Number of stable digits of any integer tetration

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Abstract: In the present paper we provide a general formula which let us easily calculate the number of stable digits of any integer tetration base $a \in \mathbb{N}_0$. The number of stable digits, at the given height of the power tower, indicates how many of the last digits of the (generic) tetration are frozen. Our formula is exact for any tetration base which is not coprime to 10, although a maximum gap equal to $V(a) + 1$ digits (where $V(a)$ indicates the congruence speed of a) can occur, in the worst-case scenario, between the given upper and lower bound.

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1 Introduction

The aim of this paper is to give a general formula which returns the number of stable digits [2, 5, 9] of the tetration ${}^{b}a := \begin{cases} a & \text{if } b = 1 \\ a^{\left({}^{(b-1)}a\right)} & \text{if } b \geq 2 \end{cases}$, for any $a \in \mathbb{N}_0$, at any given height $b \in \mathbb{N} - \{0\}$ [3]. Thus, we are interested in an easy way to find the value of $n \in \mathbb{N}_0$ such that ${}^{b}a \equiv {}^{b+1}a (\text{mod } 10^n) \wedge {}^{b}a \not\equiv {}^{b+1}a (\text{mod } 10^{n+1})$.

In order to simplify the notations, let us invoke the definition of the congruence speed of ${}^{b}a$ from Reference [8], and then (Definition 2) we will extend it to the base $a = 0$.

Definition 1. *Let $n \in \mathbb{N}_0$ and assume that $a \in \mathbb{N} - \{0, 1\}$ is not a multiple of 10. Then, given ${}^{b-1}a \equiv {}^{b}a (\text{mod } 10^n) \wedge {}^{b-1}a \not\equiv {}^{b}a (\text{mod } 10^{n+1})$, $\forall b > a$, $V(a, b)$ returns the strictly positive integer such that ${}^{b}a \equiv {}^{b+1}a \left(\text{mod } 10^{n+V(a)}\right) \wedge {}^{b}a \not\equiv {}^{b+1}a \left(\text{mod } 10^{n+V(a)+1}\right)$, and we define $V(a, b)$ as the "congruence speed" of the base a at the given height of its hyperexponent $b \in \mathbb{N} - \{0\}$.*

Consequently, if $a = 2$, the tetrations for b from 1 to 5 are ${}^{1}2 = 2$, ${}^{2}2 = 4$, ${}^{3}2 = 16$, ${}^{4}2 = 65536$, and ${}^{5}2 = \ldots 19156736$ (respectively), so we can see that $\mathrm{V}(2,1) = \mathrm{V}(2,2) = 0$, whereas $\mathrm{V}(2,3) = \mathrm{V}(2,4) = 1$.

Definition 2. *Let $a = 1$, then $V(1,1) = 1$ and $V(1,b) = 0 = V(1)$ for any $b \geq 2$. We also define $V(0) = 0$ for any $b \in \mathbb{N} - \{0\}$, since it is possible to extend the domain of tetration by considering that $\lim\limits_{a \to 0} {}^{b}a := {}^{b}0$ implies ${}^{b}0 = 1$ if and only if b is even and ${}^{b}0 = 1$ otherwise (see* [1]*). Thus, for any $b \geq 1$, ${}^{b}0$ does not produce any stable digit, and $V(0,b) = V(0) = 0$ by Definition 1.*

Since, in general, n depends on a and b, from here on, let us indicate the number of stable digits of all the bases belonging to the congruence class $c(\text{mod } 10)$ as $\#S_c(a,b)$ (e.g., if we consider only tetration bases which have 3 or 7 as their rightmost digit, we will indicate the number of their stable digits, at height b, as $\#S_{\{3,7\}}(a,b)$).

From [7, 8] we know that, for any given a which is not a multiple of 10, exists a unique "optimal" value, $\bar{b} := \min_b\{b \in \mathbb{N} - \{0\} : V(a,b) = V(a,b+k), \forall k \in \mathbb{N}_0\}$, of the hyperexponent which guarantees $V(a, \bar{b} + k) = V(a)$ for any $k \in \mathbb{N}_0$ [5], and reaching a height of $a + 1$ represents a sufficient but not necessary condition for the constancy of the congruence speed, since $V(a, a+1) = V(a)$ is always true. Improved bounds for $\bar{b}(a)$ will be introduced in the next section.

