On the calculation of the ripple voltage in half–wave rectifier circuits

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Abstract

In this article we propose a computational algorithm written in Matlab as well as a mathematical formula to calculate the magnitude of the ripple voltage encountered in halfwave rectifying circuits. After rectification of a symmetric AC harmonic voltage signal, this ripple voltage remains, frequently measurable, superimposed on the resulting DC voltage signal. Both, the algorithm as well as the derived formula enable us to calculate the magnitude of the ripple voltage to a degree of precision any better than 1 to 10^6, i.e. 1 ppm. We conclude this article by comparing the accuracy of the proposed algorithm to simulated findings obtained with LTspice® [1]. The technique discussed here for calculating the magnitude of the remaining ripple voltage can easily be generalized and extended to the calculation of the magnitude of the ripple voltage encountered in full wave rectifying circuits.

1 Introduction

Rectification of AC waveforms is a basic technique to generate DC signals. Since this technique is well–known and has been well-documented in literature i.e. [2], [3], [4], [5], [6] we emphasize the most important properties and features of a halfwave rectifying circuit only. Briefly, to establish rectification in its simplest form, a harmonic voltage source $u_s(t) = u_0 \cdot \sin(2\pi f t)$, with period $T = 1/f$, is connected to a diode in series with a load resistor R, see Figure 1.

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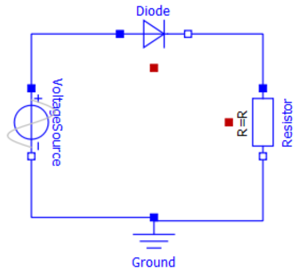

Figure 1: Graph of the most basic rectifying circuit consisting of a harmonic voltage source, a diode and an ohmic resistor only.

As can be seen from Figure 2, depicting the rectified voltage over the resistor, according to Kirchhoff's circuit laws (voltage law, abbreviated KVL, and current law, abbreviated KCL), the diode conducts or is forward biased during the time interval $0 \leq t \leq \frac{T}{4}$ only when the time dependent voltage over the diode $u_d\left(t\right)$ is lower than the knee voltage or cut–in voltage u_k. In case of a silicon diode the knee voltage $u_k \approx 0.7$ V. So the load voltage $u_R\left(t\right) > 0$ if and only if $u_d\left(t\right) > u_k$. It is clear from the graph of Figure 2 that, when using the basic circuit of Figure 1 to rectify a sinusoidal voltage, the generated ripple voltage is undesirably large in magnitude, i.e. $u_{ripple} = u_0 - u_k$. In the folowing, the objective is to achieve a ripple voltage which is as small as possible or negligible in magnitude.

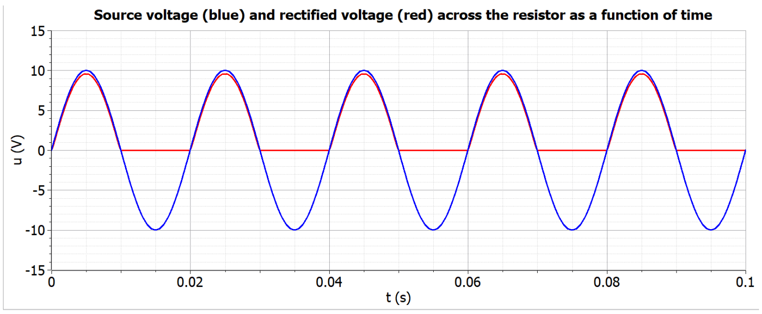

Figure 2: Graph of the rectified voltage $u_R\left(t\right) = u_{ripple}$ measured over the load resistor R of the circuit displayed in Figure 1.

To improve upon the performance of rectification, i.e. to reduce the ripple voltage in magnitude, a capacitor C, acting as a charge buffer, is added in parallel to the load resistor R as shown in Figure 3.

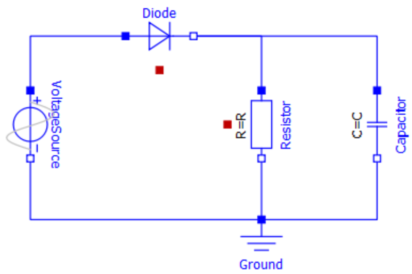

Figure 3: To reduce the magnitude of the ripple voltage a capacitor C is added in parallel to the load resistor R of the basic circuit shown in Figure 1.

