymbol{M}(0))}^{\,-1}\\ &\qquad\cdot\sum_{\mu=1}^m SymDet([\boldsymbol{M}_{(1)}(0)^\top,\,\cdots,\,\boldsymbol{M}_{(\mu-1)}(0)^\top,\,\dot{\boldsymbol{M}}_{(\mu)}(0)^\top,\,\boldsymbol{M}_{(\mu+1)}(0)^\top,\,\cdots,\,\boldsymbol{M}_{(m)}(0)^\top]^\top)\Big)\Big)d^m\boldsymbol{x}\\ &= -T_{m-1}\int_U Re\Big(Tr\Big((R^{e^{-\Phi}}[1])|_{\boldsymbol{y}^{\boldsymbol{d}}\,\boldsymbol{\cdot}(\varphi^\sharp(\boldsymbol{y}^{\boldsymbol{d}}))}\cdot\sqrt{-SymDet(\boldsymbol{M}(0))}\\ &\qquad-\frac{1}{2}e^{-\varphi^\sharp(\Phi)}\sqrt{-SymDet(\boldsymbol{M}(0))}^{\,-1}\\ &\qquad\cdot\sum_{\mu=1}^m\sum_{\sigma\in Sym_m}(-1)^\sigma\boldsymbol{M}_{1\,\sigma(1)}(0)\odot\cdots\odot\boldsymbol{M}_{(\mu-1)\,\sigma(\mu-1)}(0)\\ &\qquad\qquad\odot\dot{\boldsymbol{M}}_{\mu\,\sigma(\mu)}(0)\odot\boldsymbol{M}_{(\mu+1)\,\sigma(\mu+1)}(0)\odot\cdots\odot\boldsymbol{M}_{m\,\sigma(m)}(0)\Big)\Big)d^m\boldsymbol{x}.\end{aligned}$$

Step (2) : *Arrangement to boundary terms and the linear functional* $\delta S_{\mathrm{DBI}}(\varphi,\nabla)/\delta(\varphi,\nabla)$ *on* $(\dot{\varphi}^\sharp(y^1),\,\cdots,\,\dot{\varphi}^\sharp(y^n);\,\dot{A}_1,\,\cdots,\,\dot{A}_m)$

Summands from the first cluster

$$R^{e^{-\Phi}}[1]|_{\boldsymbol{y}^{\boldsymbol{d}}\cdot_{(\varphi^\sharp(\boldsymbol{y}^{\boldsymbol{d}}))}}\cdot\sqrt{-SymDet(\boldsymbol{M}(0))}$$

contain only $\dot{\varphi}^\sharp(y^i)$, $i=1,\,\ldots,\,n$, from $(R^{e^{-\Phi}}[1])|_{\boldsymbol{y}^{\boldsymbol{d}}\cdot_{(\varphi^\sharp(\boldsymbol{y}^{\boldsymbol{d}}))}}$. Hence, it contributes solely to the linear functional $\delta S_{\mathrm{DBI}}^{(\Phi,g,B)}(\varphi,\nabla)/\delta(\varphi,\nabla)$ on $(\dot{\varphi}^\sharp(y^1),\,\cdots,\,\dot{\varphi}^\sharp(y^n);\,\dot{A}_1,\,\cdots,\,\dot{A}_m)$ and, hence, to the equations of motion for (φ,∇).

On the other hand, summands from the expansion of the second cluster

$$-\,\frac{1}{2}\,e^{-\varphi^\sharp(\Phi)}\sqrt{-SymDet(\boldsymbol{M}(0))}^{\,-1}\cdot\sum_{\mu=1}^{m}\sum_{\sigma\in Sym_m}(-1)^\sigma\boldsymbol{M}_{1\,\sigma(1)}(0)\odot\cdots\odot\boldsymbol{M}_{(\mu-1)\,\sigma(\mu-1)}(0)\odot\dot{\boldsymbol{M}}_{\mu\,\sigma(\mu)}(0)\odot\boldsymbol{M}_{(\mu+1)\,\sigma(\mu+1)}(0)\odot\cdots\odot\boldsymbol{M}_{m\,\sigma(m)}(0)$$

are of two types:

- One contains a factor in the list $\dot{\varphi}^\sharp(y^i)$, $i=1,\,\ldots,\,n$, $\dot{A}_\mu$, $\mu=1,\,\ldots,\,m$ from some $\dot{\boldsymbol{M}}_{\mu'\nu'}(0)$, $\mu',\,\nu'=1,\,\ldots,\,m$. They contribute to the linear functional $\delta S_{\mathrm{DBI}}(\varphi,\nabla)/\delta(\varphi,\nabla)$ on $(\dot{\varphi}^\sharp(y^1),\,\cdots,\,\dot{\varphi}^\sharp(y^n);\,\dot{A}_1,\,\cdots,\,\dot{A}_m)$ and, hence, to the equations of motion for (φ,∇).
- The other contains a factor in the list $D_\mu\dot{\varphi}(y^i)$, $i=1,\,\ldots,\,n$, $\mu=1,\,\ldots,\,m$, $D_\mu\dot{A}_\nu$, $\mu,\,\nu=1,\,\ldots,\,m$, from some $\dot{\boldsymbol{M}}_{\mu'\nu'}(0)$, $\mu',\,\nu'=1,\,\ldots,\,m$. After integration by parts, each contributes a boundary term in an integral $\int_{\partial U}(\cdots)$ and a term in the linear functional $\delta S_{\mathrm{DBI}}(\varphi,\nabla)/\delta(\varphi,\nabla)$ on $(\dot{\varphi}^\sharp(y^1),\,\cdots,\,\dot{\varphi}^\sharp(y^n);\,\dot{A}_1,\,\cdots,\,\dot{A}_m)$. The latter contributes then to the equations of motion for (φ,∇).

We now proceed to study their details.

Step (3) : *Details for the first cluster*

For the first cluster,

$$\begin{aligned}
&Tr\Big(R^{e^{-\Phi}}[1]|_{\boldsymbol{y}^{\boldsymbol{d}}\cdot_{(\varphi^\sharp(\boldsymbol{y}^{\boldsymbol{d}}))}}\cdot\sqrt{-SymDet(\boldsymbol{M}(0))}\Big)\\
&=\;Tr\Big(\Big(\sum_{i'=1}^{n}\sum_{d,\,\boldsymbol{d},\,\vec{\pi};\,|\boldsymbol{d}|=d,\,i_{(\vec{\pi},\boldsymbol{d})}=i'}R^{e^{-\Phi}}[1]^{L}_{(\boldsymbol{d},\,\vec{\pi})}(\varphi^\sharp(\boldsymbol{y}))\cdot\dot{\varphi}^\sharp(y^{i'})\cdot R^{e^{-\Phi}}[1]^{R}_{(\boldsymbol{d},\,\vec{\pi})}(\varphi^\sharp(\boldsymbol{y}))\Big)\cdot\sqrt{-SymDet(\boldsymbol{M}(0))}\Big)\\
&=\;Tr\Big(\sum_{i'=1}^{n}\Big(\sum_{d,\,\boldsymbol{d},\,\vec{\pi};\,|\boldsymbol{d}|=d,\,i_{(\vec{\pi},\boldsymbol{d})}=i'}R^{e^{-\Phi}}[1]^{R}_{(\boldsymbol{d},\,\vec{\pi})}(\varphi^\sharp(\boldsymbol{y}))\cdot\sqrt{-SymDet(\boldsymbol{M}(0))}\cdot R^{e^{-\Phi}}[1]^{L}_{(\boldsymbol{d},\,\vec{\pi})}(\varphi^\sharp(\boldsymbol{y}))\Big)\cdot\dot{\varphi}^\sharp(y^{i'})\Big)\\
&=:\;Tr\Big(\sum_{i'=1}^{n}\mathbb{N}\!\mathbb{L}_{i'}^{1,\,(\Phi,g,B)}(\varphi,\nabla)\cdot\dot{\varphi}^\sharp(y^{i'})\Big)\,.
\end{aligned}$$

Step (4) : *Details for the second cluster*

For the second cluster, we have

$$\begin{aligned}
&SymDet([\boldsymbol{M}_{(1)}(0)^\top,\cdots,\boldsymbol{M}_{(\mu-1)}(0)^\top,\dot{\boldsymbol{M}}_{(\mu)}(0)^\top,\boldsymbol{M}_{(\mu+1)}(0)^\top,\cdots,\boldsymbol{M}_{(m)}(0)^\top]^\top)\;=\\
&\frac{1}{m!}\sum_{\mu'=1}^{m}\sum_{\substack{\sigma\,\in\,Sym_m\\ \sigma(\mu')\,=\,\mu}}(-1)^\sigma Det([\boldsymbol{M}_{(\sigma(1))}(0)^\top,\cdots,\boldsymbol{M}_{(\sigma(\mu'-1))}(0)^\top,\dot{\boldsymbol{M}}_{(\mu)}(0)^\top,\boldsymbol{M}_{(\sigma(\mu'+1))}(0)^\top,\cdots,\boldsymbol{M}_{(\sigma(m))}(0)^\top]^\top)\,.
\end{aligned}$$

Thus, denoting the factor $-\frac{1}{2}\,e^{-\varphi^\sharp(\Phi)}\sqrt{-SymDet(\boldsymbol{M}(0))}^{\,-1}$ by $F_2(\varphi,\nabla;\Phi,g,B)$,

$$
\begin{aligned}
& Tr\Big(-\frac{1}{2}\,e^{-\varphi^\sharp(\Phi)}\sqrt{-SymDet(\boldsymbol{M}(0))}^{\,-1} \\
&\qquad \cdot \sum_{\mu=1}^{m} SymDet([\boldsymbol{M}_{(1)}(0)^\top,\,\cdots,\,\boldsymbol{M}_{(\mu-1)}(0)^\top,\,\dot{\boldsymbol{M}}_{(\mu)}(0)^\top,\,\boldsymbol{M}_{(\mu+1)}(0)^\top,\,\cdots,\,\boldsymbol{M}_{(m)}(0)^\top]^\top)\Big) \\
&= Tr\Big(F_2(\varphi,\nabla;\Phi,g,B)\cdot \sum_{\mu=1}^{m} SymDet([\boldsymbol{M}_{(1)}(0)^\top,\,\cdots,\,\boldsymbol{M}_{(\mu-1)}(0)^\top,\,\dot{\boldsymbol{M}}_{(\mu)}(0)^\top,\,\boldsymbol{M}_{(\mu+1)}(0)^\top,\,\cdots,\,\boldsymbol{M}_{(m)}(0)^\top]^\top)\Big) \\
&= Tr\Big(\frac{1}{m!}\,F_2(\varphi,\nabla;\Phi,g,B)\cdot \sum_{\mu=1}^{m}\sum_{\mu'=1}^{m}\sum_{\substack{\sigma\,\in\,Sym_m\\ \sigma(\mu')\,=\,\mu}} \\
&\qquad (-1)^\sigma\, Det([\boldsymbol{M}_{(\sigma(1))}(0)^\top,\,\cdots,\,\boldsymbol{M}_{(\sigma(\mu'-1))}(0)^\top,\,\dot{\boldsymbol{M}}_{(\mu)}(0)^\top,\,\boldsymbol{M}_{(\sigma(\mu'+1))}(0)^\top,\,\cdots,\,\boldsymbol{M}_{(\sigma(m))}(0)^\top]^\top)\Big) \\
&= Tr\Big(\frac{1}{m!}\sum_{\mu=1}^{m}\sum_{\mu'=1}^{m}\sum_{\substack{\sigma\,\in\,Sym_m\\ \sigma(\mu')\,=\,\mu}} (-1)^\sigma \\
&\qquad \cdot Det([F_2(\varphi,\nabla;\Phi,g,B)\,\boldsymbol{M}_{(\sigma(1))}(0)^\top,\,\cdots,\,\boldsymbol{M}_{(\sigma(\mu'-1))}(0)^\top,\,\dot{\boldsymbol{M}}_{(\mu)}(0)^\top,\,\boldsymbol{M}_{(\sigma(\mu'+1))}(0)^\top,\,\cdots,\,\boldsymbol{M}_{(\sigma(m))}(0)^\top]^\top)\Big) \\
&= Tr\Big(\frac{1}{m!}\sum_{\mu=1}^{m}\sum_{\mu'=1}^{m}\sum_{\substack{\sigma\,\in\,Sym_m\\ \sigma(\mu')\,=\,\mu}} (-1)^\sigma\,(-1)^{\mu'(m-\mu')} \\
&\qquad \cdot Det([\boldsymbol{M}_{(\sigma(\mu'+1))}(0)^\top,\,\cdots,\,\boldsymbol{M}_{(\sigma(m))}(0)^\top,\,F_2(\varphi,\nabla;\Phi,g,B)\,\boldsymbol{M}_{(\sigma(1))}(0)^\top,\,\cdots,\,\boldsymbol{M}_{(\sigma(\mu'-1))}(0)^\top,\,\dot{\boldsymbol{M}}_{(\mu)}(0)^\top]^\top)\Big)
\end{aligned}
$$

(by the invariance of trace under cyclic permutations).

Note that $\dot{\boldsymbol{M}}_{\mu\nu}(0)$, $\mu,\nu=1,\,\ldots\,,m$, now appear uniformly as the last factor in the summands from the expansion of $Det([\cdots]^\top)$ above. Let $Minor(\varphi,\nabla;\Phi,g,B\,|\,\mu',\sigma)_{\mu\nu}$ be the (m,ν)-minor of $[\boldsymbol{M}_{(\sigma(\mu'+1))}(0),\,\cdots,\,\boldsymbol{M}_{(\sigma(m))}(0),\,F_2(\varphi,\nabla;\Phi,g,B)\,\boldsymbol{M}_{(\sigma(1))}(0),\,\cdots,\,\boldsymbol{M}_{(\sigma(\mu'-1))}(0),\,\dot{\boldsymbol{M}}_{(\mu)}(0)]^\top$. Then:

$$
\begin{aligned}
&= Tr\Big(\frac{1}{m!}\sum_{\mu=1}^{m}\sum_{\mu'=1}^{m}\sum_{\substack{\sigma\,\in\,Sym_m\\ \sigma(\mu')\,=\,\mu}} (-1)^\sigma\,(-1)^{\mu'(m-\mu')}\cdot\sum_{\nu=1}^{m}(-1)^{m+\nu}\,Minor(\varphi,\nabla;\Phi,g,B\,|\,\mu',\sigma)_{\mu\nu}\dot{\boldsymbol{M}}_{\mu\nu}(0)\Big) \\
&= Tr\Big(\sum_{\mu=1}^{m}\sum_{\nu=1}^{m}\Xi(\varphi,\nabla;\Phi,g,B)_{\mu\nu}\,\dot{\boldsymbol{M}}_{\mu\nu}(0)\Big), \\
&\qquad \text{where } \Xi(\varphi,\nabla;\Phi,g,B)_{\mu\nu} \\
&\qquad\qquad := \frac{1}{m!}\sum_{\mu'=1}^{m}\sum_{\substack{\sigma\,\in\,Sym_m\\ \sigma(\mu')\,=\,\mu}} (-1)^\sigma\,(-1)^{\mu'(m-\mu')+m+\nu}\,Minor(\varphi,\nabla;\Phi,g,B\,|\,\mu',\sigma)_{\mu\nu}\,, \\
&= Tr\Big(\sum_{\mu=1}^{m}\sum_{\nu=1}^{m}\Xi(\varphi,\nabla;\Phi,g,B)_{\mu\nu} \\
&\qquad\cdot\Big(\sum_{i,j} R^{E_{ij}}[1]|_{\boldsymbol{y^d}}{}^{\bullet}{}_{(\varphi^\sharp(\boldsymbol{y^d}))}\cdot D_\mu\varphi^\sharp(y^i)D_\nu\varphi^\sharp(y^j) \\
&\qquad\quad + \sum_{i,j}\varphi^\sharp(E_{ij})\cdot(D_\mu\dot{\varphi}^\sharp(y^i)\,-\,[\varphi^\sharp(y^i),\dot{A}_\mu])\cdot D_\nu\varphi^\sharp(y^j) \\
&\qquad\quad + \sum_{i,j}\varphi^\sharp(E_{ij})D_\mu\varphi^\sharp(y^i)\cdot(D_\nu\dot{\varphi}^\sharp(y^j)\,-\,[\varphi^\sharp(y^j),\dot{A}_\nu])\,+\,2\pi\alpha'\,(D_\mu\dot{A}_\nu-D_\nu\dot{A}_\mu)\Big)\Big) \\
&= \text{(I)} + \text{(II)} + \text{(III)} + \text{(IV)} \qquad (\text{defined in } Step\ (4.1) - Step\ (4.4) \text{ below}).
\end{aligned}
$$

Let us now study each of the four subclusters of the second cluster separately.

Step (4.1) : *The subcluster* (I)

$$
\text{(I)} \;:=\; Tr\Big(\sum_{\mu=1}^{m}\sum_{\nu=1}^{m}\Xi(\varphi,\nabla;\Phi,g,B)_{\mu\nu}\cdot\sum_{i,j}R^{E_{ij}}[1]|_{\boldsymbol{y^d}}{}^{\bullet}{}_{(\varphi^\sharp(\boldsymbol{y^d}))}\cdot D_\mu\varphi^\sharp(y^i)D_\nu\varphi^\sharp(y^j)\Big)
$$

$$
\begin{aligned}
&= Tr\Big(\sum_{\mu=1}^{m}\sum_{\nu=1}^{m} \Xi(\varphi,\nabla;\Phi,g,B)_{\mu\nu} \\
&\qquad \cdot \sum_{i,j} \Big(\sum_{i'=1}^{n} \sum_{d,\,\boldsymbol{d},\,\vec{\pi};\,|\boldsymbol{d}|=d,\,i_{(\vec{\pi},\boldsymbol{d})}=i'} R^{E_{ij}}[1]^{L}_{(\boldsymbol{d},\,\vec{\pi})}(\varphi^\sharp(\boldsymbol{y}))\cdot\dot{\varphi}^\sharp(y^{i'})\cdot R^{E_{ij}}[1]^{R}_{(\boldsymbol{d},\,\vec{\pi})}(\varphi^\sharp(\boldsymbol{y}))\Big)\cdot D_\mu\varphi^\sharp(y^i)D_\nu\varphi^\sharp(y^j)\Big) \\
&= Tr\Big(\sum_{i'=1}^{n}\Big(\sum_{\mu=1}^{m}\sum_{\nu=1}^{m}\sum_{i,j} \sum_{d,\,\boldsymbol{d},\,\vec{\pi};\,|\boldsymbol{d}|=d,\,i_{(\vec{\pi},\boldsymbol{d})}=i'} R^{E_{ij}}[1]^{R}_{(\boldsymbol{d},\,\vec{\pi})}(\varphi^\sharp(\boldsymbol{y}))\cdot D_\mu\varphi^\sharp(y^i)D_\nu\varphi^\sharp(y^j) \\
&\qquad \cdot\Xi(\varphi,\nabla;\Phi,g,B)_{\mu\nu}\cdot R^{E_{ij}}[1]^{L}_{(\boldsymbol{d},\,\vec{\pi})}(\varphi^\sharp(\boldsymbol{y}))\Big)\cdot\dot{\varphi}^\sharp(y^{i'})\Big) \\
&=: Tr\Big(\sum_{i'=1}^{n} \mathbb{L}_{i'}^{2.I,\,(\Phi,g,B)}(\varphi,\nabla)\cdot\dot{\varphi}^\sharp(y^{i'})\Big).
\end{aligned}
$$

Step (4.2) : *The subcluster* (II)

This subcluster contributes also to boundary terms.

$$
\begin{aligned}
\text{(II)} \;:=\;& Tr\Big(\sum_{\mu=1}^{m}\sum_{\nu=1}^{m} \Xi(\varphi,\nabla;\Phi,g,B)_{\mu\nu} \\
&\qquad \cdot\Big(\sum_{i,j}\varphi^\sharp(E_{ij})\cdot D_\mu\dot{\varphi}^\sharp(y^i)\cdot D_\nu\varphi^\sharp(y^j) + \sum_{i,j}\varphi^\sharp(E_{ij})D_\mu\varphi^\sharp(y^i)\cdot D_\nu\dot{\varphi}^\sharp(y^j)\Big)\Big) \\
=\;& Tr\Big(\sum_{\mu=1}^{m}\sum_{\nu=1}^{m}\sum_{i,j}\Big(D_\mu\varphi^\sharp(y^i)\cdot\Xi(\varphi,\nabla;\Phi,g,B)_{\nu\mu}\,\varphi^\sharp(E_{ji}) \\
&\qquad + \Xi(\varphi,\nabla;\Phi,g,B)_{\mu\nu}\varphi^\sharp(E_{ij})D_\mu\varphi^\sharp(y^i)\Big)\cdot D_\nu\dot{\varphi}^\sharp(y^j)\Big) \\
=\;& Tr\Big(\sum_{\nu=1}^{m}\sum_{\mu=1}^{m}\sum_{i,j} D_\nu\Big[\Big(D_\mu\varphi^\sharp(y^i)\,\Xi(\varphi,\nabla;\Phi,g,B)_{\nu\mu}\,\varphi^\sharp(E_{ji}) \\
&\qquad + \Xi(\varphi,\nabla;\Phi,g,B)_{\mu\nu}\,\varphi^\sharp(E_{ij})\,D_\mu\varphi^\sharp(y^i)\Big)\dot{\varphi}^\sharp(y^j)\Big]\Big) \\
&- Tr\Big(\sum_{j=1}^{n}\Big(\sum_{\mu=1}^{m}\sum_{\nu=1}^{m}\sum_{i=1}^{n} D_\nu\Big[D_\mu\varphi^\sharp(y^i)\cdot\Xi(\varphi,\nabla;\Phi,g,B)_{\nu\mu}\,\varphi^\sharp(E_{ji}) \\
&\qquad + \Xi(\varphi,\nabla;\Phi,g,B)_{\mu\nu}\varphi^\sharp(E_{ij})D_\mu\varphi^\sharp(y^i)\Big]\Big)\cdot\dot{\varphi}^\sharp(y^j)\Big) \\
=\;& \sum_{\nu=1}^{m}\partial_\nu\, Tr\Big(\sum_{\mu=1}^{m}\sum_{i,j}\Big[\Big(D_\mu\varphi^\sharp(y^i)\,\Xi(\varphi,\nabla;\Phi,g,B)_{\nu\mu}\,\varphi^\sharp(E_{ji}) \\
&\qquad + \Xi(\varphi,\nabla;\Phi,g,B)_{\mu\nu}\,\varphi^\sharp(E_{ij})\,D_\mu\varphi^\sharp(y^i)\Big)\dot{\varphi}^\sharp(y^j)\Big]\Big) \\
&- Tr\Big(\sum_{j=1}^{n}\Big(\sum_{\mu=1}^{m}\sum_{\nu=1}^{m}\sum_{i=1}^{n} D_\nu\Big[D_\mu\varphi^\sharp(y^i)\cdot\Xi(\varphi,\nabla;\Phi,g,B)_{\nu\mu}\,\varphi^\sharp(E_{ji}) \\
&\qquad + \Xi(\varphi,\nabla;\Phi,g,B)_{\mu\nu}\varphi^\sharp(E_{ij})D_\mu\varphi^\sharp(y^i)\Big]\Big)\cdot\dot{\varphi}^\sharp(y^j)\Big) \\
=:\;& \sum_{\nu=1}^{m}(-1)^{\nu-1}\,\partial_\nu(\mathbb{B}\Gamma_\nu^{2.II,\,(\varphi,\nabla;\Phi,g,B)}(\dot{\varphi}^\sharp(\boldsymbol{y}))) + Tr\Big(\sum_{j=1}^{n}\mathbb{L}_j^{2.II,\,(\Phi,g,B)}(\varphi,\nabla)\cdot\dot{\varphi}^\sharp(y^j)\Big).
\end{aligned}
$$

Step (4.3) : *The subcluster* (III)

$$
\begin{aligned}
\text{(III)} \;:=\;& Tr\Big(-\sum_{\mu=1}^{m}\sum_{\nu=1}^{m}\Xi(\varphi,\nabla;\Phi,g,B)_{\mu\nu} \\
&\qquad \cdot\Big(\sum_{i,j}\varphi^\sharp(E_{ij})\cdot[\varphi^\sharp(y^i),\dot{A}_\mu]\cdot D_\nu\varphi^\sharp(y^j) + \sum_{i,j}\varphi^\sharp(E_{ij})\,D_\mu\varphi^\sharp(y^i)\cdot[\varphi^\sharp(y^j),\dot{A}_\nu]\Big)\Big) \\
=\;& Tr\Big(\sum_{\nu=1}^{m}\sum_{\mu=1}^{m}\sum_{i,j}\Big(\Xi(\varphi,\nabla;\Phi,g,B)_{\mu\nu}\,\varphi^\sharp(E_{ij})\,D_\mu\varphi^\sharp(y^i)\,\varphi^\sharp(y^j) \\
&\qquad - \varphi^\sharp(y^j)\,\Xi(\varphi,\nabla;\Phi,g,B)_{\mu\nu}\,\varphi^\sharp(E_{ij})\,D_\mu\varphi^\sharp(y^i) \\
&\qquad - D_\mu\varphi^\sharp(y^j)\,\Xi(\varphi,\nabla;\Phi,g,B)_{\nu\mu}\,\varphi^\sharp(E_{ij})\,\varphi^\sharp(y^i) \\
&\qquad + \varphi^\sharp(y^i)\,D_\mu\varphi^\sharp(y^j)\,\Xi(\varphi,\nabla;\Phi,g,B)_{\nu\mu}\,\varphi^\sharp(E_{ij})\Big)\cdot\dot{A}_\nu\Big) \\
=:\;& Tr\Big(\sum_{\nu=1}^{m}\mathbb{L}_\nu^{2.III,\,(\Phi,g,B)}(\varphi,\nabla)\cdot\dot{A}_\nu\Big).
\end{aligned}
$$

Step (4.4) : *The subcluster* (IV)

This subcluster contributes also to boundary terms.

$$
\begin{aligned}
(\mathrm{IV}) \;\; &:= \;\; Tr\Big(2\pi\alpha' \sum_{\mu=1}^{m}\sum_{\nu=1}^{m} \Xi(\varphi,\nabla;\Phi,g,B)_{\mu\nu}\cdot(D_\mu\dot{A}_\nu - D_\nu\dot{A}_\mu)\Big) \\
&= \;\; Tr\Big(2\pi\alpha' \sum_{\mu=1}^{m}\sum_{\nu=1}^{m} \Big(\Xi(\varphi,\nabla;\Phi,g,B)_{\mu\nu} - \Xi(\varphi,\nabla;\Phi,g,B)_{\nu\mu}\Big)\cdot D_\mu\dot{A}_\nu\Big) \\
&= \;\; Tr\Big(2\pi\alpha' \sum_{\mu=1}^{m}\sum_{\nu=1}^{m} D_\mu\Big[\Big(\Xi(\varphi,\nabla;\Phi,g,B)_{\mu\nu} - \Xi(\varphi,\nabla;\Phi,g,B)_{\nu\mu}\Big)\cdot\dot{A}_\nu\Big]\Big) \\
&\qquad - Tr\Big(2\pi\alpha' \sum_{\nu=1}^{m}\Big(\sum_{\mu=1}^{m} D_\mu\Big[\Xi(\varphi,\nabla;\Phi,g,B)_{\mu\nu} - \Xi(\varphi,\nabla;\Phi,g,B)_{\nu\mu}\Big]\Big)\cdot\dot{A}_\nu\Big) \\
&= \;\; \sum_{\mu=1}^{m}\partial_\mu\, Tr\Big(2\pi\alpha' \sum_{\nu=1}^{m} \Big(\Xi(\varphi,\nabla;\Phi,g,B)_{\mu\nu} - \Xi(\varphi,\nabla;\Phi,g,B)_{\nu\mu}\Big)\cdot\dot{A}_\nu\Big) \\
&\qquad - Tr\Big(2\pi\alpha' \sum_{\nu=1}^{m}\Big(\sum_{\mu=1}^{m} D_\mu\Big[\Xi(\varphi,\nabla;\Phi,g,B)_{\mu\nu} - \Xi(\varphi,\nabla;\Phi,g,B)_{\nu\mu}\Big]\Big)\cdot\dot{A}_\nu\Big) \\
&=: \;\; \sum_{\mu=1}^{m}(-1)^{\mu-1}\,\partial_\mu\big(\mathit{BT}_\mu^{\,2.IV,\,(\varphi,\nabla;\Phi,g,B)}(\dot{\boldsymbol{A}})\big) \; + \; Tr\Big(\sum_{\nu=1}^{m} \mathit{NL}_\nu^{2.IV,\,(\Phi,g,B)}(\varphi,\nabla)\cdot\dot{A}_\nu\Big)\,.
\end{aligned}
$$

Step (5) : *The final formula*

In summary, with the notation introduced for the various nonlinear first-order and second-order differential expressions on (φ,∇) that depend on (Φ,g,B) and appear in the calculation (subject to a relabelling of the dummy i' index), one has

$$
\begin{aligned}
&\frac{d}{dt}\Big|_{t=0} S_{\mathrm{DBI}}(\varphi_t,\nabla^t) \;=\; -T_{m-1}\,\frac{d}{dt}\Big|_{t=0}\int_U Re\,\Big(Tr\,\Big(e^{-\varphi_t^\sharp(\Phi)}\sqrt{-SymDet(\boldsymbol{M}(t))}\Big)\Big)\,d^m\boldsymbol{x} \\
&= -T_{m-1}\int_U Re\Big(\sum_{\mu=1}^{m}(-1)^{\mu-1}\partial_\mu\Big(\mathit{BT}_\mu^{\,2.II,\,(\varphi,\nabla;\Phi,g,B)}(\dot{\varphi}^\sharp(\boldsymbol{y})) + \mathit{BT}_\mu^{\,2.IV,\,(\varphi,\nabla;\Phi,g,B)}(\dot{\boldsymbol{A}})\Big)\Big)\,d^m\boldsymbol{x} \\
&\qquad -T_{m-1}\int_U Re\Big(Tr\Big(\sum_{j=1}^{n}(\mathit{NL}_j^{1,\,(\Phi,g,B)}(\varphi,\nabla) + \mathit{NL}_j^{2.I,\,(\Phi,g,B)}(\varphi,\nabla) + \mathit{NL}_j^{2.II,\,(\Phi,g,B)}(\varphi,\nabla))\cdot\dot{\varphi}^\sharp(y^j) \\
&\qquad\qquad + \sum_{\nu=1}^{m}(\mathit{NL}_\nu^{2.III,\,(\Phi,g,B)}(\varphi,\nabla) + \mathit{NL}_\nu^{2.IV,\,(\Phi,g,B)}(\varphi,\nabla))\cdot\dot{A}_\nu\Big)\Big)\,d^m\boldsymbol{x} \\
&=: -T_{m-1}\int_{\partial U} Re\,(\mathit{BT}^{\,(\varphi,\nabla;\Phi,g,B)}(\dot{\varphi}^\sharp(\boldsymbol{y}),\dot{\boldsymbol{A}})) \\
&\qquad -T_{m-1}\int_U Re\Big(Tr\Big(\sum_{j=1}^{n}\mathit{NL}_j^{(\Phi,g,B);\delta\varphi}(\varphi,\nabla)\cdot\dot{\varphi}^\sharp(y^j) + \sum_{\nu=1}^{m}\mathit{NL}_\nu^{(\Phi,g,B);\delta\nabla}(\varphi,\nabla)\cdot\dot{A}_\nu\Big)\Big)\,d^m\boldsymbol{x}\,.
\end{aligned}
$$

Here,

$$
\begin{aligned}
&\mathit{BT}^{\,(\varphi,\nabla;\Phi,g,B)}(\dot{\varphi}^\sharp(\boldsymbol{y}),\dot{\boldsymbol{A}}) \\
&\quad := \sum_{\mu=1}^{m}\Big(\mathit{BT}_\mu^{\,2.II,\,(\varphi,\nabla;\Phi,g,B)}(\dot{\varphi}^\sharp(\boldsymbol{y})) + \mathit{BT}_\mu^{\,2.IV,\,(\varphi,\nabla;\Phi,g,B)}(\dot{\boldsymbol{A}})\Big)\,dx^1\wedge\cdots\wedge dx^{\mu-1}\wedge\widehat{dx^\mu}\wedge dx^{\mu+1}\cdots\wedge dx^m\,,
\end{aligned}
$$

with the $\widehat{dx^\mu}$ meaning the removal of dx^μ, is a complex-valued $(m-1)$-form on U that depends linearly on $(\dot{\boldsymbol{y}},\dot{\boldsymbol{A}})$ and whose real part gives the total boundary term (up to the factor $-T_{m-1}$) of the first variation of $S_{\mathrm{DBI}}^{(\Phi,g,B)}(\varphi,\nabla)$ with respect to (φ,∇).

2.1 The equations of motion for D-branes

Remark 2.1.1 [*effect of* $Re\,(\,\cdot\,)$ *in action to equations of motion*] Due to the operation '*Taking the real part of*' $Re\,(\,\cdot\,)$, to go from the the first variation formula to the expression for the equations of motion there is a detail that depends on how the space of pairs (φ,∇) and its tangents $(\delta\varphi,\delta\nabla)$ are parameterized; (cf. $Re\,(e^{\sqrt{-1}\theta}z)=\cos\theta\cdot Re\,(z)-\sin\theta\cdot Im\,(z)$).

(1) For the φ-part, first, caution that it is *not* that just because $\varphi^\sharp(y^i)$, $i=1,\,\ldots\,,n$, take values in a ring over C (i.e. $C^\infty(End_C(E))$) that the space $Map\,((X^{Az},E),Y)$ of all such φ's becomes a complex space. Indeed, due to the fact that all the eigenvalues of $\varphi^\sharp(f)$, $f\in C^\infty(Y)$ are real (cf. [L-Y4: Sec. 3], D(11.1)), $Map\,((X^{Az},E),Y)$ is intrinsically a real space and there is no natural complex-space structure on it (even if exists) that can be made compatible with the underlying moduli problem since if $\delta\varphi$ is an unobstructed tangent to $Map\,((X^{Az},E),Y)$, then $\sqrt{-1}\delta\varphi$ can never be an unobstructed tangent to $Map\,((X^{Az}),Y)$. So this part is good in the sense that if we fix a real presentation for φ's in the study, then $Re\,(\delta S_{\mathrm{DBI}}/\delta\varphi)$ gives the system of equations of motion for φ.

(2) For the ∇-part, if alone, the parameter space is complex in nature in our most general setting. When E is Hermitian and ∇ is required to be compatible with the Hermitian structure, the resulting parameter space becomes intrinsically real. In the latter case, depending on the convention in presenting a unitary gauge theory (mathematicians vs. physicists), one may take either $Re\,(\delta S_{\mathrm{DBI}}^{(\Phi,g,B)}/\delta\nabla)$ or $Im\,(\delta S_{\mathrm{DBI}}^{(\Phi,g,B)})/\delta\nabla$ as the system of equations for ∇. *However*, this is not the full story as we imposed the admissible condition $\nabla_{\bullet}\mathcal{A}_\varphi\subset\mathcal{A}_\varphi$ on ∇. Details on writing the equations of motion will have to depend on how we present this condition.

Not to let this additional detail to distract us in this first work in the D(13) subseries, we present for the current notes the system of equations of motion that remove the effect of $Re\,(\,\cdot\,)$ in $S_{\mathrm{DBI}}^{(\Phi,g,B)}$. In other words, a true system of equations of motion will involve only a combination of what are given below.