For any given pair (a,b) of positive integers and assuming $c \in \{1,2,3,4,5,6,7,8,9\}$, by definition, we have that

$$\#S_c(a,b) := \sum_{i=1}^{b} V(a,i) = \begin{cases} \sum_{i=1}^{b} V(a,i) & \text{if} \quad b < \bar{b} \\ \sum_{j=1}^{\bar{b}-1} V(a,j) + \left(b - \bar{b} + 1\right) \cdot V(a) & \text{if} \quad b \geq \bar{b} \end{cases}. \tag{1}$$

Now, in the rest of the present paper, let us assume that $a \in \mathbb{N} : a \not\equiv 0(\text{mod } 10)$ does not belong to the congruence class 0 modulo 10, since, for any $b \geq 1$, if $a \equiv 0(\text{mod } 10)$, then the number of stable digits of ${}^{b}a$ corresponds to 0 if and only if $a = 0$ (by Definitions 1&2), and to the number of trailing zeros which appear at the end of ${}^{b}\big((k+1) \cdot 10\big)$ otherwise (e.g., if $k = 1$ and $b = 2$, we have ${}^{b}a = {}^{2}20 = 2^{20} \cdot 10^{20}$, so that $\#S_0(20,2) = 20$).

In subsection 2.1 we show an easy formula that returns the exact value of $\#S_c(a,b)$ for any c which is not coprime to 10, whereas subsection 2.2 is devoted to study the four remaining cases.

2 A formula for the number of stable digits of ${}^{b}a : a \not\equiv 0(\text{mod } 10)$

In this section we study $\#S_c(a,b)$ assuming that the last digit of the tetration base is not equal to zero, so that the residues modulo 10 of c cover the whole set $\{1,2,3,4,5,6,7,8,9\}$.

For this purpose, let us indicate the p-adic valuation of $d \in \mathbb{N} - \{0\}$ as $v_p(d)$. By definition, $v_p(d)$ returns the highest exponent v_p such that p^{v_p} divides d [4]. Now, from [8], we know that the constant congruence speed of any given base a which is not congruent to 0 modulo 5 is (always) less than or equal to the 5-adic valuation of

- $a - 1$ if $a \equiv 1(\text{mod } 5)$;
- $a^2 + 1$ if $a \equiv \{2,3\}(\text{mod } 5)$;

- $a+1$ if $a \equiv 4 \pmod 5$;

while, if $a : a \equiv 5 \pmod{10}$, we have that $V(a)+1 = v_2(a^2-1)$ (see [8], Corollary 2, pp. 55-56).

Moreover, $V(10\cdot k+2 \ \vee \ 10\cdot k+8) = v_5(a^2+1)$, $V(10\cdot k+4) = v_5(a+1)$, and $V(10\cdot k+6) = v_5(a-1)$ are true for any $k \in \mathbb{N}_0$.

Definition 3. *Let* $a \not\equiv 0 \pmod{10}$. *We define* $\tilde{v}(a) := \begin{cases} v_5(a-1) & \text{iff } \ a \equiv 1 \pmod 5 \\ v_5(a^2+1) & \text{iff } \ a \equiv \{2,3\} \pmod 5 \\ v_5(a+1) & \text{iff } \ a \equiv 4 \pmod 5 \\ v_2(a^2-1)-1 & \text{iff } \ a \equiv 5 \pmod{10} \end{cases}$.

2.1 The exact value of $\#S_{\{2,4,5,6,8\}}(a,b)$

Assuming radix-10 [2], as usual, we describe the structure $\#S(a,b)$ providing an exact formula for any pair (a,b) such that $a \equiv \{2,4,5,6,8\} \pmod{10} \ \wedge \ b \geq 3$, and very tight bounds which hold for all the bases $a : a \equiv \{1,3,7,9\} \pmod{10}$.