According to KVL, as soon as $u_d(t) > u_k$ the diode is forward biased and the voltage over the load $u_{R,C}(t) > 0$ until the sinusoidal voltage source reaches its maximum at $t = T/4$, i.e. $u_s(T/4) = u_0$. Again, according to KVL, from $t > T/4$ onwards, $u_s(t)$ starts to decrease in value, resulting in the diode becoming reverse biased, subsequently, it stops conducting. Consequently, the charged capacitor discharges through the resistor at a rate

$$u_{R,C}(t) = (u_0 - u_k) \cdot e^{-(t-T/4)/\tau},\ \tau := RC$$

The improved performance of this second rectifying circuit in reducing the magnitude of the ripple voltage is clear from the graph shown in Figure 4.
In the following, we have put an at random chosen lower positive boundary value to $\tau > 7 \cdot 10^{-4}$ (s) as to avoid an intersection in the time interval $T/4 \leq t < T/2$ between the source voltage $u_s(t)$ and the discharing voltage $u_{R,C}$: in this time interval the proposed Matlab program miscalculates the time stamp t_{fb} and thus, the ripple voltage. The numerical lower boundary value is approximated from the inequality

$$(u_0 - u_k) \cdot e^{-(t-T/4)/\tau} > \epsilon = 10^{-3}\ (V)$$

in which we have set a lowest value for ϵ. The numerical constraint on τ was calculated with $u_0 = 10\ (V)\,,\ u_k = 0.7\ (V)$.

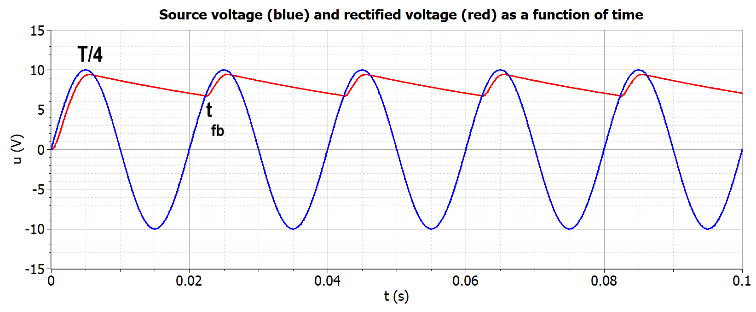

Figure 4: Graph of the decreased magnitude of the ripple voltage due to having added a capacitor C in parallel to the load resistor R. In red is depicted the source signal $u_s(t)$ and in blue the voltage $u_{R,C}(t)$. The time t_{fb} at which the diode becomes forward biased during the second cycle is shown as well. The time difference $\Delta t = t_{fb} - T/4$ marks the period of the diode being reverse biased during which the loaded capacitor discharges.

2 Calculation of the ripple voltage in a sinusoidal driven rectifying $RC-$circuit

To calculate the ripple voltage generated in a rectifying circuit as depicted in Figure 4, define the time-dependent ripple voltage as

$$u_{ripple} := (u_0 - u_k) - u_s(t_{fb})$$

with $u_s(t_{fb})$ being the voltage of the sinusoidal source at the time stamp when the diode turns on again in the second cycle $T < t < 2T$. For $t > T/4$, the ripple voltage is described by

$$u_{R,C}(t) = (u_0 - u_k) \cdot e^{-(t-T/4)/\tau}$$

Applying KVL once more, from the time $t > T/4$ onwards, the ripple voltage decreases continuously until the voltage drop across the diode exceeds u_k. This occurs at a time $T < t_{fb} < 5T/4$ when the diode becomes forward biased and thus, conducts again. The magnitude of the voltage of $u_{R,C}(t_{fb})$ at the time $t = t_{fb}$ is

$$u_{R,C}(t_{fb}) = (u_0 - u_k) \cdot e^{-(t_{fb}-T/4)/\tau}$$

According to KVL, the time t_{fb} may be calculated from the following transcendental equation

$$u_0 \sin(\omega t_{fb}) - u_k = (u_0 - u_k) \cdot \mathrm{e}^{-(t_{fb}-T/4)/\tau}$$

As this transcendental equation cannot be solved in close form, the following program in Matlab is proposed to achieve this goal numerically.