It follows from the study in Sec. 5.2 that the equations of motion for D-branes from the Dirac-Born-Infeld action, with the D-brane world-volume modelled in the current context as an admissible map

$$\varphi:(X^{Az},E;\nabla)\;\longrightarrow\;(Y,\Phi,g,B)$$

from an Azumaya/matrix manifold with a fundamental module with a connection $(X^{Az},E;\nabla)$ to a space-time Y with massless background fields (Φ,g,B) from closed string excitations, are given by the following system of second-order nonlinear partial differential equations on (φ,∇):

$$\left\{\begin{array}{lll} \mathbb{N}\!\mathbb{L}_j^{(\Phi,g,B);\delta\varphi}(\varphi,\nabla)\;=\;0\,, & \text{for } j=1,\,\ldots\,,\,n\,;\\ \mathbb{N}\!\mathbb{L}_\nu^{(\Phi,g,B);\delta\nabla}(\varphi,\nabla)\;=\;0\,, & \text{for } \nu=1,\,\ldots\,,\,m\,. \end{array}\right.$$

Here, for the first subsystem,

$$\mathbb{N}\!\mathbb{L}_j^{(\Phi,g,B);\delta\varphi}(\varphi,\nabla)\;=\;\mathbb{N}\!\mathbb{L}_j^{1,\,(\Phi,g,B)}(\varphi,\nabla)\,+\,\mathbb{N}\!\mathbb{L}_j^{2.I,\,(\Phi,g,B)}(\varphi,\nabla)+\,\mathbb{N}\!\mathbb{L}_j^{2.II,\,(\Phi,g,B)}(\varphi,\nabla)$$

with

$$\begin{aligned}
\mathbb{N}\!\mathbb{L}_j^{1,\,(\Phi,g,B)}(\varphi,\nabla)\;&=\;\sum_{d,\,\boldsymbol{d},\,\vec{\pi};\,|\boldsymbol{d}|=d,\,i_{(\vec{\pi},\boldsymbol{d})}=j} R^{\,e^{-\Phi}}[1]^{\,R}_{(\boldsymbol{d},\,\vec{\pi})}(\varphi^\sharp(\boldsymbol{y}))\cdot\sqrt{-SymDet\,(\boldsymbol{M}(0))}\cdot R^{\,e^{-\Phi}}[1]^{\,L}_{(\boldsymbol{d},\,\vec{\pi})}(\varphi^\sharp(\boldsymbol{y}))\,,\\
\mathbb{N}\!\mathbb{L}_j^{2.I,\,(\Phi,g,B)}(\varphi,\nabla)\;&=\;\sum_{\mu=1}^m\sum_{\nu=1}^m\sum_{i',j'}\sum_{d,\,\boldsymbol{d},\,\vec{\pi};\,|\boldsymbol{d}|=d,\,i_{(\vec{\pi},\boldsymbol{d})}=j} R^{\,E_{i'j'}}[1]^{\,R}_{(\boldsymbol{d},\,\vec{\pi})}(\varphi^\sharp(\boldsymbol{y}))\cdot D_\mu\varphi^\sharp(y^i)D_\nu\varphi^\sharp(y^j)\\
&\qquad\cdot\,\Xi(\varphi,\nabla;\Phi,g,B)_{\mu\nu}\cdot R^{\,E_{i'j'}}[1]^{\,L}_{(\boldsymbol{d},\,\vec{\pi})}(\varphi^\sharp(\boldsymbol{y}))\,,\\
\mathbb{N}\!\mathbb{L}_j^{2.II,\,(\Phi,g,B)}(\varphi,\nabla)\;&=\;-\sum_{\mu=1}^m\sum_{\nu=1}^m\sum_{i=1}^n D_\nu\Big(D_\mu\varphi^\sharp(y^i)\cdot\Xi(\varphi,\nabla;\Phi,g,B)_{\nu\mu}\,\varphi^\sharp(E_{ji})\\
&\qquad+\,\Xi(\varphi,\nabla;\Phi,g,B)_{\mu\nu}\varphi^\sharp(E_{ij})D_\mu\varphi^\sharp(y^i)\Big)\,;
\end{aligned}$$

and, for the second subsystem,

$$\mathbb{N}\!\!\mathbb{L}_\nu^{(\Phi,g,B);\delta\nabla}(\varphi,\nabla) \;=\; \mathbb{N}\!\!\mathbb{L}_\nu^{2.III,\,(\Phi,g,B)}(\varphi,\nabla) +\; \mathbb{N}\!\!\mathbb{L}_\nu^{2.IV,\,(\Phi,g,B)}(\varphi,\nabla)$$

with

$$\begin{aligned}
\mathbb{N}\!\!\mathbb{L}_\nu^{2.III,\,(\Phi,g,B)}(\varphi,\nabla) \;&=\; \sum_{\mu=1}^{m}\sum_{i,j}\Big(\Xi(\varphi,\nabla;\Phi,g,B)_{\mu\nu}\,\varphi^\sharp(E_{ij})\,D_\mu\varphi^\sharp(y^i)\,\varphi^\sharp(y^j)\\
&\qquad -\varphi^\sharp(y^j)\,\Xi(\varphi,\nabla;\Phi,g,B)_{\mu\nu}\,\varphi^\sharp(E_{ij})\,D_\mu\varphi^\sharp(y^i)\\
&\qquad -D_\mu\varphi^\sharp(y^j)\,\Xi(\varphi,\nabla;\Phi,g,B)_{\nu\mu}\,\varphi^\sharp(E_{ij})\,\varphi^\sharp(y^i)\\
&\qquad +\varphi^\sharp(y^i)\,D_\mu\varphi^\sharp(y^j)\,\Xi(\varphi,\nabla;\Phi,g,B)_{\nu\mu}\,\varphi^\sharp(E_{ij})\Big)\,,\\
\mathbb{N}\!\!\mathbb{L}_\nu^{2.IV,\,(\Phi,g,B)}(\varphi,\nabla) \;&=\; 2\pi\alpha^\prime\sum_{\mu=1}^{m}D_\mu(\Xi(\varphi,\nabla;\Phi,g,B)_{\nu\mu}\,-\,\Xi(\varphi,\nabla;\Phi,g,B)_{\mu\nu})\,.
\end{aligned}$$

In both subsystems,

$$\Xi(\varphi,\nabla;\Phi,g,B)_{\mu\nu} \;=\; \frac{1}{m!}\sum_{\mu'=1}^{m}\sum_{\substack{\sigma\,\in\,Sym_m\\ \sigma(\mu')\,=\,\mu}}(-1)^\sigma\,(-1)^{\mu'(m-\mu')+m+\nu}\,Minor(\varphi,\nabla;\Phi,g,B\,|\,\mu',\sigma)_{\mu\nu}\,,$$

where

$$\begin{aligned}
&Minor(\varphi,\nabla;\Phi,g,B\,|\,\mu',\sigma)_{\mu\nu} \;=\; \text{the } (m,\nu)\text{-minor of}\\
&\quad[\boldsymbol{M}_{(\sigma(\mu'+1))}(0)^\top,\,\cdots,\,\boldsymbol{M}_{(\sigma(m))}(0)^\top,\,F_2(\varphi,\nabla;\Phi,g,B)\,\boldsymbol{M}_{(\sigma(1))}(0)^\top,\,\cdots,\,\boldsymbol{M}_{(\sigma(\mu'-1))}(0)^\top,\,\dot{\boldsymbol{M}}_{(\mu)}(0)^\top]^\top
\end{aligned}$$

with

$$\begin{aligned}
F_2(\varphi,\nabla;\Phi,g,B) \;&=\; -\frac{1}{2}\,e^{-\varphi^\sharp(\Phi)}\,\sqrt{-SymDet(\boldsymbol{M}(0))}^{\,-1}\,,\\
\boldsymbol{M}_{(\,\bullet\,)}(0) \;&=\; \text{the }\bullet\text{-th row vector of } \boldsymbol{M}(0)\,,\\
\boldsymbol{M}_{\mu\nu}(0) \;&=\; \text{the } (\mu,\nu)\text{-entry of } \boldsymbol{M}(0) \;=\; \sum_{i',j'}\varphi^\sharp(E_{i'j'})\,D_\mu\varphi^\sharp(y^{i'})\,D_\nu\varphi^\sharp(y^{j'})\,+\,2\pi\alpha'\,[\nabla_\mu,\nabla_\nu]\,,\\
\dot{\boldsymbol{M}}_{\mu\nu}(0) \;&=\; \sum_{i',j'}R^{\,E_{i'j'}}[1]|_{\boldsymbol{y^d}\,\bullet_{(\varphi^\sharp(\boldsymbol{y^d}))}}\cdot D_\mu\varphi^\sharp(y^{i'})D_\nu\varphi^\sharp(y^{j'})\\
&\quad+\sum_{i',j'}\varphi^\sharp(E_{i'j'})\cdot(D_\mu\dot{\varphi}^\sharp(y^{i'})\,-\,[\varphi^\sharp(y^{i'}),\dot{A}_\mu])\cdot D_\nu\varphi^\sharp(y^{j'})\\
&\quad+\sum_{i',j'}\varphi^\sharp(E_{i'j'})D_\mu\varphi^\sharp(y^{i'})\cdot(D_\nu\dot{\varphi}^\sharp(y^{j'})\,-\,[\varphi^\sharp(y^{j'}),\dot{A}_\nu])\,+\,2\pi\alpha'\,(D_\mu\dot{A}_\nu-D_\nu\dot{A}_\mu)\,.
\end{aligned}$$

Remark 2.1.2 [*origin/correction from anomaly equations for open strings*] From the string-theory point of view, it is very important to understand further how such systems of differential equations on the pair (φ,∇) can arise from or be correced/improved by the anomaly-free conditions in open-string theory.

Remark 2.1.3 [*the case of Hermitian/unitary D-branes*] When in addition E is equipped with a Hermitian structure and φ is Hermitian and ∇ is unitary, the Dirac-Born-Infeld action functional $S_{(\varphi,\nabla)}^{(\Phi,g,B)}$ and, hence, the resulting equations of motion can be simplified. The detail should be studied further.

3 Remarks on the Chern-Simons/Wess-Zumino term

In view of Polchinski's realization ([Po1]) that a D-brane world-volume can couple to a Ramond-Ramond field in superstring theory (cf. FIGURE 6-0-1), the Chern-Simons/Wess-Zumino term $S_{CS/WZ}$ for D-branes is also an indispensable part to understand the dynamics of D-branes. With the same essence as for the construction of $S_{\mathrm{DBI}}^{(\Phi,g,B)}(\varphi,\nabla)$, we construct in this section the Chern-Simons/Wess-Zumino action $S_{CS/WZ}(\varphi,\nabla)$ for lower-dimensional D-branes, in which cases anomaly issues do not occur, derive their first variation formula and, hence, obtain their contribution to the equations of motions for D-branes.

To begin, with anomalies taken into account, the coupling of a simple embedded D-brane

$$f\;:\;X\hookrightarrow Y$$

with the Ramond-Ramond field C on Y (with a B-field background B), is encoded in the Chern-Simons/Wess-Zumino action for D-branes, which takes the form

$$S_{CS/WZ}^{(C,B)}(f,\nabla)\;=\;T_{m-1}\int_X\left(f^*C\wedge e^{2\pi\alpha' F_\nabla+f^*B}\wedge\sqrt{\hat{A}(X)/\hat{A}(N_{X/Y})}\right)_{(m)}\,,$$

where

- $m=dim\,X$, T_{m-1} the D$(m-1)$-brane tension, $\hat{A}(\,\cdot\,)$ the $\hat{A}$-class of the bundle in question, $N_{X/Y}$ the normal bundle of X in Y along f,
- $(\cdots)_{(m)}$ is the degree-m component of a differential form $(\cdots)$ on X.

The fact that the over coupling strength is identical with the D-brane tension T_{m-1} is a consequence of supersymmetry.

With the lesson already learned from studying the Dirac-Born-Infeld action, formally the Chern-Simons/Wess-Zumino action generalizes to the case of coincident D-brane in our setting

$$\varphi:(X^{\!A\!z},E;\nabla)\;\longrightarrow\;Y\,,$$

as

$$S_{CS/WZ}^{(C,B)}(f,\nabla)\;\stackrel{\text{formally}}{=}\;T_{m-1}\int_X Re\left(Tr\left(\varphi^\diamond C\wedge e^{2\pi\alpha' F_\nabla+\varphi^\diamond B}\wedge\sqrt{\hat{A}(X^{\!A\!z})/\hat{A}(N_{X^{\!A\!z}/Y})}\right)\right)_{(m)}\,.$$

One now has to resolve in addition the following issues:

(8) the anomaly factor "$\sqrt{\hat{A}(X^{\!A\!z})/\hat{A}(N_{X^{\!A\!z}/Y})}$", which presumably is an $End_C(E)$-valued differential form on X;

(9) wedging of of $End_C(E)$-valued differential forms on X: $\varphi^\diamond C\wedge e^{2\pi\alpha' F_\nabla+\varphi^\diamond B}\wedge\sqrt{\hat{A}(X)/\hat{A}(N_{X^{\!A\!z}/Y})}$.

3.1 Resolution of issues in the Chern-Simons/Wess-Zumino term

We address in this subsection the resolution of Issue (9) in a way that is compatible with how we treat/interpret the Dirac-Born-Infled action in Sec. 3. This gives us a version of the Chern-Simons/Wess-Zumino term $S_{CS/WZ}^{(C,B)}$ for D-branes of dimension -1, 0, 1, and 2 that matches the Dirac-Born-Infeld action $S_{\mathrm{DBI}}^{(\Phi,g,B)}$ constructed.

From determinant function to wedge product of differential forms

For an ordinary differentiable manifold M, the wedge product of differential forms is determined by the wedge product of a collection of 1-forms and the latter is set by the determinat function through the following rule

$$(\omega^1 \wedge \cdots \wedge \omega^s)(e_1 \wedge \cdots \wedge e_s) \;=\; Det(\omega^i(e_j))\,.$$

Here, $e_1, \cdots, e_s$ are vector fields on M, $\omega^1, \cdots, \omega^s$ are 1-forms on M, $e_1\wedge\cdots\wedge e_s := \sum_{\sigma\in Sym_s}(-1)^\sigma e_{\sigma(1)}\otimes \cdots \otimes e_{\sigma(s)}$, and $(\omega^i(e_j))$ is the $s\times s$ matrix with the (i,j)-entry $\omega^i(e_j)$. When $\omega^1, \cdots, \omega^s$ are enhanced to 1-forms with value in a noncommutative ring R, the original determinant function $Det(\,\cdot\,)$ needs to be enhanced/generalized as well to a determinant function for matrices with entries in R since now $\omega^j(e_i)\in R$, for $i,j = 1, \ldots, s$.

Recall that in the study of non-Abelian Dirac-Born-Infeld action for the pair (φ, ∇), we ran into the need for such a generalization, too, and introduced the notion of symmetrized determinant *SymDet*; cf. Definition 3.1.3.6. There, we propose an Ansatz that this is the determinant function for the construction of the non-Abelian Dirac-Born-Infeld action, cf. Ansatz 3.1.3.11. It is very natural to suggest that the same notion of determinant function is applied to both the Dirac-Born-Infeld term and the Chern/Simons/Wess-Zumino term in the full action for D-branes:

Ansatz 3.1.1 [wedge product in the Chern-Simons/Wess-Zumino action] We interpret the wedge products that appear in the formal expresion for the Chern-Simons/Wess-Zumino term $S^{(C,B)}_{CS/WZ}$ through the symmetrized determinant that applies to the above defining identities for wedge product; namely, we require that

$$(\omega^1 \wedge \cdots \wedge \omega^s)(e_1 \wedge \cdots \wedge e_s) \;=\; SymDet(\omega^i(e_j))$$

for $End_C(E)$-valued 1-forms $\omega^1, \cdots, \omega^s$ on X. Denote this generalized wedge product by $\overset{\odot}{\wedge}$.

Example 3.1.2 $[C_{(1)} \overset{\odot}{\wedge} F \overset{\odot}{\wedge} F]$ Let $C_{(1)} = \sum_\mu C_\mu dx^\mu$ and $F = \sum_{\mu',\nu'} F_{\mu'\nu'}dx^{\mu'}\wedge dx^{\nu'}$ be an $End_C(E)$-valued 1-form and 2-form respectively, then

$$C_{(1)} \overset{\odot}{\wedge} F \overset{\odot}{\wedge} F \;=\; \sum_{\mu,\mu',\nu',\mu'',\nu''} (C_\mu \odot F_{\mu'\nu'} \odot F_{\mu''\nu''})\, dx^\mu \wedge dx^{\mu'} \wedge dx^{\nu'} \wedge dx^{\mu''} \wedge dx^{\nu''}\,,$$

where, recall that, $C_\mu \odot F_{\mu'\nu'} \odot F_{\mu''\nu''}$ is the symmetrized product of the triple $(C_\mu, F_{\mu'\nu'}, F_{\mu''\nu''})$.

Remark 3.1.3 [*on the ring* $(C^\infty(\bigwedge^\bullet T^*X \otimes_R End_C(E)), +, \overset{\odot}{\wedge})$] (Cf. Remark 3.1.3.10.) Properties of $\overset{\odot}{\wedge}$ follow from properties of $\odot$ on $C^\infty(End_C(E))$ and properties of $\wedge$ on $C^\infty(\bigwedge^\bullet T^*X)$. In particular, for example, $C_{(1)} \overset{\odot}{\wedge} F \overset{\odot}{\wedge} F$ is directly defined for the triple $(C_{(1)}, F, F)$ of $End_C(E)$-valued differential forms on X, rather than through a train of applications of a binary operation. The three elements in $\bigwedge^5 T^*X \otimes_R End_C(E)$

$$C_{(1)} \overset{\odot}{\wedge} F \overset{\odot}{\wedge} F\,, \qquad (C_{(1)} \overset{\odot}{\wedge} F) \overset{\odot}{\wedge} F\,, \qquad C_{(1)} \overset{\odot}{\wedge} (F \overset{\odot}{\wedge} F)$$

in general are all different. The ring $(C^\infty(\bigwedge^\bullet T^*X \otimes_R End_C(E)), +, \overset{\odot}{\wedge})$ is Z_2-graded, Z_2-commutative, but not associative.

Lemma 3.1.4 [$\varphi^\diamond$, $\wedge$, **and** $\overset{\odot}{\wedge}$] *Let* $\varphi : (X^{Az}, E, \nabla) \to Y$ *be an admissible map and* $\zeta_1, \cdots, \zeta_k$ *differential forms on* Y. *Then*

$$\varphi^\diamond\zeta_1 \overset{\odot}{\wedge} \cdots \overset{\odot}{\wedge} \varphi^\diamond\zeta_k \;=\; \varphi^\diamond(\zeta_1 \wedge \cdots \wedge \zeta_k)\,.$$

Recall the surrogate X_φ of X^{Az} specified by φ and the built-in maps

$$X_\varphi[rr]^- f_\varphi[d]^- \pi_\varphi Y X \qquad .$$

Since the function-ring $A_\varphi := C^\infty(X)\langle Im\,\varphi^\sharp\rangle$ of X_φ is commutative, for differential forms $\zeta_1', \cdots, \zeta_k'$ on X_φ,

$$\pi_{\varphi*}\zeta_1' \overset{\odot}{\wedge} \cdots \overset{\odot}{\wedge} \pi_{\varphi*}\zeta_k' \;=\; \pi_{\varphi*}\zeta_1' \wedge \cdots \wedge \pi_{\varphi*}\zeta_k' \;=\; \pi_{\varphi*}(\zeta_1' \wedge \cdots \wedge \zeta_k')\,.$$

It follows that

$$\begin{aligned}\varphi^\diamond\zeta_1 \overset{\odot}{\wedge} \cdots \overset{\odot}{\wedge} \varphi^\diamond\zeta_k \;&=\; \pi_{\varphi*}(f_\varphi^*\zeta_1) \overset{\odot}{\wedge} \cdots \overset{\odot}{\wedge} \pi_{\varphi*}(f_\varphi^*\zeta_k)\\ &=\; \pi_{\varphi*}(f_\varphi^*\zeta_1 \wedge \cdots \wedge f_\varphi^*\zeta_k) \;=\; \pi_{\varphi*}(f_\varphi^*(\zeta_1 \wedge \cdots \wedge \zeta_k)) \;=\; \varphi^\diamond(\zeta_1 \wedge \cdots \wedge \zeta_k)\,.\end{aligned}$$

The Chern-Simons/Wess-Zumino action for lower dimensional D-branes

For a simple D-brane world-volume $f : X \hookrightarrow Y$, the anomaly factor $\sqrt{\hat{A}(X)/\hat{A}(N_{X/Y})} = 1$, for $dim\,X = m \leq 3$. This may not hold for φ since $\varphi(X^{Az})$ can have fuzzy/nilpotent structure of nilpotency $\leq r$ (the rank of E as a complex vector bundle on X), which can be large even when the dimension m of X is small. However, if one formally assume that the same is true, then for lower dimensional D-branes (i.e. D(-1)-, D0-, D1-, D2-branes), one has: (Assuming that $B = \sum_{i,j} B_{ij}dy^i \otimes dy^j$, $B_{ji} = -B_{ij}$)

- For $D(-1)$-*brane* world-point $(m = 0)$:

$$S_{CS/WZ}^{(C_{(0)})}(\varphi) \;=\; T_{-1}\,\cdot\, Tr(\varphi^\diamond C_{(0)}) \;=\; T_{-1}\,\cdot\, Tr(\varphi^\sharp(C_{(0)}))\,.$$

- For *D-particle* world-line $(m = 1)$: Assume that $C_{(1)} = \sum_{i=1}^n C_i\,dy^i$ locally; then

$$S_{CS/WZ}^{(C_{(1)})}(\varphi, \nabla) \;=\; T_0\int_X Tr(\varphi^\diamond C_{(1)}) \quad \overset{\text{locally}}{=} \;T_0\int_U Tr\Big(\sum_{i=1}^n \varphi^\sharp(C_i)\cdot D_x\varphi^\sharp(y^i)\Big)\,dx\,.$$

  Here, $D_x := D_{\partial/\partial x}$.

- For *D-string* world-sheet $(m = 2)$: Assume that $C_{(2)} = \sum_{i,j=1}^n C_{ij}\,dy^i \otimes dy^j$ locally, with $C_{ij} = -C_{ji}$; then

$$\begin{aligned}S_{CS/WZ}^{(C_{(0)},C_{(2)},B)}(\varphi, \nabla) \;&=\; T_1\int_X Re\,(Tr(\varphi^\diamond C_{(2)} \,+\, \varphi^\diamond(C_{(0)}B) + \,2\pi\alpha'\varphi^\sharp(C_{(0)}) \odot F_\nabla))\\ &=\; T_1\int_X Re\,(Tr(\varphi^\diamond(C_{(2)} + C_{(0)}B) \,+\, \pi\alpha'\varphi^\sharp(C_{(0)})F_\nabla \,+\, \pi\alpha' F_\nabla\varphi^\sharp(C_{(0)})))\\ &\overset{\text{locally}}{=}\; T_1\int_U Re\Big(Tr\Big(\sum_{i,j=1}^n \varphi^\sharp(C_{ij} + C_{(0)}B_{ij})\,D_{x^1}\varphi^\sharp(y^i)\,D_{x^2}\varphi^\sharp(y^j)\\ &\qquad +\pi\alpha'\varphi^\sharp(C_{(0)})\,[\nabla_{x^1}, \nabla_{x^2}] \,+\, \pi\alpha'[\nabla_{x^1}, \nabla_{x^2}]\,\varphi^\sharp(C_{(0)})\,\Big)\Big)\,d^2\boldsymbol{x}\,.\end{aligned}$$

  Here, $D_{x^1} := D_{\partial/\partial x^1}$, $D_{x^2} := D_{\partial/\partial x^2}$ and $\nabla_{x^1} := \nabla_{\partial/\partial x^1}$, $\nabla_{x^2} := \nabla_{\partial/\partial x^2}$.

- For *D-membrane* world-volume ($m = 3$): Assume that $C_{(1)} = \sum_{i=1}^n C_i\, dy^i$ and $C_{(3)} = \sum_{i,j,k=1}^n C_{ijk}\, dy^i \otimes dy^j \otimes dy^k$ locally, with C_{ijk} alternating with respect to ijk; then

$$
\begin{aligned}
S_{CS/WZ}^{(C_{(1)},C_{(3)},B)}(\varphi,\nabla) \;&=\; T_2 \int_X Re\,(\,Tr(\varphi^\diamond C_{(3)} \,+\, \varphi^\diamond(C_{(1)}\wedge B) \,+\, 2\pi\alpha'\, \varphi^\diamond C_{(1)} \overset{\odot}{\wedge} F_\nabla\,))\\
&\overset{\text{locally}}{=}\; T_2 \int_U Re\Big(Tr\Big(\sum_{i,j,k=1}^n \varphi^\sharp(C_{ijk} + C_iB_{jk} + C_jB_{ki} + C_kB_{ij})\, D_{x^1}\varphi^\sharp(y^i)\, D_{x^2}\varphi^\sharp(y^j)\, D_{x^3}\varphi^\sharp(y^k)\\
&\qquad + \pi\alpha' \sum_{(\lambda\mu\nu)\in Sym_3} \sum_{i=1}^n (-1)^{(\lambda\mu\nu)} \Big(\varphi^\sharp(C_i)\, D_{x^\lambda}(\varphi^\sharp(y^i))\, [\nabla_{x^\mu}, \nabla_{x^\nu}] \,+\, [\nabla_{x^\mu}, \nabla_{x^\nu}]\, \varphi^\sharp(C_i)\, D_{x^\lambda}\varphi^\sharp(y^i) \Big)\Big)\Big)\, d^3\boldsymbol{x}\,.
\end{aligned}
$$

The technical issue of anomaly is the focus of another work. For the moment, we will take the above as our *working Anzatz* for the Chern-Simons/Wess-Zumino action for lower-dimensional D-branes.

3.2 The first variation and the contribution to the equations of motion

Under the same setup as in Sec. 5.2, we derive in this subsection the first variation of the Chern-Simons/Wess-Zumino action $S_{CS/WZ}^{(C,B)}$ for lower-dimensional D-brane world-volumes. The additional contribution to the equations of motion for such lower-dimensional D-branes due to the additional term $S_{CS/WZ}^{(C,B)}$ in the total action for D-brane world-volume would then follow.

3.2.1 D(-1)-brane world-point ($m = 0$)

For a D(-1)-brane world-point, $dim\,X = 0$, $\nabla = 0$, and $S_{CS/WZ}^{(C_{(0)})}(\varphi) = T_{-1} \cdot Tr(\varphi^\sharp(C_{(0)}))$. It follows that

$$
\begin{aligned}
\frac{d}{dt}\Big|_{t=0} S_{CS/WZ}^{(C_{(0)})}(\varphi_T) \;&=\; T_{-1}\, \frac{d}{dt}\Big|_{t=0} Tr(\varphi_T^\sharp(C_{(0)})) \;=\; T_{-1}\, Tr\Big(\frac{d}{dt}\Big|_{t=0} \varphi_T^\sharp(C_{(0)}) \Big)\\
&=\; T_{-1}\, Tr\Big(\sum_{j=1}^n \Big(\sum_{d,\,\boldsymbol{d},\,\vec{\pi}} R^{C_{(0)}}[1]^R_{(\boldsymbol{d},\,\vec{\pi})}(\varphi^\sharp(\boldsymbol{y})) \cdot R^{C_{(0)}}[1]^L_{(\boldsymbol{d},\,\vec{\pi})}(\varphi^\sharp(\boldsymbol{y})) \Big) \cdot \dot{\varphi}^\sharp(y^j) \Big)\\
&=:\; T_{-1}\, Tr\Big(\sum_{j=1}^n \mathbb{L}_j^{(C_{(0)});\delta\varphi}(\varphi) \cdot \dot{\varphi}^\sharp(y^j) \Big)\,.
\end{aligned}
$$

Here, the following identities are employed:

$$
\begin{aligned}
\frac{d}{dt}\Big|_{t=0} \Big(\varphi_T^\sharp(C_{(0)})\Big) \;&=\; R^{C_{(0)}}[1]\big|_{\boldsymbol{y}^{\boldsymbol{d}}}\,{}^{\bullet}(\varphi^\sharp(\boldsymbol{y}^{\boldsymbol{d}}))\\
&=\; \sum_{j=1}^n \dot{\sum_{d=0}} \sum_{\boldsymbol{d},\,|\boldsymbol{d}|=d} \sum_{\vec{\pi}\in \vec{Ptn}(1,d),\, i_{(\vec{\pi},\,\boldsymbol{d})}=j}\\
&\qquad ([\partial_{y^j}^{\vec{\pi}}] R^{C_{(0)}}[1]_{(\boldsymbol{d})})^L(\varphi^\sharp(\boldsymbol{y})) \cdot \dot{\varphi}^\sharp(y^j) \cdot ([\partial_{y^j}^{\vec{\pi}}] R^{C_{(0)}}[1]_{(\boldsymbol{d})})^R(\varphi^\sharp(\boldsymbol{y}))\\
&=:\; \sum_{j=1}^n \sum_{d,\,\boldsymbol{d},\,\vec{\pi}} R^{C_{(0)}}[1]^L_{(\boldsymbol{d},\,\vec{\pi})}(\varphi^\sharp(\boldsymbol{y})) \cdot \dot{\varphi}^\sharp(y^j) \cdot R^{C_{(0)}}[1]^R_{(\boldsymbol{d},\,\vec{\pi})}(\varphi^\sharp(\boldsymbol{y}))\,.
\end{aligned}
$$

In this case, $S_{\text{DBI}}^{(\Phi,g,B)}(\varphi) = 0$ always and the full action $S_{\text{DBI}}^{(\Phi,g,B)} + S_{CS/WZ}^{(C_{(0)})}$ is simply $S_{CS/WZ}^{(C_{(0)})}$. The full system of equations of motion is thus

$$
\mathbb{L}_j^{(\Phi,g,B,C_{(0)});\delta\varphi}(\varphi) \;:=\; \mathbb{L}_j^{(C_{(0)});\delta\varphi}(\varphi) \;=\; 0\,,
$$

$j = 1, \ldots, n$, for D(-1)-brane. Such world-points give rise to instantons in space-time.

3.2.2 D-particle world-line $(m = 1)$

For a D-particle world-line, $dim\, X = 1$ and

$$S^{(C_{(1)})}_{CS/WZ}(\varphi, \nabla) \;=\; T_0 \int_U Tr\Big(\sum_{i=1}^n \varphi^\sharp(C_i)\cdot D_x\varphi^\sharp(y^i)\Big)\, dx$$

locally over X. It follows that

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} S^{(C_{(1)})}_{CS/WZ}(\varphi_T, \nabla^T) \;&=\; T_0 \int_U Tr\Big(\sum_{i=1}^n \dot\varphi^\sharp(C_i)\cdot D_x\varphi^\sharp(y^i) \,+\, \varphi^\sharp(C_i)\cdot (D_x\dot\varphi^\sharp(y^i) \,-\, [\varphi^i(\sharp)(y^i), \dot A_x])\Big)\, dx \\
&= T_0\, Tr\Big(\sum_{i=1}^n \varphi^\sharp(C_i)\,\dot\varphi^\sharp(y^i)\Big)\Big|_{\partial U} \\
&\qquad + T_0 \int_U Tr\Big(\sum_{i=1}^n \dot\varphi^\sharp(C_i)\cdot D_x\varphi^\sharp(y^i) \,-\, D_x\varphi^\sharp(C_i)\cdot\dot\varphi^\sharp(y^i) \,-\, \varphi^\sharp(C_i)\cdot[\varphi^i(\sharp)(y^i), \dot A_x]\Big)\, dx \\
&= T_0\, B\!\varGamma^{(\varphi; C_{(1)})}(\dot\varphi^\sharp(\boldsymbol{y}))|_{\partial U} \,+\, T_0 \int_U Tr\,\Big(\sum_{j=1}^n \mathit{N\!L}^{(C_{(1)});\delta\varphi}_j(\varphi,\nabla)\cdot\dot\varphi^\sharp(y^j) \,+\, \mathit{N\!L}^{(C_{(1)});\delta\nabla}_x(\varphi)\cdot\dot A_x\Big)\, dx\,,
\end{aligned}$$

where

$$\begin{aligned}
B\!\varGamma^{(\varphi;C_{(1)})}(\dot\varphi^\sharp(\boldsymbol{y})) \;&=\; Tr\Big(\sum_{i=1}^n \varphi^\sharp(C_i)\,\dot\varphi^\sharp(y^i)\Big)\,, \\
\mathit{N\!L}^{(C_{(1)});\delta\varphi}_j&(\varphi,\nabla) \\
&= \;-\,D_x\varphi^\sharp(C_j) \,+\, \sum_{i=1}^n \sum_{d,\boldsymbol{d},\vec\pi;\ |\boldsymbol{d}|=d,\, i_{(\vec\pi,\boldsymbol{d})}=j} R^{C_i}[1]^R_{(\boldsymbol{d},\vec\pi)}(\varphi^\sharp(\boldsymbol{y}))\cdot D_x\varphi^\sharp(y^i)\cdot R^{C_i}[1]^L_{(\boldsymbol{d},\vec\pi)}(\varphi^\sharp(\boldsymbol{y}))\,, \\
\mathit{N\!L}^{(C_{(1)});\delta\nabla}_x(\varphi) \;&=\; \sum_{i=1}^n [\varphi^\sharp(y^i)\,,\,\varphi^\sharp(C_i)] \;=\; 0\,.
\end{aligned}$$

The full action $S^{(\Phi,g,B)}_{\mathrm{DBI}}(\varphi,\nabla) + S^{(C_{(1)})}_{CS/WZ}(\varphi,\nabla)$ gives the system of equations of motion for a D-particle moving in Y:

$$\begin{aligned}
\mathit{N\!L}^{(\Phi,g,B,C_{(1)});\delta\varphi}_j(\varphi,\nabla) \;&:=\; \mathit{N\!L}^{(\Phi,g,B);\delta\varphi}_j(\varphi,\nabla) \,+\, \mathit{N\!L}^{(C_{(1)});\delta\varphi}_j(\varphi,\nabla) \;=\; 0\,, \\
\mathit{N\!L}^{(\Phi,g,B,C_{(1)});\delta\nabla}_x(\varphi,\nabla) \;&:=\; \mathit{N\!L}^{(\Phi,g,B);\delta\nabla}_x(\varphi,\nabla) \;=\; 0\,,
\end{aligned}$$

$j = 1,\,\ldots\,,n$.

For the current case, the curvature F_∇ of ∇ is zero and the above system may still involves A_x but not its differentials with respect to x. I.e. it is a system of differential equations on φ but non-differential equations on ∇. ∇ is thus non-dynamical, as is anticipated. Thus, after a re-trivialization fo the fundamental module E on X, one may assume that $A_x \equiv 0$ and the above system is reduced to a system

$$\mathit{N\!L}^{(\Phi,g,B,C_{(1)});\delta\varphi}(\varphi) \;=\; 0\,, \qquad j = 1,\,\ldots\,,n\,,$$

of second-order nonlinear differential equations that involve φ alone.

3.2.3 D-string world-sheet $(m = 2)$

Denote

$$\breve C_{(2)} \;:=\; C_{(2)} + C_{(0)}B \;=\; \sum_{ij}(C_{ij} + C_{(0)}B_{ij})\, dy^i\otimes dy^j \;=\; \sum_{i,j}\breve C_{ij} dy^i\otimes dy^j$$

in local coordinates of Y. Then, for a D-string world-sheet, $dim\, X = 2$ and

$$\begin{aligned}
S^{(C_{(0)},C_{(2)},B)}_{CS/WZ}(\varphi,\nabla) \;=\; T_1\int_U Re\Big(Tr\Big(\sum_{i,j=1}^n &\varphi^\sharp(\breve C_{ij})\,D_1\varphi^\sharp(y^i)\,D_2\varphi^\sharp(y^j) \\
&+\pi\alpha'\varphi^\sharp(C_{(0)})\,F_{12} \,+\, \pi\alpha' F_{12}\,\varphi^\sharp(C_{(0)})\Big)\Big)\, d^2\boldsymbol{x}
\end{aligned}$$

locally over X. (Here, $D_1 := D_{\partial/\partial x^1}$, $D_2 := D_{\partial/\partial x^2}$, and $F_{12} := [\nabla_{x^1}, \nabla_{x^2}]$ is the curvature of ∇.) It follows then from a straightforward computation that

$$
\begin{aligned}
&\frac{d}{dt}\Big|_{t=0} S^{(C_{(0)},C_{(2)},B)}_{CS/WZ}(\varphi_T, \nabla^T) \\
&= T_1 \int_U Re\Big(Tr\Big(\sum_{i,j=1}^{n} \Big(\dot{\varphi}^\sharp(\breve{C}_{ij})\, D_1\varphi^\sharp(y^i)\, D_2\varphi^\sharp(y^j) \,+\, \varphi^\sharp(\breve{C}_{ij})\, (D_1\dot{\varphi}^\sharp(y^i) - [\varphi^\sharp(y^i), \dot{A}_1])\, D_2\varphi^\sharp(y^j) \\
&\qquad\qquad + \varphi^\sharp(\breve{C}_{ij})\, D_1\varphi^\sharp(y^i)\, (D_2\dot{\varphi}^\sharp(y^j) - [\varphi^\sharp(y^j), \dot{A}_2]) \Big) \\
&\qquad + \pi\alpha'\dot{\varphi}^\sharp(C_{(0)})\, F_{12} \,+\, \pi\alpha'\varphi^\sharp(C_{(0)})\, (D_1\dot{A}_2 - D_2\dot{A}_1) \\
&\qquad + \pi\alpha'(D_1\dot{A}_2 - D_2\dot{A}_1)\, \varphi^\sharp(C_{(0)}) \,+\, \pi\alpha' F_{12}\, \dot{\varphi}^\sharp(C_{(0)}) \Big)\Big)\, d^2\boldsymbol{x} \\
&= T_1 \int_{\partial U} Re\, (BT^{(\varphi,\nabla;C_{(0)},C_{(2)},B)}(\dot{\varphi}^\sharp(\boldsymbol{y}), \dot{\boldsymbol{A}})) \\
&\qquad + T_1 \int_U Re\Big(Tr\Big(\sum_{j=1}^{n} \mathcal{N\!L}_j^{(C_{(0)},C_{(2)},B);\delta\varphi}(\varphi,\nabla) \cdot \dot{\varphi}^\sharp(y^j) + \sum_{\nu=1}^{2} \mathcal{N\!L}_\nu^{(C_{(0)});\delta\nabla}(\varphi,\nabla) \cdot \dot{A}_\nu \Big)\Big)\, d^2\boldsymbol{x}\,,
\end{aligned}
$$

where

- the *boundary term* is given by

$$
\begin{aligned}
&BT^{(\varphi,\nabla;C_{(0)},C_{(2)},B)}(\dot{\varphi}^\sharp(\boldsymbol{y}), \dot{\boldsymbol{A}})\,, \quad \text{a 1-form on } U\,, \\
&= Tr\Big(\sum_{j=1}^{n} \Big(\sum_{i=1}^{n} D_2\varphi^\sharp(y^i)\, \varphi^\sharp(\breve{C}_{ji}) \Big) \cdot \dot{\varphi}^\sharp(y^j) \,+\, 2\pi\alpha'\varphi^\sharp(C_{(0)}) \cdot \dot{A}_2 \Big)\, dx^2 \\
&\qquad - Tr\Big(\sum_{j=1}^{n} \Big(\sum_{i=1}^{n} \varphi^\sharp(\breve{C}_{ij})\, D_1\varphi^\sharp(y^i) \Big) \cdot \dot{\varphi}^\sharp(y^j) \,-\, 2\pi\alpha'\varphi^\sharp(C_{(0)}) \cdot \dot{A}_1 \Big)\, dx^1\,,
\end{aligned}
$$

- the *subsystem associated to variations of* φ:

$$
\begin{aligned}
&\mathcal{N\!L}_j^{(C_{(0)},C_{(2)},B);\delta\varphi}(\varphi,\nabla) \\
&= \sum_{i',j'=1}^{n} \sum_{d,\boldsymbol{d},\vec{\pi};\, |\boldsymbol{d}|=d,\, i_{(\vec{\pi},\boldsymbol{d})}=j} R^{\breve{C}_{i'j'}}[1]^R_{(\boldsymbol{d},\vec{\pi})}(\varphi^\sharp(\boldsymbol{y}))\, D_1\varphi^\sharp(y^{i'})\, D_2\varphi^\sharp(y^{j'})\, R^{\breve{C}_{i'j'}}[1]^L_{(\boldsymbol{d},\vec{\pi})}(\varphi^\sharp(\boldsymbol{y})) \\
&\quad - \sum_{i=1}^{n} \Big(D_2\varphi^\sharp(y^i)\, D_1\varphi^\sharp(\breve{C}_{ji}) \,+\, D_2\varphi^\sharp(\breve{C}_{ij})\, D_1\varphi^\sharp(y^i) \,+\, [F_{12}, \varphi^\sharp(y^i)] \cdot \varphi^\sharp(\breve{C}_{ji}) \Big) \\
&\quad + 2\pi\alpha' \sum_{d,\boldsymbol{d},\vec{\pi};\, |\boldsymbol{d}|=d,\, i_{(\vec{\pi},\boldsymbol{d})}=j} R^{C_{(0)}}[1]^R_{(\boldsymbol{d},\vec{\pi})}(\varphi^\sharp(\boldsymbol{y}))\, F_{12}\, R^{C_{(0)}}[1]^L_{(\boldsymbol{d},\vec{\pi})}(\varphi^\sharp(\boldsymbol{y}))\,,
\end{aligned}
$$

- the *subsystem associated to variations of* ∇:

$$
\mathcal{N\!L}_1^{(C_{(0)});\delta\nabla}(\varphi,\nabla) \;=\; 2\pi\alpha' D_2\varphi^\sharp(C_{(0)})\,, \qquad \mathcal{N\!L}_2^{(C_{(0)});\delta\nabla}(\varphi,\nabla) \;=\; -\,2\pi\alpha' D_1\varphi^\sharp(C_{(0)})\,.
$$

  Note that, as a consequence of Leibniz rule or integration by parts, there are at first summands

$$
\begin{aligned}
&-D_2\varphi^\sharp(y^j)\, \varphi^\sharp(\breve{C}_{ij})\, \varphi^\sharp(y^i) \,+\, \varphi^\sharp(y^i)\, D_2\varphi^\sharp(y^j)\, \varphi^\sharp(\breve{C}_{ij}) \qquad \text{in } \mathcal{N\!L}_1^{(C_{(0)});\delta\nabla}(\varphi,\nabla)\,, \\
&-\varphi^\sharp(\breve{C}_{ij})\, D_1\varphi^\sharp(y^i)\, \varphi^\sharp(y^j) \,+\, \varphi^\sharp(y^j)\, \varphi^\sharp(\breve{C}_{ij})\, D_1\varphi^\sharp(y^i) \qquad \text{in } \mathcal{N\!L}_2^{(C_{(0)});\delta\nabla}(\varphi,\nabla)\,,
\end{aligned}
$$

  respectively. However, they vanish for (φ,∇) admissible. Thus, the 2-forms $C_{(2)}$ and B has no consequence to the variation of $S^{(C_{(0)},C_{(2)},B)}_{CS/WZ}$ with respect to ∇. This is anticipated since there is no coupling term between $C_{(2)}$, B and ∇ in $S^{(C_{(0)},C_{(2)},B)}_{CS/WZ}$.