Let $k \in \mathbb{N}_0$ and assume that $a = (20\cdot k+2 \ \vee \ 20\cdot k+18)$. Then, for any $a : a \equiv \{2,18\} \pmod{20}$, ${}^1a \equiv \{2,8\} \pmod{10}$, ${}^2a \equiv 4 \pmod{10}$, and finally ${}^3a \equiv {}^4a \pmod{10}$ since ${}^3a \equiv 6 \pmod{10}$. It follows that

$$\#S_{\{2,8\}}(20\cdot k+2 \ \vee \ 20\cdot k+18, b) = \begin{cases} 0 & \text{if } \ b=1 \\ (b-2)\cdot V(a) & \text{if } \ b \geq 2 \end{cases} =$$

$$\begin{cases} 0 & \text{if } \ b=1 \\ (b-2)\cdot v_5(a^2+1) & \text{if } \ b \geq 2 \end{cases}. \tag{2}$$

If $a : a \equiv \{12,8\} \pmod{20}$, then

$$\#S_{\{2,8\}}(20\cdot k+12 \ \vee \ 20\cdot k+8, b) = (b-1)\cdot V(a) = (b-1)\cdot v_5(a^2+1), \forall b \in \mathbb{N}-\{0\}. \tag{3}$$

Even if the cases $a : a \equiv 4 \pmod{10}$ and $a : a \equiv 6 \pmod{10}$ have already been fully described in References [6, 8], "repetita iuvant", and so (for any b) we have

$$\#S_4(a,b) = (b-1)\cdot V(a) = (b-1)\cdot v_5(a+1); \tag{4}$$

while $a : a \equiv 6 \pmod{10}$ trivially implies $V(a,1) \geq 1 \Rightarrow V(a,b) \geq 1$, so that (for any b)

$$\#S_6(a, b \geq 2) = (b+1)\cdot V(a) = (b+1)\cdot v_5(a-1) \tag{5}$$

immediately follows from $V(a \equiv 6 \pmod{10}, 1) + V(a \equiv 6 \pmod{10}, 2) = 3\cdot V(a \equiv 6 \pmod{10}, b \geq 3) = 3\cdot v_5(a-1)$.

If $a : a \equiv 5 \pmod{10}$, then $V(a) = \min\left(v_2(a-1),\ v_2(a+1)\right) = v_2(a^2-1)-1$, and $\bar{b}(a)$ is always equal to 3, with the only exception of the base $a=5$ (i.e., $\bar{b}(5) = 4 \neq 3 = \bar{b}(10\cdot k+15)$, $\forall k \in \mathbb{N}_0$). It follows that

$$\#S_5(20\cdot k-5, b \geq 2) = b\cdot(v_2(a^2-1)-1)+1; \tag{6}$$

$$\#S_5(20\cdot k+5, b \geq 2) = (b+1)\cdot(v_2(a^2-1)-1); \tag{7}$$

$$\#S_5(5,b) = \begin{cases} 1 & \text{iff } \ b=1 \\ 4 & \text{iff } \ b=2 \\ 8+2\cdot(b-3) & \text{iff } \ b \geq 3 \end{cases}. \tag{8}$$

In order to complete the $\#S(a, b)$ map, we need to study all the tetration bases which are not coprime to 10, and this will be the goal of the next subsection.

2.2 Bounding $\#S_{\{1,3,7,9\}}(a, b)$

Let $a : a \not\equiv 0 \pmod{10} \wedge a \neq 1$ be given (bearing in mind that $V(1,1) = 1$, whereas $V(1, \bar{b}(1)) = V(1,2) = V(1) = 0$, and also $V(0, \bar{b}(0)) = V(0,1) = V(0) = 0$), so that, as shown in Reference [8], Section 2, $V(a) \leq \tilde{v}(a)$.

Under the abovementioned condition $a \neq 1$, we note that if $V(a \equiv \{1,3,7,9\} \pmod{10},\ b) = 0$, then $a = (20 \cdot k + 3\ \vee\ 20 \cdot k + 7)\ \wedge\ b = 1$, for any $k \in \mathbb{N}_0$.