2.1 Computational algorithm written in Matlab

```
clear
clf
clc
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
% R: load in ohm
% C: capacitance in Farad
% f: frequency of sinusoidal voltage source in Hz
% u_0: magnitude of source in V
% u_k: knee voltage of diode in V
% omega: radial frequency in 1/s
% tau: response time constant in s, product of R and C
% t_fb: time in s at which diode starts to conduct
% magnitude_u_ripple: magnitude of ripple voltage in V
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
R = 1e3;
C = 5e-6;
f = 50;
u_0 = 10;
u_k = 0.7;
syms t
omega = 2*pi*f;
tau = R*C;
u_s = u_0 * sin(omega*t);
u_ripple = (u_0 - u_k)*exp(-(t-1/(4*f))/(tau));
eq = u_0*sin(omega*t)- u_k - (u_0 - u_k).*exp(-(t-1/(4*f))/(tau));
t_fb = vpasolve(eq, t, [0, 5/(4*f)])
t_fb = double(t_fb)
t_bf = tau*log((u_0-u_k)/u_k) + 1/(4*f);
magnitude_u_ripple = (u_0 - u_k)*(1 - exp(-(t_fb-1/(4*f))/(tau)));
fprintf('Amplitude of the ripple voltage u_{ripple} = %2.4g V\n',
magnitude_u_ripple)
fprintf('Time difference between forward and reverse biasing
\x0394t = %2.5g s\n', t_fb - 1/(4*f))
fprintf('Ratio of u_{ripple} to u0 = %2.4g%% \n',
100*magnitude_u_ripple/u_0)
fplot(u_s, [0, 0.04], 'LineWidth', 2)
hold on
fplot(u_ripple, [1/(4*f), t_fb], 'LineWidth', 2)
grid on
grid minor
xlabel(' t(s)')
ylabel('u(V)')
title(' Graph of the voltage source and the ripple voltage')
```

2.2 Derivation of an approximation for the ripple voltage

In literature, see [2], [3], [4], [5], [6], the following formula

$$u_{ripple} = (u_0 - u_k) \cdot \frac{T}{\tau}$$

is frequently encountered to calculate the magnitude of the ripple voltage. For this formula to be accurately applicable, strict constraints, as there are, $\tau \gg t_{fb} - T/4 \approx T$ are frequently not met at all in daily electrical engineering practice. Therefore, from the mathematical point of view it remains to be seen, whether or not a mathematical approximation for calculation of the magnitude of the ripple voltage can be derived with lesser on no constraints at all. This subject we address in the following.

Lemma:

$$\frac{d^n}{dW^n}\left(e^{W\cdot\cos(\varphi)} \cdot \sin\left(Z + W \cdot \sin\left(\varphi\right)\right)\right) =$$
$$e^{W\cdot\cos(\varphi)} \cdot \sin\left(Z + W \cdot \sin\left(\varphi\right) + n \cdot \varphi\right)$$

Equation for $Z_{fb} \coloneqq \omega \cdot t_{fb},\ \omega \coloneqq \frac{2\pi}{T}$

$$\left(\sin\left(Z_{fb}\right) - \rho\right) \cdot e^{Z_{fb}/\sigma} = (1-\rho) \cdot e^{\frac{\pi}{2}/\sigma} = \left(\sin\left(Z\right) - \rho\right) \cdot e^{2\pi/\sigma}$$
$$Z \coloneqq 2\pi + \arcsin\left(\rho + (1-\rho) \cdot e^{\frac{-3\pi}{2}/\sigma}\right)$$

with

$$\sigma \coloneqq \omega \cdot \tau,\ \rho \coloneqq \frac{u_k}{u_0}$$

Construction of a mathematical approximation for calculation of the magnitude of the ripple voltage

Let

$$Z_{fb} \coloneqq Z + W \cdot \sin\left(\varphi\right)$$

with

$$\varphi \coloneqq \arctan\left(\sigma\right)$$

Then

$$F\left(W\right) \coloneqq \left(\sin\left(Z + W \cdot \sin\left(\varphi\right)\right) - \rho\right) \cdot e^{W\cdot\cos(\varphi)} - X \cdot \left(\sin\left(Z\right) - \rho\right) = 0$$

with

$$X \coloneqq e^{(2\pi - Z)/\sigma}$$

The Taylor series

$$F(W) = F(0) + F'(0) \cdot W + \frac{1}{2} \cdot F''(0) \cdot W^2 + \cdots = 0$$

truncated after the second derivative is invoked to derive the following quadratic equation as an approximation for the variable W

$$\frac{1}{2} \cdot F''(0) \cdot W^2 + F'(0) \cdot W + F(0) = 0$$

Using the previous lemma with

$$\begin{aligned}
F(0) &= (1 - X) \cdot (\sin(Z) - \rho), \\
F'(0) &= \sin(Z + \varphi) - \rho \cdot \cos(\varphi), \\
F''(0) &= \sin(Z + 2\varphi) - \rho \cdot \cos^2(\varphi)
\end{aligned}$$