The contribution of the Chern-Simon/Wess-Zumino term $S^{(C_{(0)},C_{(2)},B)}_{CS/WZ}$ to the equations of motion for a D-string follows immediately.

3.2.4 D-membrane world-volume ($m = 3$)

Denote

$$\begin{aligned}\breve{C}_{(3)} &:= C_{(3)} + C_{(1)} \wedge B \\ &= \sum_{i,j,k} (C_{ijk} + C_i B_{jk} + C_j B_{ki} + C_k B_{ij})\, dy^i \otimes dy^j \otimes dy^k \;=\; \sum_{i,j,k} \breve{C}_{ijk} dy^i \otimes dy^j \otimes dy^k\end{aligned}$$

in local coordinates of Y. Then, for D-membrane world-volume, $dim\, X = 3$ and

$$\begin{aligned}&S^{(C_{(1)},C_{(3)},B)}_{CS/WZ}(\varphi, \nabla) \\ &= T_2 \int_U Re\Big(Tr\Big(\sum_{i,j,k=1}^{n} \varphi^\sharp(\breve{C}_{ijk})\, D_1\varphi^\sharp(y^i)\, D_2\varphi^\sharp(y^j)\, D_3\varphi^\sharp(y^k) \\ &\qquad + 2\pi\alpha' \sum_{(\lambda\mu\nu)\in Sym_3} \sum_{i=1}^{n} (-1)^{(\lambda\mu\nu)} \Big(\varphi^\sharp(C_i)\, D_\lambda \varphi^\sharp(y^i)\, F_{\mu\nu} \Big)\Big)\Big)\, d^3\boldsymbol{x}\end{aligned}$$

locally over X. (Here, $F_{\mu\nu} := [\nabla_{x^\mu}, \nabla_{x^\nu}]$ is the curvature of ∇.) It follows then from a straightforward computation that

$$\begin{aligned}&\frac{d}{dt}\Big|_{t=0} S^{(C_{(1)},C_{(3)},B)}_{CS/WZ}(\varphi_T, \nabla^T) \\ &= T_2 \int_U Re\Big(Tr\Big(\sum_{i,j,k=1}^{n} \Big(\dot{\varphi}^\sharp(\breve{C}_{ijk})\, D_1\varphi^\sharp(y^i)\, D_2\varphi^\sharp(y^j)\, D_3\varphi^\sharp(y^k) \\ &\qquad + \varphi^\sharp(\breve{C}_{ijk} \cdot (D_1\dot{\varphi}^\sharp(y^i) - [\varphi^\sharp(y^i), \dot{A}_1]) \cdot D_2\varphi^\sharp(y^j)\, D_3\varphi^\sharp(y^k) \\ &\qquad + \varphi^\sharp(\breve{C}_{ijk})\, D_1\varphi^\sharp(y^i) \cdot (D_2\dot{\varphi}^\sharp(y^j) - [\varphi^\sharp(y^j), \dot{A}_2]) \cdot D_3\varphi^\sharp(y^k) \\ &\qquad + \varphi^\sharp(\breve{C}_{ijk})\, D_1\varphi^\sharp(y^i)\, D_2\varphi^\sharp(y^j) \cdot (D_3\dot{\varphi}^\sharp(y^k) - [\varphi^\sharp(y^k), \dot{A}_3]) \Big) \\ &\quad + 2\pi\alpha' \sum_{(\lambda\mu\nu)\in Sym_3} \sum_{i=1}^{n} (-1)^{(\lambda\mu\nu)} \Big(\dot{\varphi}^\sharp(C_i)\, D_\lambda\varphi^\sharp(y^i)\, F_{\mu\nu} \\ &\qquad + \varphi^\sharp(C_i) \cdot (D_\lambda\dot{\varphi}^\sharp(y^i) - [\varphi^\sharp(y^i), \dot{A}_\lambda]) \cdot F_{\mu\nu} \;+\; \varphi^\sharp(C_i)\, D_\lambda\varphi^\sharp(y^i) \cdot (D_\mu\dot{A}_\nu - D_\nu\dot{A}_\mu) \Big) \Big)\Big)\, d^3\boldsymbol{x} \\ &= T_2 \int_{\partial U} Re\,(BT^{(\varphi,\nabla;C_{(1)},C_{(3)},B)}(\dot{\varphi}^\sharp(\boldsymbol{y}), \dot{\boldsymbol{A}})) \\ &\quad + T_2 \int_U Re\Big(Tr\Big(\sum_{j=1}^{n} NL^{(C_{(1)},C_{(3)},B);\delta\varphi}_j(\varphi, \nabla) \cdot \dot{\varphi}^\sharp(y^j) + \sum_{\nu=1}^{3} NL^{(C_{(1)});\delta\nabla}_\nu(\varphi, \nabla) \cdot \dot{A}_\nu \Big)\Big)\, d^3\boldsymbol{x}\,,\end{aligned}$$

where

· the *boundary term* is given by

$$\begin{aligned}&BT^{(\varphi,\nabla;C_{(1)},C_{(3)},B)}(\dot{\varphi}^\sharp(\boldsymbol{y}), \dot{\boldsymbol{A}})\,, \quad \text{a 2-form on } U\,, \\ &= Tr\Big(\sum_{j=1}^{n} \Big(\sum_{i,k=1}^{n} D_2\varphi^\sharp(y^i)\, D_3\varphi^\sharp(y^k)\, \varphi^\sharp(\breve{C}_{jik}) \;+\; 4\pi\alpha'\, F_{23}\, \varphi^\sharp(C_j) \Big) \cdot \dot{\varphi}^\sharp(y^j) \\ &\qquad + 4\pi\alpha' \Big(\sum_{i=1}^{n} \varphi^\sharp(C_i)\, D_3\varphi^\sharp(y^i) \Big) \cdot \dot{A}_2 \;-\; 4\pi\alpha' \Big(\sum_{i=1}^{n} \varphi^\sharp(C_i)\, D_2\varphi^\sharp(y^i) \Big) \cdot \dot{A}_3 \Big)\, d^2 \wedge dx^3 \\ &\quad - Tr\Big(\sum_{j=1}^{n} \Big(\sum_{i,k=1}^{n} D_3\varphi^\sharp(y^k)\, \varphi^\sharp(\breve{C}_{ijk})\, D_1\varphi^\sharp(y^i) \;-\; 4\pi\alpha'\, F_{13}\, \varphi^\sharp(C_j) \Big) \cdot \dot{\varphi}^\sharp(y^j) \\ &\qquad - 4\pi\alpha' \Big(\sum_{i=1}^{n} \varphi^\sharp(C_i)\, D_3\varphi^\sharp(y^i) \Big) \cdot \dot{A}_1 \;+\; 4\pi\alpha' \Big(\sum_{i=1}^{n} \varphi^\sharp(C_i)\, D_1\varphi^\sharp(y^i) \Big) \cdot \dot{A}_3 \Big)\, d^1 \wedge dx^3 \\ &\quad + Tr\Big(\sum_{j=1}^{n} \Big(\sum_{i,k=1}^{n} \varphi^\sharp(\breve{C}_{ikj})\, D_1\varphi^\sharp(y^i)\, D_2\varphi^\sharp(y^k) \;+\; 4\pi\alpha'\, F_{12}\, \varphi^\sharp(C_j) \Big) \cdot \dot{\varphi}^\sharp(y^j) \\ &\qquad + 4\pi\alpha' \Big(\sum_{i=1}^{n} \varphi^\sharp(C_i)\, D_2\varphi^\sharp(y^i) \Big) \cdot \dot{A}_1 \;-\; 4\pi\alpha' \Big(\sum_{i=1}^{n} \varphi^{C_i}\, D_1\varphi^\sharp(y^i) \Big) \cdot \dot{A}_2 \Big)\, dx^1 \wedge dx^2\,,\end{aligned}$$

- the *subsystem associated to variations of* φ :

$$
\begin{aligned}
&\mathbb{N}\!L_j^{(C_{(1)},C_{(3)},B);\delta\varphi}(\varphi,\nabla)\\
&= \sum_{i',j',k'=1}^{n}\ \sum_{d,\boldsymbol{d},\vec{\pi};\,|\boldsymbol{d}|=d,i_{(\vec{\pi},\boldsymbol{d})}=j} R^{\breve{C}_{i'j'k'}}[1]^R_{(\boldsymbol{d},\vec{\pi})}(\varphi^\sharp(\boldsymbol{y}))\,D_1\varphi^\sharp(y^{i'})\,D_2\varphi^\sharp(y^{j'})\,D_3\varphi^\sharp(y^{k'})R^{\breve{C}_{i'j'k'}}[1]^L_{(\boldsymbol{d},\vec{\pi})}(\varphi^\sharp(\boldsymbol{y}))\\
&\quad-\sum_{i,k=1}^{n}\Big(\,D_2\varphi^\sharp(y^i)\,D_3\varphi^\sharp(y^k)\,D_1\varphi^\sharp(\breve{C}_{jik})\,+\,D_3\varphi^\sharp(y^k)\,D_2\varphi^\sharp(\breve{C}_{ijk})\,D_1\varphi^\sharp(y^i)\\
&\qquad+D_3\varphi^\sharp(\breve{C}_{ikj})\,D_1\varphi^\sharp(y^i)\,D_2\varphi^\sharp(y^k)\,+\,[F_{12},\varphi^\sharp(y^i)]\cdot D_3\varphi^\sharp(y^k)\,\varphi^\sharp(\breve{C}_{jik})\\
&\qquad\quad+[F_{23},\varphi^\sharp(y^k)]\cdot\varphi^\sharp(\breve{C}_{ijk})\,D_1\varphi^\sharp(y^i)\,+\,[F_{31},\varphi^\sharp(y^i)]\cdot D_2\varphi^\sharp(y^k)\,\varphi^\sharp(\breve{C}_{ikj})\,\Big)\\
&\quad+2\pi\alpha'\sum_{i=1}^{n}\ \sum_{(\lambda\mu\nu)\in Sym_3}\ \sum_{d,\boldsymbol{d},\vec{\pi};|\boldsymbol{d}|=d,i_{(\vec{\pi},\boldsymbol{d})}=j}(-1)^{(\lambda\mu\nu)}R^{C_i}[1]^R_{(\boldsymbol{d},\vec{\pi})}(\varphi^\sharp(\boldsymbol{y}))\,D_\lambda\varphi^\sharp(y^i)\,F_{\mu\nu}\,R^{C_i}[1]^L_{(\boldsymbol{d},\vec{\pi})}(\varphi^\sharp(\boldsymbol{y}))\\
&\quad-2\pi\alpha'\sum_{(\lambda\mu\nu)\in Sym_3}(-1)^{(\lambda\mu\nu)}\Big(\,F_{\mu\nu}\,D_\lambda\varphi^\sharp(C_j)\,+\,D_\lambda F_{\mu\nu}\varphi^\sharp(C_j)\,\Big)\,,
\end{aligned}
$$

- the *subsystem associated to variations of* ∇ :

$$
\begin{aligned}
&\mathbb{N}\!L_\lambda^{(C_{(1)});\delta\nabla}(\varphi,\nabla)\\
&\quad=\ 2\pi\alpha'\sum_{i=1}^{n}[\varphi^\sharp(y^i)\,,\,F_{\mu\nu}\varphi^\sharp(C_i)]\\
&\qquad+4\pi\alpha'\sum_{i=1}^{n}\Big(\,D_\mu\varphi^\sharp(C_i)\,D_\nu\varphi^\sharp(y^i)\,-\,D_\nu\varphi^\sharp(C_i)\,D_\mu\varphi^\sharp(y^i)\,+\,\varphi^\sharp(C_i)\cdot[F_{\mu\nu},\varphi^\sharp(y^i)]\,\Big)\,,
\end{aligned}
$$

where $(\lambda\mu\nu)\ =\ (123),\,(231),\,(312)\,.$

Note that, as a consequence of Leibniz rule or integration by parts, there are at first summands

$$
\begin{aligned}
&\sum_{i,j,k=1}^{n}[\varphi^\sharp(y^i),D_2\varphi^\sharp(y^j)D_3\varphi^\sharp(y^k)\varphi^\sharp(\breve{C}_{ijk})] &&\text{in }\ \mathbb{N}\!L_1^{(C_{(1)});\delta\nabla}(\varphi,\nabla)\,,\\
&\sum_{i,j,k=1}^{n}[\varphi^\sharp(y^j),D_3\varphi^\sharp(y^k)\varphi^\sharp(\breve{C}_{ijk})D_1\varphi^\sharp(y^i)] &&\text{in }\ \mathbb{N}\!L_2^{(C_{(1)});\delta\nabla}(\varphi,\nabla)\,,\\
&\sum_{i,j,k=1}^{n}[\varphi^\sharp(y^k),\varphi^\sharp(\breve{C}_{ijk})D_1\varphi^\sharp(y^i)D_2\varphi^\sharp(y^j)] &&\text{in }\ \mathbb{N}\!L_3^{(C_{(1)});\delta\nabla}(\varphi,\nabla)
\end{aligned}
$$

respectively. However, they vanish for (φ,∇) admissible. Thus, the 3-forms $C_{(3)}$ and $C_{(1)}\wedge B$ have no consequence to the variation of $S_{CS/WZ}^{(C_{(1)},C_{(3)},B)}$ with respect to ∇. This is anticipated since there is no coupling term between $C_{(3)}$, $C_{(1)}\wedge B$ and ∇ in $S_{CS/WZ}^{(C_{(1)},C_{(3)},B)}$.

The contribution of the Chern-Simon/Wess-Zumino term $S_{CS/WZ}^{(C_{(1)},C_{(3)},B)}$ to the equations of motion for a D-membrane follows immediately.

Remark 3.2.4.1 [*contribution only to first-order terms in EOM*] As observed from these examples, for lower dimensional D-branes, the Chern-Simons/Wess-Zumino term $S_{CS/WZ}^{(C,B)}$ in the action contributes an additional set of *first-order* nonlinear differential-expression terms to the system of equations of motion fo D-branes. In particular, they preserve the signature of the original system from the Dirac-Born-Infeld term $S_{\mathrm{DBI}}^{(\Phi,g,B)}$ in the action.

The current notes lay down some foundation toward the dynamics of D-branes along the line of our D-project. Solutions to the system of equations of motion from the total action $S_{\mathrm{DBI}}^{(\Phi,g,B)}(\varphi,\nabla)+S_{CS/WZ}^{(C,B)}(\varphi,\nabla)$ for a D-brane world-volume should be thought of as an Azumaya/matrix version of minimal submanifolds or harmonic maps, twisted/bent, on one hand, by the (dynamical) gauge field ∇ on the domain manifold X with a (noncommutative) endomorphism/matrix function-ring and, on the other hand, by the background field (Φ, g, B, C), created by closed (super)strings, on the target space(-time) Y. Further details, issues, and examples are the focus of the sequels.

4 The standard action for D-branes

We introduce in this section the standard action, which is to D-branes as the (Brink-Di Vecchia-Howe/Deser-Zumino/) Polyakov action is to fundamental superstrings. Abstractly, it is an enhanced non-Abelian gauged sigma model based on maps $\varphi:(X^{Az},\mathcal{E};\nabla)\rightarrow Y$.

The gauge-symmetry group $C^{\infty}(Aut_{\mathbb{C}}(E))$

Let $Aut_{\mathbb{C}}(E)$ be the *automorphism bundle* of the complex vector bundle E (of rank r) over E. $Aut_{\mathbb{C}}(E)\subset End_{\mathbb{C}}(E)$ canonically as the bundle of invertible endomorphisms; it is a principal $GL_r(\mathbb{C})$-bundle over X. The set

$$\mathcal{G}_{gauge}\;:=\;C^{\infty}(Aut_{\mathbb{C}}(E))$$

of smooth sections of $Aut_{\mathbb{C}}(E)$ forms an infinite-dimensional Lie group and acts on the space of pairs (φ,∇) as a gauge-symmetry group:

$$\begin{array}{rcl} g'\in\mathcal{G}_{gauge}\;:\quad (\varphi,\nabla=d+A) & \longmapsto & ({}^{g'}\!\varphi\,,\,{}^{g'}\!\nabla=d+{}^{g'}\!A)\\ & & :=\;(g'\varphi g'^{-1}\,,\,d-(dg')g'^{-1}+g'Ag'^{-1})\quad. \end{array}$$

The induced action of $\mathcal{G}_{gauge}$ on other basic objects are listed in the lemma below:

Lemma 4.1 [induced action of $\mathcal{G}_{gauge}$ on other basic objects] *(All the $\mathcal{G}_{gauge}$-actions are denoted by a representation ρ_{gauge} of $\mathcal{G}_{gauge}$, if in need.)*

(0_1) *on* $\mathcal{O}_X^{Az}$: $\qquad\rho_{gauge}(g')(m)\;=\;g'mg'^{-1}$ *for* $m\in\mathcal{O}_X^{Az}$.

(0_2) *on induced connections*: $\qquad D=d+[A,\,\bullet\,]\;\longmapsto\;{}^{g'}\!D\;:=\;d+[\,{}^{g'}\!A\,,\,\bullet\,]$.

(1) *on* $\mathcal{T}^*X^{\mathbb{C}}\otimes_{\mathcal{O}_X^{\mathbb{C}}}\mathcal{O}_X^{Az}$: $\qquad\rho_{gauge}(g')(\omega\otimes m)\;=\;\omega\otimes(g'mg'^{-1})=:g'(\omega\otimes m)g'^{-1}$.

(2) *for* $\varphi^*\mathcal{T}_*Y$:

$$\begin{array}{rcl}\varphi^*\mathcal{T}_*Y & \longrightarrow & {}^{g'}\!\varphi^*\mathcal{T}_*Y\\ m\otimes v & \longmapsto & (g'mg'^{-1})\otimes v\;=:\;g'(m\otimes v)g'^{-1}\quad.\end{array}$$

.

(3) *for* $\mathcal{T}^*X\otimes_{\mathcal{O}_X}\varphi^*\mathcal{T}_*Y$:

$$\begin{array}{rcl}\mathcal{T}^*X\otimes_{\mathcal{O}_X}\varphi^*\mathcal{T}_*Y & \longrightarrow & \mathcal{T}^*X\otimes_{\mathcal{O}_X}{}^{g'}\!\varphi^*\mathcal{T}_*Y\\ \omega\otimes m\otimes v & \longmapsto & \omega\otimes(g'mg'^{-1})\otimes v\;=:\;g'(\omega\otimes m\otimes v)g'^{-1}\quad.\end{array}$$

.

(4) *for covariant differential*: $D\varphi \longmapsto {}^{g'}\!D\,{}^{g'}\!\varphi = g' D\varphi\, g'^{-1}$.

(5) *for pull-push*: $({}^{g'}\!\varphi)^\diamond\alpha = g'\,\varphi^\diamond\alpha\, g'^{-1}$.

The proof is elementary. Let us demonstrate Item (2) as an example.
For $m\otimes v \in \varphi^*\mathcal{T}_*Y := \mathcal{O}_X^{Az}\otimes_{\varphi^\sharp,\mathcal{O}_Y}\mathcal{T}_*Y$,

$$\rho_{gauge}(g')(m\otimes v) \;=\; \rho_{gauge}(g')(m)\otimes v \;=\; (g'mg'^{-1})\otimes v$$

since $\mathcal{G}_{gauge}$ acts on $\mathcal{T}_*Y$ trivially (i.e. by by the identity map Id_Y). The only issue is: Where does $(g'mg'^{-1})\otimes v$ now live? To answer this, note that, for $f\in C^\infty(Y)$, on one hand

$$\rho_{gauge}(g')(m\otimes fv) \;=\; \rho_{gauge}(g')(m\varphi^\sharp(f)\otimes v) \;=\; (g'm\varphi^\sharp(f)g'^{-1})\otimes v\,,$$

while on the other hand

$$\rho_{gauge}(g')(m\otimes fv) \;=\; (g'mg'^{-1})\otimes fv\,,$$

It follows that

$$\begin{aligned}&(g'mg'^{-1})\otimes fv\\ &\quad= (g'm\varphi^\sharp(f)g'^{-1})\otimes v \;=\; (g'mg'^{-1}\cdot g'\varphi^\sharp(f)g'^{-1})\otimes v \;=\; (g'mg'^{-1}\cdot {}^{g'}\!\varphi^\sharp(f))\otimes v\,.\end{aligned}$$

Which says that our section $(g'mg'^{-1})\otimes v$ now lives in ${}^{g'}\!\varphi^*\mathcal{T}_*Y$.

The standard action for D-branes

Fix a (dilaton field ρ, metric h) on the underlying smooth manifold X (of dimension m) of the Azumaya/matrix manifold with a fundamental module $(X^{Az},\mathcal{E})$. Fix a background (dilaton field Φ, metric g , B-field B, Ramond-Ramond field C) on the target space(-time) Y (of dimension n). Here, h and g can be either Riemannian or Lorentzian.

Definition 4.2 [standard action = enhanced non-Abelian gauged sigma model] With the given background fields (ρ,h) on X and (Φ,g,B,C) on Y, the *standard action* $S^{(\rho,h;\Phi,g,B,C)}_{standard}(\varphi,\nabla)$ for $(*_1)$-admissible pairs (φ,∇) is defined to be the functional

$$\begin{aligned}S^{(\rho,h;\Phi,g,B,C)}_{standard}(\varphi,\nabla) \;&:=\; S^{(\rho,h;\Phi,g,B,C)}_{nAGSM^+}(\varphi,\nabla)\\ &:=\; S^{(\rho,h;\Phi,g)}_{map:kinetic^+}(\varphi,\nabla) \;+\; S^{(h;B)}_{gauge\,/\,YM}(\varphi,\nabla) \;+\; S^{(C,B)}_{CS/WZ}(\varphi,\nabla)\end{aligned}$$

with the *enhanced kinetic term for maps*

$$S^{(\rho,h;\Phi,g)}_{map:kinetic^+}(\varphi,\nabla) \;:=\; \frac{1}{2}\,T_{m-1}\int_X Re\Big(Tr\langle D\varphi\,,\,D\varphi\rangle_{(h,g)}\Big)\,vol_h \;+\; \int_X Re\Big(Tr\langle d\rho,\varphi^\diamond d\Phi\rangle_h\Big)\,vol_h\,,$$

the *gauge/Yang-Mills term*

$$S^{(h;B)}_{gauge\,/\,YM}(\varphi,\nabla) \;:=\; -\frac{1}{2}\int_X Re\Big(Tr\,\|2\pi\alpha' F_\nabla+\varphi^\diamond B\|_h^2\Big)\,vol_h$$

and the *Chern-Simons/Wess-Zumino term*
(if (φ,∇) is furthermore $(*_2)$-admissible, cf. Remark 2.1.13)

$$S^{(C,B)}_{CS/WZ}(\varphi,\nabla) \;\stackrel{\text{formally}}{=}\; T_{m-1}\int_X Re\Big(Tr\Big(\varphi^\diamond C\wedge e^{2\pi\alpha' F_\nabla+\varphi^\diamond B}\wedge\sqrt{\hat{A}(X^{Az})/\hat{A}(N_{X^{Az}/Y})}\,\Big)\Big)_{(m)}\,.$$

Here,

(0) *On Re* Note that while eigenvalues of $\varphi^\sharp(f)$ are all real ([L-Y5: Sec. 3.1] (D(11.1))) for $f \in \mathcal{O}_Y$, the eigenvalues of $D_\xi\varphi^\sharp(f)$, $\xi \in \mathcal{T}_*X$, may not be so under the $(*_1)$-Admissible Condition. Thus, $Tr(\cdots\cdots)$ in the integrand of terms in $S_{standard}^{(\rho,h;\Phi,g,B,C)}(\varphi,\nabla)$ are in general C-valued and we take the *real part* $Re\,Tr(\cdots\cdots)$ of it.

(1) *The enhanced kinetic term for maps* The first summand of $S_{map:kinetic^+}^{(\rho,h;\Phi,g)}$ defines the *kinetic energy*

$$E^\nabla(\varphi) \;:=\; S_{map:kinetic}^{(h;g)}(\varphi,\nabla) \;:=\; \frac{1}{2}\,T_{m-1}\int_X Re\Big(Tr\,\langle D\varphi\,,\,D\varphi\rangle_{(h,g)}\Big)\,vol_h$$

of the map φ for a given ∇ and, hence, will be called the *kinetic term* for maps in the standard action $S_{standard}^{(\rho,h;\Phi,g,B,C)}(\varphi,\nabla)$. When the metric g on Y is Lorentzian, then depending on the convention of its signature $(-,+,\cdots+)$ vs. $(+,-,\cdots-)$, one needs to add an overall minus $-$ vs. plus $+$ sign. In this note, for simplicity of presentation, we choose h and g to be both Riemannian (i.e. for Euclideanized/Wick-rotated D-branes and space-time).

- The world-volume X^{Az} of D-brane is m-dimensional; T_{m-1} is the tension of $(m-1)$-dimensional D-branes. Like the tension of the fundamental string, it is a fixed constant of nature.

- The second summand of $S_{map:kinetic^+}^{(\rho,h;\Phi,g)}$

$$S_{dilaton}^{(\rho,h;\Phi)}(\varphi) \;:=\; \int_X Re\Big(Tr\,\langle d\rho,\varphi^\diamond d\Phi\rangle_h\Big)\,vol_h\,,$$

will be called the *dilaton term* of the standard action $S_{standard}^{(\rho,h;\Phi,g,B,C)}(\varphi,\nabla)$.

Note that if let U be small enough and fix a local trivialization of $E|_U$. and assume that $\nabla = d + A$ with respect to this local trivialization. Then $D = d + [A,\,\bullet\,]$ and, over U with an orthonormal frame $(e_\mu)_\mu$,

$$\begin{aligned} Tr\,\langle d\rho,\varphi^\diamond d\Phi\rangle_h \;&=\; \sum_\mu Tr\,(d\rho(e_\mu)D_{e_\mu}\varphi^\sharp(\Phi)) \\ &=\; \sum_\mu Tr\Big(d\rho(e_\mu)\big(e_\mu\varphi^\sharp(\Phi)+[A(e_\mu),\varphi^\sharp(\Phi)]\big)\Big) \;=\; \sum_\mu Tr\Big(d\rho(e_\mu)(e_\mu\varphi^\sharp(\Phi))\Big)\,. \end{aligned}$$

Thus, while $\varphi^\diamond d\Phi$ depends on the connection ∇, the integrand $(Tr\,\langle d\rho,\varphi^\diamond d\Phi\rangle_h)\,vol_h$ does not. This justifies the dilaton term as a functional of φ alone.

In contrast, over U with the above setting, $Tr\,\langle D\varphi, D\varphi\rangle_{(h,g)}$ contains summand

$$\sum_\mu\sum_{i,j} Tr\Big([A(e_\mu),\varphi^\sharp(y^i)]\,[A(e_\mu),\varphi^\sharp(y^j)]\,\varphi^\sharp(g_{ij})\Big)\,,$$

which does not vanish in general. Thus, $Tr\,\langle D\varphi, D\varphi\rangle_{(h,g)}$ does depend on the pair (φ,∇).

(2) *The gauge/Yang-Mills term* $S_{gauge/YM}^{(h;B)}(\varphi,\nabla)$ α' is the Regge slope; $2\pi\alpha'$ is the inverse to the tension of a fundamental string.

- F_∇ is the curvature tensor of the connection ∇ on E; $2\pi\alpha' F_\nabla + \varphi^\diamond B$ is an $\mathcal{O}_X^{Az}$-valued 2-tensor on X; and

$$\|2\pi\alpha' F_\nabla + \varphi^\diamond B\|_h^2 \;:=\; \langle 2\pi\alpha' F_\nabla + \varphi^\diamond B\,,\,2\pi\alpha' F_\nabla + \varphi^\diamond B\rangle_h$$

from Sec. 3.2.1. Up to the shift by $\varphi^\diamond B$, this is a norm-squared of the field strength of the gauge field, and hence the name *Yang-Mills term*. Note that in $S_{gauge/YM}^{(h;B)}(\varphi,\nabla)$, ∇ couples with φ only through the background B-field B. When $B=0$, this is simply a functional $S_{gauge/YM}^{(h)}(\nabla)$ of ∇ alone.

- In the current bosonic case, the *Yang-Mills functional* for the gauge term $S^{(h;B)}_{gauge\,/\,YM}(\varphi,\nabla)$ can be replaced any other standard action functional, e.g. Chern-Simons functional, in gauge theories.

(3) *The Chern-Simons/Wess-Zumino term* $S^{(C,B)}_{CS/WZ}(\varphi,\nabla)$ The coupling constant of Ramond-Ramond fields with D-branes is taken to be equal to the D-brane tension T_{m-1}. This choice is adopted from the situation of the Dirac-Born-Infeld action. However, in the current bosonic case, one may take a different constant. As given here, $S^{(C,B)}_{CS/WZ}(\varphi,\nabla)$ is only formal; the *anomaly factor* $\sqrt{\hat{A}(X^{Az})/\hat{A}(N_{X^{Az}/Y})}$ in its integrand remains to be understood in the current situation.

- The wedge product of $\mathcal{O}_X^{Az}$-valued differential forms was discussed in [L-Y8: Sec.6.1] (D(13.1)). An Ansatz was proposed there in accordance with the notion of 'symmetrized determinant' for an $\mathcal{O}_X^{Az}$-valued 2-tensor on X in the construction of the non-Abelian Dirac-Born-Infeld action ibidem. Here, we no longer have a direct guide from the construction of the kinetic term $S^{(h;g)}_{map:kinetic}(\varphi,\nabla)$ for maps as to how to define such wedge products. However, just like Polyakov string should be thought of as being equivalent to Nambu-Goto string (at least at the classical level) but technically more robust, here we would think that 'standard D-branes' should be equivalent to 'Dirac-Born-Infeld D-branes' (at least classically) and, hence, will take the same Ansatz:

  Ansatz [wedge product in the Chern-Simons/Wess-Zumino action] We interpret the wedge products that appear in the formal expression for the Chern-Simons/Wess-Zumino term $S^{(C,B)}_{CS/WZ}(\varphi,\nabla)$ through the symmetrized determinant that applies to the above defining identities for wedge product; namely, we require that

$$(\omega^1\wedge\,\cdots\,\wedge\omega^s)(e_1\wedge\,\cdots\,\wedge e_s)\;=\;\mathit{SymDet}(\omega^i(e_j))$$

  for $\mathcal{O}_X^{Az}$-valued 1-forms $\omega^1,\,\cdots\,,\,\omega^s$ on X. Denote this generalized wedge product by $\overset{\odot}{\wedge}$.

  Then, for lower-dimensional D-branes $m=0,1,2,3$, it is reasonable to assume that the anomaly factor is 1 (i.e. no anomaly) and $S^{(C,B)}_{CS/WZ}(\varphi,\nabla)$ can be written out precisely.

  Locally in terms of a local frame $(e_\mu)_\mu$ on an open set $U\subset X$ and a coordinate $(y^1,\,\cdots\,,\,y^n)$ on a local chart of Y, one has: (Assuming that $B=\sum_{i,j}B_{ij}dy^i\otimes dy^j$, $B_{ji}=-B_{ij}$.)

  - For *D(−1)-brane* world-point $(m=0)$:

$$S^{(C_{(0)})}_{CS/WZ}(\varphi)\;=\;T_{-1}\,\cdot\,\mathit{Tr}(\varphi^\diamond C_{(0)})\;=\;T_{-1}\,\cdot\,\mathit{Tr}(\varphi^\sharp(C_{(0)}))\,.$$

  - For *D-particle* world-line $(m=1)$: Assume that $C_{(1)}=\sum_{i=1}^n C_i\,dy^i$ locally; then

$$S^{(C_{(1)})}_{CS/WZ}(\varphi)\;=\;T_0\int_X \mathit{Re}\,(\mathit{Tr}(\varphi^\diamond C_{(1)}))\;\;\overset{\text{locally}}{=}\;\;T_0\int_U \mathit{Re}\Big(\mathit{Tr}\Big(\sum_{i=1}^n\varphi^\sharp(C_i)\cdot D_{e_1}\varphi^\sharp(y^i)\Big)\Big)de^1\,.$$

    Note that as in the case of the dilaton term $S^{(\rho,h;\Phi)}_{dilaton}(\varphi)$, this is a functional of φ alone.

  - For *D-string* world-sheet $(m=2)$: Assume that $C_{(2)}=\sum_{i,j=1}^n C_{ij}\,dy^i\otimes dy^j$ locally, with $C_{ij}=-C_{ji}$; then

$$\begin{aligned}S^{(C_{(0)},C_{(2)},B)}_{CS/WZ}(\varphi,\nabla)\;&=\;T_1\int_X \mathit{Re}\,(\mathit{Tr}(\varphi^\diamond C_{(2)}\,+\,\varphi^\diamond(C_{(0)}B)\,+\,2\pi\alpha'\varphi^\sharp(C_{(0)})\odot F_\nabla))\\ &=\;T_1\int_X \mathit{Re}\,(\mathit{Tr}(\varphi^\diamond(C_{(2)}+C_{(0)}B)\,+\,\pi\alpha'\varphi^\sharp(C_{(0)})F_\nabla\,+\,\pi\alpha' F_\nabla\varphi^\sharp(C_{(0)})))\end{aligned}$$

$$\begin{aligned}
&\overset{\text{locally}}{=} \;\; T_1\int_U Re\Big(Tr\Big(\sum_{i,j=1}^n \varphi^\sharp(C_{ij}+C_{(0)}B_{ij})\,D_{e_1}\varphi^\sharp(y^i)\,D_{e_2}\varphi^\sharp(y^j)\\
&\qquad\qquad + \pi\alpha'\varphi^\sharp(C_{(0)})\,F_\nabla(e_1,e_2)\;+\;\pi\alpha' F_\nabla(e_1,e_2)\,\varphi^\sharp(C_{(0)})\Big)\Big)\,e^1\wedge e^2\\
&= \;\; T_1\int_U Re\Big(Tr\Big(\sum_{i,j=1}^n \varphi^\sharp(C_{ij}+C_{(0)}B_{ij})\,D_{e_1}\varphi^\sharp(y^i)\,D_{e_2}\varphi^\sharp(y^j)\\
&\qquad\qquad + 2\,\pi\alpha'\varphi^\sharp(C_{(0)})\,F_\nabla(e_1,e_2)\Big)\Big)\,e^1\wedge e^2\,.
\end{aligned}$$

Here, the last identity comes from the effect of the trace map Tr.

- For *D-membrane* world-volume $(m=3)$: Assume that $C_{(1)}=\sum_{i=1}^n C_i\,dy^i$ and $C_{(3)}=\sum_{i,j,k=1}^n C_{ijk}\,dy^i\otimes dy^j\otimes dy^k$ locally, with C_{ijk} alternating with respect to ijk; then

$$\begin{aligned}
&S_{CS/WZ}^{(C_{(1)},C_{(3)},B)}(\varphi,\nabla) \;=\; T_2\int_X Re\big(Tr(\varphi^\diamond C_{(3)}\;+\;\varphi^\diamond(C_{(1)}\wedge B)\;+\;2\pi\alpha'\,\varphi^\diamond C_{(1)}\overset{\odot}{\wedge} F_\nabla\,)\big)\\
&\overset{\text{locally}}{=} \;\; T_2\int_U Re\Big(Tr\Big(\sum_{i,j,k=1}^n \varphi^\sharp(C_{ijk}+C_iB_{jk}+C_jB_{ki}+C_kB_{ij})\\
&\qquad\qquad\qquad \cdot D_{e_1}\varphi^\sharp(y^i)\,D_{e_2}\varphi^\sharp(y^j)\,D_{e_3}\varphi^\sharp(y^k)\\
&\qquad + \pi\alpha'\sum_{(\lambda\mu\nu)\in Sym_3}\sum_{i=1}^n(-1)^{(\lambda\mu\nu)}\Big(\varphi^\sharp(C_i)\,D_{e_\lambda}(\varphi^\sharp(y^i))\,F_\nabla(e_\mu,e_\nu)\\
&\qquad\qquad\qquad + F_\nabla(e_\mu,e_\nu)\,\varphi^\sharp(C_i)\,D_{e_\lambda}\varphi^\sharp(y^i)\Big)\Big)\Big)\,e^1\wedge e^2\wedge e^3\\
&= \;\; T_2\int_U Re\Big(Tr\Big(\sum_{i,j,k=1}^n \varphi^\sharp(C_{ijk}+C_iB_{jk}+C_jB_{ki}+C_kB_{ij})\\
&\qquad\qquad\qquad \cdot D_{e_1}\varphi^\sharp(y^i)\,D_{e_2}\varphi^\sharp(y^j)\,D_{e_3}\varphi^\sharp(y^k)\\
&\qquad + 2\pi\alpha'\sum_{(\lambda\mu\nu)\in Sym_3}\sum_{i=1}^n(-1)^{(\lambda\mu\nu)}\Big(\varphi^\sharp(C_i)\,D_{e_\lambda}(\varphi^\sharp(y^i))\,F_\nabla(e_\mu,e_\nu)\Big)\Big)\Big)\,e^1\wedge e^2\wedge e^3\,.
\end{aligned}$$

Here, the last identity comes from the effect of the trace map Tr.

Their partial study was done in [L-Y8 : Sec. 6.2] (D(13.1)).

(4) *The background B-field* The coupling of (φ,∇) with the background B-field B on Y in the part

$$S_{gauge\,/\,YM}^{(h;B)}(\varphi,\nabla)\;+\;S_{CS/WZ}^{(C,B)}(\varphi,\nabla)$$

of the standard action means that we have to adjust the fundamental module $\mathcal{E}$ on X by a compatible "twisting" governed by φ and B. With this "twisting", $\mathcal{E}$ now lives on a gerb over X. See [L-Y2] (D(5)) for details and further references. However, since the study of the variational problems in this note is mainly local and focuses on the enhanced kinetic term for maps $S_{map:kinetic^+}^{(\rho,h;\Phi,g)}$, we'll ignore this twisting for the current note to keep the language and expressions simple.