Thus,

$$\#S_{\{3,7\}}\big(a = (20 \cdot k + 3\ \vee\ 20 \cdot k + 7),\ b \geq \bar{b}(a) - 1\big) = \begin{cases} (b-1) \cdot V(a) & \text{iff } V(a,2) = V(a) \\ b \cdot V(a) + 1 & \text{iff } V(a,2) > V(a) \end{cases}, \quad (9)$$

(e.g., $V(6907922943, 2) = 11 > 9 = v_5(6907922943^2 + 1) \Rightarrow \#S_3\big(a = (20 \cdot 345396147 + 3),\ b \geq \bar{b}(a)\big) = \#S_3(6907922943,\ b \geq 6) = b \cdot V(a) + 1$, while $V(107,2) = 2 = v_5(107^2 + 1) \Rightarrow \#S_7\big(a = (20 \cdot 5 + 7),\ b \geq \bar{b}(a) - 1\big) \Rightarrow \#S_7(107,\ b \geq 1) = \big((b-1) \cdot V(a)\big)$).

In addition, for any b, the above also implies the bound (10)

$$(b-1) \cdot V(a) \leq \#S_{\{3,7\}}(a = (20 \cdot k + 3\ \vee\ 20 \cdot k + 7),\ b) \leq b \cdot V(a) + 1\,, \quad (10)$$

and the (weaker) relation (11) follows

$$(b-1) \cdot \big(v_5(a^2+1)\big) \leq \#S_{\{3,7\}}(a = (20 \cdot k + 3\ \vee\ 20 \cdot k + 7),\ b \geq 2) \leq b \cdot \big(v_5(a^2+1)\big). \quad (11)$$

Finally, for any $a \equiv \{1,3,7,9\} \pmod{10}$ which cannot be written as $20 \cdot k + 3\ \vee 20 \cdot k + 7$, the number of stable digits of ${}^{b}a$ at height $b \geq \bar{b}(a) - 1$ is $b \cdot V(a)$, or $b \cdot V(a) + 1$, or $(b+1) \cdot V(a)$.

We can also derive the following general bound which holds for any $b \geq 2$,

$$b \cdot V(a) \leq \#S_{\{1,3,7,9\}}(a \neq (20 \cdot k + 3\ \vee\ 20 \cdot k + 7),\ b \geq 2) \leq (b+1) \cdot V(a)\,, \quad (12)$$

and we additionally state that $\bar{b}(a) \leq v_5(a^2+1) + 2$ is valid for every tetration base a which is congruent to $\{3,7\} \pmod{10}$. The aforementioned limit on $\bar{b}(a)$ arises by combining the upper bounds by Equations (10)&(12) with the general constraint from Equation (14) (see Section 3), taking also into account that if $a \not\equiv \{0,2,8\} \pmod{10}$, then $V(a,2)$ always assumes a strictly positive value.

Furthermore, if $a \not\equiv \{3,7\} \pmod{20}$, then $\bar{b}(a) \leq \tilde{v}(a) + 1$, since we have not to worry about the case $V(a,1) = 0$, which cannot happen (the only a which is characterized by $V(a,2) > 0$ and such that $\bar{b}(a) > \tilde{v}(a) + 1$ is the base 5, but we already know that $\bar{b}(5) = \tilde{v}(5) + 2$). In general, assuming $a \neq 5$, only a maximum of $\tilde{v}(a)$ additional iterations can occur from the first time that the congruence speed assumes a strictly positive value (i.e., the first step or the second one for any a which is coprime to 10) to the last time that $V(a,b) > V(a)$. Thus, for any a which is not congruent to 0 modulo 10, the maximum theoretical value of $\bar{b}(a)$ is bounded above by $1 + \tilde{v}(a) + 1$.

Therefore, $\bar{b}(a) \leq \tilde{v}(a) + 2$ for every $a : a \not\equiv 0 \pmod{10}$ (let us observe that $a = 1 \Rightarrow \tilde{v}(1) = v_5(0) = \infty$ and $\bar{b}(1) = 2$ by definition), and this result confirms also Conjecture 1 of Reference [8].