we arrive at the following explicit expression for the variable W:

$$W = \frac{\sqrt{A^2 + B} - A}{C}$$

with

$$\begin{aligned}
A &\coloneqq \sin(Z + \varphi) - \rho \cdot \cos(\varphi) \\
B &\coloneqq 2(X - 1) \cdot (\sin(Z) - \rho) \cdot \big(\sin(Z + 2\varphi) - \rho \cdot \cos^2(\varphi)\big) \\
C &\coloneqq \sin(Z + 2\varphi) - \rho \cdot \cos^2(\varphi)
\end{aligned}$$

In the following, we compute an approximation for the time t_{fb}, the time stamp when the diode is forward–biased again: assuming that the period T, the time constant τ and ρ are known, we then have

$$t_{fb} = \frac{Z + W \cdot \sin(\varphi)}{\omega}$$

with W as defined as above.

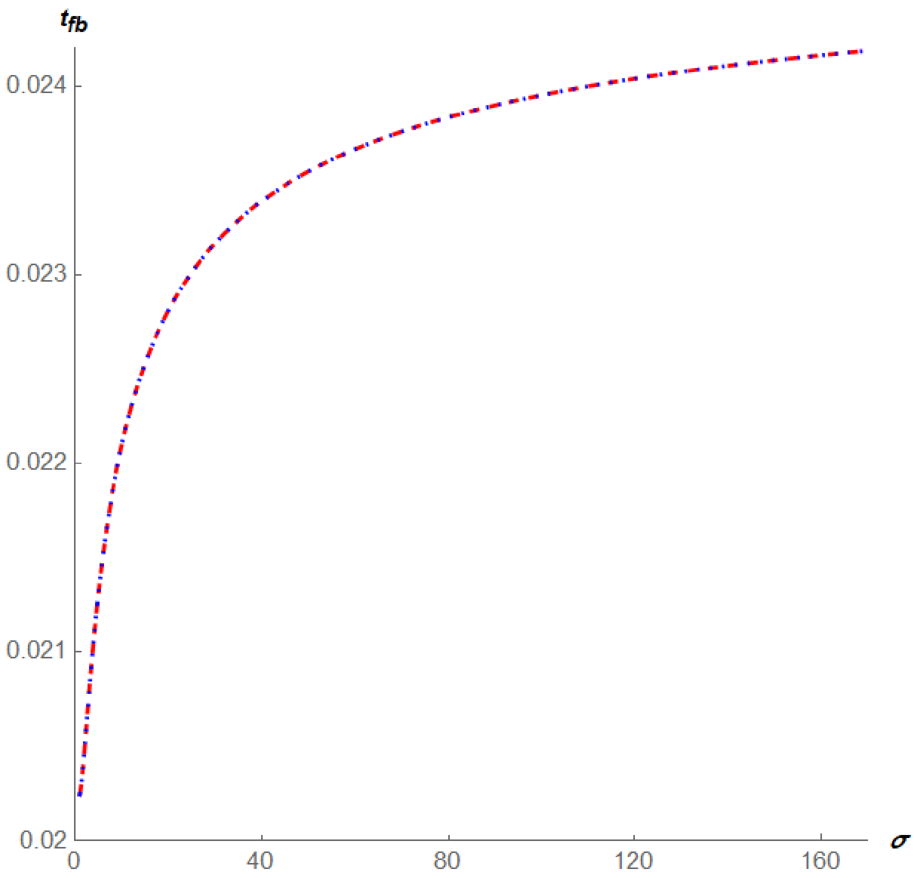

Figure 5: Graph in blue of the approximating function t_{fb} and in red of the exact function t_{fb}.

Ideal diode

$$\begin{cases} C \cdot \dfrac{du_{R,C}}{dt} + \dfrac{u_{R,C}}{R} \coloneqq i_d = 0, & \text{for } u_s - u_{R,C} \coloneqq u_d < u_k \\ u_{R,C} = u_s - u_k, & \text{for } u_s - u_{R,C} \coloneqq u_d = u_k \end{cases}$$