Remark 4.3 [*other effects from B-field and Ramond-Ramond field*] There are other effects to D-branes beyond just mentioned above from the background B-field and Ramond-Ramond field that have not yet been taken into account in this project so far; e.g. [H-M1], [H-M2], and [H-Y]. They can influence the action for D-branes as well. Such additional effects should be investigated in the future.

Theorem 4.4 [well-defined gauge-symmetry-invariant action] *Except the anomaly factor in the Chen-Simons/Wess-Zumino term,* which is yet to be understood*, the standard action $S_{standard}^{(\rho,h;\Phi,g,B,C)}(\varphi,\nabla)$ as given in Definition 4.2 for $(*_1)$-admissible pairs (φ,∇) (and $S_{CS/WZ}^{(C,B)}(\varphi,\nabla)$ for $(*_2)$-admissible (φ,∇)) is well-defined. Assume that the anomaly factor in the Chen-Simons/*

Wess-Zumino term transforms also by conjugation as for $\mathcal{O}_X^{Az}$ *under a gauge symmetry, then* $S_{standard}^{(\rho,h;\Phi,g,B,C)}(\varphi,\nabla)$ *is invariant under gauge symmetries:*

$$S_{standard}^{(\rho,h;\Phi,g,B,C)}(\varphi,\nabla) \;=\; S_{standard}^{(\rho,h;\Phi,g,B,C)}({}^{g'}\!\varphi,\,{}^{g'}\nabla)$$

for $g'\in\mathcal{G}_{gauge}:=C^\infty(Aut_C(E))$.

For the kinetic term for maps

$$S_{map:kinetic}^{(h;g)}(\varphi,\nabla) \;:=\; \frac{1}{2}\,T_{m-1}\int_X Re\Big(Tr\langle D\varphi\,,\,D\varphi\rangle_{(h,g)}\Big)\,vol_h\,,$$

that it is well-defined follows Lemma 3.2.2.4. Under a gauge transformation $g'\in\mathcal{G}_{gauge}:=C^\infty(Aut_C(E))$ and in terms of local coordinates $(x^1,\cdots,x^m)$ on X and $(y^1,\cdots,y^n)$ on Y,

$${}^{g'}\!D\,{}^{g'}\!\varphi \;=\; \sum_\mu dx^\mu\otimes\sum_i {}^{g'}\!D_{\frac{\partial}{\partial x^\mu}}\,{}^{g'}\!\varphi^\sharp(\frac{\partial}{\partial y^i})\otimes_{{}^{g'}\!\varphi}\frac{\partial}{\partial y^i} \;=\; \sum_\mu dx^\mu\otimes\sum_i\Big(g'\Big(D_{\frac{\partial}{\partial x^\mu}}\varphi^\sharp(\frac{\partial}{\partial y^i})\Big)g'^{-1}\Big)\otimes_{{}^{g'}\!\varphi}\frac{\partial}{\partial y^i}\,.$$

Thus,

$$\begin{aligned}
&\langle\,{}^{g'}\!D\,{}^{g'}\!\varphi\,,\,{}^{g'}\!D\,{}^{g'}\!\varphi\rangle_{(h,g)}\\
&\quad=\;\sum_{\mu,\nu}\sum_{i,j}h^{\mu\nu}\otimes\Big(g'\Big(D_{\frac{\partial}{\partial x^\mu}}\varphi^\sharp(\frac{\partial}{\partial y^i})\Big)g'^{-1}\cdot g'\Big(D_{\frac{\partial}{\partial x^\nu}}\varphi^\sharp(\frac{\partial}{\partial y^j})\Big)g'^{-1}\Big)\otimes_{{}^{g'}\!\varphi}g_{ij}\\
&\quad=\;\sum_{\mu,\nu}\sum_{i,j}h^{\mu\nu}\cdot\Big(g'\Big(D_{\frac{\partial}{\partial x^\mu}}\varphi^\sharp(\frac{\partial}{\partial y^i})\Big)g'^{-1}\cdot g'\Big(D_{\frac{\partial}{\partial x^\nu}}\varphi^\sharp(\frac{\partial}{\partial y^j})\Big)g'^{-1}\Big)\cdot g'\varphi^\sharp(g_{ij})g'^{-1}\\
&\quad=\;g'\Big(\sum_{\mu,\nu}\sum_{i,j}h^{\mu\nu}\cdot D_{\frac{\partial}{\partial x^\mu}}\varphi^\sharp(\frac{\partial}{\partial y^i})\cdot D_{\frac{\partial}{\partial x^\nu}}\varphi^\sharp(\frac{\partial}{\partial y^j})\cdot\varphi^\sharp(g_{ij})\Big)g'^{-1}\\
&\quad=\;g'\,\langle D\varphi\,,\,D\varphi\rangle_{(h,g)}\,g'^{-1}\,.
\end{aligned}$$

It follows that $Tr\langle\,{}^{g'}\!D\,{}^{g'}\!\varphi\,,\,{}^{g'}\!D\,{}^{g'}\!\varphi\rangle_{(h,g)} = Tr\langle D\varphi\,,\,D\varphi\rangle_{(h,g)}$ and, hence,

$$S_{map:kinetic}^{(h;g)}({}^{g'}\!\varphi,\,{}^{g'}\nabla) \;=\; S_{map:kinetic}^{(h;g)}(\varphi,\nabla)\,.$$

The other terms in $S_{standard}^{(\rho,h;\Phi,g,B,C)}(\varphi,\nabla)$ do not involve a partially-defined inner product and hence are all defined. That the integrand inside Tr all transform by conjugation under a gauge symmetry as for $\mathcal{O}_X^{Az}$ follows Lemma 4.1.

This proves the theorem.

Remark 4.5 [*gauge-fixing condition*] As in any gauge field theory (e.g. [P-S]), understanding how to fix the gauge is an important part of understanding $S_{standard}^{(\rho,h;\Phi,g,B,C)}(\varphi,\nabla)$.

The standard action as an enhanced non-Abelian gauged sigma model

Recall that, in an updated language and in a form for easy comparison, a *sigma model* (σ-model, SM) on a (Riemannian or Lorentzian) manifold (Y,g) (of dimension n) is a field theory on a (Riemannian or Lorentzian) manifold (X,h) (of some dimension m) with

- *Field*: differentiable maps $f:X\to Y$,
- *Action functional*:

$$\begin{aligned}
S_{sigma\ model}^{(h,g)}(f) \;&:=\; \pm\frac{1}{2}\int_X\langle df\,,\,df\rangle_{(g,h)}\,vol_h \;=\; \pm\frac{1}{2}\int_X\|f^*g\|_h^2\,vol_h\\
&:=\; \pm\frac{1}{2}\int_X\sum_{\mu,\nu=1}^m\sum_{i,j=1}^n h^{\mu\nu}(\boldsymbol{x})g_{ij}(f(\boldsymbol{x}))\frac{\partial f^i}{\partial x^\mu}(\boldsymbol{x})\,\frac{\partial f^j}{\partial x^\nu}(\boldsymbol{x})\,\sqrt{|det\,h(\boldsymbol{x})|}\,d^m\boldsymbol{x}\,,
\end{aligned}$$

in terms of local coordinates $\boldsymbol{x} = (x^1, \cdots, x^m)$ on X and $\boldsymbol{y} = (y^1, \cdots, y^n)$ on Y; cf. [GM-L] and see e.g. [C-T] for modern update and further references. (The $\pm$ sign depends on the signature of the metric.) At the classical level, this is a theory of harmonic maps; cf. [E-L], [E-S], [Ma], [Sm].

Back to our situation. To begin with, the kinetic term

$$S^{(h;g)}_{map:kinetic}(\varphi, \nabla) \;:=\; \frac{1}{2}\, T_{m-1} \int_X Re\,(\,Tr\,\langle D\varphi\,,\, D\varphi\rangle_{(h,g)})\, vol_h$$

qualifies the standard action $S^{(\rho,h;\Phi,g,B,C)}_{standard}(\varphi,\nabla)$ to be regarded as a sigma model, now based on

- *Field*: $(*_1)$-admissible differentiable maps $\varphi : (X^{Az}, \mathcal{E}; \nabla) \to Y$.

The fact that $S^{(\rho,h;\Phi,g,B,C)}_{standard}(\varphi,\nabla)$ is invariant under the gauge symmetry group $\mathcal{G}_{gauge} := C^\infty(Aut_C(E))$ and that the latter is non-Abelian justify that this sigma model is indeed a *non-Abelian gauged sigma model* (nAGSM). However, compared with, for example, the well-studied $d=2$, $N=(2,2)$ (Abelian) gauged linear sigma model, e.g. [H-V] and [Wi1], the gauge symmetry of $S^{(\rho,h;\Phi,g,B,C)}_{standard}(\varphi,\nabla)$ does not arise from gauging a global group-action on the target space Y. (For this reason, one may call $S^{(\rho,h;\Phi,g,B,C)}_{standard}(\varphi,\nabla)$ a *sigma model with non-Abelian gauge symmetry* as well.) For D-branes, its additional coupling to the background Ramond-Ramond field C on Y is essential ([Po1]) and, hence, the Chern-Simons/Wess-Zumino term $S^{(C,B)}_{CS/WZ}(\varphi,\nabla)$. Also, we like our dynamical field (φ,∇) coupled to the background dilaton field Φ on Y as well so that the essence of the other important action — the Dirac-Born-Infeld action — for D-branes can be retained as much as we can. This motivates the dilaton term $S^{(\rho,h;\Phi)}_{dilaton}(\varphi)$. In summary,

$$\begin{aligned} S^{(\rho,h;\Phi,g,B,C)}_{standard}(\varphi,\nabla) \;&:=\; S^{(\rho,h;\Phi,g,B)}_{nAGSM}(\varphi,\nabla) \;+\; S^{(C,B)}_{CS/WZ}(\varphi,\nabla) \;+\; S^{(\rho,h;\Phi)}_{dilaton}(\varphi) \\ &=:\; S^{(\rho,h;\Phi,g,B,C)}_{nAGSM^+}(\varphi,\nabla)\,, \end{aligned}$$

which explains the name *enhanced non-Abelian gauged sigma model* (nAGSM$^+$).

5 Admissible family of admissible pairs (φ_T, ∇^T)

In this section we introduce the notion of one-parameter admissible families of admissible pairs and rephrase the basic settings and results in Sec. 3.2 in a relative format for such a family. Some curvature tensor computations are given for later use. The natural generalization (without work) to two-parameter admissible families of admissible pairs is remarked in the last theme of the section. This prepares us for the study of the variational problem of the enhanced kinetic term for maps $S^{(\rho,h;\Phi,g)}_{map:kinetic^+}(\varphi,\nabla)$ in the standard action $S^{(\rho,h;\Phi,g,B,C)}_{standard}(\varphi,\nabla)$ for D-branes.

Basic setup and the notion of admissible families of admissible pairs (φ_T, ∇^T)

Let $T = (-\varepsilon, \varepsilon) \subset R^1$, with coordinate t and $\varepsilon > 0$ small, be the one-parameter space and $\partial_t := \partial/\partial t$ and dt be respectively the tangent vector field and the 1-form determined by the coordinate t on T. Let (X, E) be a manifold X of dimension m with a complex vector bundle E of rank r. Recall the structure sheaf $\mathcal{O}_X$ of X and the $\mathcal{O}_X$-module $\mathcal{E}$ from E.

Consider the following families of objects over T:

- $X_T := X \times T$, with the structure sheaf $\mathcal{O}_{X_T}$ and regarded as the constant family of manifolds over T determined by X. X_T is equipped with the built-in projection maps $pr_X : X_T \to X$ and $pr_T : X_T \times T \to T$. For $U \subset X$ an open set, we will denote by U_T the corresponding open set $U \times T \subset X \times T$ over T.

- $T_*X_T :=$ the *tangent bundle* of X_T and $\mathcal{T}_*X_T :=$ the *tangent sheaf* of X_T;
  $T^*X_T :=$ the *cotangent bundle* of X_T and $\mathcal{T}^*X_T :=$ the *cotangent sheaf* of X_T;
  $T_*(X_T/T) :=$ the *relative tangent bundle* of X_T over T and
  $\mathcal{T}_*(X_T/T) :=$ the *relative tangent sheaf* of X_T over T;
  $T^*(X_T/T) :=$ the *relative cotangent bundle* of X_T over T and
  $\mathcal{T}^*(X_T/T) :=$ the *relative cotangent sheaf* of X_T over T.
  When X is endowed with a (Riemannain or Lorentzian) metric h, h induces canonically an inner-product structure on fibers of $T_*(X_T/T)$ and its dual, $T^*(X_T/T)$, over T. These induced inner-product structure will be denoted by $\langle\,\cdot\,,\,\cdot\rangle_h$.

- $E_T := pr_X^*E$ the pull-back vector bundle of E to X_T, regarded as the constant T-family of vector bundles over X determined by E; and $\mathcal{E}_T := pr_X^*\mathcal{E}$ the corresponding $\mathcal{O}_{X_T}$-module, regarded as the constant T-family of $\mathcal{O}_X$-modules determined by $\mathcal{E}$.
  The projection map $pr_X : X_T \to X$ induces a projection map $pr_E : E_T \to E$ between the total space of bundles in question. T_*E_T (resp. $\mathcal{T}_*E_T$) denotes the tangent space (resp. the tangent sheaf) of the total space of E_T.

- $(X_T^{Az}, \mathcal{E}_T) := (X_T, \mathcal{O}_{X_T}^{Az} := \mathcal{E}nd_{\mathcal{O}_{X_T}^C}(\mathcal{E}_T), \mathcal{E}_T)$, regarded as the constant T-family of Azumaya/matrix manifolds with a fundamental module determined by $(X^{Az}, \mathcal{E})$. There is a *trace map*
  $$Tr\;:\;\mathcal{O}_{X_T}^{Az}\;\longrightarrow\;\mathcal{O}_{X_T}^{C}$$
  as $\mathcal{O}_{X_T}$-modules, which takes $Id_{\mathcal{E}_T}$ to r.

and take the following notational conventions:

- Through the product structure $X_T = X \times T$, a vector field ξ (resp. 1-form ω) on X and the vector field ∂_t on T lift canonically to a vector field (resp. 1-form) on X_T, which will still be denoted by ξ (resp. ω) and ∂_t respectively.

- For referral, the restriction of X_T, X_T^{Az}, E_T, $\cdots_T$ to over $t \in T$ will be denoted X_t, X_t^{Az}, E_t, $\cdots_t$ respectively.

Definition 5.1 [connection/covariant derivation trivially flat over T] A connection ∇^T on E_T (equivalently, connection/covariant derivative ∇^T on $\mathcal{E}_T$) is said to be *trivially flat over* T if the horizontal lifting of ∂_t to T_*E_T lies in the kernel of the map $pr_{E*} : T_*E_T \to T_*E$. For such a ∇^T, we will denote the covariant derivative $\nabla^T_{\partial_t}$ simply by ∂_t. The *curvature tensor* of ∇^T will be denoted by F_{∇^T}.

Note that any connection on E_T is flat over T and hence, due to the topology of T, can be made trivially flat over T after a bundle-isomorphism. Thus the notion of 'trivially flat' is only a notational convenience for our variational problem, not a true constraint. However, caution that while ∇^T is always flat over T, its restriction ∇^t to X_t varies as t varies in T. Thus, in general, $F_{\nabla^T}(\partial_t, \cdot\,) \neq 0$.

Definition 5.2 [admissible family of admissible pairs (φ_T, ∇^T)] A *T-family of maps with varying connections* from $(X^{Az}, \mathcal{E})$ to Y is a pair (φ_T, ∇^T), where

$$\varphi_T\;:\;(X^{Az}, \mathcal{E}_T)\;\longrightarrow\;Y$$

is a map from $(X_T^{Az}, \mathcal{E}_T)$ to Y defined contravariantly by a ring-homomorphism

$$\varphi_T^\sharp\;:\;C^\infty(Y)\;\longrightarrow\;C^\infty(End_C(E_T))$$

over $R \subset C$ and ∇^T is a connection on $\mathcal{E}_T$ that is trivially flat over T. $\varphi_T^\sharp$ induces a homomorphism

$$\mathcal{O}_Y \longrightarrow \mathcal{O}_{X_T}^{Az}$$

between equivalence classes of gluing systems of rings, which will still be denoted by $\varphi_T^\sharp$.

Let $\mathcal{A}_{\varphi_T} \subset \mathcal{O}_{X_T}^{Az} = \mathcal{O}_{X_T}\langle Im\,\varphi_T^\sharp\rangle$. Then (φ_T, ∇^T) is said to be a $(*_i)$-*admissible T-family of $(*_j)$-admissible pairs* if (φ_T, ∇^T) satisfies Admissible Condition $(*_i)$ along T and Admissible Condition $(*_j)$ along X, for $i, j = 1, 2, 3$.

Example 5.3 [**$(*_2)$-admissible T-family of $(*_1)$-admissible pairs**] A $(*_2)$-admissible T-family of $(*_1)$-admissible pairs (φ_T, ∇^T) is a T-family of maps φ_T with a varying connection ∇^T trivially flat over T such that

$$(*_2)\ :\ \ \partial_t Comm(\mathcal{A}_{\varphi_T}) \subset Comm(\mathcal{A}_{\varphi_T}) \qquad \text{and} \qquad (*_1)\ :\ \ \nabla_\xi^T \mathcal{A}_{\varphi_T} \subset Comm(\mathcal{A}_{\varphi_T})$$

for all $\xi \in \mathcal{T}_*(X_T/T)$. Here, $Comm(\mathcal{A}_{\varphi_T})$ is the commutant of $\mathcal{A}_{\varphi_T}$ in $\mathcal{O}_{X_T}^{Az}$.

Three basic $\mathcal{O}_{X_T}$-modules with induced structures

Let X be endowed with a (Riemannian or Lorentzian) metric h and Y be endowed with a (Riemannian or Lorentzian) metric g. Denote the canonically induced inner-product structure from h and g on whatever bundle applicable by $\langle\,\cdot\,,\,\cdot\,\rangle_h$ and $\langle\,\cdot\,,\,\cdot\,\rangle_g$ respectively. Denote the induced connection on $\mathcal{T}_*(X_T/T)$ and $\mathcal{T}^*(X_T/T)$ by ∇^h and the Levi-Civita connection on $\mathcal{T}_*Y$ by ∇^g. The associated Riemann curvature tensor is denoted by R^h and R^g respectively.

Let (φ_T, ∇^T) be a $(*_1)$-admissible T-family of $(*_1)$-admissible pairs. The basic $\mathcal{O}_{X_T}^C$-modules with induced structures from the setting, as in Sec. 3.2, are listed below to fix notations.

(0) $\mathcal{O}_{X_T}^{Az}$: *the noncommutative structure sheaf on X_T*

- The induced connection D^T from ∇^T, which is also trivially flat over T,
- An $\mathcal{O}_{X_T}^{Az}$-valued, $\mathcal{O}_X^C$-bilinear (nonsymmetric) inner product from the multiplication in $\mathcal{O}_{X_T}^{Az}$;
  an $\mathcal{O}_X^C$-valued, $\mathcal{O}_X^C$-bilinear (symmetric) inner product after the post-composition with Tr.
- Both inner products are covariantly constant with respect to D^T and one has the Leizniz rules

$$\begin{aligned} D^T(m_T^1 m_T^2) &= (D^T m_T^1)m_T^2 + m_T^1\, D^T m_T^2\,; \\ d\,Tr(m_T^1 m_T^2) &= Tr\, D^T(m_T^1 m_T^2) \\ &= Tr((D^T m_T^1)m_T^2) + Tr(m_T^1\, D^T m_T^2)\,. \end{aligned}$$

(1) $\mathcal{T}^*(X_T/T)\otimes_{\mathcal{O}_{X_T}}\mathcal{O}_{X_T}^{Az}$: $\mathcal{O}_{X_T}^{Az}$*-valued relative 1-forms on X_T/T*

- The induced connection $\nabla^{T,(h,D^T)} := \nabla^h\otimes Id + Id\otimes D^T$, trivially flat over T.
- An $\mathcal{O}_{X_T}^{Az}$-valued, $\mathcal{O}_{X_T}^C$-bilinear (nonsymmetric) inner product $\langle\,\bullet\,,\,\bullet\,\rangle_h$;
  an $\mathcal{O}_X^C$-valued, $\mathcal{O}_X^C$-bilinear (symmetric) inner product $Tr\langle\,\bullet\,,\,\bullet\,\rangle_h$.
- Both inner products are covariantly constant with respect to $\nabla^{T,(h,D^T)}$ and one has the Leibniz rules

$$\begin{aligned} D^T\langle\,\bullet\,,\,\bullet'\rangle_h &= \langle\,\nabla^{T,(h,D^T)}\,\bullet\,,\,\bullet'\rangle_h + \langle\,\bullet\,,\,\nabla^{T,(h,D^T)}\,\bullet'\rangle_g\,, \\ d\,Tr\langle\,\bullet\,,\,\bullet'\rangle_h &= Tr(D^T\langle\,\bullet\,,\,\bullet'\rangle_h) \;=\; Tr\langle\,\nabla^{T,(h,D^T)}\,\bullet\,,\,\bullet'\rangle_h + Tr\langle\,\bullet\,,\,\nabla^{T,(h,D^T)}\,\bullet'\rangle_h \end{aligned}$$

  for $\bullet, \bullet' \in \mathcal{T}^*(X_T/T)\otimes_{\mathcal{O}_{X_T}}\mathcal{O}_{X_T}^{Az}$.

(2) $\varphi_T^*\mathcal{T}_*Y := \mathcal{O}_{X_T}^{Az}\otimes_{\varphi_T^\sharp,\mathcal{O}_Y}\mathcal{T}_*Y$: *$\mathcal{O}_{X_T}^{Az}$-valued derivations on $\mathcal{O}_Y$*

- The induced connection $\nabla^{T,(\varphi_T,g)} := D^T\otimes Id + Id\cdot\sum_{i=1}^n D^T\varphi_T^\sharp(y^i)\otimes\nabla^g_{\frac{\partial}{\partial y^i}}$ (in local expression), trivially flat over T.
- A *partially defined* $\mathcal{O}_{X_T}^{Az}$-valued, $\mathcal{O}_X^{C}$-bilinear (nonsymmetric) inner product $\langle\,\bullet\,,\,\bullet\,\rangle_g$; a *partially defined* $\mathcal{O}_X^{C}$-valued, $\mathcal{O}_X^{C}$-bilinear (symmetric) inner product $Tr\langle\,\bullet\,,\,\bullet\,\rangle_g$.
- Both inner products, when defined, are covariantly constant with respect to $\nabla^{T,(\varphi_T,g)}$ and one has the Leibniz rules

$$\begin{aligned} D^T\langle -\,,\,-'\rangle_g &= \langle\,\nabla^{T,(\varphi_T,g)}-\,,\,-'\rangle_g + \langle -\,,\,\nabla^{T,(\varphi_T,g)}-'\rangle_g\,,\\ d\,Tr\langle -\,,\,-'\rangle_g &= Tr(D^T\langle -\,,\,-'\rangle_g) = Tr\langle\,\nabla^{T,(\varphi_T,g)}-\,,\,-'\rangle_g + Tr\langle -\,,\,\nabla^{T,(\varphi_T,g)}-'\rangle_g\,, \end{aligned}$$

whenever all $\langle -''\,,\,-'''\rangle_g$ and $Tr\langle -''\,,\,-'''\rangle_g$ involved are defined.

(3) $\mathcal{T}^*(X_T/T)\otimes_{\mathcal{O}_{X_T}}\varphi_T^*\mathcal{T}_*Y$: *($\mathcal{O}_{X_T}^{Az}$-valued relative 1-form)-valued derivations on $\mathcal{O}_Y$*

This is a combination of the construction in Item (1) and in Item (2).

- The induced connection

$$\nabla^{T,(h,\varphi_T,g)} = \nabla^h\otimes Id\otimes Id + Id\otimes D^T\otimes Id + Id\otimes Id\cdot\sum_{i=1}^n D^T\varphi_T^\sharp(y^i)\otimes\nabla^g_{\frac{\partial}{\partial y^i}}$$

(in local expression), trivially flat over T.
- A *partially defined* $\mathcal{O}_{X_T}^{Az}$-valued, $\mathcal{O}_X^{C}$-bilinear (nonsymmetric) inner product $\langle\,\bullet\,,\,\bullet\,\rangle_{(h,g)}$; a *partially defined* $\mathcal{O}_X^{C}$-valued, $\mathcal{O}_X^{C}$-bilinear (symmetric) inner product $Tr\langle\,\bullet\,,\,\bullet\,\rangle_{(h,g)}$.
- Both inner products, when defined, are covariantly constant with respect to $\nabla^{T,(h,\varphi_T,g)}$ and one has the Leibniz rules

$$\begin{aligned} D^T\langle\sim\,,\,\sim'\rangle_{(h,g)} &= \langle\,\nabla^{T,(h,\varphi_T,g)}\sim\,,\,\sim'\rangle_{(h,g)} + \langle\sim\,,\,\nabla^{T,(h,\varphi_T,g)}\sim'\rangle_{(h,g)}\,,\\ d\,Tr\langle\sim\,,\,\sim'\rangle_{(h,g)} &= Tr(D^T\langle\sim\,,\,\sim'\rangle_{(h,g)})\\ &= Tr\langle\,\nabla^{T,(h,\varphi_T,g)}\sim\,,\,\sim'\rangle_{(h,g)} + Tr\langle\sim\,,\,\nabla^{T,(h,\varphi_T,g)}\sim'\rangle_{(h,g)}\,, \end{aligned}$$

whenever the $\langle\sim''\,,\,\sim'''\rangle_{(h,g)}$ and $Tr\langle\sim''\,,\,\sim'''\rangle_{(h,g)}$ involved are defined.

Curvature tensors with ∂_t and other order-switching formulae

Let (φ_T,∇^T) be a $(*_1)$-admissible T-family of $(*_1)$-admissible pairs. A very basic step in (particularly the second) variational problem involves passing ∂_t over a differential operator on X_t's. In general, a curvature term appears whenever such passing occurs. In this theme, we collect and prove such formulae we need.

First, passing ∂_t over a differential operator usually means the appearance of a curvature term by the very definition of a curvature tensor:

Lemma 5.4 [curvature tensor with ∂_t] *Let (φ_T,∇^T) be a $(*_1)$-admissible T-family of $(*_1)$-admissible pairs. Let ξ be a vector field on an open set $U\subset X$ small enough so that $\varphi_T(U_T^{Az})$ is contained in a coordinate chart on Y, with coordinates $(y^1,\cdots,y^n)$. The standard lifting of ξ to U_T is denoted also by ξ. Note that, by construction, $[\partial_t,\xi]=0$ and all our connection $\nabla^\bullet$ in Theme 'Three basic $\mathcal{O}_X^{Az}$-modules with induced structures' are trivially flat; hence, $F_{\nabla^\bullet}(\partial_t,\xi)=\partial_t\nabla^\bullet_\xi-\nabla^\bullet_\xi\partial_t$. One has the following curvature expressions with ∂_t on the basic $\mathcal{O}_{X_T}$-modules: (Below we adopt the convention that the Riemann curvature tensor from a metric is denoted by R while the curvature tensor of a connection in all other bundle situations is denoted by F.)*

(0_1) *For sections* ω_T *of* $\mathcal{T}^*(X_T/T)$ *:* $R_{\nabla^h}(\partial_t,\xi)\,\omega_T \;=\; \partial_t\nabla^h_\xi\omega_T \;-\; \nabla^h_\xi\partial_t\omega_T \;=\; 0.$

(0_2) *For sections* m_T *of* $\mathcal{O}^{Az}_{X_T}$ *:* $F_{D^T}(\partial_t,\xi)\,m_T \;=\; \partial_t D^T_\xi m_T \;-\; D^T_\xi\partial_t m_T \;=\; [(\partial_t\nabla^T)(\xi), m_T]\,.$

As a consequence of this, if (φ_T,∇^T) *is furthermore a* $(*_2)$*-admissible* T*-family of* $(*_2)$*-admissible pairs, then*

$$(\partial_t\nabla^T)(\xi)\;\in\; Inn^{\varphi}_{(*_1)}(\mathcal{O}^{Az}_X) \qquad i.e.\quad [(\partial_t\nabla^T)(\xi),\mathcal{A}_{\varphi_T}]\;\subset\; Comm(\mathcal{A}_{\varphi_T})\,.$$

(1) *For sections* $\omega_T\otimes m_T$ *of* $\mathcal{T}^*(X_T/T)\otimes_{\mathcal{O}_{X_T}}\mathcal{O}^{Az}_{X_T}$ *:*

$$\begin{aligned}&F_{\nabla^{T,(h,D^T)}}(\partial_t,\xi)\,(\omega_T\otimes m_T)\\ &\quad=\;\partial_t\nabla^{T,(h,D^T)}_\xi(\omega_T\otimes m_T)\;-\;\nabla^{T,(h,D^T)}_\xi\partial_t(\omega_T\otimes m_T)\;=\;\omega_T\otimes[(\partial_t\nabla^T)(\xi),m_T]\,.\end{aligned}$$

(2) *For sections* $m_T\otimes v$ *of* $\varphi_T^*\mathcal{T}_*Y := \mathcal{O}^{Az}_{X_T}\otimes_{\varphi_T^\sharp,\mathcal{O}_Y}\mathcal{T}_*Y$ *:*
(v *on the coordinate chart of* Y *above, with coordinates* $(y^1,\,\cdots,\,y^n)$*)*

$$\begin{aligned}F_{\nabla^{T,(\varphi_T,g)}}(\partial_t,\xi)(m_T\otimes v)\;&=\;\partial_t\nabla^{T,(\varphi_T,g)}_\xi(m_T\otimes v)\;-\;\nabla^{T,(\varphi_T,g)}_\xi\partial_t(m_T\otimes v)\\ =\;[(\partial_t\nabla^T)(\xi),m_T]\otimes v\;&+\;m_T\sum_{i=1}^n[(\partial_t\nabla^T)(\xi),\varphi_T^\sharp(y^i)]\otimes\nabla^g_{\frac{\partial}{\partial y^i}}v\\ +\;m_T\sum_{i,j=1}^n\Big(D^T_\xi\varphi_T^\sharp(y^i)\,\partial_t\varphi_T^\sharp(y^j)&\otimes\nabla^g_{\frac{\partial}{\partial y^j}}\nabla^g_{\frac{\partial}{\partial y^i}}v\;-\;\partial_t\varphi_T^\sharp(y^j)\,D^T_\xi\varphi_T^\sharp(y^i)\otimes\nabla^g_{\frac{\partial}{\partial y^i}}\nabla^g_{\frac{\partial}{\partial y^j}}v\Big).\end{aligned}$$

If (φ_T,∇^T) *is furthermore a* $(*_2)$*-admissible* T*-family of* $(*_1)$*-admissible pairs, then the last term has a* Y*-coordinate-free form*

$$\text{The last term}\;=\;m_T\sum_{i,j}\partial_t\varphi_T^\sharp(y^j)\,D_\xi\varphi_T^\sharp(y^i)\otimes R^g(\tfrac{\partial}{\partial y^j},\tfrac{\partial}{\partial y^i})\,v\;=\;m_T\,((\varphi_T^\diamond R^g)(\partial_t,\xi))v\,.$$

(3) *For sections* $\omega_T\otimes m_T\otimes v$ *of* $\mathcal{T}^*(X_T/T)\otimes\varphi_T^*\mathcal{T}_*Y := \mathcal{T}^*(X_T/T)\otimes_{\mathcal{O}_{X_T}}\mathcal{O}^{Az}_{X_T}\otimes_{\varphi_T^\sharp,\mathcal{O}_Y}\mathcal{T}_*Y$ *:*
(v *on the coordinate chart of* Y *above, with coordinates* $(y^1,\,\cdots,\,y^n)$ *)*

$$\begin{aligned}&F_{\nabla^{T,(h,\varphi_T,g)}}(\partial_t,\xi)(\omega_T\otimes m_T\otimes v)\\ &\quad=\;\partial_t\nabla^{T,(h,\varphi_T,g)}_\xi(\omega_T\otimes m_T\otimes v)\;-\;\nabla^{T,(h,\varphi_T,g)}\partial_t(\omega_T\otimes m_T\otimes v)\\ &\quad=\;\omega_T\otimes\Big(F_{\nabla^{T,(\varphi_T,g)}}(\partial_t,\xi)(m_T\otimes v)\Big)\,.\end{aligned}$$

Statement (0_1) follows from the fact that X_T is a constant family over T. Statement (0_2), First Part, follows from a computation with respect to an induced local trivialization of $\mathcal{E}_T$ from a local trivialization of $\mathcal{E}$

$$\begin{aligned}\partial_t D^T_\xi m_T\;&=\;\partial_t(\xi m_T+[A_{\nabla^T}(\xi),m_T])\\ &=\;\xi\partial_t m_T+[\partial_t A_{\nabla^T}(\xi),m_T]\,+\,[A_{\nabla^T}(\xi),\partial_t m_T]\;=\;D^T_\xi\partial_t m_T+[(\partial_t\nabla^T)(\xi),m_T]\,.\end{aligned}$$

For Second Part, if (φ_T,∇^T) is furthermore a $(*_2)$-admissible T-family of $(*_2)$-admissible pairs, then for $f_1,f_2\in\mathcal{O}_Y$, by First Part and the $(*_2)$-Admissible Condition,

$$[[(\partial_t\nabla^T)(\xi),\varphi_T^\sharp(f_1)],\varphi_T^\sharp(f_2)]\;=\;[\partial_t D^T_\xi\varphi_T^\sharp(f_1),\varphi_T^\sharp(f_2)]\,-\,[D^T_\xi\partial_t\varphi_T^\sharp(f_1),\varphi_T^\sharp(f_2)]\;=\;0\,.$$

Which says that $(\partial_t\nabla^T)(\xi)\in Inn^{\varphi_T}_{(*_1)}(\mathcal{O}^{Az}_{X_T})$.

Statement (1) is a consequence of Statement (0_1) and Statement (0_2). Statement (3) is a consequence of Statement (0_1) and a property of the induced connection on a tensor product of

$\mathcal{O}_{X_T}^C$-modules with a connection. Let us carry out Statement (2) as a demonstration of the covariant differential calculus involved.

Let $m_T\otimes v\in\varphi_T^*\mathcal{T}_*Y$. Then, by Statement ($0_2$),

$$\begin{aligned}
\partial_t\nabla_\xi^{T,(\varphi_T,g)}(m_T\otimes v) \;&=\; \partial_t\Big(D_\xi^T m_T\otimes v\;+\;m_T\sum_i D_\xi^T\varphi_T^\sharp(y^i)\otimes\nabla^g_{\frac{\partial}{\partial y^i}}v\Big)\\
&=\;(D_\xi^T\partial_t m_T+[(\partial_t\nabla^T)(\xi),m_T])\otimes v\;+\;(D_\xi^T m_T)\sum_i\partial_t\varphi_T^\sharp(y^i)\otimes\nabla^g_{\frac{\partial}{\partial y^i}}v\\
&\quad+(\partial_t m_T)\sum_i D_\xi^T\varphi_T^\sharp(y^i)\otimes\nabla^g_{\frac{\partial}{\partial y^i}}v\;+\;m_T\sum_i(D_\xi^T\partial_t\varphi_T^\sharp(y^i)+[(\partial_t\nabla^T)(\xi),\varphi_T^\sharp(y^i)])\otimes\nabla^g_{\frac{\partial}{\partial y^i}}v\\
&\quad+m_T\sum_{i,j}D_\xi^T\varphi_T^\sharp(y^i)\partial_t\varphi_T^\sharp(y^j)\otimes\nabla^g_{\frac{\partial}{\partial y^j}}\nabla^g_{\frac{\partial}{\partial y^i}}v
\end{aligned}$$

while

$$\begin{aligned}
\nabla_\xi^{T,(\varphi_T,g)}\partial_t(m_T\otimes v)\;&=\;\nabla_\xi^{T,(\varphi_T,g)}\Big(\partial_t m_T\otimes v\;+\;m_T\sum_i\partial_t\varphi_T^\sharp(y^i)\otimes\nabla^g_{\frac{\partial}{\partial y^i}}v\Big)\\
&=\;D_\xi^T\partial_t m_T\otimes v\;+\;(\partial_t m_T)\sum_i D_\xi^T\varphi_T^\sharp(y^i)\otimes\nabla^g_{\frac{\partial}{\partial y^i}}v\;+\;(D_\xi^T m_T)\sum_i\partial_t\varphi_T^\sharp(y^i)\otimes\nabla_{\frac{\partial}{\partial y^i}}v\\
&\quad+m_T\sum_i D_\xi^T\partial_t\varphi_T^\sharp(y^i)\otimes\nabla^g_{\frac{\partial}{\partial y^i}}v\;+\;m_T\sum_{i,j}\partial_t\varphi_T^\sharp(y^i)D_\xi^T\varphi_T^\sharp(y^j)\otimes\nabla^g_{\frac{\partial}{\partial y^j}}\nabla^g_{\frac{\partial}{\partial y^i}}v\,.
\end{aligned}$$

Thus,

$$\begin{aligned}
F_{\nabla^{T,(\varphi_T,g)}}(\partial_t,\xi)(m_T\otimes v)\;&=\;(\partial_t\nabla^{T,(\varphi_T,g)}-\nabla^{T,(\varphi_T,g)}\partial_t)(m_T\otimes v)\\
&=\;[(\partial_t\nabla^T)(\xi),m_T]\otimes v\;+\;m_T\sum_i[(\partial_t\nabla^T)(\xi),\varphi_T^\sharp(y^i)]\otimes\nabla^g_{\frac{\partial}{\partial y^i}}v\\
&\quad+m_T\sum_{i,j}\Big(D_\xi^T\varphi_T^\sharp(y^i)\partial_t\varphi_T^\sharp(y^j)\otimes\nabla^g_{\frac{\partial}{\partial y^j}}\nabla^g_{\frac{\partial}{\partial y^i}}v\;-\;\partial_t\varphi_T^\sharp(y^j)D_\xi^T\varphi_T^\sharp(y^i)\otimes\nabla^g_{\frac{\partial}{\partial y^i}}\nabla^g_{\frac{\partial}{\partial y^j}}v\Big)
\end{aligned}$$

as claimed, after a relabeling of i,j.