We can take a look at the congruence speed of the base $a = 163574218751$ as a random check on the upper bound provided by (11). $a = 163574218751$ is characterized by $\tilde{v}(163574218751) = v_5(163574218751 - 1) = 13 = V(163574218751)$, so we have $V(a,1) = 12$, $V(a,2) = 19$,

$V(a,3) = V(a,4) = V(a,5) = V(a,6) = 15$, and $V(a, b \geq 7) = V(a) = 13$. Hence, by Equation (1), $\#S_1\left(163574218751,\ b \geq \bar{b}\right) = 12 + 19 + 15 \cdot 4 + \left(b - \left(\bar{b} - 1\right)\right) \cdot 13 = 91 + \left(b - \bar{b} + 1\right) \cdot 13 = (6+1) \cdot 13 + (b-6) \cdot 13 = (b+1) \cdot V(163574218751)$.

In addition, some more bases from each one of the four critical congruence classes modulo 10, whose $\#S_{\{1,3,7,9\}}\left(a,\ b \geq \bar{b}(a)\right)$ is uniquely given by $(b-1) \cdot V(a)$, or $b \cdot V(a)$, or $b \cdot V(a) + 1$, or $(b+1) \cdot V(a)$, are shown below:

- $\#S_1(74218751,\ b \geq 3) = b \cdot V(a) + 1 = b \cdot 8 + 1$,
- $\#S_1(45215487480163574218751,\ b \geq 13) = (b+1) \cdot V(a) = (b+1) \cdot 25$;

- $\#S_3(143,\ b \geq 2) = (b-1) \cdot V(a) = (b-1) \cdot 2$,
- $\#S_3(133,\ b \geq 1) = b \cdot V(a) = b$,
- $\#S_3(847288609443,\ b \geq 5) = b \cdot V(a) + 1 = b \cdot 2 + 1$,
- $\#S_3(2996418333704193,\ b \geq 17) = (b+1) \cdot V(a) = (b+1) \cdot 16$;

- $\#S_7(907,\ b \geq 2) = (b-1) \cdot V(a) = b \cdot 2$,
- $\#S_7(177,\ b \geq 1) = b \cdot V(a) = b$,
- $\#S_7(807,\ b \geq 6) = b \cdot V(a) + 1 = b \cdot 3 + 1$,
- $\#S_7(23418092077057,\ b \geq 15) = (b+1) \cdot V(a) = (b+1) \cdot 14$;

- $\#S_9(599,\ b \geq 1) = b \cdot V(a) = b \cdot 2$,
- $\#S_9(499,\ b \geq 2) = b \cdot V(a) + 1 = b \cdot 2 + 1$,
- $\#S_9(781249,\ b \geq 4) = (b+1) \cdot V(a) = (b+1) \cdot 6$.

The rules which let us anticipate the value of every $\#S_{\{1,3,7,9\}}\left(a,\ b \geq \bar{b}(a)\right)$ (including all the examples above), can be derived from Reference [8], Equation (2) for $i = 1,3,4,9,10,12$, since critical bases are originated by those digits of α'_1, α'_3, α''_3, α'_7, α''_7, α'_9, and α''_9 which are congruent to $0 (\mathrm{mod}\ 5)$ (e.g.,, let us select one of the aforementioned decadic integers and perform a surgical $mod\ 10^n$ cut on that string, just at the right of a casual digit 5, so that the number $\breve{a}$ we get is a pretty special tetration base characterized by $\tilde{v}(\breve{a}) > V(\breve{a})$, as long as $\breve{a} \neq 7$ – since $\alpha'_7 \equiv 7 (\mathrm{mod}\ 10^2)$ and $\alpha''_7 \equiv 57 (\mathrm{mod}\ 10^2)$).