If (φ_T,∇^T) is furthermore a $(*_2)$-admissible T-family of $(*_1)$-admissible pairs, then $D_\xi^T\varphi_T^\sharp(y^i)$ and $\partial_t\varphi_T^\sharp(y^j)$ commute since $[D_\xi^T\varphi_T^\sharp(y^i),\varphi_T^\sharp(y^j)]=0$ by the $(*_1)$-Admissible Condition along X and, hence,

$$\begin{aligned}
0\;&=\;\partial_t[D_\xi^T\varphi_T^\sharp(y^i),\varphi_T^\sharp(y^j)]\\
&=\;[\partial_t D_\xi^T\varphi_T^\sharp(y^i),\varphi_T^\sharp(y^j)]\;+\;[D_\xi^T\varphi_T^\sharp(y^i),\partial_t\varphi_T^\sharp(y^j)]\;=\;[D_\xi^T\varphi_T^\sharp(y^i),\partial_t\varphi_T^\sharp(y^j)]
\end{aligned}$$

by the $(*_1)$-Admissible Condition along X and the $(*_2)$-Admissible Condition along T. The last summand of $F_{\nabla^{T,(\varphi_T,g)}}(\partial_t,\xi)(m_T\otimes v)$ is then equal to

$$m_T\sum_{i,j}\partial_t\varphi_T^\sharp(y^j)D_\xi^T\varphi_T^\sharp\otimes\Big(\nabla^g_{\frac{\partial}{\partial y^j}}\nabla^g_{\frac{\partial}{\partial y^i}}\;-\;\nabla^g_{\frac{\partial}{\partial y^i}}\nabla^g_{\frac{\partial}{\partial y^j}}\Big)v\;=\;m_T\,((\varphi_T^\diamond R^g)(\partial_t,\xi))\,v\,.$$

This proves the lemma.

The following lemma addresses the issue of passing ∂_t over the covariant differential $D\varphi_T$ of φ_T. Though such passing is not a curvature issue in the conventional sense, it does carry a taste of curvature calculations.

Lemma 5.5 $[\partial_t D^T\varphi_T$ **versus** $\nabla^{T,(\varphi_T,g)}\partial_t\varphi_T]$ *Let (φ_T,∇^T) be a $(*_1)$-admissible T-family of $(*_1)$-admissible pairs. With the above notation and convention, let ξ be a vector field on X. Then, for a chart of Y with coordinates $(y^1,\cdots,y^n)$, one has*

$$\partial_t D_\xi^T\varphi_T\;=\;\nabla_\xi^{T,(\varphi_T,g)}\partial_t\varphi_T\;-\;(ad\otimes\nabla^g)_{\partial_t\varphi_T}D_\xi^T\varphi_T\;+\;\sum_{i=1}^n[(\partial_t\nabla^T)(\xi),\varphi_T^\sharp(y^i)]\otimes\frac{\partial}{\partial y^i}\,.$$

Here, only as a compact notation,

$$
\begin{aligned}
(ad\otimes\nabla^g)_{\partial_t\varphi_T} D^T_\xi\varphi_T \;&:=\; \sum_{i,j=1}^n [\partial_t\varphi^\sharp(y^i), D^T_\xi\varphi^\sharp(y^j)]\otimes\nabla^g_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j}\\
&=\; -\sum_{i,j=1}^n [D^T_\xi\varphi^\sharp(y^j), \partial_t\varphi^\sharp(y^i)]\otimes\nabla^g_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^i} \;=:\; -(ad\otimes\nabla^g)_{D^T_\xi\varphi_T}\partial_t\varphi_T\,.
\end{aligned}
$$

If (φ_T,∇^T) is furthermore a $(*_2)$-admissible T-family of $(*_2)$-admissible pairs, then the last term has a Y-coordinate-free expression

$$
ad_{\,(\partial_t\nabla^T)(\xi)}\varphi_T\,.
$$

Under the given setting and by Lemma 5.4 (0_2),

$$
\begin{aligned}
\partial_t D^T_\xi\varphi_T \;&=\; \partial_t\Big(\sum_i D^T_\xi\varphi^\sharp_T(y^i)\otimes\frac{\partial}{\partial y^i}\Big)\\
&=\; \sum_i \big(D^T_\xi\partial_t\varphi^\sharp_T(y^i)+[(\partial_t\nabla^T)(\xi),\varphi^\sharp_T(y^i)]\big)\otimes\frac{\partial}{\partial y^i} \;+\; \sum_{i,j} D^T_\xi\varphi^\sharp_T(y^i)\partial_t\varphi^\sharp_T(y^j)\otimes\nabla^g_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^i}
\end{aligned}
$$

while

$$
\begin{aligned}
\nabla^{T,(\varphi_T,g)}_\xi\partial_t\varphi_T \;&=\; \nabla^{T,(\varphi_T,g)}_\xi\Big(\sum_i\partial_t\varphi^\sharp_T(y^i)\otimes\frac{\partial}{\partial y^i}\Big)\\
&=\; \sum_i D^T_\xi\partial_t\varphi^\sharp_T(y^i)\otimes\frac{\partial}{\partial y^i} \;+\; \sum_{i,j}\partial_t\varphi^\sharp_T(y^i)D^T_\xi\varphi^\sharp_T(y^j)\otimes\nabla^g_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^i}\,.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\partial_t D^T_\xi\varphi \;-\; \nabla^{T,(\varphi_T,g)}_\xi\partial_t\varphi_T\\
&\quad=\; \sum_i[(\partial_t\nabla^T)(\xi),\varphi^\sharp_T(y^i)]\otimes\frac{\partial}{\partial y^i}\\
&\qquad+\sum_{i,j}D^T_\xi\varphi^\sharp_T(y^i)\partial_t\varphi^\sharp_T(y^j)\otimes\nabla^g_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^i} \;-\; \sum_{i,j}\partial_t\varphi^\sharp_T(y^i)D^T_\xi\varphi^\sharp_T(y^j)\otimes\nabla^g_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^i}
\end{aligned}
$$

Either apply the identity $\nabla^g_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^i}=\nabla^g_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j}$ to the second term and relabeling i,j of the third, or apply the identity $\nabla^g_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^i}=\nabla^g_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j}$ to the third term and relabeling i,j of the second,

$$
\begin{aligned}
&\sum_{i,j}D^T_\xi\varphi^\sharp_T(y^i)\partial_t\varphi^\sharp_T(y^j)\otimes\nabla^g_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^i} \;-\; \sum_{i,j}\partial_t\varphi^\sharp_T(y^i)D^T_\xi\varphi^\sharp_T(y^j)\otimes\nabla^g_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^i}\\
&\quad=\; \sum_{i,j}[D^T_\xi\varphi^\sharp_T(y^i)\,,\,\partial_t\varphi^\sharp_T(y^j)]\otimes\nabla^g_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j} \qquad\Big(\;:=\;(ad\otimes\nabla^g)_{D^T_\xi\varphi}\partial_t\varphi_T\qquad\Big)\\
&\quad=\; -\sum_{i,j}[\partial_t\varphi^\sharp_T(y^i)\,,\,D^T_\xi\varphi^\sharp_T(y^j)]\otimes\nabla^g_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j} \qquad\Big(\;:=\;-(ad\otimes\nabla^g)_{\partial_t\varphi_T}D^T_\xi\varphi_T\Big)\,.
\end{aligned}
$$

This proves the First Statement in Lemma.

The Second Statement in Lemma is a consequence of Corollary 3.1.10 and Lemma 5.4 (0_2).

This proves the lemma.

Before continuing the discussion, we introduce a notion that is needed in the next lemma.

Definition 5.6 [half-torsion tensor $Tor^{\frac{1}{2},\bullet}_{\nabla^g}$] Recall the torsion tensor $Tor_{\nabla'}$ of a connection ∇' on Y

$$Tor_{\nabla'}(v_1,v_2) \;:=\; \nabla'_{v_1}v_2 \,-\, \nabla'_{v_2}v_1 \,-\, [v_1,v_2]$$

for $v_1,v_2\in\mathcal{T}_*Y$. For the Levi-Civita connection ∇^g associated to a metric g on Y, $Tor_{\nabla^g}\equiv 0$ by construction. Thus, in this case, for a $\Phi\in C^\infty(Y)$,

$$(\nabla^g_{v_1}v_2 \,-\, v_1v_2)\Phi = (\nabla^g_{v_2}v_1 \,-\, v_2v_1)\Phi$$

for $v_1,v_2\in\mathcal{T}_*Y$. This defines a symmetric 2-tensor on Y

$$\begin{array}{cccc} Tor^{\frac{1}{2},\Phi}_{\nabla^g} & : & \mathcal{T}_*Y\times_Y\mathcal{T}_*Y & \longrightarrow & \mathcal{O}_Y \\ & & (v_1,v_2) & \longmapsto & (\nabla^g_{v_1}v_2 \,-\, v_1v_2)\,\Phi \end{array},$$

called the *half-torsion tensor of* (the torsion-free connection) ∇^g *associated to* $\Phi\in C^\infty(Y)$.

The following lemma addresses the issue of passing ∂_t over 'evaluation of an $\mathcal{O}^{Az}_{X_T}$-valued derivation on $C^\infty(Y)$', and another similar situation:

Lemma 5.7 [$\partial_t((D^T_\xi\varphi_T)\Phi)$ versus $(\partial_tD^T_\xi\varphi_T)\Phi$; $D^T_\xi((\partial_t\varphi_T)\Phi)$ versus $(\nabla^{T,(\varphi_T,g)}_\xi\partial_t\varphi_T)\Phi$] *Let (φ_T,∇^T) be a $(*_1)$-admissible T-family of $(*_1)$-admissible pairs. Continue the notation and convention in Lemma 5.4. Under the canonical isomorphism $\mathcal{O}^{Az}_{X_T}\otimes_{\varphi_T^\sharp,\mathcal{O}_Y}\mathcal{O}_Y\simeq\mathcal{O}^{Az}_{X_T}$,*

$$\begin{aligned} &\partial_t((D^T_\xi\varphi_T)\Phi)\\ &\quad=\;(\partial_tD^T_\xi\varphi_T)\Phi \;+\; \sum_{i,j=1}^n D^T_\xi\varphi_T^\sharp(y^i)\,\partial_t\varphi_T^\sharp(y^j)\,\otimes\Big(\frac{\partial}{\partial y^i}\frac{\partial}{\partial y^j}\Phi \,-\, \Big(\nabla^g_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j}\Big)\Phi\Big)\\ &\quad=\;(\partial_tD^T_\xi\varphi_T)\Phi \;-\; (\varphi_T^\diamond Tor^{\frac{1}{2},\Phi}_{\nabla^g})(\xi,\partial_t)\,; \end{aligned}$$

and

$$\begin{aligned} &D^T_\xi((\partial_t\varphi_T)\Phi)\\ &\quad=\;(\nabla^{T,(\varphi_T,g)}_\xi\partial_t\varphi_T)\Phi \;+\; \sum_{i,j=1}^n \partial_t\varphi_T^\sharp(y^i)\,D^T_\xi\varphi_T^\sharp(y^j)\,\otimes\Big(\frac{\partial}{\partial y^i}\frac{\partial}{\partial y^j}\Phi \,-\, \Big(\nabla^g_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j}\Big)\Phi\Big)\\ &\quad=\;(\nabla^{T,(\varphi_T,g)}_\xi\partial_t\varphi_T)\Phi \;-\; (\varphi_T^\diamond Tor^{\frac{1}{2},\Phi}_{\nabla^g})(\partial_t,\xi)\,. \end{aligned}$$

For the first identity,

$$\begin{aligned} \partial_t((D^T_\xi\varphi_T)\Phi) \;&=\; \partial_t\Big(\sum_i D^T_\xi\varphi_T^\sharp(y^i)\otimes\frac{\partial}{\partial y^i}\Phi\Big)\\ &=\;\sum_i\partial_tD^T_\xi\varphi_T^\sharp\otimes\frac{\partial}{\partial y^i}\Phi \;+\; \sum_{i,j}D^T_\xi\varphi_T^\sharp(y^i)\partial_t\varphi_T^\sharp(y^j)\otimes\frac{\partial}{\partial y^j}\frac{\partial}{\partial y^i}\Phi \end{aligned}$$

while

$$\begin{aligned} (\partial_tD^T_\xi\varphi_T)\Phi \;&=\; \Big(\partial_t\sum_i D^T_\xi\varphi_T^\sharp(y^i)\otimes\frac{\partial}{\partial y^i}\Big)\Phi\\ &=\;\Big(\sum_i\partial_tD^T_\xi\varphi_T^\sharp(y^i)\otimes\frac{\partial}{\partial y^i} \;+\; \sum_{i,j}D^T_\xi\varphi_T^\sharp(y^i)\partial_t\varphi_T^\sharp(y^j)\otimes\nabla^g_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^i}\Big)\Phi\,. \end{aligned}$$

Thus,

$$
\begin{aligned}
&\partial_t((D^T_\xi \varphi_T)\Phi) \;-\; (\partial_t D^T_\xi \varphi_T)\Phi \\
&\quad=\; \sum_{i,j} D^T_\xi \varphi_T^\sharp(y^i)\partial_t\varphi_T^\sharp(y^j)\otimes\Big(\frac{\partial}{\partial y^j}\frac{\partial}{\partial y^i} \;-\; \nabla^g_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^i}\Big)\Phi \\
&\quad=\; \sum_{i,j} D^T_\xi \varphi_T^\sharp(y^i)\partial_t\varphi_T^\sharp(y^j)\otimes\Big(\frac{\partial}{\partial y^i}\frac{\partial}{\partial y^j} \;-\; \nabla^g_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j}\Big)\Phi \;=\; -\,(\varphi_T^\diamond\, Tor^{\frac{1}{2},\Phi}_{\nabla^g})(\xi,\partial_t)
\end{aligned}
$$

and the first identity follows.

For the second identity,

$$
\begin{aligned}
D^T_\xi((\partial_t\varphi_T)\Phi) \;&=\; D^T_\xi\Big(\sum_i \partial_t\varphi_T^\sharp(y^i)\otimes\frac{\partial}{\partial y^i}\Phi\Big) \\
&=\; \sum_i D^T_\xi\partial_t\varphi_T^\sharp(y^i)\otimes\frac{\partial}{\partial y^i}\Phi \;+\; \sum_{i,j}\partial_t\varphi_T^\sharp(y^i)D^T_\xi\varphi_T^\sharp(y^j)\otimes\frac{\partial}{\partial y^j}\frac{\partial}{\partial y^i}\Phi
\end{aligned}
$$

while

$$
\begin{aligned}
(\nabla^{T,(\varphi_T,g)}_\xi\partial_t\varphi_T)\Phi \;&=\; \Big(\nabla^{T,(\varphi_T,g)}_\xi\sum_i\partial_t\varphi_T^\sharp(y^i)\otimes\frac{\partial}{\partial y^i}\Big)\Phi \\
&=\; \Big(\sum_i D^T_\xi\partial_t\varphi_T^\sharp(y^i)\otimes\frac{\partial}{\partial y^i} \;+\; \sum_{i,j}\partial_t\varphi_T^\sharp(y^i)D^T_\xi\varphi_T^\sharp(y^j)\otimes\nabla^g_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^i}\Big)\Phi\,.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&D^T_\xi((\partial_t\varphi_T)\Phi) \;-\; (\nabla^{T,(\varphi_T,g)}_\xi\partial_t\varphi_T)\Phi \\
&\quad=\; \sum_{i,j}\partial_t\varphi_T^\sharp(y^i)D^T_\xi\varphi_T^\sharp(y^j)\otimes\Big(\frac{\partial}{\partial y^j}\frac{\partial}{\partial y^i} \;-\; \nabla^g_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^i}\Big)\Phi \\
&\quad=\; \sum_{i,j}\partial_t\varphi_T^\sharp(y^i)D^T_\xi\varphi_T^\sharp(y^j)\otimes\Big(\frac{\partial}{\partial y^i}\frac{\partial}{\partial y^j} \;-\; \nabla^g_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j}\Big)\Phi \;=\; -\,(\varphi_T^\diamond\, Tor^{\frac{1}{2},\Phi}_{\nabla^g})(\partial_t,\xi)
\end{aligned}
$$

and the second identity follows.

This proves the lemma.

Remark 5.8 [for $(*_2)$-admissible family of $(*_1)$-admissible pairs] If (φ_T,∇^T) is furthermore a $(*_2)$-admissible T-family of $(*_1)$-admissible pairs, then, as in the proof of Lemma 5.4 (2), $D^T_\xi\varphi_T^\sharp(y^i)$ and $\partial_t\varphi_T^\sharp(y^j)$ commute for all i,j. In this case,

$$
(\varphi^\diamond\ Tor^{\frac{1}{2},\Phi}_{\nabla^g})(\xi,\partial_t) \;=\; (\varphi^\diamond\ Tor^{\frac{1}{2},\Phi}_{\nabla^g})(\partial_t,\xi)\,.
$$

Two-parameter admissible families of admissible pairs

Let $T=(-\varepsilon,\varepsilon)^2\subset R^2$, $\varepsilon>0$ small, be a two-parameter space with coordinates (s,t). The setting and results above for one-parameter admissible families of admissible pairs generalizes without work to two-parameter admissible of admissible pairs. In particular,

Definition 5.9 [two-parameter admissible family of admissible pairs] A $(*_2)$-*admissible T-family of* $(*_1)$-*admissible maps* is a $(*_1)$-admissible map $\varphi_T:(X_T^{Az},\mathcal{E}_T;\nabla^T)\to Y$, where $\mathcal{E}_T$ is trivially flat over T, such that $\partial_s Comm\,\mathcal{A}_{\varphi_T}\subset Comm(\mathcal{A}_{\varphi_T})$ and $\partial_t Comm\,\mathcal{A}_{\varphi_T}\subset Comm(\mathcal{A}_{\varphi_T})$.

The following is a consequence of the proof of Lemma 3.2.2.5:

Lemma 5.10 [symmetry property of $Tr\langle F_{\nabla^{T,(\varphi_T,g)}}(\partial_s,\xi_2)\partial_t\varphi_T, D^T_{\xi_4}\varphi_T\rangle$**]** *Let* $\varphi_T:(X_T^{Az},\mathcal{E}_T;\nabla^T)\to Y$ *be a* $(*_2)$*-admissible* T*-family of* $(*_1)$*-admissible maps. Let* $\xi_2,\xi_4\in\mathcal{T}_*X$ *and denote the same for their respective lifting to* $\mathcal{T}_*(X_T/T)$*. Then,*

$$
\begin{aligned}
Tr\langle F_{\nabla^{T,(\varphi_T,g)}}(\partial_s,\xi_2)\partial_t\varphi_T\,,\,D^T_{\xi_4}\varphi_T\rangle_g &= -\,Tr\langle\partial_t\varphi_T\,,\,F_{\nabla^{T,(\varphi_T,g)}}(\partial_s,\xi_2)D_{\xi_4}\varphi_T\rangle_g\\
&= -\,Tr\langle F_{\nabla^{T,(\varphi_T,g)}}(\partial_s,\xi_2)D^T_{\xi_4}\varphi_T\,,\,\partial_t\varphi_T\rangle_g \;=\; Tr\langle F_{\nabla^{T,(\varphi_T,g)}}(\xi_2,\partial_s)D^T_{\xi_4}\varphi_T\,,\,\partial_t\varphi_T\rangle_g\,.
\end{aligned}
$$

Let ξ be ξ_2 or ξ_4. Since $\partial_s\,Comm(\mathcal{A}_{\varphi_T})\subset Comm(\mathcal{A}_{\varphi_T})$ and and $\partial_\xi\mathcal{A}_{\varphi_T}\subset Comm(\mathcal{A}_{\varphi_T})$, both $\partial_sD^T_\xi\varphi_T$ and $\partial_s\partial_t\varphi_T$ lie in $Comm(\mathcal{A}_{\varphi_T})\otimes_{\varphi^\sharp,\mathcal{O}_Y}\mathcal{T}_*Y$. Locally explicitly,

$$
\begin{aligned}
\partial_sD^T_\xi\varphi_T &= \sum_i\partial_sD^T_\xi\varphi_T^\sharp(y^i)\otimes\frac{\partial}{\partial y^i}+\sum_{i,j}D^T_\xi\varphi_T^\sharp(y^i)\,\partial_s\varphi_T^\sharp(y^j)\otimes\nabla^g_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^i}\,;\\
\partial_s\partial_t\varphi_T &= \sum_i\partial_s\partial_t\varphi_T^\sharp(y^i)\otimes\frac{\partial}{\partial y^i}+\sum_{i,j}\partial_t\varphi_T^\sharp(y^i)\,\partial_s\varphi_T^\sharp(y^j)\otimes\nabla^g_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^i}\,.
\end{aligned}
$$

Now follow the proof of Lemma 3.2.2.5, but under only the $(*_1)$-Admissible Condition on (φ_T,∇^T), to convert $Tr\langle F_{\nabla^{T,(\varphi_T,g)}}(\partial_s,\xi_2)\partial_t\varphi_T\,,\,D^T_{\xi_4}\varphi_T\rangle_g$ to $Tr\langle\partial_t\varphi_T\,,\,F_{\nabla^{T,(\varphi_T,g)}}(\partial_s,\xi_2)D_{\xi_4}\varphi_T\rangle_g$. Since $Tr\langle -,-'\rangle_g$ is defined as long as one of $-$, $-'$ is in $Comm(\mathcal{A}_{\varphi_T})\otimes_{\varphi_T^\sharp,\mathcal{O}_Y}\mathcal{T}_*Y$, one realizes that all the terms that appear in the process via the Leibniz rule are defined *except*

$$
-\,Tr\langle\nabla^{T,(\varphi_T,g)}_{\xi_2}\partial_t\varphi_T\,,\,\partial_sD^T_{\xi_4}\varphi_T\rangle_g+Tr\langle\partial_s\partial_t\varphi_T\,,\,\nabla^{T,(\varphi_T,g)}_{\xi_2}D^T_{\xi_4}\varphi_T\rangle_g\,.
$$

Under the additional $(*_2)$-Admissible Condition on (φ_T,∇^T) along T, both $\partial_sD^T_{\xi_4}\varphi_T$ and $\partial_s\partial_t\varphi_T$ now lie in $Comm(\mathcal{A}_{\varphi_T})\otimes_{\varphi^\sharp,\mathcal{O}_Y}\mathcal{T}_*Y$; and the above two exceptional terms become defined.

The lemma follows.

6 The first variation of the enhanced kinetic term for maps and ......

Let (φ,∇) be a $(*_1)$-admissible pair. Recall the setup in Sec. 5. Let $T=(-\varepsilon,\varepsilon)\subset R^1$, for some $\varepsilon>0$ small, and (φ_T,∇^T) be a $(*_1)$-admissible T-family of $(*_1)$-admissible pairs that deforms $(\varphi,\nabla)=(\varphi_T,\nabla^T)|_{t=0}$. We derive in Sec. 6.1 and Sec. 6.2 the first variation formula of the newly introduced enhanced kinetic term for maps

$$
S^{(\rho,h;\Phi,g)}_{map:kinetic^+}(\varphi,\nabla)\;:=\;\frac12\,T_{m-1}\int_X Re\,Tr\langle D\varphi\,,\,D\varphi\rangle_{(h,g)}\,vol_h\;+\;\int_X Re\,Tr\langle d\rho,\varphi^\diamond d\Phi\rangle_h\,vol_h
$$

in the standard action for D-branes. As the '*taking the real part*' operation $Re(\cdots\cdots)$ is a $\mathcal{O}_X$-linear operation and can always be added back in the end, we will consider

$$
S^{(\rho,h;\Phi,g)}_{map:kinetic^+}(\varphi,\nabla)^C\;:=\;\frac12\,T_{m-1}\int_X Tr\langle D\varphi\,,\,D\varphi\rangle_{(h,g)}\,vol_h\;+\;\int_X Tr\langle d\rho,\varphi^\diamond d\Phi\rangle_h\,vol_h
$$

so that we don't have to carry Re around.

The first variation of the gauge/Yang-Mills term is analogous to that in the ordinary Yang-Mills theory and the first variation of the Chern-Simons/Wess-Zumino term is an update from [L-Y8: Sec. 6] (D(13.1)). Both are given in Sec. 6.3 under the stronger $(*_2)$-Admissible Condition.

6.1 The first variation of the kinetic term for maps

Recall the (complexified) kinetic energy $E^{\nabla^t}(\varphi_t)^C$ of φ_t for a given ∇^t, $t \in T := (-\varepsilon, \varepsilon)$,

$$E^{\nabla^t}(\varphi_t)^C \;:=\; S^{(h;g)}_{map:kinetic}(\varphi_t, \nabla^t)^C \;:=\; \frac{1}{2}\, T_{m-1}\int_X Tr\,\langle D^t\varphi_t\,,\, D^t\varphi^t\rangle_{(h,g)}\, vol_h\,.$$

As t varies, with a slight abuse of notation, denote the resulting function of t by

$$E^{\nabla^T}(\varphi_T)^C \;:=\; S^{(h;g)}_{map:kinetic}(\varphi_T, \nabla^T)^C \;:=\; \frac{1}{2}\, T_{m-1}\int_X Tr\,\langle D^T\varphi_T\,,\, D^T\varphi_T\rangle_{(h,g)}\, vol_h\,,$$

with the understanding that all expressions are taken on X_t with t varying in T.

Let $U \subset X$ be an open set with an orthonormal frame $(e_\mu)_{\mu=1,\cdots,m}$. Let $(e^\mu)_{\mu=1,\cdots,m}$ be the dual co-frame. Assume that U is small enough so that $\varphi_T(U_T^{Az})$ is contained in a coordinate chart of Y, with coordinates $(y^1, \cdots, y^n)$. Then, over U,

$$\begin{aligned}
\frac{d}{dt}E^{\nabla^T}(\varphi_T)^C &= \frac{1}{2}\, T_{m-1}\int_U \partial_t\, Tr\,\langle D^T\varphi_T\,,\, D^T\varphi_T\rangle_{(h,g)}\, vol_h\\
&= \frac{1}{2}\, T_{m-1}\int_U Tr\,\partial_t\langle D^T\varphi_T\,,\, D^T\varphi_T\rangle_{(h,g)}\, vol_h\\
&= \frac{1}{2}\, T_{m-1}\int_U Tr\,\partial_t\sum_{\mu=1}^m\langle D^T_{e_\mu}\varphi_T\,,\, D^T_{e_\mu}\varphi_T\rangle_g\, vol_h\\
&= T_{m-1}\int_U Tr\sum_{\mu=1}^m\langle \partial_t D^T_{e_\mu}\varphi_T\,,\, D^T_{e_\mu}\varphi_T\rangle_g\, vol_h\\
&= T_{m-1}\int_U Tr\sum_\mu\langle \nabla^{T,(\varphi_T,g)}_{e_\mu}\partial_t\varphi_T\,,\, D^T_{e_\mu}\varphi_T\rangle_g\, vol_h\\
&\qquad + T_{m-1}\int_U Tr\sum_\mu\langle (ad\otimes\nabla^g)_{D^T_{e_\mu}\varphi_T}\partial_t\varphi_T\,,\, D^T_{e_\mu}\varphi_T\rangle_g\, vol_h\\
&\qquad\qquad + T_{m-1}\int_U \sum_\mu\langle\sum_{i=1}^n[(\partial_t\nabla^T)(e_\mu), \varphi_T^\sharp(y^i)]\otimes\tfrac{\partial}{\partial y^i}\,,\, D^T_{e_\mu}\varphi_T\rangle_g\, vol_h\\
&= (\mathrm{I.1}) \;+\; (\mathrm{I.2}) \;+\; (\mathrm{I.3})\,.
\end{aligned}$$

$$\begin{aligned}
(\mathrm{I.1}) &= T_{m-1}\int_U \sum_\mu Tr\,\Big(D^T_{e_\mu}\langle\partial_t\varphi_T\,,\, D^T_{e_\mu}\varphi_T\rangle_g \;-\; \langle\partial_t\varphi_T\,,\, \nabla^{T,(\varphi_T,g)}_{e_\mu}D^T_{e_\mu}\varphi_T\rangle_g\Big)\, vol_h\\
&= T_{m-1}\int_U \sum_\mu e_\mu\, Tr\,\langle\partial_t\varphi_T\,,\, D^T_{e_\mu}\varphi_T\rangle_g\, vol_h \;+\; T_{m-1}\int_U Tr\,\langle\partial_t\varphi_T\,,\, -\textstyle\sum_\mu\nabla^{T,(\varphi_T,g)}_{e_\mu}D^T_{e_\mu}\varphi_T\rangle_g\, vol_h\\
&= (\mathrm{I.1.1}) \;+\; (\mathrm{I.1.2})\,.
\end{aligned}$$

Summand (I.1.1) suggests a boundary term. To really extract the boundary term from it, consider the T-family of C-valued 1-forms on U

$$\alpha^T_{(\mathrm{I},\,\partial_t\varphi_T)} \;:=\; Tr\,\langle\partial_t\varphi_T\,,\, D^T\varphi_T\rangle_g\,,$$

which depends $C^\infty(U)^C$-linearly on $\partial_t\varphi_T$. Let

$$\xi^T_{(\mathrm{I},\,\partial_t\varphi_T)} \;:=\; \sum_{\mu=1}^m\Big(Tr\,\langle\partial_t\varphi_T\,,\, D^T_{e_\mu}\varphi_T\rangle_g\Big)\, e_\mu$$

be the T-family of dual C-valued vector fields of $\alpha^T_{(\mathrm{I},\,\partial_t\varphi_T)}$ on U with respect to the metric h. Note that $\xi^T_{(\mathrm{I},\,\partial_t\varphi_T)}$ depends $C^\infty(U)^C$-linearly on $\partial_t\varphi_T$ as well. Then

$$\begin{aligned}
(\mathrm{I}.1.1) \;&=\; T_{m-1}\int_U \sum_\mu e_\mu\langle \xi^T_{(\mathrm{I},\,\partial_t\varphi_T)}\,,\, e_\mu\rangle_h\, vol_h \\
&=\; T_{m-1}\int_U \sum_\mu \langle \nabla^h_{e_\mu}\xi^T_{(\mathrm{I},\,\partial_t\varphi_T)}\,,\, e_\mu\rangle_h\, vol_h \;+\; T_{m-1}\int_U \langle \xi^T_{(\mathrm{I},\,\partial_t\varphi_T)}\,,\, \textstyle\sum_\mu \nabla^h_{e_\mu}e_\mu\rangle_h\, vol_h\,.
\end{aligned}$$

The first term is equal to

$$T_{m-1}\int_U (-\,div\,\xi^T_{(\mathrm{I},\,\partial_t\varphi_T)})\,vol_h \;=\; T_{m-1}\int_U d\,i_{\xi^T_{(\mathrm{I},\,\partial_t\varphi_T)}}vol_h \;=\; T_{m-1}\int_{\partial U} i_{\xi^T_{(\mathrm{I},\,\partial_t\varphi_T)}}vol_h\,,$$

which is the sought-for boundary term, whose integrand satisfies the requirement that it be $C^\infty(U)^C$-linear on $\partial_t\varphi_T$. The second term is equal to

$$T_{m-1}\int_U Tr\,\langle \partial_t\varphi_T\,,\, D^T_{\sum_\mu \nabla^h_{e_\mu}e_\mu}\varphi_T\rangle_g\, vol_h$$

by construction, which is $C^\infty(U)^C$-linear in $\partial_t\varphi_T$ and hence in a final form.

The integrand of Summand (I.1.2) is already $C^\infty(U)^C$-linear in $\partial_t\varphi_T$ and hence in a final form.

Summand (I.2) can be re-written as

$$(\mathrm{I}.2) \;=\; -\,T_{m-1}\int_U Tr\sum_\mu \langle (ad\otimes\nabla^g)_{\partial_t\varphi_T}D^T_{e_\mu}\varphi_T\,,\, D^T_{e_\mu}\varphi_T\rangle_g\, vol_h\,.$$

Thus, its integrand is already $C^\infty(U)^C$-linear in $\partial_t\varphi_T$ and hence in a final form.

Finally, since the built-in inclusion $\mathcal{O}_U^C\subset\mathcal{O}_U^{Az}$ identifies $\mathcal{O}_U^C$ with the center of $\mathcal{O}_U^{Az}$, Summand (I.3) is $C^\infty(U)^C$-linear and hence in its final fom.

Altogether, we almost complete the calculation except the issue of whether all the inner products $Tr\langle\,\bullet\,,\,\bullet\,\rangle_g$ that appear in the procedure are truly defined. For this, one notices that wherever such an inner product appears above, at least one of its arguments is either $\partial_t\varphi_T$ or $D^T_{e_\mu}\varphi_T$, for some μ. It follows from Lemma 3.2.2.4 that they are indeed defined.

In summary,

Proposition 6.1.1 [first variation of kinetic term for maps] *Let (φ_T,∇^T) be a $(*_1)$-admissible T-family of $(*_1)$-admissible pairs. Then,*

$$\begin{aligned}
\frac{d}{dt}E^{\nabla^T}(\varphi_T)^C \;&=\; \frac{d}{dt}\left(\frac{1}{2}\,T_{m-1}\int_U Tr\,\langle D^T\varphi_T\,,\, D^T\varphi_T\rangle_{(h,g)}\, vol_h\right) \\
&=\; T_{m-1}\int_{\partial U} i_{\xi^T_{(\mathrm{I},\,\partial_t\varphi_T)}}vol_h \\
&\quad +\; T_{m-1}\int_U Tr\,\langle \partial_t\varphi_T\,,\, (D^T_{\sum_{\mu=1}^m\nabla^h_{e_\mu}e_\mu} - \textstyle\sum_{\mu=1}^m \nabla^{T,(\varphi_T,g)}_{e_\mu}D^T_{e_\mu})\,\varphi_T\rangle_g\, vol_h \\
&\qquad -\; T_{m-1}\int_U Tr\sum_{\mu=1}^m \langle (ad\otimes\nabla^g)_{\partial_t\varphi_T}D^T_{e_\mu}\varphi_T\,,\, D^T_{e_\mu}\varphi_T\rangle_g\, vol_h \\
&\quad +\; T_{m-1}\int_U \sum_{\mu=1}^m \langle \sum_{i=1}^n [(\partial_t\nabla^T)(e_\mu), \varphi_T^\sharp(y^i)]\otimes\frac{\partial}{\partial y^i}\,,\, D^T_{e_\mu}\varphi_T\rangle_g\, vol_h\,.
\end{aligned}$$

Here, the first summand is the boundary term with $\xi^T_{(\mathrm{I},\,\partial_t\varphi_T)} := \sum_{\mu=1}^m (Tr\,\langle\partial_t\varphi_T, D^T_{e_\mu}\varphi_T\rangle_g)e_\mu$ $C^\infty(U)^C$-linear in $\partial_t\varphi_T$; the integrand of the second and the third terms are $C^\infty(U)^C$-linear in

$\partial_t\varphi_T$ and their real part contribute first-order and second-order terms to the equations of motion for (φ,∇); the integrand of the last term is $C^\infty(U)^C$-linear in $\partial_t\nabla^T$ and its real part contributes terms, first order in φ but zeroth order in the connection 1-from of ∇, to the equations of motion for (φ,∇) in addition to those from the first variation of the rest part of $S_{standard}^{(\rho,h;\Phi,g,B,C)}(\varphi,\nabla)$. These lower-order terms contribute to the equations of motion for (φ,∇) but do not change the signature of the system.

Remark 6.1.2 [for $(*_2)$-admissible T-family of $(*_2)$-admissible pairs] If furtheremore (φ,∇) is $(*_2)$-admissible and (φ_T,∇^T) is a $(*_2)$-admissible T-family of $(*_2)$-admissible pairs that deforms (φ,∇), then the third summand of the first variation formula in Proposition 6.1.1 vanishes and the fourth/last summand has a Y-coordinate-free form

$$T_{m-1}\int_U \sum_{\mu=1}^m \langle ad_{(\partial_t\nabla^T)(e_\mu)}\varphi_T\,,\, D^T_{e_\mu}\varphi_T\rangle_g\, vol_h\,.$$

In this case, the first variation with respect to φ alone (i.e. setting $\partial_t\nabla^T=0$), cf. the first two summands, takes the form of a direct formal generalization of the first variation formula in the study of harmonic maps; e.g. [E-L], [E-S], [Ma], [Sm].

6.2 The first variation of the dilaton term

We now turn to the (complexified) dilaton term in $S_{standard}^{(\rho,h;\Phi,g,B,C)}(\varphi,\nabla)^C$.