To be fair, as stated in Reference [8], Proposition 6, p. 47, there is also one last fundamental intersection which arises from the solution $y_{15}(t) = 1$ of $y^t = y$ over the commutative ring of decadic integers, considering the corresponding decimal integers modulo 10^n (by the well-known ring homomorphism). For this purpose, as a clarifying example, let us show how $y_{15}(5)$: $1^5 = 1$ works (see [8], pp. 47-48). Let $n \in \mathbb{N} - \{0,1\}$, and let $a(n) \coloneqq \sum_{j=1}^{n} s_j \cdot 10^j$ be such that $s_{j=1} = 1$, $s_{1<j<n} \in \{0,1,2,3,4,5,6,7,8,9\}$, and $s_{j=n} \in \{1,2,3,4,5,6,7,8,9\}$. When $s_2 = 0$, for any given set $\{s_3, s_4, \dots, s_{n-1}, s_n\}$ as above, we can verify that $\#S_1(a(n), b \geq 2) = (b+1) \cdot V(a(n))$ is always true, whereas, if $s_2 = s_n$ is an arbitrary element of the set $\{1,2,3,4,5,6,7,8,9\}$, then $\#S_1(a(2), b \geq 2) = (b+1) \cdot V(a(2)) = b \cdot V(a(2)) + 1$ if and only if $s_2 \neq 5$ (where $5 = \frac{\alpha'_1 (\mathrm{mod}\ 10^2) - \alpha'_1 (\mathrm{mod}\ 10)}{10}$ by Equation (2) from Reference [8]). Since $V(51,1) = 2$, $V(51,2) = 3$, and $V(51,3) = V(51) = 2$, it follows that $\#S_1(51, b \geq 2) = b \cdot V(51) + 1$ is not equal to $(b+1) \cdot V(51)$ (i.e., $V(a) \neq 1 \Rightarrow b \cdot V(a) + 1 \neq (b+1) \cdot V(a)$).

In the end, it is possible to use Equations (6)-(7)-(14)-(15) from Reference [8] to compute the exact value of $V(a)$ for any a which is coprime to 10, even if the map of all the bases with a congruence speed below 3 can be immediately known by simply looking at (15) of Section 3 of this paper.

3 Some useful properties of the congruence speed

The regularity features of the congruence speed [7, 8] can be very useful when performing peculiar mental calculations, finding also the precise value of $\#S(a,b)$ by Equation (1).

We start by saying that, for any $a : a \not\equiv 0 \pmod{10} \wedge a \neq 1$, $V(a,1) \leq V(a,2)$ always holds, so let a be such that $V(a,2) = 0$ (i.e., assuming $a > 1$, $V(a,2) = 0 \Leftrightarrow a = \big((20 \cdot k + 2) \vee (20 \cdot k + 18)\big)$, $\forall k \in \mathbb{N}_0$). Thus,

$$V(a,b) \geq V(a,b+1),\ \ \forall b \geq 3\ . \tag{13}$$

If $a : a \not\equiv \{0,2,10,18\} \pmod{20} \wedge a \neq 5$ (i.e., $a \not\equiv \{0,2,10,18\} \pmod{20} \Rightarrow V(a,2) \neq 0$), then

$$V(a,b) \geq V(a,b+1),\ \ \forall b \geq 2\ . \tag{14}$$

A general rule which is very easy to keep in mind is that $V(a,1) + V(a,2) \leq 3 \cdot V(a) \leq 3 \cdot \tilde{v}(a)$, with the unique exception represented by the very special base $a = 1$ (since $V(1) = 0$, whereas $V(1,1) > 0$). Furthermore, for any $k \in \mathbb{N}_0$, let us underline that $V(a,b) = 0$ if and only if $b = 1$ and $a \equiv \{2,3,7,12,4,14,8,18\} \pmod{20} \ \vee\ a = 0$, or if $b = 2$ and $a \equiv \{2,18\} \pmod{20} \ \vee\ a = 1 \ \vee\ a = 0$, or if $b \geq 2$ and $a = 1 \ \vee\ a = 0$ (see Equations (2)-(3)-(4)-(9)).