Let $\varphi_T:(X^{Az},\mathcal{E}_T;\nabla^T)\to Y$ be a $(*_1)$-admissible T-family of $(*_1)$-admissible pairs. Then, over an open set $U\subset X$,

$$\begin{aligned}
S_{dilaton}^{(\rho,h;\Phi)}(\varphi_T)^C &= \int_U Tr\langle d\rho\,,\,\varphi_T^\diamond d\Phi\rangle_h\, vol_h\\
&= \int_U Tr\sum_{\mu=1}^m\Big(d\rho(e_\mu)\sum_{i=1}^n D^T_{e_\mu}\varphi_T^\sharp(y^i)\,\varphi_T^\sharp\big(\frac{\partial\Phi}{\partial y^i}\big)\Big)\, vol_h\\
&= \int_U Tr\Big(\sum_\mu d\rho(e_\mu)\,((D^T_{e_\mu}\varphi_T)\Phi)\Big)\, vol_h\,.
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}S_{dilaton}^{(\rho,h;\Phi)}(\varphi_T)^C &= \int_U Tr\sum_{\mu=1}^m d\rho(e_\mu)\,\partial_t\big((D^T_{e_\mu}\varphi_T)\Phi\big)\, vol_h\\
&= \int_U Tr\sum_\mu d\rho(e_\mu)\,\big((\partial_t D^T_{e_\mu}\varphi_T)\Phi\big)\, vol_h\\
&\quad + \int_U Tr\sum_\mu d\rho(e_\mu)\sum_{i,j=1}^n D^T_{e_\mu}\varphi_T^\sharp(y^i)\partial_t\varphi_T^\sharp(y^j)\otimes\Big(\tfrac{\partial}{\partial y^j}\tfrac{\partial}{\partial y^i}\Phi-\big(\nabla^g_{\frac{\partial}{\partial y^j}}\tfrac{\partial}{\partial y^i}\big)\Phi\Big)\, vol_h\\
&= \text{(II.1)} + \text{(II.2)}\,.
\end{aligned}$$

The integrand of Summand (II.2) is $C^\infty(U)^C$-linear in $\partial_t\varphi_T$ and hence in a final form.

$$\begin{aligned}
\text{(II.1)} &= \int_U Tr\sum_\mu d\rho(e_\mu)\,\big((\nabla^{T,(\varphi_T,g)}_{e_\mu}\partial_t\varphi_T)\Phi\big)\, vol_h\\
&\quad - \int_U Tr\sum_\mu d\rho(e_\mu)\,\big(((ad\otimes\nabla^g)_{\partial_t\varphi_T}D^T_{e_\mu}\varphi_T)\Phi\big)\, vol_h\\
&\quad + \int_U Tr\sum_\mu d\rho(e_\mu)\,\Big(\Big(\sum_i[(\partial_t\nabla^T)(e_\mu)\,,\,\varphi_T^\sharp(y^i)]\otimes\tfrac{\partial}{\partial y^i}\Big)\Phi\Big)\, vol_h\\
&= \text{(II.1.1)} + \text{(II.1.2)} + \text{(II.1.3)}\,.
\end{aligned}$$

Both Summand (II.1.2) and Summand (II.1.3) vanish since

$$Tr([a,b]c) \;=\; 0 \qquad \text{if}\;\; [b,c] \;=\; 0$$

for $r \times r$ matrices a, b, c.

$$\begin{aligned}
\text{(II.1.1)} \;&=\; \int_U Tr \sum_\mu d\rho(e_\mu)\, D^T_{e_\mu}((\partial_t\varphi_T)\Phi))\, vol_h \\
&\quad - \int_U Tr \sum_\mu d\rho(e_\mu) \sum_{i,j=1}^n \partial_t\varphi_T^\sharp(y^i)\, D^T_{e_\mu}\varphi_T^\sharp(y^j) \otimes \left(\frac{\partial}{\partial y^i}\frac{\partial}{\partial y^j}\Phi \;-\; \left(\nabla^g_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j}\right)\Phi \right) vol_h \\
&=\; \text{(II.1.1.1)} \;+\; \text{(II.1.1.2)}\,.
\end{aligned}$$

The integrand of Summand (II.1.1.2) is $C^\infty(U)^C$-linear in $\partial_t\varphi_T$ and hence in a final form. It can be combined with Summand (II.2) to give

$$\begin{aligned}
&\text{(II.1.1.2)} \;+\; \text{(II.2)} \\
&\quad=\; -\int_U Tr \sum_\mu d\rho(e_\mu) \sum_{i,j=1}^n [\partial_t\varphi_T^\sharp(y^i)\,,\, D^T_{e_\mu}\varphi_T^\sharp(y^j)] \otimes \left(\frac{\partial}{\partial y^i}\frac{\partial}{\partial y^j}\Phi \;-\; \left(\nabla^g_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j}\right)\Phi \right) vol_h\,,
\end{aligned}$$

which again vanishes due to Tr.

$$\begin{aligned}
\text{(II.1.1.1)} \;&=\; \int_U \sum_\mu d\rho(e_\mu)\, Tr\, D^T_{e_\mu}((\partial_t\varphi_T)\Phi)\, vol_h \\
&=\; \int_U \sum_\mu d\rho(e_\mu)\, e_\mu\, Tr((\partial_t\varphi_T)\Phi)\, vol_h \\
&=\; \int_U \sum_\mu e_\mu \left(d\rho(e_\mu)\; Tr((\partial_t\varphi_T)\Phi) \right) vol_h \;-\; \int_U \left(\sum_\mu e_\mu d\rho(e_\mu) \right) Tr((\partial_t\varphi_T)\Phi)\, vol_h \\
&=\; \text{(II.1.1.1.1)} \;+\; \text{(II.1.1.1.2)}
\end{aligned}$$

The integrand of Summand (II.1.1.1.2) is $C^\infty(U)^C$-linear in $\partial_t\varphi_T$ and hence in a final form. To extract the boundary term from Summand (II.1.1.1.1), consider the T-family of C-valued 1-forms on U

$$\alpha^T_{(\text{II},\partial_t\varphi_T)} \;:=\; d\rho\; Tr((\partial_t\varphi_T)\Phi)\,,$$

which depends $C^\infty(U)^C$-linearly on $\partial_t\varphi_T$. Let

$$\xi^T_{(\text{II},\partial_t\varphi_T)} \;=\; \sum_{\mu=1}^m \left(d\rho(e_\mu)\; Tr((\partial_t\varphi_T)\Phi) \right) e_\mu$$

be the T-family of dual C-valued vector fields of $\alpha^T_{(\text{II},\partial_t\varphi_T)}$ on U with respect to the metric h. Note that $\xi^T_{(\text{II},\partial_t\varphi_T)}$ depends $C^\infty(U)^C$-linearly on $\partial_t\varphi_T$ as well. Then

$$\begin{aligned}
\text{(II.1.1.1.1)} \;&=\; \int_U \sum_\mu e_\mu \langle \xi^T_{(\text{II},\partial_t\varphi_T)}\,,\, e_\mu\rangle_h\, vol_h \\
&=\; \int_U \sum_\mu \langle \nabla^h_{e_\mu}\xi^T_{(\text{II},\partial_t\varphi_T)}\,,\, e_\mu\rangle_h\, vol_h \;+\; \int_U \langle \xi^T_{(\text{II},\partial_t\varphi_T)}\,,\, \textstyle\sum_\mu \nabla^h_{e_\mu} e_\mu\rangle_h\, vol_h
\end{aligned}$$

The first term is equal to

$$\int_U (-\,div\; \xi^T_{(\text{II},\,\partial_t\varphi_T)})\, vol_h \;=\; \int_U d\, i_{\xi^T_{(\text{II},\,\partial_t\varphi_T)}} vol_h \;=\; \int_{\partial U} i_{\xi^T_{(\text{II},\,\partial_t\varphi_T)}} vol_h\,,$$

which is the sought-for boundary term, whose integrand satisfies the requirement that it be $C^\infty(U)^C$-linear in $\partial_t\varphi_T$. The second term is equal to

$$\int_U d\rho(\sum_\mu \nabla^h_{e_\mu} e_\mu)\; Tr((\partial_t\varphi_T)\Phi)\, vol_h$$

by construction, which is $C^\infty(U)^C$-linear in $\partial_t\varphi_T$ and hence in a final form.

In summary,

Proposition 6.2.1 [first variation of dilaton term] *Let (φ_T, ∇^T) be a $(*_1)$-admissible T-family of $(*_1)$-admissible pairs. Then,*

$$\begin{aligned}
\frac{d}{dt} S^{(\rho,h;\Phi)}_{dilaton}(\varphi_T)^C \;&=\; \frac{d}{dt}\int_U Tr\langle d\rho\,,\,\varphi_T^\diamond d\Phi\rangle_h\, vol_h \\
&=\; \int_{\partial U} i_{\xi^T_{(\mathrm{II},\,\partial_t\varphi_T)}} vol_h \\
&\quad +\; \int_U \Big(d\rho(\textstyle\sum_{\mu=1}^m \nabla^h_{e_\mu} e_\mu) \,-\, \sum_{\mu=1}^m e_\mu d\rho(e_\mu)\Big)\, Tr((\partial_t\varphi_T)\Phi)\, vol_h\,.
\end{aligned}$$

Here, the first summand is the boundary term with $\xi^T_{(\mathrm{II},\partial_t\varphi_T)} := \sum_{\mu=1}^m (d\rho(e_\mu)\; Tr((\partial_t\varphi_T)\Phi))e_\mu$ $C^\infty(U)^C$-linear in $\partial_t\varphi_T$; the integrand of the second summand $C^\infty(U)^C$-linear in $\partial_t\varphi_T$ and they contribute additional zeroth-order terms to the equations of motion for (φ,∇). In particular, while the dilaton term of the standard action modifies the equations of motion for (φ,∇), it does not change the signature of the system.

6.3 The first variation of the gauge/Yang-Mills term and the Chern-Simons/ Wess-Zumino term

To make sure that differential forms on Y of rank ≥ 2 are pull-pushed to ($\mathcal{O}_X^{Az}$-valued-)differential forms on X (cf. Lemma 2.1.11), we assume in this subsection that $\varphi_T : (X_T^{Az}, \mathcal{E}_T; \nabla^T) \to Y$ is a $(*_2)$-family of $(*_2)$-admissible maps. (Note that as the gauge/Yang-Mills term is defined through a norm-squared, $(*_1)$-admissible family of $(*_1)$-admissible (φ_T, ∇^T) is enough for the derivation of the first variation formula of the gauge/Yang-Mills term but the result will be slightly messier.)

6.3.1 The first variation of the gauge/Yang-Mills term

Let $(e_1, \cdots, e_m)$ be an orthonormal frame on U. Then, over U,

$$\begin{aligned}
S^{(h;B)}_{gauge/YM}(\varphi_T, \nabla^T)^C \;&:=\; -\frac{1}{2}\int_U Tr\|2\pi\alpha' F_{\nabla^T} + \varphi_T^\diamond B\|_h^2\, vol_h \\
&=\; -\frac{1}{2}\int_U Tr\sum_{\mu,\nu}\Big((2\pi\alpha' F_{\nabla^T} + \varphi_T^\diamond B)(e_\mu, e_\nu)\Big)^2 vol_h\,.
\end{aligned}$$

Applying the following basic identities:

$$\begin{aligned}
\partial_t F_{\nabla^T}(e_\mu, e_\nu) \;&=\; D^T_{e_\mu}((\partial_t\nabla^T)(e_\nu)) \,-\, D^T_{e_\nu}((\partial_t\nabla^T)(e_\mu)) \,-\, (\partial_t\nabla^T)([e_\mu, e_\nu])\,, \\
\partial_t((\varphi_T^\diamond B)(e_\mu, e_\nu)) \;&=\; \sum_{i,j}\partial_t(\varphi_T^\sharp(B_{ij}))D^T_{e_\mu}\varphi_T^\sharp(y^i)D^T_{e_\nu}\varphi_T^\sharp(y^j) \\
&\quad +\sum_{i,j}\varphi_T^\sharp(B_{ij})\Big(D^T_{e_\mu}\partial_t\varphi_T^\sharp(y^i) + [(\partial_t\nabla^T)(e_\mu), \varphi_T^\sharp(y^i)]\Big)D^T_{e_\nu}\varphi_T^\sharp(y^j) \\
&\quad +\sum_{i,j}\varphi_T^\sharp(B_{ij})D^T_{e_\mu}\varphi_T^\sharp(y^i)\Big(D^T_{e_\nu}\partial_t\varphi_T^\sharp(y^j) + [(\partial_t\nabla^T)(e_\nu), \varphi_T^\sharp(y^j)]\Big)\,.
\end{aligned}$$

and proceeding similarly to Sec. 6.1, one has the following results.

$$
\begin{aligned}
\frac{d}{dt} S^{(h;B)}_{\mathit{gauge}\,/\,YM}(\varphi_T, \nabla^T)^C \;&=\; -\frac{1}{2}\int_U Tr\,\partial_t \sum_{\mu,\nu} \Big((2\pi\alpha' F_{\nabla^T} + \varphi_T^\diamond B)(e_\mu, e_\nu)\Big)^2 vol_h \\
&=\; -\int_U Tr \sum_{\mu,\nu} \partial_t\Big((2\pi\alpha' F_{\nabla^T} + \varphi_T^\diamond B)(e_\mu, e_\nu)\Big)\cdot\Big((2\pi\alpha' F_{\nabla^T} + \varphi_T^\diamond B)(e_\mu, e_\nu)\Big) vol_h \\
&=\; -\int_U Tr \sum_{\mu,\nu} 2\pi\alpha' \partial_t(F_{\nabla^T}(e_\mu, e_\nu))\cdot\Big((2\pi\alpha' F_{\nabla^T} + \varphi_T^\diamond B)(e_\mu, e_\nu)\Big) vol_h \\
&\qquad -\int_U Tr \sum_{\mu,\nu} \partial_t(\varphi_T^\diamond B(e_\mu, e_\nu))\cdot\Big((2\pi\alpha' F_{\nabla^T} + \varphi_T^\diamond B)(e_\mu, e_\nu)\Big) vol_h \\
&=\; (\mathrm{III.1}) \;+\; (\mathrm{III.2})\,.
\end{aligned}
$$

$$
\begin{aligned}
(\mathrm{III.1}) \;:=&\; -\int_U Tr \sum_{\mu,\nu} 2\pi\alpha' \partial_t(F_{\nabla^T}(e_\mu, e_\nu))\cdot\Big((2\pi\alpha' F_{\nabla^T} + \varphi_T^\diamond B)(e_\mu, e_\nu)\Big) vol_h \\
=&\; -2\pi\alpha' \int_U Tr \sum_{\mu,\nu} \Big(D^T_{e_\mu}((\partial_t\nabla^T)(e_\nu)) \;-\; D^T_{e_\nu}((\partial_t\nabla^T)(e_\mu)) \;-\; (\partial_t\nabla^T)([e_\mu, e_\nu])\Big) \\
&\qquad\qquad \cdot\Big((2\pi\alpha' F_{\nabla^T} + \varphi_T^\diamond B)(e_\mu, e_\nu)\Big) vol_h \\
=&\; -4\pi\alpha' \int_{\partial U} i_{\xi^T_{(\mathrm{III}, \partial_t\nabla^T)}}\, vol_h \\
&\; -4\pi\alpha' \int_U Tr \sum_\nu (\partial_t\nabla^T)(e_\nu)\cdot\Big(\, (2\pi\alpha' F_{\nabla^T} + \varphi_T^\diamond B)(\textstyle\sum_\mu \nabla^h_{e_\mu} e_\mu, e_\nu) \\
&\qquad\qquad - \displaystyle\sum_\mu D^T_{e_\mu}((2\pi\alpha' F_{\nabla^T} + \varphi_T^\diamond B)(e_\mu, e_\nu)) \\
&\qquad\qquad - \frac{1}{2}\sum_{\mu,\lambda} e^\nu([e_\mu, e_\lambda])(2\pi\alpha' F_{\nabla^T} + \varphi_T^\diamond B)(e_\mu, e_\lambda)\,\Big) vol_h\,.
\end{aligned}
$$

Here,

$$
\xi^T_{(\mathrm{III}, \partial_t\nabla^T)} \;:=\; \sum_{\mu,\nu} Tr\Big((\partial_t\nabla^T)(e_\nu)\cdot(2\pi\alpha' F_{\nabla^T} + \varphi_T^\diamond B)(e_\mu, e_\nu)\Big)\, e_\mu \qquad \in\; \mathcal{T}_*(U_T/T)^C
$$

is $\mathcal{O}_U^C$-linear in $\partial_t\nabla^T$; and the second summand contributes to the equations of motion for (φ, ∇). The latter are standard terms from non-Abelian Yang-Mills theory with additional terms from $\varphi^\diamond B$.

$$
\begin{aligned}
(\mathrm{III.2}) \;:=&\; -\int_U Tr \sum_{\mu,\nu} \partial_t(\varphi_T^\diamond B(e_\mu, e_\nu))\cdot\Big((2\pi\alpha' F_{\nabla^T} + \varphi_T^\diamond B)(e_\mu, e_\nu)\Big) vol_h \\
=&\; -\int_U Tr \sum_{\mu,\nu} \Big(\sum_{i,j} \partial_t(\varphi_T^\sharp(B_{ij})) D^T_{e_\mu}\varphi_T^\sharp(y^i) D^T_{e_\nu}\varphi_T^\sharp(y^j) \\
&\qquad + \sum_{i,j} \varphi_T^\sharp(B_{ij})\Big(D^T_{e_\mu}\partial_t\varphi_T^\sharp(y^i) + [(\partial_t\nabla^T)(e_\mu), \varphi_T^\sharp(y^i)]\Big) D^T_{e_\nu}\varphi_T^\sharp(y^j) \\
&\qquad + \sum_{i,j} \varphi_T^\sharp(B_{ij}) D^T_{e_\mu}\varphi_T^\sharp(y^i)\Big(D^T_{e_\nu}\partial_t\varphi_T^\sharp(y^j) + [(\partial_t\nabla^T)(e_\nu), \varphi_T^\sharp(y^j)]\Big)\Big) \\
&\qquad \cdot\Big((2\pi\alpha' F_{\nabla^T} + \varphi_T^\diamond B)(e_\mu, e_\nu)\Big) vol_h \\
=&\; (\mathrm{III.2.1}) \;++\; ((\mathrm{III.2.2.1}) \;+\; (\mathrm{III.2.2.2})) \;+\; (\mathrm{III.2.3.1}) \;+\; (\mathrm{III.2.3.2})
\end{aligned}
$$

in the order of the appearance of the five summands after the expansion.

$$
(\mathrm{III.2.1}) \;:=\; -\int_U Tr \sum_{\mu,\nu}\sum_{i,j} \partial_t(\varphi_T^\sharp(B_{ij})) D^T_{e_\mu}\varphi_T^\sharp(y^i) D^T_{e_\nu}\varphi_T^\sharp(y^j)
$$

$$\cdot\Big((2\pi\alpha' F_{\nabla^T} + \varphi_T^\diamond B)(e_\mu, e_\nu)\Big) vol_h$$

$$:= \; - \int_U Tr \sum_{\mu,\nu} \sum_{i,j} ((\partial_t \varphi_T) B_{ij}) D^T_{e_\mu} \varphi_T^\sharp (y^i) D^T_{e_\nu} \varphi_T^\sharp (y^j) \cdot\Big((2\pi\alpha' F_{\nabla^T} + \varphi_T^\diamond B)(e_\mu, e_\nu)\Big) vol_h$$

has an integrand $\mathcal{O}_U^C$-linear in $\partial_t \varphi_T$ and hence in a final form.

$$\begin{aligned}
&(\mathrm{III}.2.2.1) + (\mathrm{III}.2.3.1) \\
&:= -\int_U Tr \sum_{\mu,\nu} \Big(\sum_{i,j} \varphi_T^\sharp(B_{ij}) D^T_{e_\mu} \partial_t \varphi_T^\sharp(y^i) D^T_{e_\nu} \varphi_T^\sharp (y^j) \\
&\qquad\qquad + \sum_{i,j} \varphi_T^\sharp(B_{ij}) D^T_{e_\mu} \varphi_T^\sharp(y^i) D^T_{e_\nu} \partial_t \varphi_T^\sharp (y^j) \Big) \\
&\qquad\qquad \cdot\Big((2\pi\alpha' F_{\nabla^T} + \varphi_T^\diamond B)(e_\mu, e_\nu)\Big) vol_h \\
&= -2 \int_U Tr \sum_{\mu,\nu} \sum_{i,j} D^T_{e_\mu} \partial_t \varphi_T^\sharp (y^i)\, \varphi_T^\sharp(B_{ij})\, D^T_{e_\nu} \varphi_T^\sharp(y^j) \\
&\qquad\qquad \cdot((2\pi\alpha' F_{\nabla^T} + \varphi_T^\diamond B)(e_\mu, e_\nu)) vol_h \\
&= -2 \int_{\partial U} i_{\xi^T_{(\mathrm{III},\partial_t\varphi_T)}} vol_h \\
&\quad -2 \int_U Tr \sum_\nu \sum_{i,j} \partial_t \varphi_T^\sharp (y^i) \\
&\qquad \Big(\varphi_T^\sharp(B_{ij})\, D^T_{e_\nu} \varphi_T^\sharp(y^j) \cdot ((2\pi\alpha' F_{\nabla^T} + \varphi_T^\diamond B)(\textstyle\sum_\mu \nabla^h_{e_\mu} e_\mu, e_\nu)) \\
&\qquad\quad - \sum_\mu D^T_{e_\mu} \Big(\varphi_T^\sharp(B_{ij})\, D^T_{e_\nu} \varphi_T^\sharp(y^j) \cdot ((2\pi\alpha' F_{\nabla^T} + \varphi_T^\diamond B)(e_\mu, e_\nu)) \Big) \Big) vol_h \,.
\end{aligned}$$

Here,

$$\xi^T_{(\mathrm{III},\partial_t\varphi_T)} := \sum_\mu \Big(\sum_\nu \sum_{i,j} \partial_t \varphi_T^\sharp(y^i)\, \varphi_T^\sharp(B_{ij})\, D^T_{e_\nu} \varphi_T^\sharp(y^j) \cdot ((2\pi\alpha' F_{\nabla^T} + \varphi_T^\diamond B)(e_\mu, e_\nu)) \Big) e_\mu$$

in $\mathcal{T}_*(U_T/T)^C$ is $\mathcal{O}_U^C$-linear in $\partial_t \varphi_T$; and the second summand contributes to $\delta S^{(\rho,h;\Phi,g,B,C)}_{standard}(\varphi,\nabla)/\delta\varphi$-part of the equations of motion for (φ, ∇).

$$\begin{aligned}
&(\mathrm{III}.2.2.2) + (\mathrm{III}.2.3.2) \\
&:= -\int_U Tr \sum_{\mu,\nu} \Big(\sum_{i,j} \varphi_T^\sharp(B_{ij}) [(\partial_t \nabla^T)(e_\mu), \varphi_T^\sharp(y^i)] D^T_{e_\nu} \varphi_T^\sharp(y^j) \\
&\qquad\qquad + \sum_{i,j} \varphi_T^\sharp(B_{ij}) D^T_{e_\mu} \varphi_T^\sharp(y^i) [(\partial_t \nabla^T)(e_\nu), \varphi_T^\sharp(y^j)] \Big) \\
&\qquad\qquad \cdot\Big((2\pi\alpha' F_{\nabla^T} + \varphi_T^\diamond B)(e_\mu, e_\nu)\Big) vol_h \,.
\end{aligned}$$

has an integrand $\mathcal{O}_U^C$-linear in $\partial_t \nabla^T$ and hence in a final form.

In summary,

Proposition 6.3.1.1 [first variation of gauge/Yang-Mills term] *Let (φ_T, ∇^T) be a $(*_2)$-admissible family of $(*_2)$-admissible pairs. Then*

$$\begin{aligned}
&\frac{d}{dt} S^{(h;B)}_{gauge/YM}(\varphi_T, \nabla^T)^C \;=\; -\frac{1}{2} \frac{d}{dt} \int_U Tr \| 2\pi\alpha' F_{\nabla^T} + \varphi_T^\diamond B \|_h^2 \, vol_h \\
&= -4\pi\alpha' \int_{\partial U} i_{\xi^T_{(\mathrm{III},\partial_t\nabla^T)}} vol_h - 2 \int_{\partial U} i_{\xi^T_{(\mathrm{III},\partial_t\varphi_T)}} vol_h \\
&\quad - 4\pi\alpha' \int_U Tr \sum_\nu (\partial_t \nabla^T)(e_\nu) \cdot \Big((2\pi\alpha' F_{\nabla^T} + \varphi_T^\diamond B)(\textstyle\sum_\mu \nabla^h_{e_\mu} e_\mu, e_\nu)
\end{aligned}$$

$$
\begin{aligned}
&\qquad - \sum_{\mu} D^T_{e_\mu}((2\pi\alpha' F_{\nabla^T} + \varphi_T^\diamond B)(e_\mu, e_\nu)) \\
&\qquad - \frac{1}{2}\sum_{\mu,\lambda} e^\nu([e_\mu, e_\lambda])(2\pi\alpha' F_{\nabla^T} + \varphi_T^\diamond B)(e_\mu, e_\lambda)\Big) vol_h\,. \\
&- \int_U Tr \sum_{\mu,\nu}\Big(\sum_{i,j}\varphi_T^\sharp(B_{ij})[(\partial_t\nabla^T)(e_\mu), \varphi_T^\sharp(y^i)]D^T_{e_\nu}\varphi_T^\sharp(y^j) \\
&\qquad + \sum_{i,j}\varphi_T^\sharp(B_{ij})D^T_{e_\mu}\varphi_T^\sharp(y^i)[(\partial_t\nabla^T)(e_\nu), \varphi_T^\sharp(y^j)]\Big) \\
&\qquad \cdot\Big((2\pi\alpha' F_{\nabla^T} + \varphi_T^\diamond B)(e_\mu, e_\nu)\Big) vol_h \\
&- \int_U Tr \sum_{\mu,\nu}\sum_{i,j}((\partial_t\varphi_T)B_{ij})D^T_{e_\mu}\varphi_T^\sharp(y^i)D^T_{e_\nu}\varphi_T^\sharp(y^j) \\
&\qquad \cdot\Big((2\pi\alpha' F_{\nabla^T} + \varphi_T^\diamond B)(e_\mu, e_\nu)\Big) vol_h \\
&- 2\int_U Tr \sum_{\nu}\sum_{i,j}\partial_t\varphi_T^\sharp(y^i) \\
&\qquad \Big(\varphi_T^\sharp(B_{ij})\, D^T_{e_\nu}\varphi_T^\sharp(y^j)\cdot((2\pi\alpha' F_{\nabla^T} + \varphi_T^\diamond B)(\textstyle\sum_\mu \nabla^h_{e_\mu}e_\mu, e_\nu)) \\
&\qquad - \sum_{\mu} D^T_{e_\mu}\Big(\varphi_T^\sharp(B_{ij})\, D^T_{e_\nu}\varphi_T^\sharp(y^j)\cdot((2\pi\alpha' F_{\nabla^T} + \varphi_T^\diamond B)(e_\mu, e_\nu))\Big)\Big) vol_h\,.
\end{aligned}
$$

Here,

$$
\begin{aligned}
\xi^T_{(\mathrm{III},\partial_t\nabla^T)} &:= \sum_{\mu,\nu} Tr\Big((\partial_t\nabla^T)(e_\nu)\cdot(2\pi\alpha' F_{\nabla^T} + \varphi_T^\diamond B)(e_\mu, e_\nu)\Big)\, e_\mu\,, \\
\xi^T_{(\mathrm{III},\partial_t\varphi_T)} &:= \sum_{\mu}\Big(\sum_{\nu}\sum_{i,j}\partial_t\varphi_T^\sharp(y^i)\,\varphi_T^\sharp(B_{ij})\, D^T_{e_\nu}\varphi_T^\sharp(y^j)\cdot((2\pi\alpha' F_{\nabla^T} + \varphi_T^\diamond B)(e_\mu, e_\nu))\Big)e_\mu
\end{aligned}
$$

in $\mathcal{T}_*(U_T/T)^C$, with the first $\mathcal{O}_U^C$-linear in $\partial_t\nabla^T$ and the second $\mathcal{O}_U^C$-linear in $\partial_t\varphi_T$.

6.3.2 The first variation of the Chern-Simons/Wess-Zumino term for lower dimensional D-branes

This is an update of [L-Y8: Sec.6.2] (D(13.1)) in the current setting. Let $\varphi_T : (X^{Az}, \mathcal{E}_T; \nabla^T) \to Y$ be an $(*_2)$-family of $(*_2)$-admissible maps. We work out the first variation of the Chern-Simons/Wess-Zumino term $S^{(C,B)}_{CS/WZ}(\varphi, \nabla)$ for the cases where $m := dim\, X = 0, 1, 2, 3$. As the details involve no identities or techniques that have not yet been used in Sec. 6.1, Sec. 6.2, and/or Sec. 6.3.1, we only summarize the final results below.

6.3.2.1 D(-1)-brane world-point ($m = 0$)

For a D(-1)-brane world-point, $dim\, X = 0$, $\nabla = 0$, and $S^{(C_{(0)})}_{CS/WZ}(\varphi_T) = T_{-1}\cdot Tr(\varphi_T^\sharp(C_{(0)}))$. It follows that

$$
\frac{d}{dt}S^{(C_{(0)})}_{CS/WZ}(\varphi_T) \;=\; T_{-1}\, Tr\,\partial_t(\varphi_T^\sharp(C_{(0)})) \;=\; T_{-1}\, Tr((\partial_t\varphi_T)C_{(0)})\,.
$$

6.3.2.2 D-particle world-line ($m = 1$)

For a D-particle world-line, $dim\, X = 1$. Let e_1 be the orthonormal frame on an open set $U \subset X$; e^1 its dual co-frame. Then, over U,

$$S_{CS/WZ}^{(C_{(1)})}(\varphi_T)^C \;=\; T_0 \int_U Tr\, \varphi_T^{\diamond} C_{(1)} \;=\; T_0 \int_U Tr \Big(\sum_{i=1}^n \varphi_T^\sharp(C_i) \cdot D_{e_1}^T \varphi_T^\sharp(y^i) \Big)\, e^1 \,.$$

It follows that

$$\begin{aligned} \frac{d}{dt} S_{CS/WZ}^{(C_{(1)})}(\varphi) \;&=\; T_0 \Big(Tr \sum_i \partial_t \varphi_T^\sharp(y^i) \varphi_T^\sharp(C_i) \Big)\big|_{\partial U} \\ &\quad - T_0 \int_U Tr \Big(\sum_i \partial_t \varphi_T^\sharp(y^i) D_{e_1}^T \varphi_T^\sharp(C_i) \Big) e^1 \;+\; T_0 \int_U Tr \Big(\sum_i D_{e_1} \varphi_T^\sharp(y^i) \cdot (\partial_t \varphi_T) C_i \Big) e^1 \,. \end{aligned}$$

6.3.2.3 D-string world-sheet $(m = 2)$

Denote

$$\breve{C}_{(2)} \;:=\; C_{(2)} + C_{(0)} B \;=\; \sum_{ij} (C_{ij} + C_{(0)} B_{ij})\, dy^i \otimes dy^j \;=\; \sum_{i,j} \breve{C}_{ij} dy^i \otimes dy^j$$

in a local coordinate $(y^1, \cdots, y^n)$ of Y. For a D-string world-sheet, $dim\, X = 2$. Let (e_1, e_2) be an orthonormal frame on an open set $U \subset X$; (e^1, e^2) its dual co-frame. Then, over U,

$$\begin{aligned} & S_{CS/WZ}^{(C_{(0)}, C_{(2)}, B)}(\varphi_T, \nabla^T)^C \\ &= T_1 \int_U Tr \Big(\sum_{i,j=1}^n \varphi_T^\sharp(\breve{C}_{ij})\, D_{e_1}^T \varphi_T^\sharp(y^i)\, D_{e_2}^T \varphi_T^\sharp(y^j) \\ &\qquad\qquad + \pi\alpha' \varphi_T^\sharp(C_{(0)})\, F_{\nabla^T}(e_1, e_2) \;+\; \pi\alpha' F_{\nabla^T}(e_1, e_2)\, \varphi_T^\sharp(C_{(0)}) \Big)\, e^1 \wedge e^2 \\ &= T_1 \int_U Tr \Big(\sum_{i,j=1}^n \varphi_T^\sharp(\breve{C}_{ij})\, D_{e_1}^T \varphi_T^\sharp(y^i)\, D_{e_2}^T \varphi_T^\sharp(y^j) \;+\; 2\,\pi\alpha' \varphi_T^\sharp(C_{(0)})\, F_{\nabla^T}(e_1, e_2) \Big)\, e^1 \wedge e^2 \,. \end{aligned}$$

It follows that

$$\begin{aligned} & \frac{d}{dt} S_{CS/WZ}^{(C_{(0)}, C_{(2)}, B)}(\varphi_T, \nabla^T)^C \\ &= T_1 \int_U Tr\, \partial_t \Big(\sum_{i,j=1}^n \varphi_T^\sharp(\breve{C}_{ij})\, D_{e_1}^T \varphi_T^\sharp(y^i)\, D_{e_2}^T \varphi_T^\sharp(y^j) \;+\; 2\,\pi\alpha' \varphi_T^\sharp(C_{(0)})\, F_{\nabla^T}(e_1, e_2) \Big)\, e^1 \wedge e^2 \\ &= T_1 \int_{\partial U} i_{\xi^T_{(\mathrm{IV}, \partial_t \varphi_T)}} (e^1 \wedge e^2) \;+\; 2\pi\alpha' T_1 \int_{\partial U} i_{\xi^T_{(\mathrm{IV}, \partial_t \nabla^T)}} (e^1 \wedge e^2) \\ &\quad + T_1 \int_U Tr \Big(\sum_{i,j=1}^n \partial_t \varphi^\sharp(y^i) \Big(D_{e_2}^T \varphi_T^\sharp(y^j) \varphi_T^\sharp(\breve{C}_{ij}) e^1 \;-\; D_{e_1}^T \varphi_T^\sharp(y^j) \varphi_T^\sharp(\breve{C}_{ij}) e^2 \Big) (\nabla_{e_1}^h e_1 + \nabla_{e_2}^h e_2) \Big)\, e^1 \wedge e^2 \\ &\qquad - T_1 \int \int_U Tr \Big(\sum_{i,j} \partial_t \varphi_T^\sharp(y^i) \Big(D_{e_1}^T \big(D_{e_2}^T \varphi_T^\sharp(y^j) \cdot \varphi_T^\sharp(\breve{C}_{ij}) \big) \;-\; D_{e_2}^T \big(D_{e_1}^T \varphi_T^\sharp(y^j) \cdot \varphi_T^\sharp(\breve{C}_{ij}) \big) \Big) \Big)\, e^1 \wedge e^2 \\ &\qquad + T_1 \int_U Tr \Big(\sum_{i,j} \partial_t \varphi_T^\sharp(\breve{C}_{ij}) D_{e_1}^T \varphi_T^\sharp(y^i) D_{e_2}^T \varphi_T^\sharp(y^j) \Big)\, e^1 \wedge e^2 \\ &\qquad + 2\pi\alpha'\, T_1 \int_U Tr \Big(\partial_t \varphi_T^\sharp(C_{(0)}) \cdot F_{\nabla^T}(e_1, e_2) \Big)\, e^1 \wedge e^2 \\ &\quad + 2\pi\alpha'\, T_1 \int_U Tr \Big(\varphi_T^\sharp(C_{(0)}) \Big(\big((\partial_t \nabla^T)(e_2)\, e^1 \;-\; (\partial_t \nabla^T)(e_1)\, e^2 \big) (\nabla_{e_1}^h e_1 + \nabla_{e_2}^h e_2) \;-\; (\partial_t \nabla^T)([e_1, e_2]) \Big) \Big)\, e^1 \wedge e^2 \\ &\qquad - 2\pi\alpha'\, T_1 \int_U Tr \Big(D_{e_1}^T \varphi_T^\sharp(C_{(0)}) \cdot (\partial_t \nabla^T)(e_2) \;-\; D_{e_2}^T \varphi_T^\sharp(C_{(0)}) \cdot (\partial_t \nabla^T)(e_1) \Big)\, e^1 \wedge e^2 \,. \end{aligned}$$

Here,

$$\begin{aligned} \xi^T_{(\mathrm{IV}, \partial_t \varphi_T)} \;&:=\; Tr\big(\textstyle\sum_{i,j} \partial_t \varphi_T^\sharp(y^i) D_{e_2}^T \varphi_T^\sharp(y^j) \varphi_T^\sharp(\breve{C}_{ij}) \big)\, e_1 \;-\; Tr\big(\textstyle\sum_{i,j} \partial_t \varphi_T^\sharp(y^i) D_{e_1}^T \varphi_T^\sharp(y^j) \varphi_T^\sharp(\breve{C}_{ij}) \big)\, e_2 \,, \\ \xi^T_{(\mathrm{IV}, \partial_t \nabla^T)} \;&:=\; Tr\big(\varphi_T^\sharp(C_{(0)}) \cdot (\partial_t \nabla^T)(e_2) \big)\, e_1 \;-\; Tr\big(\varphi_T^\sharp(C_{(0)}) \cdot (\partial_t \nabla^T)(e_1) \big)\, e_2 \end{aligned}$$

in $\mathcal{T}_*(U_T/T)^C$, with the first $\mathcal{O}_U^C$-linear in $\partial_t \varphi_T$ and the second $\mathcal{O}_U^C$-linear in $\partial_t \nabla^T$.

6.3.2.4 D-membrane world-volume ($m = 3$)

Denote

$$\begin{aligned}
\breve{C}_{(3)} &:= C_{(3)} + C_{(1)} \wedge B \\
&= \sum_{i,j,k} (C_{ijk} + C_i B_{jk} + C_j B_{ki} + C_k B_{ij})\, dy^i \otimes dy^j \otimes dy^k \;=\; \sum_{i,j,k} \breve{C}_{ijk} dy^i \otimes dy^j \otimes dy^k
\end{aligned}$$

in a local coordinate $(y^1, \cdots, y^n)$ of Y. For D-membrane world-volume, $dim\, X = 3$. Let (e_1, e_2, e_3) be an orthonormal frame on an open set $U \subset X$; (e^1, e^2, e^3) its dual co-frame. Then, over U,

$$\begin{aligned}
& S^{(C_{(1)}, C_{(3)}, B)}_{CS/WZ}(\varphi_T, \nabla^T)^C \\
&= T_2 \int_U Tr\Big(\sum_{i,j,k=1}^{n} \varphi^\sharp(\breve{C}_{ijk})\, D^T_{e_1}\varphi^\sharp_T(y^i)\, D^T_{e_2}\varphi^\sharp_T(y^j)\, D^T_{e_3}\varphi^\sharp_T(y^k) \\
&\qquad + 2\pi\alpha' \sum_{(\lambda\mu\nu)\in Sym_3} \sum_{i=1}^{n} (-1)^{(\lambda\mu\nu)} (\varphi^\sharp_T(C_i)\, D^T_\lambda \varphi^\sharp_T(y^i)\, F_{\nabla^T}(e_\mu, e_\nu)\,)\Big)\, e^1 \wedge e^2 \wedge e^3
\end{aligned}$$

It follows that

$$\begin{aligned}
& \frac{d}{dt} S^{(C_{(1)}, C_{(3)}, B)}_{CS/WZ}(\varphi_T, \nabla^T)^C \\
&= T_2 \int_U Tr\, \partial_t\Big(\sum_{i,j,k=1}^{n} \varphi^\sharp(\breve{C}_{ijk})\, D^T_{e_1}\varphi^\sharp_T(y^i)\, D^T_{e_2}\varphi^\sharp_T(y^j)\, D^T_{e_3}\varphi^\sharp_T(y^k) \\
&\qquad + 2\pi\alpha' \sum_{(\lambda\mu\nu)\in Sym_3} \sum_{i=1}^{n} (-1)^{(\lambda\mu\nu)} \big(\varphi^\sharp_T(C_i)\, D^T_{e_\lambda} \varphi^\sharp_T(y^i)\, F_{\nabla^T}(e_\mu, e_\nu)\,\big)\Big)\, e^1 \wedge e^2 \wedge e^3 \\
&= T_2 \int_{\partial U} i_{\xi^T_{(\mathrm{IV}, \partial_t\varphi_T; \breve{C}_{(3)})}} (e^1 \wedge e^2 \wedge e^3) \;+\; 4\pi\alpha'\, T_2 \int_{\partial U} i_{\xi^T_{(\mathrm{IV}, \partial_t\varphi_T; C_{(1)})}} (e^1 \wedge e^2 \wedge e^3) \\
&\qquad + 2\pi\alpha'\, T_2 \int_{\partial U} i_{\xi^T_{(\mathrm{IV}, \partial_t\nabla^T)}} (e^1 \wedge e^2 \wedge e^3) \\
&\quad + T_2 \int_U \bigg(\Big(Tr \sum_{i,j,k} \varphi^\sharp_T(\breve{C}_{ijk}) \partial_t\varphi^\sharp_T(y^i) D^T_{e_2}\varphi^\sharp_T(y^j) D^T_{e_3}\varphi^\sharp_T(y^k)\Big)\, e^1 \\
&\qquad - \Big(Tr \sum_{i,j,k} \varphi^\sharp_T(\breve{C}_{ijk}) \partial_t\varphi^\sharp_T(y^i) D^T_{e_1}\varphi^\sharp_T(y^j) D^T_{e_3}\varphi^\sharp_T(y^k)\Big)\, e^2 \\
&\qquad - \Big(Tr \sum_{i,j,k} \varphi^\sharp_T(\breve{C}_{ijk}) \partial_t\varphi^\sharp_T(y^i) D^T_{e_2}\varphi^\sharp_T(y^j) D^T_{e_1}\varphi^\sharp_T(y^k)\Big)\, e^3 \bigg)\, (\textstyle\sum_{\mu=1}^3 \nabla^h_{e_\mu} e_\mu)\, e^1 \wedge e^2 \wedge e^3 \\
&\quad - T_2 \int_U Tr \sum_{i,j,k} \partial_t\varphi^\sharp_T(y^i) \Big(D^T_{e_1}\big(\varphi^\sharp_T(\breve{C}_{ijk}) D^T_{e_2}\varphi^\sharp_T(y^j) D^T_{e_3}\varphi^\sharp_T(y^k)\big) \;-\; D^T_{e_2}\big(\varphi^\sharp_T(\breve{C}_{ijk}) D^T_{e_1}\varphi^\sharp_T(y^j) D^T_{e_3}\varphi^\sharp_T(y^k)\big) \\
&\qquad - D^T_{e_3}\big(\varphi^\sharp_T(\breve{C}_{ijk}) D^T_{e_2}\varphi^\sharp_T(y^j) D^T_{e_1}\varphi^\sharp_T(y^k)\big)\Big)\, e^1 \wedge e^2 \wedge e^3 \\
&\quad + T_2 \int_U Tr \sum_{i,j,k} \partial_t\varphi^\sharp_T(\breve{C}_{ijk}) D^T_{e_1}\varphi^\sharp_T(y^i) D^T_{e_2}\varphi^\sharp_T(y^j) D^T_{e_3}\varphi^\sharp_T(y^k)\, e^1 \wedge e^2 \wedge e^3 \\
&\quad + 4\pi\alpha'\, T_2 \int_U \bigg(\Big(Tr \sum_i \varphi^\sharp_T(C_i) \partial_t\varphi^\sharp_T(y^i) F_{\nabla^T}(e_2, e_3)\Big)\, e^1 \\
&\qquad - \Big(Tr \sum_i \varphi^\sharp_T(C_i) \partial_t\varphi^\sharp_T(y^i) F_{\nabla^T}(e_1, e_3)\Big)\, e^2 \\
&\qquad + \Big(Tr \sum_i \varphi^\sharp_T(C_i) \partial_t\varphi^\sharp_T(y^i) F_{\nabla^T}(e_1, e_2)\Big)\, e^3 \bigg)\, (\textstyle\sum_{\mu=1}^3 \nabla^h_{e_\mu} e_\mu)\, e^1 \wedge e^2 \wedge e^3 \\
&\quad - 4\pi\alpha'\, T_2 \int_U Tr \sum_i \partial_t\varphi^\sharp_T(y^i) \Big(D^T_{e_1}\big(\varphi^\sharp_T(C_i) F_{\nabla^T}(e_2, e_3)\big) \;-\; D^T_{e_2}\big(\varphi^\sharp_T(C_i) F_{\nabla^T}(e_1, e_3)\big) \\
&\qquad + D^T_{e_3}\big(\varphi^\sharp_T(C_i) F_{\nabla^T}(e_1, e_2)\big)\Big)\, e^1 \wedge e^2 \wedge e^3
\end{aligned}$$

$$+\,2\pi\alpha'\,T_2\int_U Tr\sum_{(\lambda\mu\nu)\in Sym_3}\sum_i(-1)^{(\lambda\mu\nu)}\partial_t\varphi_T^\sharp(C_i)\,D^T_{e_\lambda}\varphi_T^\sharp(y^i)\,F_{\nabla^T}(e_\mu,e_\nu)\,e^1\wedge e^2\wedge e^3$$

$$+\,2\pi\alpha'\,T_2\int_U\Bigg(\Big(Tr\sum_i\varphi_T^\sharp(C_i)\Big(D^T_{e_3}\varphi_T^\sharp(y^i)(\partial_t\nabla^T)(e_2)\,-\,D^T_{e_2}\varphi_T^\sharp(y^i)(\partial_t\nabla^T)(e_3)\Big)\Big)\,e^1$$
$$+\Big(Tr\sum_i\varphi_T^\sharp(C_i)\Big(D^T_{e_3}\varphi_T^\sharp(y^i)(\partial_t\nabla^T)(e_1)\,+\,D^T_{e_1}\varphi_T^\sharp(y^i)(\partial_t\nabla^T)(e_3)\Big)\Big)\,e^2$$
$$+\Big(Tr\sum_i\varphi_T^\sharp(C_i)\Big(D^T_{e_1}\varphi_T^\sharp(y^i)(\partial_t\nabla^T)(e_2)\,-\,D^T_{e_2}\varphi_T^\sharp(y^i)(\partial_t\nabla^T)(e_1)\Big)\Big)\,e^3\Bigg)\,(\textstyle\sum_{\mu=1}^3\nabla^h_{e_\mu}e_\mu)$$
$$e^1\wedge e^2\wedge e^3$$

$$+\,2\pi\alpha'\,T_2\int_U Tr\sum_{(\lambda\mu\nu)\in Sym_3}\sum_i(-1)^{(\lambda\mu\nu)}\Big(\varphi_T^\sharp(C_i)\,\big[(\partial_t\nabla^T)(e_\lambda),\varphi_T^\sharp(y^i)\big]\,F_{\nabla^T}(e_\mu,e_\nu)$$
$$-\,\varphi_T^\sharp(C_i)\,D^T_{e_\lambda}\varphi_T^\sharp(y^i)\,(\partial_t\varphi_T)([e_\mu,e_\nu])\Big)\,e^1\wedge e^2\wedge e^3\,.$$

Here,

$$\xi^T_{(\mathrm{IV},\partial_t\varphi_T;\check{C}_{(3)})}\;:=\;\Big(Tr\sum_{i,j,k}\varphi_T^\sharp(\check{C}_{ijk})\partial_t\varphi_T^\sharp(y^i)D^T_{e_2}\varphi_T^\sharp(y^j)D^T_{e_3}\varphi_T^\sharp(y^k)\Big)\,e_1$$
$$-\Big(Tr\sum_{i,j,k}\varphi_T^\sharp(\check{C}_{ijk})\partial_t\varphi_T^\sharp(y^i)D^T_{e_1}\varphi_T^\sharp(y^j)D^T_{e_3}\varphi_T^\sharp(y^k)\Big)\,e_2$$
$$-\Big(Tr\sum_{i,j,k}\varphi_T^\sharp(\check{C}_{ijk})\partial_t\varphi_T^\sharp(y^i)D^T_{e_2}\varphi_T^\sharp(y^j)D^T_{e_1}\varphi_T^\sharp(y^k)\Big)\,e_3\;,$$

$$\xi^T_{(\mathrm{IV},\partial_t\varphi_T;C_{(1)})}\;:=\;\Big(Tr\sum_i\varphi_T^\sharp(C_i)\partial_t\varphi_T^\sharp(y^i)F_{\nabla^T}(e_2,e_3)\Big)\,e_1$$
$$-\Big(Tr\sum_i\varphi_T^\sharp(C_i)\partial_t\varphi_T^\sharp(y^i)F_{\nabla^T}(e_1,e_3)\Big)\,e_2\;+\;\Big(Tr\sum_i\varphi_T^\sharp(C_i)\partial_t\varphi_T^\sharp(y^i)F_{\nabla^T}(e_1,e_2)\Big)\,e_3\;,$$

$$\xi^T_{(\mathrm{IV},\partial_t\nabla^T)}\;:=\;\Big(Tr\sum_i\varphi_T^\sharp(C_i)\Big(D^T_{e_3}\varphi_T^\sharp(y^i)(\partial_t\nabla^T)(e_2)\,-\,D^T_{e_2}\varphi_T^\sharp(y^i)(\partial_t\nabla^T)(e_3)\Big)\Big)\,e_1$$
$$+\Big(Tr\sum_i\varphi_T^\sharp(C_i)\Big(D^T_{e_3}\varphi_T^\sharp(y^i)(\partial_t\nabla^T)(e_1)\,+\,D^T_{e_1}\varphi_T^\sharp(y^i)(\partial_t\nabla^T)(e_3)\Big)\Big)\,e_2$$
$$+\Big(Tr\sum_i\varphi_T^\sharp(C_i)\Big(D^T_{e_1}\varphi_T^\sharp(y^i)(\partial_t\nabla^T)(e_2)\,-\,D^T_{e_2}\varphi_T^\sharp(y^i)(\partial_t\nabla^T)(e_1)\Big)\Big)\,e_3$$

in $\mathcal{T}_*(U_T/T)^C$, with the first two $\mathcal{O}_U^C$-linear in $\partial_t\varphi_T$ and the third $\mathcal{O}_U^C$-linear in $\partial_t\nabla^T$.

7 The second variation of the enhanced kinetic term for maps

Let $T=(-\varepsilon,\varepsilon)^2\subset R^2$, with coordinate (s,t), and (φ_T,∇^T) be an $(*_2)$-admissible family of $(*_1)$-admissible pairs, with $(\varphi_{(0,0)},\nabla_{(0,0)})=(\varphi,\nabla)$. Assume further that

$$D_\xi\partial_s\mathcal{A}_{\varphi_T}\;\subset\;Comm(\mathcal{A}_{\varphi_T})\qquad\text{for all }\xi\in\mathcal{T}_*(X_T/T)\,.$$

We work out in this section the second variation formula of the enhanced kinetic term $S^{(\rho,h;\Phi,g)}_{map:kinetic^+}(\varphi,\nabla)$ in the standard action $S^{(\rho,h;\Phi,g,B,C)}_{standard}(\varphi,\nabla)$.

7.1 The second variation of the kinetic term for maps

Recall

$$E^{\nabla^T}(\varphi_T)\;:=\;S^{(h;g)}_{map:kinetic}(\varphi_T,\nabla^T)\;:=\;\frac{1}{2}\,T_{m-1}\int_X Tr\,\langle D^T\varphi_T\,,\,D^T\varphi_T\rangle_{(h,g)}\,vol_h\,,$$

with the understanding that all expressions are taken on $X_{(s,t)}$ with (s,t) varying in T.

Let $U\subset X$ be an open set with an orthonormal frame $(e_\mu)_{\mu=1,\cdots,m}$. Let $(e^\mu)_{\mu=1,\cdots,m}$ be the dual co-frame. Assume that U is small enough so that $\varphi_T(U_T^{Az})$ is contained in a coordinate chart of Y, with coordinates $(y^1,\cdots,y^n)$. Then, as in Sec. 6.1, over U,

$$\begin{aligned}\frac{\partial}{\partial t}E^{\nabla^T}(\varphi_T) \;&=\; T_{m-1}\int_U Tr\sum_{\mu=1}^m\Big\langle \nabla^{T,(\varphi_T,g)}_{e_\mu}\partial_t\varphi_T\,,\,D^T_{e_\mu}\varphi_T\Big\rangle_g\,vol_h\\ &\qquad+T_{m-1}\int_U Tr\sum_{\mu=1}^m\Big\langle (ad\otimes\nabla^g)_{D^T_{e_\mu}\varphi_T}\partial_t\varphi_T\,,\,D^T_{e_\mu}\varphi_T\Big\rangle_g\,vol_h\\ &\qquad\qquad+T_{m-1}\int_U\sum_{\mu=1}^m\Big\langle\sum_{i=1}^n[(\partial_t\nabla^T)(e_\mu),\varphi_T^\sharp(y^i)]\otimes\tfrac{\partial}{\partial y^i}\,,\,D^T_{e_\mu}\varphi_T\Big\rangle_g\,vol_h\\ &=:\;(\mathrm{I}^2.1)\;+\;(\mathrm{I}^2.2)\;+\;(\mathrm{I}^2.3)\,;\end{aligned}$$

and

$$\frac{\partial^2}{\partial s\,\partial t}E^{\nabla^T}(\varphi_T)\;=\;\frac{\partial}{\partial s}(\mathrm{I}^2.1)\;+\;\frac{\partial}{\partial \mathrm{s}}(\mathrm{I}^2.2)\;+\;\frac{\partial}{\partial \mathrm{s}}(\mathrm{I}^2.3)\,.$$

Which we now compute term by term.

The term $\frac{\partial}{\partial s}(\mathrm{I}^2.1)$

$$\begin{aligned}\frac{\partial}{\partial s}(\mathrm{I}^2.1)\;&=\;T_{m-1}\frac{\partial}{\partial s}\int_U Tr\sum_{\mu=1}^m\Big\langle\nabla^{T,(\varphi_T,g)}_{e_\mu}\partial_t\varphi_T\,,\,D^T_{e_\mu}\varphi_T\Big\rangle_g\,vol_h\\ &=\;T_{m-1}\int_U Tr\sum_\mu\partial_s\Big\langle\nabla^{T,(\varphi_T,g)}_{e_\mu}\partial_t\varphi_T\,,\,D^T_{e_\mu}\varphi_T\Big\rangle_g\,vol_h\\ &=\;T_{m-1}\int_U Tr\sum_\mu\Big\langle\partial_s\nabla^{T,(\varphi_T,g)}_{e_\mu}\partial_t\varphi_T\,,\,D^T_{e_\mu}\varphi_T\Big\rangle_g\,vol_h\\ &\qquad\qquad+T_{m-1}\int_U Tr\sum_\mu\Big\langle\nabla^{T,(\varphi_T,g)}_{e_\mu}\partial_t\varphi_T\,,\,\partial_s D^T_{e_\mu}\varphi_T\Big\rangle_g\,vol_h\\ &=\;(\mathrm{I}^2.1.1)\;+\;(\mathrm{I}^2.1.2)\,.\end{aligned}$$

(a) Term $(\mathrm{I}^2.1.1)$

$$\begin{aligned}(\mathrm{I}^2.1.1)\;&:=\;T_{m-1}\int_U Tr\sum_\mu\Big\langle\partial_s\nabla^{T,(\varphi_T,g)}_{e_\mu}\partial_t\varphi_T\,,\,D^T_{e_\mu}\varphi_T\Big\rangle_g\,vol_h\\ &=\;T_{m-1}\int_U Tr\sum_\mu\Big\langle\nabla^{T,(\varphi_T,g)}_{e_\mu}\partial_s\partial_t\varphi_T\,,\,D^T_{e_\mu}\varphi_T\Big\rangle_g\,vol_h\\ &\qquad+T_{m-1}\int_U Tr\sum_\mu\Big\langle F_{\nabla^{T,(\varphi_T,g)}}(\partial_s,e_\mu)\,\partial_t\varphi_T\,,\,D^T_{e_\mu}\varphi_T\Big\rangle_g\,vol_h\\ &=\;(\mathrm{I}^2.1.1.1)\;+\;(\mathrm{I}^2.1.1.2)\,.\end{aligned}$$

For Term $(\mathrm{I}^2.1.1.1)$, as in Sec. 6.1 for Summand (I.1.1), consider the 1-form on U_T/T

$$\alpha^T_{(\mathrm{I}^2,\partial_\mathrm{s}\partial_\mathrm{t}\varphi_\mathrm{T})}\;:=\;Tr\langle\partial_s\partial_t\varphi_T\,,\,D^T\varphi_T\rangle_g$$

and let

$$\xi^T_{(\mathrm{I}^2,\partial_\mathrm{s}\partial_\mathrm{t}\varphi_\mathrm{T})}\;:=\;\sum_{\mu=1}^n Tr\langle\partial_s\partial_t\varphi_T\,,\,D^T_{e_\mu}\varphi_T\rangle_g\,e_\mu$$

be its dual on U_T/T with respect to h. Then,

$$\begin{aligned}
(\mathrm{I}^2.1.1.1) \;=\;& T_{m-1}\int_{\partial U} i_{\xi^T_{(\mathrm{I}^2,\partial_s\partial_t\varphi_T)}} vol_h \\
&+\; T_{m-1}\int_U Tr\Big\langle \partial_s\partial_t\varphi_T\,,\,(D^T_{\sum_\mu \nabla^h_{e_\mu}e_\mu} - \textstyle\sum_\mu \nabla^{T,(\varphi_T,g)}_{e_\mu} D^T_{e_\mu})\varphi_T\Big\rangle_g\, vol_h\,.
\end{aligned}$$

For Term (I^2.1.1.2), recall Lemma 3.2.2.5. Then,

$$\begin{aligned}
(\mathrm{I}^2.1.1.2) \;=\;& -\,T_{m-1}\int_U Tr\Big\langle \partial_t\varphi_T\,,\,\sum_\mu F_{\nabla^{T,(\varphi_T,g)}}(\partial_s,e_\mu)D^T_{e_\mu}\varphi_T\Big\rangle_g\, vol_h \\
&+ T_{m-1}\int_U Tr\sum_\mu\,[F_\nabla(\partial_s,e_\mu)\,,\,\langle\partial_t\varphi_T, D^T_{e_\mu}\varphi_T\rangle_g]\, vol_h \\
=\;& -\,T_{m-1}\int_U Tr\Big\langle \partial_t\varphi_T\,,\,\sum_\mu F_{\nabla^{T,(\varphi_T,g)}}(\partial_s,e_\mu)D^T_{e_\mu}\varphi_T\Big\rangle_g\, vol_h\,.
\end{aligned}$$

Here,

$$\begin{aligned}
F_{\nabla^{T,(\varphi_T,g)}}(\partial_s,e_\mu)\,D^T_{e_\mu}\varphi_T \;=\;& (\partial_s\nabla^{T,(\varphi_T,g)}_{e_\mu} \;-\; \nabla^{T,(\varphi_T,g)}_{e_\mu}\partial_s)\textstyle\sum_{i=1}^n D_{e_\mu}\varphi^\sharp_T(y^i)\otimes\dfrac{\partial}{\partial y^i} \\
=\sum_{i=1}^n[(\partial_s\nabla^T)(e_\mu), D_{e_\mu}\varphi^\sharp_T(y^i)]\otimes\frac{\partial}{\partial y^i} \;&+\; \sum_{i=1}^n D_{e_\mu}\varphi^\sharp_T(y^i)\sum_{j=1}^n[(\partial_s\nabla^T)(e_\mu),\varphi^\sharp_T(y^j)]\otimes\nabla^g_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^i} \\
+\;\sum_{i=1}^n D_{e_\mu}\varphi^\sharp_T(y^i)\sum_{j,k=1}^n\Big(&D^T_{e_\mu}\varphi^\sharp_T(y^j)\,\partial_s\varphi^\sharp_T(y^k)\otimes\nabla^g_{\frac{\partial}{\partial y^k}}\nabla^g_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^i} \\
&-\partial_s\varphi^\sharp_T(y^k)\,D^T_{e_\mu}\varphi^\sharp_T(y^j)\otimes\nabla^g_{\frac{\partial}{\partial y^j}}\nabla^g_{\frac{\partial}{\partial y^k}}\frac{\partial}{\partial y^i}\Big)
\end{aligned}$$

explicitly.

(*b*) *Term* (I^2.1.2)

$$\begin{aligned}
(\mathrm{I}^2.1.2) \;:=\;& T_{m-1}\int_U Tr\sum_\mu\Big\langle\nabla^{T,(\varphi_T,g)}_{e_\mu}\partial_t\varphi_T\,,\,\partial_s D^T_{e_\mu}\varphi_T\Big\rangle_g\, vol_h \\
=\;& T_{m-1}\int_U Tr\sum_\mu\Big\langle\nabla^{T,(\varphi_T,g)}_{e_\mu}\partial_t\varphi_T\,,\,\nabla^{T,(\varphi_T,g)}_{e_\mu}\partial_s\varphi_T\Big\rangle_g\, vol_h \\
&+\; T_{m-1}\int_U Tr\sum_\mu\Big\langle\nabla^{T,(\varphi_T,g)}_{e_\mu}\partial_t\varphi_T\,,\,(ad\otimes\nabla^g)_{D^T_{e_\mu}\varphi_T}\partial_s\varphi_T\Big\rangle_g\, vol_h \\
&+\; T_{m-1}\int_U Tr\sum_\mu\Big\langle\nabla^{T,(\varphi_T,g)}_{e_\mu}\partial_t\varphi_T\,,\,\textstyle\sum_{i=1}^n[(\partial_s\nabla^T)(e_\mu),\varphi^\sharp_T(y^i)]\otimes\dfrac{\partial}{\partial y^i}\Big\rangle_g\, vol_h\,.
\end{aligned}$$

As in Sec. 6.1, consider the 1-forms on U_T/T,

$$\begin{aligned}
\alpha^T_{(\mathrm{I}^2,\partial_t\varphi_T,\nabla^{T,(\varphi_T,g)})} &= Tr\langle\partial_t\varphi_T\,,\,\nabla^{T,(\varphi_T,g)}\partial_s\varphi_T\rangle_g\,, \\
\alpha^T_{(\mathrm{I}^2,\partial_t\varphi_T,D^T\varphi_T)} &= Tr\langle\partial_t\varphi_T\,,\,(ad\otimes\nabla^g)_{D^T\varphi_T}\partial_s\varphi_T\rangle_g\,, \\
\alpha^T_{(\mathrm{I}^2,\partial_t\varphi_T,\partial_s\nabla^T)} &= Tr\langle\partial_t\varphi_T\,,\,\textstyle\sum_{i=1}^n[\partial_s\nabla^T,\varphi^\sharp_T(y^i)]\otimes\dfrac{\partial}{\partial y^i}\rangle_g
\end{aligned}$$

and let

$$\begin{aligned}
\xi^T_{(\mathrm{I}^2,\partial_t\varphi_T,\nabla^{T,(\varphi_T,g)})} &= \sum_\mu Tr\langle\partial_t\varphi_T\,,\,\nabla^{T,(\varphi_T,g)}_{e_\mu}\partial_s\varphi_T\rangle_g\, e_\mu\,, \\
\xi^T_{(\mathrm{I}^2,\partial_t\varphi_T,D^T\varphi_T)} &= \sum_\mu Tr\langle\partial_t\varphi_T\,,\,(ad\otimes\nabla^g)_{D^T_{e_\mu}\varphi_T}\partial_s\varphi_T\rangle_g\, e_\mu\,, \\
\xi^T_{(\mathrm{I}^2,\partial_t\varphi_T,\partial_s\nabla^T)} &= \sum_\mu Tr\langle\partial_t\varphi_T\,,\,\textstyle\sum_{i=1}^n[(\partial_s\nabla^T)(e_\mu),\varphi^\sharp_T(y^i)]\otimes\dfrac{\partial}{\partial y^i}\rangle_g\, e_\mu
\end{aligned}$$

be their respective dual on U_T/T with respect to h. Then,

$$
\begin{aligned}
(\mathrm{I}^2.1.2) \;=\;& T_{m-1}\int_{\partial U} i_{\xi^T_{(\mathrm{I}^2,\partial_t\varphi_T,\nabla^{T,(\varphi_T,g)})}+\xi^T_{(\mathrm{I}^2,\partial_t\varphi_T,D^T\varphi_T)}+\xi^T_{(\mathrm{I}^2,\partial_t\varphi_T,\partial_s\nabla^T)}}\, vol_h \\
&+ T_{m-1}\int_U Tr\Big\langle \partial_t\varphi_T\,,\,\Big(\nabla^{T,(\varphi_T,g)}_{\sum_\mu \nabla^h_{e_\mu}e_\mu} - \textstyle\sum_\mu \nabla^{T,(\varphi_T,g)}_{e_\mu}\nabla^{T,(\varphi_T,g)}_{e_\mu}\Big)\partial_s\varphi_T\Big\rangle_g vol_h \\
&+ T_{m-1}\int_U Tr\Big\langle \partial_t\varphi_T\,,\,\Big((ad\otimes\nabla^g)_{D^T_{\sum_\mu \nabla^h_{e_\mu}e_\mu}\varphi_T} \\
&\qquad\qquad - \textstyle\sum_\mu \nabla^{T,(\varphi_T,g)}_{e_\mu}(ad\otimes\nabla^g)_{D^T_{e_\mu}\varphi_T}\Big)\partial_s\varphi_T\Big\rangle_g vol_h \\
&+ T_{m-1}\int_U Tr\Big\langle \partial_t\varphi_T\,,\,\textstyle\sum_{i=1}^n\Big([(\partial_s\nabla^T)(\sum_\mu \nabla^h_{e_\mu}e_\mu),\varphi_T^\sharp(y^i)] \\
&\qquad\qquad - \textstyle\sum_\mu \nabla^{T,(\varphi_T,g)}_{e_\mu}[(\partial_s\nabla^T)(e_\mu),\varphi^\sharp_T(y^i)]\Big)\otimes\dfrac{\partial}{\partial y^i}\Big\rangle_g vol_h\,.
\end{aligned}
$$

The term $\frac{\partial}{\partial s}\,(\mathrm{I}^2.2)$

$$
\begin{aligned}
\frac{\partial}{\partial s}\,(\mathrm{I}^2.2) \;=\;& T_{m-1}\frac{\partial}{\partial s}\int_U Tr\sum_{\mu=1}^m\Big\langle (ad\otimes\nabla^g)_{D^T_{e_\mu}\varphi_T}\partial_t\varphi_T\,,\,D^T_{e_\mu}\varphi_T\Big\rangle_g vol_h \\
=\;& T_{m-1}\int_U Tr\sum_\mu\Big\langle \partial_s((ad\otimes\nabla^g)_{D_{e_\mu}\varphi_T}\partial_t\varphi_T)\,,\,D^T_{e_\mu}\varphi_T\Big\rangle_g vol_h \\
&\qquad + T_{m-1}\int_U Tr\sum_\mu\Big\langle (ad\otimes\nabla^g)_{D_{e_\mu}\varphi_T}\partial_t\varphi_T\,,\,\partial_s D^T_{e_\mu}\varphi_T\Big\rangle_g vol_h \\
=\;& (\mathrm{I}^2.2.1) + (\mathrm{I}^2.2.2)\,.
\end{aligned}
$$

(*a*) *Term* $(\mathrm{I}^2.2.1)$

$$
\begin{aligned}
(\mathrm{I}^2.2.1) \;=\;& T_{m-1}\int_U Tr\sum_\mu\Big\langle (ad\otimes\nabla^g)_{D^T_{e_\mu}}(\partial_s\partial_t\varphi_T)\,,\,D^T_{e_\mu}\varphi_T\Big\rangle_g vol_h \\
&+ T_{m-1}\int_U Tr\sum_\mu\Big\langle \sum_{i,j}[D^T_{e_\mu}\partial_s\varphi^\sharp_T(y^j),\partial_t\varphi^\sharp_T(y^i)]\otimes\nabla^g_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^i}\,,\,D^T_{e_\mu}\varphi_T\Big\rangle_g vol_h \\
&+ T_{m-1}\int_U Tr\sum_\mu\Big\langle \sum_{i,j,k}\Big([D^T_{e_\mu}\varphi^\sharp_T(y^j),\partial_t\varphi^\sharp_T(y^i)]\partial_s\varphi^\sharp_T(y^k)\otimes R^g(\frac{\partial}{\partial y^k},\frac{\partial}{\partial y^j})\frac{\partial}{\partial y^i} \\
&\qquad - \partial_t\varphi^\sharp_T(y^i)[D^T_{e_\mu}\varphi^\sharp_T(y^j),\partial_s\varphi^\sharp_T(y^k)]\otimes\nabla^g_{\frac{\partial}{\partial y^j}}\nabla^g_{\frac{\partial}{\partial y^k}}\frac{\partial}{\partial y^i}\Big)\,, \\
&\qquad D^T_{e_\mu}\varphi_T\Big\rangle_g vol_h \\
&+ T_{m-1}\int_U Tr\sum_\mu\Big\langle \sum_{i,j}[ad_{(\partial_s\nabla^T)(e_\mu)}\varphi^\sharp_T(y^j),\partial_t\varphi^\sharp_T(y^i)]\otimes\nabla^g_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^i}\,,\,D^T_{e_\mu}\varphi_T\Big\rangle_g vol_h\,.
\end{aligned}
$$

The integrand of the first summand captures a related part in the system of equations of motion for (φ,∇). The integrand of the second summand is tensorial in $\partial_t\varphi_T$ and first-order differential operatorial in $\partial_s\varphi_T$. The integrand of the third summand is tensorial in both $\partial_t\varphi_T$ and $\partial_s\varphi_T$. The integrand of the fourth summand is tensorial in $\partial_t\varphi_T$ and $\partial_s\nabla^T$.

(*b*) *Term* $(\mathrm{I}^2.2.2)$

$$
(\mathrm{I}^2.2.2) \;=\; T_{m-1}\int_U Tr\sum_\mu\Big\langle (ad\otimes\nabla^g)_{D_{e_\mu}\varphi_T}\partial_t\varphi_T\,,\,\partial_s D^T_{e_\mu}\varphi_T\Big\rangle_g vol_h
$$

$$
\begin{aligned}
&= T_{m-1}\int_U Tr\sum_\mu \Big\langle (ad\otimes\nabla^g)_{D_{e_\mu}\varphi_T}\partial_t\varphi_T\,,\,\nabla^{T,(\varphi_T,g)}_{e_\mu}\partial_s\varphi_T\Big\rangle_g\, vol_h\\
&\quad - T_{m-1}\int_U Tr\sum_\mu \Big\langle (ad\otimes\nabla^g)_{D_{e_\mu}\varphi_T}\partial_t\varphi_T\,,\,(ad\otimes\nabla^g)_{\partial_s\varphi_T}D^T_{e_\mu}\varphi_T\Big\rangle_g\, vol_h\\
&\quad + T_{m-1}\int_U Tr\sum_\mu \Big\langle (ad\otimes\nabla^g)_{D_{e_\mu}\varphi_T}\partial_t\varphi_T\,,\,\sum_i[(\partial_s\nabla^T)(e_\mu),\varphi_T^\sharp(y^i)]\otimes\tfrac{\partial}{\partial y^i}\Big\rangle_g\, vol_h\,.
\end{aligned}
$$

The integrand of the first summand is tensorial in $\partial_t\varphi_T$ and first-order differential operatorial in $\partial_s\varphi_T$. The integrand of the second summand is tensorial in both $\partial_t\varphi_T$ and $\partial_s\varphi_T$. The integrand of the third summand is tensorial in $\partial_t\varphi_T$ and $\partial_s\nabla^T$.

The term $\frac{\partial}{\partial s}\,(\mathrm{I}^2.3)$

$$
\begin{aligned}
\frac{\partial}{\partial s}\,(\mathrm{I}^2.3) &= T_{m-1}\frac{\partial}{\partial s}\int_U\sum_{\mu=1}^m\Big\langle\sum_{i=1}^n[(\partial_t\nabla^T)(e_\mu),\varphi_T^\sharp(y^i)]\otimes\tfrac{\partial}{\partial y^i}\,,\,D^T_{e_\mu}\varphi_T\Big\rangle_g\, vol_h\\
&= T_{m-1}\int_U\sum_{\mu=1}^m\Big\langle\sum_{i=1}^n\partial_s\Big([(\partial_t\nabla^T)(e_\mu),\varphi_T^\sharp(y^i)]\otimes\tfrac{\partial}{\partial y^i}\Big)\,,\,D^T_{e_\mu}\varphi_T\Big\rangle_g\, vol_h\\
&\qquad + T_{m-1}\int_U\sum_{\mu=1}^m\Big\langle\sum_{i=1}^n[(\partial_t\nabla^T)(e_\mu),\varphi_T^\sharp(y^i)]\otimes\tfrac{\partial}{\partial y^i}\,,\,\partial_s D^T_{e_\mu}\varphi_T\Big\rangle_g\, vol_h\\
&= (\mathrm{I}^2.3.1)\;+\;(\mathrm{I}^2.3.2)\,.
\end{aligned}
$$

(*a*) *Term* $(\mathrm{I}^2.3.1)$

$$
\begin{aligned}
(\mathrm{I}^2.3.1) &= T_{m-1}\int_U\sum_{\mu=1}^m\Big\langle\sum_{i=1}^n\partial_s\Big([(\partial_t\nabla^T)(e_\mu),\varphi_T^\sharp(y^i)]\otimes\frac{\partial}{\partial y^i}\Big)\,,\,D^T_{e_\mu}\varphi_T\Big\rangle_g\, vol_h\\
&= T_{m-1}\int_U\sum_{\mu=1}^m\Big\langle\sum_{i=1}^n[(\partial_s\partial_t\nabla^T)(e_\mu),\varphi_T^\sharp(y^i)]\otimes\frac{\partial}{\partial y^i}\,,\,D^T_{e_\mu}\varphi_T\Big\rangle_g\, vol_h\\
&\quad + T_{m-1}\int_U\sum_{\mu=1}^m\Big\langle\sum_{i=1}^n\Big([(\partial_t\nabla^T)(e_\mu),\partial_s\varphi_T^\sharp(y^i)]\otimes\frac{\partial}{\partial y^i}\\
&\qquad\qquad + [(\partial_t\nabla^T)(e_\mu),\varphi_T^\sharp(y^i)]\textstyle\sum_j\partial_s\varphi_T^\sharp(y^j)\otimes\nabla^g_{\frac{\partial}{\partial y^j}}\displaystyle\frac{\partial}{\partial y^i}\Big)\,,\,D^T_{e_\mu}\varphi_T\Big\rangle_g\, vol_h\,.
\end{aligned}
$$

The integrand of the first summand captures a related part in the system of equations of motion for (φ,∇). The integrand of the second summand is tensorial in $\partial_s\varphi_T$ and $\partial_t\nabla^T$.

(*b*) *Term* $(\mathrm{I}^2.3.2)$

$$
\begin{aligned}
(\mathrm{I}^2.3.2) &= T_{m-1}\int_U\sum_{\mu=1}^m\Big\langle\sum_{i=1}^n[(\partial_t\nabla^T)(e_\mu),\varphi_T^\sharp(y^i)]\otimes\frac{\partial}{\partial y^i}\,,\,\partial_s D^T_{e_\mu}\varphi_T\Big\rangle_g\, vol_h\\
&= T_{m-1}\int_U\sum_{\mu=1}^m\Big\langle\sum_{i=1}^n[(\partial_t\nabla^T)(e_\mu),\varphi_T^\sharp(y^i)]\otimes\frac{\partial}{\partial y^i}\,,\,\nabla^{T,(\varphi_T,g)}_{e_\mu}\partial_s\varphi_T\Big\rangle_g\, vol_h\\
&\quad - T_{m-1}\int_U\sum_{\mu=1}^m\Big\langle\textstyle\sum_{i=1}^n[(\partial_t\nabla^T)(e_\mu),\varphi_T^\sharp(y^i)]\otimes\displaystyle\frac{\partial}{\partial y^i}\,,\,(ad\otimes\nabla^g)_{\partial_s\varphi_T}D^T_{e_\mu}\varphi_T\Big\rangle_g\, vol_h\\
&\quad + T_{m-1}\int_U\sum_{\mu=1}^m\sum_{i,j=1}^n\Big\langle[(\partial_t\nabla^T)(e_\mu),\varphi_T^\sharp(y^i)]\otimes\frac{\partial}{\partial y^i}\,,\,[(\partial_s\nabla^T)(e_\mu),\varphi_T^\sharp(y^j)]\otimes\frac{\partial}{\partial y^j}\Big\rangle_g\, vol_h\,.
\end{aligned}
$$

The integrand of the first summand is tensorial in $\partial_t\nabla^T$ and first-order differential operatorial in $\partial_s\varphi_T$. The integrand of the second summand is tensorial in $\partial_s\varphi_T$ and $\partial_t\nabla^T$. The integrand of the third summand is tensorial in both $\partial_t\nabla^T$ and $\partial_s\nabla^T$.

Finally, recall Lemma 3.2.2.4 and note that with the additional assumption at the beginning of this section, all the inner products $Tr\langle\,\cdot\,,\,\cdot\,\rangle_g$ that appear in the calculation above are defined.