Moreover, for any $a : a \not\equiv 0 \pmod{10} \wedge a \neq 1$, the periodicity properties of $V(a)$ (see [8], Equation (18)) let us immediately detect if $V(a) \geq 2$ or not, by simply checking the congruence $a \equiv \{2,3,4,6,8,9,11,12,13,14,16,17,19,21,22,23\} \pmod{25}$; if so, $V(a) = 1$, and $V(a) \geq 2$ otherwise. We can go even further and try to memorize the next set of 900 values, $1 \leq a \not\equiv 0 \pmod{10} < 1000$, in order to answer in less than one second (without writing or calculating anything) if $V(a) = 0$, $V(a) = 1$, $V(a) = 2$, or even $V(a) \geq 3$ (see [7], pp. 252). Knowing that $V(1) = 0$ by definition, $\forall a \in \mathbb{N} - \{1\} : a \not\equiv 0 \pmod{10}$, we have

$$\begin{cases} V(a) = 1 \Leftrightarrow a \pmod{25} \in \mathbb{C}^{\complement}, \\ \text{where } \mathbb{C}^{\complement} := \{2,3,4,6,8,9,11,12,13,14,16,17,19,21,22,23\}; \\ V(a) = 2 \Leftrightarrow a \pmod{40} \in \{5,35\} \vee \\ (a \pmod{25} \in \{1,7,18,24\} \wedge a \pmod{1000} \notin \mathbb{Q}^{\complement}); \\ V(a) \geq 3 \Leftrightarrow a \pmod{40} \in \{15,25\} \vee a \pmod{1000} \in \mathbb{Q}^{\complement}, \\ \text{where } \mathbb{Q}^{\complement} := \begin{Bmatrix} 1,57,68,124,126,182,193,249,318,374,376,432,568, \\ 624,626,682,751,807,818,874,876,932,943,999 \end{Bmatrix}. \end{cases} \tag{15}$$

We can also take $\#S_c(a,b)$ and check the stable digits ratio of any integer tetration whose base is not congruent to 0 modulo 10. For any given ${}^{b}a$, the stable digits ratio of is

$$R(a,b) := \frac{\#S_c(a,b)}{\lceil \log_{10}({}^{b}a) \rceil}\,, \tag{16}$$

where the ceiling $\lceil q \rceil$ denotes the function which takes the rational number q as input and returns as output the least integer greater than or equal to q.

Lastly, given any tetration base $a : a \not\equiv 0 (\mathrm{mod}\ 10) \wedge a \neq 1$, if we choose beforehand the desired number of stable digits (let us indicate it as $\#T(a) \in \mathbb{N}_0$) of ${}^{b}a$, we will easily calculate which is the smallest hyperexponent $\bar{\bar{b}}(a) := \min_b\left\{b \in \mathbb{N} - \{0\} : \sum_{i=1}^{b} V(a, i) \geq \#T(a)\right\}$ such that ${}^{\bar{\bar{b}}}a$ originates at least $\#T(a)$ stable digits (see [6], pp. 13-14).

Thus, ${}^{b}a \equiv {}^{\bar{\bar{b}}}a\left(\mathrm{mod}\ 10^{\#T(a)}\right)$ for any $b(a) \geq \bar{\bar{b}}(a)$, and $\sum_{i=1}^{b} V(a, i)$ can be simplified using the relations shown in the present paper (e.g., by Equation (4), for any $k \in \mathbb{N}_0$, $a = 10 \cdot k + 4 \Rightarrow \sum_{i=1}^{b} V(a, i) = (b - 1) \cdot v_5(a + 1) \Rightarrow \bar{\bar{b}}(a) = \min_b\left\{b \in \mathbb{N} - \{0\} : b \geq \left\lceil\frac{\#T(a)}{v_5(a+1)}\right\rceil + 1\right\}$).

4 Conclusion

The number of stable digits of any integer tetration ${}^{b}a$ such that a is not a multiple of 10 is strongly related to the constant congruence speed of the base, and $\bar{b}(a) \leq \tilde{v}(a) + 2$ is a sufficient condition to guarantee the constancy of the congruence speed of a for any hyperexponent at or above $\bar{b}(a)$, so that $V\left(a,\ \bar{b}(a) + k\right) = V(a)$ for any $k \in \mathbb{N}_0$. Finally, by combining the $V(a)$ map shown in Reference [8] with a compact set of equations which allows an easy calculation of $\#S(a, b)$, we are starting to see some symmetrical harmony in the fascinating, chaotic, behaviour of hyper-4.

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