In summary,

Proposition 7.1.1 [second variation of kinetic term for maps] *Let* (φ_T, ∇^T) *be a* $(*_2)$*-admissible* T*-family of* $(*_1)$*-admissible pairs with the additional assumption that* $D_\xi\partial_s\mathcal{A}_{\varphi_T} \subset Comm(\mathcal{A}_{\varphi_T})$ *for all* $\xi \in \mathcal{T}_*(X_T/T)$*. Then,*

$$
\begin{aligned}
\frac{\partial}{\partial s}\frac{\partial}{\partial t}E^{\nabla^T}(\varphi_T)^C \;&=\; \frac{\partial}{\partial s}\frac{\partial}{\partial t}\Big(\frac{1}{2}\,T_{m-1}\int_U Tr\langle D^T\varphi_T\,,\,D^T\varphi_T\rangle_{(h,g)}\,vol_h\Big)\\
&= T_{m-1}\int_{\partial U} i_{\xi^T_{(\mathrm{I}^2,\partial_s\partial_t\varphi_T)}}\,vol_h\\
&\quad + T_{m-1}\int_{\partial U} i_{\,\xi^T_{(\mathrm{I}^2,\partial_t\varphi_T,\nabla^{T,(\varphi_T,g)})}+\xi^T_{(\mathrm{I}^2,\partial_t\varphi_T,D^T\varphi_T)}+\xi^T_{(\mathrm{I}^2,\partial_t\varphi_T,\partial_s\nabla^T)}}\,vol_h\\
&\quad + T_{m-1}\int_U Tr\Big\langle\partial_s\partial_t\varphi_T\,,\,\big(D^T_{\sum_\mu\nabla^h_{e_\mu}e_\mu} - \textstyle\sum_\mu\nabla^{T,(\varphi_T,g)}_{e_\mu}D^T_{e_\mu}\big)\varphi_T\Big\rangle_g\,vol_h\\
&\quad + T_{m-1}\int_U Tr\sum_\mu\Big\langle(ad\otimes\nabla^g)_{D^T_{e_\mu}}(\partial_s\partial_t\varphi_T)\,,\,D^T_{e_\mu}\varphi_T\Big\rangle_g\,vol_h\\
&\quad + T_{m-1}\int_U\sum_{\mu=1}^m\Big\langle\sum_{i=1}^n\big[(\partial_s\partial_t\nabla^T)(e_\mu),\varphi_T^\sharp(y^i)\big]\otimes\frac{\partial}{\partial y^i}\,,\,D^T_{e_\mu}\varphi_T\Big\rangle_g\,vol_h\\
&\quad + T_{m-1}\int_U Tr\Big\langle\partial_t\varphi_T\,,\,\Big(\nabla^{T,(\varphi_T,g)}_{\sum_\mu\nabla^h_{e_\mu}e_\mu} - \textstyle\sum_\mu\nabla^{T,(\varphi_T,g)}_{e_\mu}\nabla^{T,(\varphi_T,g)}_{e_\mu}\Big)\partial_s\varphi_T\Big\rangle_g\,vol_h\\
&\quad + T_{m-1}\int_U Tr\Big\langle\partial_t\varphi_T\,,\,\Big((ad\otimes\nabla^g)_{D^T_{\sum_\mu\nabla^h_{e_\mu}e_\mu}\varphi_T}\\
&\qquad\qquad - \textstyle\sum_\mu\nabla^{T,(\varphi_T,g)}_{e_\mu}(ad\otimes\nabla^g)_{D^T_{e_\mu}\varphi_T}\Big)\partial_s\varphi_T\Big\rangle_g\,vol_h\\
&\quad + T_{m-1}\int_U Tr\sum_\mu\Big\langle\sum_{i,j}\big[D^T_{e_\mu}\partial_s\varphi_T^\sharp(y^j),\partial_t\varphi_T^\sharp(y^i)\big]\otimes\nabla^g_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^i}\,,\,D^T_{e_\mu}\varphi_T\Big\rangle_g\,vol_h\\
&\quad + T_{m-1}\int_U Tr\sum_\mu\Big\langle\sum_{i,j,k}\Big(\big[D^T_{e_\mu}\varphi_T^\sharp(y^j),\partial_t\varphi_T^\sharp(y^i)\big]\partial_s\varphi_T^\sharp(y^k)\otimes R^g\big(\frac{\partial}{\partial y^k},\frac{\partial}{\partial y^j}\big)\frac{\partial}{\partial y^i}\\
&\qquad\qquad - \partial_t\varphi_T^\sharp(y^i)\big[D^T_{e_\mu}\varphi_T^\sharp(y^j),\partial_s\varphi_T^\sharp(y^k)\big]\otimes\nabla^g_{\frac{\partial}{\partial y^j}}\nabla^g_{\frac{\partial}{\partial y^k}}\frac{\partial}{\partial y^i}\Big)\,,\\
&\qquad\qquad D^T_{e_\mu}\varphi_T\Big\rangle_g\,vol_h\\
&\quad + T_{m-1}\int_U Tr\sum_\mu\Big\langle(ad\otimes\nabla^g)_{D_{e_\mu}\varphi_T}\partial_t\varphi_T\,,\,\nabla^{T,(\varphi_T,g)}_{e_\mu}\partial_s\varphi_T\Big\rangle_g\,vol_h\\
&\quad - T_{m-1}\int_U Tr\sum_\mu\Big\langle(ad\otimes\nabla^g)_{D_{e_\mu}\varphi_T}\partial_t\varphi_T\,,\,(ad\otimes\nabla^g)_{\partial_s\varphi_T}D^T_{e_\mu}\varphi_T\Big\rangle_g\,vol_h\\
&\quad - T_{m-1}\int_U Tr\Big\langle\partial_t\varphi_T\,,\,\sum_\mu F_{\nabla^{T,(\varphi_T,g)}}(\partial_s,e_\mu)D^T_{e_\mu}\varphi_T\Big\rangle_g\,vol_h\\
&\quad + T_{m-1}\int_U Tr\Big\langle\partial_t\varphi_T\,,\,\textstyle\sum_{i=1}^n\Big(\big[(\partial_s\nabla^T)(\sum_\mu\nabla^h_{e_\mu}e_\mu),\varphi_T^\sharp(y^i)\big]\\
&\qquad\qquad - \textstyle\sum_\mu\nabla^{T,(\varphi_T,g)}_{e_\mu}\big[(\partial_s\nabla^T)(e_\mu),\varphi_T^\sharp(y^i)\big]\Big)\otimes\frac{\partial}{\partial y^i}\Big\rangle_g\,vol_h\,.\\
&\quad + T_{m-1}\int_U Tr\sum_\mu\Big\langle\sum_{i,j}\big[ad_{(\partial_s\nabla^T)(e_\mu)}\varphi_T^\sharp(y^j),\partial_t\varphi_T^\sharp(y^i)\big]\otimes\nabla^g_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^i}\,,\,D^T_{e_\mu}\varphi_T\Big\rangle_g\,vol_h\\
&\quad + T_{m-1}\int_U Tr\sum_\mu\Big\langle(ad\otimes\nabla^g)_{D_{e_\mu}\varphi_T}\partial_t\varphi_T\,,\,\sum_i\big[(\partial_s\nabla^T)(e_\mu),\varphi_T^\sharp(y^i)\big]\otimes\frac{\partial}{\partial y^i}\Big\rangle_g\,vol_h\\
&\quad + T_{m-1}\int_U\sum_{\mu=1}^m\Big\langle\sum_{i=1}^n\Big(\big[(\partial_t\nabla^T)(e_\mu),\partial_s\varphi_T^\sharp(y^i)\big]\otimes\frac{\partial}{\partial y^i}\\
&\qquad\qquad + \big[(\partial_t\nabla^T)(e_\mu),\varphi_T^\sharp(y^i)\big]\textstyle\sum_j\partial_s\varphi_T^\sharp(y^j)\otimes\nabla^g_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^i}\Big)\,,\,D^T_{e_\mu}\varphi_T\Big\rangle_g\,vol_h
\end{aligned}
$$

$$+\ T_{m-1}\int_U \sum_{\mu=1}^m \Big\langle \sum_{i=1}^n \big[(\partial_t\nabla^T)(e_\mu),\varphi_T^\sharp(y^i)\big]\otimes\frac{\partial}{\partial y^i}\,,\ \nabla^{T,(\varphi_T,g)}_{e_\mu}\partial_s\varphi_T\Big\rangle_g\, vol_h$$

$$-\ T_{m-1}\int_U \sum_{\mu=1}^m \Big\langle \textstyle\sum_{i=1}^n \big[(\partial_t\nabla^T)(e_\mu),\varphi_T^\sharp(y^i)\big]\otimes\frac{\partial}{\partial y^i}\,,\ (ad\otimes\nabla^g)_{\partial_s\varphi_T} D^T_{e_\mu}\varphi_T\Big\rangle_g\, vol_h$$

$$+\ T_{m-1}\int_U \sum_{\mu=1}^m \sum_{i,j=1}^n \Big\langle \big[(\partial_t\nabla^T)(e_\mu),\varphi_T^\sharp(y^i)\big]\otimes\frac{\partial}{\partial y^i}\,,\ \big[(\partial_s\nabla^T)(e_\mu),\varphi_T^\sharp(y^j)\big]\otimes\frac{\partial}{\partial y^j}\Big\rangle_g\, vol_h\,.$$

Here,

$$\begin{aligned}
\xi^T_{(\mathrm{I}^2,\partial_s\partial_t\varphi_T)} &:= \sum_{\mu=1}^n Tr\langle \partial_s\partial_t\varphi_T\,,\ D^T_{e_\mu}\varphi_T\rangle_g\, e_\mu\,,\\
\xi^T_{(\mathrm{I}^2,\partial_t\varphi_T,\nabla^{T,(\varphi_T,g)})} &= \sum_\mu Tr\langle \partial_t\varphi_T\,,\ \nabla^{T,(\varphi_T,g)}_{e_\mu}\partial_s\varphi_T\rangle_g\, e_\mu\,,\\
\xi^T_{(\mathrm{I}^2,\partial_t\varphi_T,\mathrm{D}^T\varphi_T)} &= \sum_\mu Tr\langle \partial_t\varphi_T\,,\ (ad\otimes\nabla^g)_{D^T_{e_\mu}\varphi_T}\partial_s\varphi_T\rangle_g\, e_\mu\,,\\
\xi^T_{(\mathrm{I}^2,\partial_t\varphi_T,\partial_s\nabla^T)} &= \sum_\mu Tr\langle \partial_t\varphi_T\,,\ \textstyle\sum_{i=1}^n[(\partial_s\nabla^T)(e_\mu),\varphi_T^\sharp(y^i)]\otimes\frac{\partial}{\partial y^i}\rangle_g\, e_\mu
\end{aligned}$$

and

$$F_{\nabla^{T,(\varphi_T,g)}}(\partial_s,e_\mu)\,D^T_{e_\mu}\varphi_T = \big(\partial_s\nabla^{T,(\varphi_T,g)}_{e_\mu} - \nabla^{T,(\varphi_T,g)}_{e_\mu}\partial_s\big)\textstyle\sum_{i=1}^n D_{e_\mu}\varphi_T^\sharp(y^i)\otimes\frac{\partial}{\partial y^i}$$

$$= \sum_{i=1}^n[(\partial_s\nabla^T)(e_\mu),D_{e_\mu}\varphi_T^\sharp(y^i)]\otimes\frac{\partial}{\partial y^i} + \sum_{i=1}^n D_{e_\mu}\varphi_T^\sharp(y^i)\sum_{j=1}^n[(\partial_s\nabla^T)(e_\mu),\varphi_T^\sharp(y^j)]\otimes\nabla^g_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^i}$$

$$+ \sum_{i=1}^n D_{e_\mu}\varphi_T^\sharp(y^i)\sum_{j,k=1}^n\Big(D^T_{e_\mu}\varphi_T^\sharp(y^j)\,\partial_s\varphi_T^\sharp(y^k)\otimes\nabla^g_{\frac{\partial}{\partial y^k}}\nabla^g_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^i} - \partial_s\varphi_T^\sharp(y^k)\,D^T_{e_\mu}\varphi_T^\sharp(y^j)\otimes\nabla^g_{\frac{\partial}{\partial y^j}}\nabla^g_{\frac{\partial}{\partial y^k}}\frac{\partial}{\partial y^i}\Big)$$

The summands

$$+\ T_{m-1}\int_U Tr\Big\langle \partial_s\partial_t\varphi_T\,,\ \big(D^T_{\sum_\mu\nabla^h_{e_\mu}e_\mu} - \textstyle\sum_\mu\nabla^{T,(\varphi_T,g)}_{e_\mu}D^T_{e_\mu}\big)\varphi_T\Big\rangle_g\, vol_h$$

$$+\ T_{m-1}\int_U Tr\sum_\mu\Big\langle (ad\otimes\nabla^g)_{D^T_{e_\mu}}(\partial_s\partial_t\varphi_T)\,,\ D^T_{e_\mu}\varphi_T\Big\rangle_g\, vol_h$$

$$+\ T_{m-1}\int_U \sum_{\mu=1}^m\Big\langle \sum_{i=1}^n[(\partial_s\partial_t\nabla^T)(e_\mu),\varphi_T^\sharp(y^i)]\otimes\frac{\partial}{\partial y^i}\,,\ D^T_{e_\mu}\varphi_T\Big\rangle_g\, vol_h$$

will vanish when imposing the equations of motion for (φ,∇).

If (φ_T,∇^T) is furthermore a $(*_2)$-admissible T-family of $(*_2)$-admissible pairs, Then, the above expression reduces to

$$\frac{\partial}{\partial s}\frac{\partial}{\partial t}E^{\nabla^T}(\varphi_T)^C = \frac{\partial}{\partial s}\frac{\partial}{\partial t}\Big(\frac12 T_{m-1}\int_U Tr\langle D^T\varphi_T\,,\ D^T\varphi_T\rangle_{(h,g)}\, vol_h\Big)$$

$$= T_{m-1}\int_{\partial U} i_{\xi^T_{(\mathrm{I}^2,\partial_s\partial_t\varphi_T)}}\, vol_h$$

$$+ T_{m-1}\int_{\partial U} i_{\xi^T_{(\mathrm{I}^2,\partial_t\varphi_T,\nabla^{T,(\varphi_T,g)})}+\xi^T_{(\mathrm{I}^2,\partial_t\varphi_T,\mathrm{D}^T\varphi_T)}+\xi^T_{(\mathrm{I}^2,\partial_t\varphi_T,\partial_s\nabla^T)}}\, vol_h$$

$$+\ T_{m-1}\int_U Tr\Big\langle \partial_s\partial_t\varphi_T\,,\ \big(D^T_{\sum_\mu\nabla^h_{e_\mu}e_\mu} - \textstyle\sum_\mu\nabla^{T,(\varphi_T,g)}_{e_\mu}D^T_{e_\mu}\big)\varphi_T\Big\rangle_g\, vol_h$$

$$+\ T_{m-1}\int_U \sum_{\mu=1}^m\Big\langle \sum_{i=1}^n[(\partial_s\partial_t\nabla^T)(e_\mu),\varphi_T^\sharp(y^i)]\otimes\frac{\partial}{\partial y^i}\,,\ D^T_{e_\mu}\varphi_T\Big\rangle_g\, vol_h$$

$$+\ T_{m-1}\int_U Tr\Big\langle \partial_t\varphi_T\,,\ \Big(\nabla^{T,(\varphi_T,g)}_{\sum_\mu\nabla^h_{e_\mu}e_\mu} - \textstyle\sum_\mu\nabla^{T,(\varphi_T,g)}_{e_\mu}\nabla^{T,(\varphi_T,g)}_{e_\mu}\Big)\partial_s\varphi_T\Big\rangle_g\, vol_h$$

$$
\begin{aligned}
&+\ T_{m-1}\int_U Tr\Big\langle \partial_t\varphi_T\,,\,\Big((ad\otimes\nabla^g)_{D^T_{\sum_\mu \nabla^h_{e_\mu}e_\mu}\varphi_T} \\
&\qquad\qquad -\textstyle\sum_\mu \nabla^{T,(\varphi_T,g)}_{e_\mu}(ad\otimes\nabla^g)_{D^T_{e_\mu}\varphi_T}\Big)\partial_s\varphi_T\Big\rangle_g\, vol_h \\
&-\ T_{m-1}\int_U Tr\Big\langle \partial_t\varphi_T\,,\,\sum_\mu F_{\nabla^{T,(\varphi_T,g)}}(\partial_s,e_\mu)D^T_{e_\mu}\varphi_T\Big\rangle_g\, vol_h \\
&+\ T_{m-1}\int_U Tr\Big\langle \partial_t\varphi_T\,,\,\textstyle\sum_{i=1}^n\Big([(\partial_s\nabla^T)(\sum_\mu\nabla^h_{e_\mu}e_\mu),\varphi_T^\sharp(y^i)] \\
&\qquad\qquad -\textstyle\sum_\mu\nabla^{T,(\varphi_T,g)}_{e_\mu}[(\partial_s\nabla^T)(e_\mu),\varphi_T^\sharp(y^i)]\Big)\otimes\dfrac{\partial}{\partial y^i}\Big\rangle_g\, vol_h\,. \\
&+\ T_{m-1}\int_U\sum_{\mu=1}^m\Big\langle\sum_{i=1}^n\Big([(\partial_t\nabla^T)(e_\mu),\partial_s\varphi_T^\sharp(y^i)]\otimes\frac{\partial}{\partial y^i} \\
&\qquad\qquad +[(\partial_t\nabla^T)(e_\mu),\varphi_T^\sharp(y^i)]\textstyle\sum_j\partial_s\varphi_T^\sharp(y^j)\otimes\nabla^g_{\frac{\partial}{\partial y^j}}\dfrac{\partial}{\partial y^i}\Big)\,,\,D^T_{e_\mu}\varphi_T\Big\rangle_g\, vol_h \\
&+\ T_{m-1}\int_U\sum_{\mu=1}^m\Big\langle\sum_{i=1}^n[(\partial_t\nabla^T)(e_\mu),\varphi_T^\sharp(y^i)]\otimes\frac{\partial}{\partial y^i}\,,\,\nabla^{T,(\varphi_T,g)}_{e_\mu}\partial_s\varphi_T\Big\rangle_g\, vol_h \\
&-\ T_{m-1}\int_U\sum_{\mu=1}^m\Big\langle\textstyle\sum_{i=1}^n[(\partial_t\nabla^T)(e_\mu),\varphi_T^\sharp(y^i)]\otimes\dfrac{\partial}{\partial y^i}\,,\,(ad\otimes\nabla^g)_{\partial_s\varphi_T}D^T_{e_\mu}\varphi_T\Big\rangle_g\, vol_h \\
&+\ T_{m-1}\int_U\sum_{\mu=1}^m\sum_{i,j=1}^n\Big\langle[(\partial_t\nabla^T)(e_\mu),\varphi_T^\sharp(y^i)]\otimes\frac{\partial}{\partial y^i}\,,\,[(\partial_s\nabla^T)(e_\mu),\varphi_T^\sharp(y^j)]\otimes\frac{\partial}{\partial y^j}\Big\rangle_g\, vol_h\,.
\end{aligned}
$$

If one further imposes the equations of motion on (φ,∇)*, then the expression reduces further to*

$$
\begin{aligned}
\frac{\partial}{\partial s}\frac{\partial}{\partial t}E^{\nabla^T}(\varphi_T)^C \;&=\; \frac{\partial}{\partial s}\frac{\partial}{\partial t}\Big(\frac12\,T_{m-1}\int_U Tr\langle D^T\varphi_T\,,\,D^T\varphi_T\rangle_{(h,g)}\,vol_h\Big) \\
=\;& T_{m-1}\int_{\partial U} i_{\xi^T_{(\mathrm{I}^2,\partial_s\partial_t\varphi_T)}}vol_h \\
&+T_{m-1}\int_{\partial U} i_{\xi^T_{(\mathrm{I}^2,\partial_t\varphi_T,\nabla^{T,(\varphi_T,g)})}+\xi^T_{(\mathrm{I}^2,\partial_t\varphi_T,D^T\varphi_T)}+\xi^T_{(\mathrm{I}^2,\partial_t\varphi_T,\partial_s\nabla^T)}}vol_h \\
&+\ T_{m-1}\int_U Tr\Big\langle\partial_t\varphi_T\,,\,\Big(\nabla^{T,(\varphi_T,g)}_{\sum_\mu\nabla^h_{e_\mu}e_\mu}-\textstyle\sum_\mu\nabla^{T,(\varphi_T,g)}_{e_\mu}\nabla^{T,(\varphi_T,g)}_{e_\mu}\Big)\partial_s\varphi_T\Big\rangle_g\, vol_h \\
&+\ T_{m-1}\int_U Tr\Big\langle \partial_t\varphi_T\,,\,\Big((ad\otimes\nabla^g)_{D^T_{\sum_\mu \nabla^h_{e_\mu}e_\mu}\varphi_T} \\
&\qquad\qquad -\textstyle\sum_\mu \nabla^{T,(\varphi_T,g)}_{e_\mu}(ad\otimes\nabla^g)_{D^T_{e_\mu}\varphi_T}\Big)\partial_s\varphi_T\Big\rangle_g\, vol_h \\
&-\ T_{m-1}\int_U Tr\Big\langle \partial_t\varphi_T\,,\,\sum_\mu F_{\nabla^{T,(\varphi_T,g)}}(\partial_s,e_\mu)D^T_{e_\mu}\varphi_T\Big\rangle_g\, vol_h \\
&+\ T_{m-1}\int_U Tr\Big\langle \partial_t\varphi_T\,,\,\textstyle\sum_{i=1}^n\Big([(\partial_s\nabla^T)(\sum_\mu\nabla^h_{e_\mu}e_\mu),\varphi_T^\sharp(y^i)] \\
&\qquad\qquad -\textstyle\sum_\mu\nabla^{T,(\varphi_T,g)}_{e_\mu}[(\partial_s\nabla^T)(e_\mu),\varphi_T^\sharp(y^i)]\Big)\otimes\dfrac{\partial}{\partial y^i}\Big\rangle_g\, vol_h\,. \\
&+\ T_{m-1}\int_U\sum_{\mu=1}^m\Big\langle\sum_{i=1}^n\Big([(\partial_t\nabla^T)(e_\mu),\partial_s\varphi_T^\sharp(y^i)]\otimes\frac{\partial}{\partial y^i} \\
&\qquad\qquad +[(\partial_t\nabla^T)(e_\mu),\varphi_T^\sharp(y^i)]\textstyle\sum_j\partial_s\varphi_T^\sharp(y^j)\otimes\nabla^g_{\frac{\partial}{\partial y^j}}\dfrac{\partial}{\partial y^i}\Big)\,,\,D^T_{e_\mu}\varphi_T\Big\rangle_g\, vol_h \\
&+\ T_{m-1}\int_U\sum_{\mu=1}^m\Big\langle\sum_{i=1}^n[(\partial_t\nabla^T)(e_\mu),\varphi_T^\sharp(y^i)]\otimes\frac{\partial}{\partial y^i}\,,\,\nabla^{T,(\varphi_T,g)}_{e_\mu}\partial_s\varphi_T\Big\rangle_g\, vol_h \\
&-\ T_{m-1}\int_U\sum_{\mu=1}^m\Big\langle\textstyle\sum_{i=1}^n[(\partial_t\nabla^T)(e_\mu),\varphi_T^\sharp(y^i)]\otimes\dfrac{\partial}{\partial y^i}\,,\,(ad\otimes\nabla^g)_{\partial_s\varphi_T}D^T_{e_\mu}\varphi_T\Big\rangle_g\, vol_h \\
&+\ T_{m-1}\int_U\sum_{\mu=1}^m\sum_{i,j=1}^n\Big\langle[(\partial_t\nabla^T)(e_\mu),\varphi_T^\sharp(y^i)]\otimes\frac{\partial}{\partial y^i}\,,\,[(\partial_s\nabla^T)(e_\mu),\varphi_T^\sharp(y^j)]\otimes\frac{\partial}{\partial y^j}\Big\rangle_g\, vol_h\,.
\end{aligned}
$$

7.2 The second variation of the dilaton term

We now work out the second variation of the (complexified) dilaton term

$$\begin{aligned} S^{(\rho,h;\Phi)}_{dilaton}(\varphi_T)^C &= \int_U Tr\langle d\rho\,,\, \varphi_T^\diamond d\Phi\rangle_h\, vol_h \\ &= \int_U Tr\Big(\sum_\mu d\rho(e_\mu)\,((D^T_{e_\mu}\varphi_T)\Phi)\Big)\, vol_h\,. \end{aligned}$$

for an $(*_1)$-admissible family of $(*_1)$-admissible pairs (φ_T, ∇^T), $T := (-\varepsilon,\varepsilon)^2 \subset R^2$ with coordinate (s,t).

It follows from Sec. 6.2 that, due to the effect of the trace map Tr,

$$\begin{aligned} \frac{\partial}{\partial t} S^{(\rho,h;\Phi)}_{dilaton}(\varphi_T)^C &= \int_U Tr \sum_{\mu=1}^m d\rho(e_\mu)\,\partial_t((D^T_{e_\mu}\varphi_T)\Phi)\, vol_h \\ &= \int_U Tr \sum_\mu d\rho(e_\mu) D^T_{e_\mu}((\partial_t\varphi_T)\Phi)\, vol_h\,. \end{aligned}$$

Thus, due to the effect of the trace map Tr again,

$$\begin{aligned} \frac{\partial}{\partial s}\frac{\partial}{\partial t} S^{(\rho,h;\Phi)}_{dilaton}(\varphi_T)^C &= \int_U Tr \sum_\mu d\rho(e_\mu)\partial_s D^T_{e_\mu}((\partial_t\varphi_T)\Phi)\, vol_h \\ &= \int_U Tr \sum_\mu d\rho(e_\mu) D^T_{e_\mu}\partial_s((\partial_t\varphi_T)\Phi)\, vol_h \\ &= \int_U Tr \sum_\mu d\rho(e_\mu) D^T_{e_\mu}((\partial_s\partial_t\varphi_T)\Phi)\, vol_h \\ &\quad + \int_U Tr \sum_\mu d\rho(e_\mu) D^T_{e_\mu}\Big(\sum_{i,j}\partial_t\varphi_T^\sharp(y^i)\partial_s\varphi_T^\sharp(y^j)\otimes\Big(\frac{\partial}{\partial y^i}\frac{\partial}{\partial y^j} - \nabla^g_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j}\Big)\Phi\Big)\, vol_h \\ &= (\mathrm{II}^2.1) + (\mathrm{II}^2.2)\,. \end{aligned}$$

For Summand $(\mathrm{II}^2.1)$, repeating the same argument in Sec. 6.1 for Summand (I.1.1), one concludes that

$$\begin{aligned} (\mathrm{II}^2.1) &= \int_{\partial U} i_{\xi^T_{(\mathrm{II}^2,\partial_s\partial_t\varphi_T)}}\, vol_h \\ &\quad + \int_U \Big(\sum_\mu \nabla^h_{e_\mu}e_\mu - \sum_\mu e_\mu d\rho(e_\mu)\Big)\, Tr((\partial_s\partial_t\varphi_T)\Phi)\, vol_h\,, \end{aligned}$$

where

$$\xi^T_{(\mathrm{II}^2,\partial_s\partial_t\varphi_T)} := \sum_\mu \Big(d\rho(e_\mu)\, Tr((\partial_s\partial_t\varphi_T)\Phi)\Big)e_\mu \qquad \in \mathcal{T}_*(U_T/T)\,.$$

The second summand of Summand $(\mathrm{II}^2.1)$ above is the term that captures the $S^{(\rho,h;\Phi)}_{dilaton}(\varphi)$-contribution to the system of equations of motion for (φ,∇).

With $\partial_s\partial_t\varphi_T$ replaced by $\sum_{i,j}\partial_t\varphi_T^\sharp(y^i)\partial_s\varphi_T^\sharp(y^j)\otimes(\frac{\partial}{\partial y^i}\frac{\partial}{\partial y^j} - \nabla^g_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j})\Phi$, one has similarly

$$\begin{aligned} (\mathrm{II}^2.2) &= \int_{\partial U} i_{\xi^T_{(\mathrm{II}^2,\partial_t\varphi_T,\partial_s\varphi_T)}}\, vol_h \\ &\quad + \int_U \Big(\sum_\mu \nabla^h_{e_\mu}e_\mu - \sum_\mu e_\mu d\rho(e_\mu)\Big)\cdot \\ &\qquad\qquad Tr\Big(\sum_{i,j}\partial_t\varphi_T^\sharp(y^i)\partial_s\varphi_T^\sharp(y^j)\otimes\Big(\frac{\partial}{\partial y^i}\frac{\partial}{\partial y^j} - \nabla^g_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j}\Big)\Phi\Big)\, vol_h\,, \end{aligned}$$

where

$$\xi^T_{(\mathrm{II}^2,\partial_t\varphi_T,\partial_s\varphi_T)} := \sum_\mu \Big(d\rho(e_\mu)\, Tr\Big(\sum_{i,j}\partial_t\varphi_T^\sharp(y^i)\partial_s\varphi_T^\sharp(y^j)\otimes\Big(\frac{\partial}{\partial y^i}\frac{\partial}{\partial y^j} - \nabla^g_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j}\Big)\Phi\Big)\Big)e_\mu$$

in $\mathcal{T}_*(U_T/T)$. The second summand of Summand (II^2.2) above contributes to the zeroth order terms of the differential operator on $(\partial_s\varphi_T, \partial_t\varphi_T)$ from the second variation of $S^{(\rho,h;\Phi,g,B,C)}_{standard}(\varphi,\nabla)$.

In summary,

Proposition 7.2.1 [second variation of $S^{(\rho,h;\Phi)}_{dilaton}(\varphi)^C$] *For the (complexified) dilaton term*

$$S^{(\rho,h;\Phi)}_{dilaton}(\varphi)^C \ :=\ \int_U Tr\langle d\rho\,,\,\varphi^\diamond d\Phi\rangle_h\, vol_h\,,$$

its second variation for a $(*_1)$-admissible family of $(*_1)$-admissible pairs (φ_T, ∇^T), $T := (-\varepsilon,\varepsilon)^2 \subset R^2$ with coordinate (s,t), is given by

$$\begin{aligned}
&\frac{\partial}{\partial s}\frac{\partial}{\partial t} S^{(\rho,h;\Phi)}_{dilaton}(\varphi_T)^C \\
&= \int_{\partial U} i_{\xi^T_{(\mathrm{II}^2,\partial_s\partial_t\varphi_T)}+\xi^T_{(\mathrm{II}^2,\partial_t\varphi_T\partial_s\varphi_T)}} vol_h \\
&\quad + \int_U \Big(\sum_\mu \nabla^h_{e_\mu}e_\mu - \sum_\mu e_\mu d\rho(e_\mu)\Big)\, Tr((\partial_s\partial_t\varphi_T)\Phi)\, vol_h \\
&\quad + \int_U \Big(\sum_\mu \nabla^h_{e_\mu}e_\mu - \sum_\mu e_\mu d\rho(e_\mu)\Big)\cdot \\
&\qquad\qquad Tr\Big(\sum_{i,j}\partial_t\varphi^\sharp_T(y^i)\partial_s\varphi^\sharp_T(y^j)\otimes\Big(\frac{\partial}{\partial y^i}\frac{\partial}{\partial y^j} - \nabla^g_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j}\Big)\Phi\Big)\, vol_h\,,
\end{aligned}$$

where

$$\begin{aligned}
\xi^T_{(\mathrm{II}^2,\partial_s\partial_t\varphi_T)} &:= \sum_\mu \Big(d\rho(e_\mu)\, Tr((\partial_s\partial_t\varphi_T)\Phi)\Big)e_\mu \\
\xi^T_{(\mathrm{II}^2,\partial_t\varphi_T,\partial_s\varphi_T)} &:= \sum_\mu \Big(d\rho(e_\mu)\, Tr\Big(\sum_{i,j}\partial_t\varphi^\sharp_T(y^i)\partial_s\varphi^\sharp_T(y^j)\otimes\Big(\frac{\partial}{\partial y^i}\frac{\partial}{\partial y^j} - \nabla^g_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j}\Big)\Phi\Big)\Big)e_\mu
\end{aligned}$$

in $\mathcal{T}_*(U_T/T)$. The integral

$$\int_U \Big(\sum_\mu \nabla^h_{e_\mu}e_\mu - \sum_\mu e_\mu d\rho(e_\mu)\Big)\, Tr((\partial_s\partial_t\varphi_T)\Phi)\, vol_h$$

would vanish when imposing the equations of motion of (φ,∇) after the combination with other Equations-of-Motion capturing parts from the second variation of other terms in $S^{(\rho,h;\Phi,g,B,C)}_{standard}(\varphi,\nabla)^C$.

8 Conclusion

The current notes lay down some foundation toward the dynamics of D-branes along the line of our D-project. Solutions to the system of equations of motion from the total action $S^{(\Phi,g,B)}_{\mathrm{DBI}}(\varphi,\nabla)+S^{(C,B)}_{CS/WZ}(\varphi,\nabla)$ for a D-brane world-volume should be thought of as an Azumaya/matrix version of minimal submanifolds or harmonic maps, twisted/bent, on one hand, by the (dynamical) gauge field ∇ on the domain manifold X with a (noncommutative) endomorphism/matrix function-ring and, on the other hand, by the background field (Φ,g,B,C), created by closed (super)strings, on the target space(-time) Y. Further details, issues, and examples are the focus of the sequels. At the classical level Polyakov superstring or its generalization, a sigma model, is a theory of harmonic maps on the mathematical side. We construct all the building blocks to generalize the existing theory of harmonic maps to a theory of maps $\varphi : (X^{Az},\mathcal{E};\nabla)\to Y$, which describe D-branes. It will turn out that both the connection ∇ and the Admissible Condition $(*_1)$ chosen are needed to build up a mathematically sound theory for such maps φ. We introduce a new action $S^{(\rho,h;\Phi,g,B,C)}_{standard}$ for D-branes that is to D-branes as the Polyakov action is to fundamental superstrings. This 'standard action' is abstractly a non-Abelian gauged sigma model based on maps $\varphi : (X^{Az}, E; \nabla) \to Y$ from an Azumaya/matrix manifold X^{Az} with a fundamental module E with a connection ∇ to Y enhanced by the dilaton term, the gauge-theory term, and the Chern-Simons/Wess-Zumino term that couples (φ, ∇) to Ramond-Ramond field. In a special situation, this new theory merges the theory of harmonic maps and a gauge theory, with a nilpotent type fuzzy extension. With the analysis developed for such maps and an improved understanding of the hierarchy of various admissible conditions on the pairs (φ, ∇) and how they resolve the built-in obstruction to pull-push of covariant tensors under a map from a noncommutative manifold to a commutative manifold, we develop further in this note some covariant differential calculus needed and apply them to work out the first variation and hence the corresponding equations of motion for D-branes of the standard action and the second variation of the kinetic term for maps and the dilaton term in this action. Compared with the non-Abelian Dirac-Born-Infeld action constructed in the article along the same line, the current note brings the Nambu-Goto-string-to-Polyakov-string analogue to D-branes. The current bosonic setting is the first step toward the dynamics of fermionic D-branes and their quantization as fundamental dynamical objects, in parallel to what happened to the theory of fundamental superstrings. It's by now a history that as the built-in structure of a string is far richer than that for a point, a physical theory that takes strings as fundamental objects has brought us to where a physical theory that takes only point-particles as fundamental objects cannot reach. Now that a D-brane carries even more built-in structures, are these even-richer-than-string structures all just in vain? Or is a physical theory that takes D-branes as fundamental objects going to lead us to somewhere beyond that from string theories? Besides a theory in its own right, a theory that takes D-branes as fundamental objects has deep connection with other themes outside. In particular, at low dimensions, that there should be the following connections are "obvious" but most details to realize these connections remain far from reach at the moment.

References

[1] E. Artin, C.J. Nesbitt, and R.M. Thrall, *Rings with minimum condition*, Univ. of Michigan Press, 1944.

[2] D. Bleecker, *Gauge theory and vatiational principles*, Addison-Wesley, 1981.

[3] B. Booss and D.D. Bleecker, *Topology and analysis – The Atiyah-Singer Index Formula and gauge-theoretic physics*, English ed. translated by D.D. Bleecker and A. Mader, Springer, 1985.

[4] L. Brink, P. Di Vecchia, and P. Howe, *A locally supersymmetric and reparameterization invariant action for the spinning string*, Phys. Lett. **65B** (1976), 471-474.

[5] R. Bott and L.W. Tu, *Differential forms in algebraic geometry*, GTM 82, Springer, 1982.

[6] M. Carmeli, Kh. Hulerhil, and E. Leibowitz, *Gauge fields: classiication and equations of motion*, World Scientific, 1989.

[7] C.G. Callan, Jr. and L. Thorlacius, *Sigma models and string theory*, in *Particles, strings and supernovae (TASI 88)*, A. Jevicki and C.-I. Tan eds., 795–878, World Scientific, 1989.

[8] P. Deligne, P. Etingof, D.S. Freed, L.C. Jeffrey, D. Kazhdan, J.W. Morgan, D.R. Morrison, and E. Witten eds., *Quantum fields and strings: A course for mathematicians*, vol. 1 and vol. 2, Amer. Math. Soc. and Inst. Advanced Study, 1999.

[9] S.K. Donaldson and P.B. Kronheimer, *The geometry of four-manifolds*, Oxford Univ. Press, 1990.

[10] M. Dubois-Violette and T. Masson, *$SU(n)$-connections and noncommutative differential geometry*, *J. Geom. Phys.* **25** (1998), 104–118. (arXiv:dg-ga/9612017)

[11] S. Deser and B. Zumino, *A complete action for the spinning string*, *Phys. Lett.* **65B** (1976), 369-373.

[12] D. Eisenbud, *Commutative algebra – with a view toward algebraic geometry*, GTM 150, Springer, 1995.

[13] L.P. Eisenhart, *Riemannian geometry*, 5th printing, Princeton Univ. Press, 1964.

[14] J. Eells, Jr. and L. Lemaire, *A report on harmonic maps*, *Bull. London Math. Soc.* **10** (1978), 1–68.

[15] J. Eells, Jr. and J.H. Sampson, *Harmonic mappings of Riemannian manifolds*, *Amer. J. Math.* **86** (1964), 109–160.

[16] S. Gallot, D. Hulin, and J. Lafontaine, *Riemannian geometry*, 2nd ed., Springer, 1990.

[17] M.B. Green, J.H. Schwarz, and E. Witten, *Superstring theory*, *vol. 1*: *Introduction*; *vol. 2*: *Loop amplitudes, anomalies, and phenomenology*, Cambridge Univ. Press, 1987.

[18] J.M. Gracia-Bondía, J.C. Várilly, and H. Figueroa, *Elements of noncommutative geometry*, Birkhäuser, 2001.

[19] M. Gell-Mann and M. Lévy, *The axial vector current in beta decay*, *Il Nuovo Cimento* **16** (1960), 705–726.

[20] R. Hartshorne, *Algebraic geometry*, GTM 52, Springer, 1977.

[21] N.J. Hicks, *Notes on differential geometry*, Van Nostrand Math. Studies 3, D. Van Nostrand Co., Inc., 1965.

[22] G. 't Hooft ed., *50 years of Yang-Mills theory*, World Scientific, 2005.

[23] S.W. Hawking and G.F.R. Ellis, *The large scale structure of space-time*, Cambridge Univ. Press, 1973.

[24] D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves*, 2nd ed., Cambridge Univ. Press, 2010.

[25] P.-M. Ho and C.-T. Ma, *Effective action for Dp-brane in large RR $(p-1)$-form background*, axXiv:1302.6919 [hep-th].

[26] K. Hori and C. Vafa, *Mirror symmetry*, arXiv:hep-th/0002222.

[27] P.-M. Ho and Y.-S. Wu, *Noncommutative geometry and D-branes*, *Phys. Lett.* **B398** (1997), 52 – 60. (arXiv:hep-th/9611233)

[28] P.-M. Ho and C.-H. Yeh, *D-brane in R-R field background*, arXiv:1101.4054 [hep-th].

[29] J.D. Jackson, *Classical electrodynamics*, 2nd ed., John Wiley & Sons, 1990.

[30] D. Joyce, *Algebraic geometry over C^{∞}-rings*, arXiv:1001.0023v4 [math.AG].

[31] S. Kobayashi, *Differential geometry of complex vector bundles*, Math. Soc. Japan and Princeton Univ. Press, 1987.

[32] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, vol. I and vol. II, John Wiley & Sons, 1963 and 1969.

[33] H.B. Lawson, Jr., *Lectures on minimal submanifolds*, vol. 1, Mono. Mat. 14, Instituto de Matemática Pura e Aplicada, Rio de Janeiro, 1970.

[34] R.G. Leigh, *Dirac-Born-Infeld action from Dirichlet σ-model*, *Mod. Phys. Lett.* **A4** (1989), 2767–2772.

[35] C.-H. Liu and S.-T. Yau, *Azumaya-type noncommutative spaces and morphism therefrom: Polchinski's D-branes in string theory from Grothendieck's viewpoint*, arXiv:0709.1515 [math.AG]. (D(1))

[36] S. Li, C.-H. Liu, R. Song, S.-T. Yau, *Morphisms from Azumaya prestable curves with a fundamental module to a projective variety: Topological D-strings as a master object for curves*, arXiv:0809.2121 [math.AG]. (D(2))

[37] C.-H. Liu and S.-T. Yau, *Nontrivial Azumaya noncommutative schemes, morphisms therefrom, and their extension by the sheaf of algebras of differential operators: D-branes in a B-field background à la Polchinski-Grothendieck Ansatz*, arXiv:0909.2291 [math.AG]. (D(5))

[38] E. Mazet, *La formule de la variation seconde de l'énergie au voisinage d'une application harmonique*, *J. Diff. Geom.* **8** (1973), 279–296.

[39] R.S. Millman and G.D. Parker, *Elements of differential geometry*, Prentice-Hall, 1977.

[40] J. Polchinski, *Dirichlet-branes and Ramond-Ramond charges*, *Phys. Rev. Lett.* **75** (1995), pp. 4724 - 4727 (hep-th/9510017)

[41] M.E. Peskin and D.V. Schroeder, *An introduction to quantum field theory*, Addison-Wesley, 1995.

[42] R.T. Smith, *The second variation formula for harmonic maps*, *Proc. Amer. Math. Soc.* **47** (1975), 229–236.

[43] E. Witten, *Phases of $N = 2$ theories in two dimensions*, *Nucl. Phys.* **B403** (1993), pp. 159 - 222. (hep-th/9301042)

[44] F.W. Warner, *Foundations of differentiable manifolds and Lie groups*, Scott Foresmann & Company, 1971.