For an arbitrary resistance $\widetilde{R}$:

$$i_d \cdot \widetilde{R} - u_d + u_k = |i_d \cdot \widetilde{R} + u_d - u_k|$$

Using the transformation

$$\widetilde{u}_d \coloneqq i_d \cdot \widetilde{R} + u_d - u_k,\ \widetilde{i}_d \cdot \widetilde{R} \coloneqq i_d \cdot \widetilde{R} - u_d + u_k$$

it follows

$$\widetilde{i}_d = \widetilde{f}\left(\widetilde{u}_d\right) \coloneqq \frac{|\widetilde{u}_d|}{\widetilde{R}}$$

$$u_{R,C}(t) = \begin{cases} u_0 \cdot \sin(\omega \cdot t) - u_k, & t < \frac{T}{4} \vee \left(t - \frac{T}{4}\right) \pmod T > t_{fb} - \frac{T}{4} \\ (u_0 - u_k) \cdot e^{-\left(t-\frac{T}{4}\right) \pmod T / \tau}, & \left(t - \frac{T}{4}\right) \pmod T \le t_{fb} - \frac{T}{4} \end{cases}$$

$$u_s(t) = u_d(t) + u_{R,C}(t)$$

Non–ideal diode

$$C \cdot \frac{du_{R,C}}{dt} + \frac{u_{R,C}}{R} =: i_d = f(u_s - u_{R,C})$$

Piecewise linear diode model

$$f(u_d) := \frac{(u_d - u_k) + |u_d - u_k|}{2 \cdot r_f}$$

Using the transformation

$$\widetilde{u}_d := i_d \cdot \widetilde{R} + u_d - u_k, \ \widetilde{i}_d \cdot \widetilde{R} := i_d \cdot \widetilde{R} - u_d + u_k$$

$$\widetilde{i}_d = \widetilde{f}(\widetilde{u}_d) := \frac{1}{2 \cdot \widetilde{R}} \cdot \left(|\widetilde{u}_d| - \widetilde{u}_d + \frac{1 - \frac{r_f}{\widetilde{R}}}{1 + \frac{r_f}{\widetilde{R}}} \cdot (|\widetilde{u}_d| + \widetilde{u}_d) \right)$$

Note that

$$\lim_{r_f \to 0} \widetilde{f}(\widetilde{u}_d) = \lim_{r_f \to 0} \frac{1}{2 \cdot \widetilde{R}} \cdot \left(|\widetilde{u}_d| - \widetilde{u}_d + \frac{1 - \frac{r_f}{\widetilde{R}}}{1 + \frac{r_f}{\widetilde{R}}} \cdot (|\widetilde{u}_d| + \widetilde{u}_d) \right) = \frac{|\widetilde{u}_d|}{\widetilde{R}}$$

Example ideal model

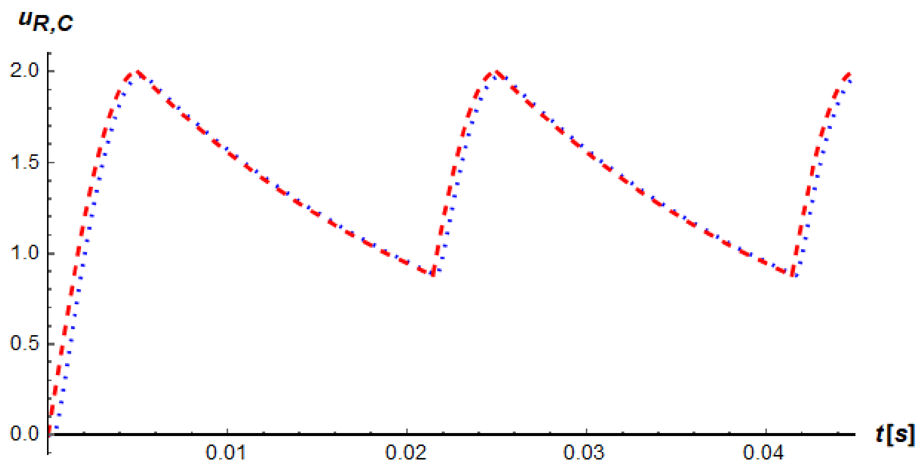

Figure 6: Graph of the piecewise linear diode model, ideal (red), piecewise linear (blue), $\frac{R}{r_f} = 10$

Lemma

Let $y(x) = f(x)$ define a monotonically increasing function and $\widetilde{x} := y + x, \widetilde{y} := y - x$, then $\widetilde{y} = \widetilde{f}(\widetilde{x})$ defines a function.

Proof

$\widetilde{x} := y + x = f(x) + x =: g(x)$. Since $g(x)$ defines a monotonically increasing function $x = g^{-1}(\widetilde{x})$, it follows that $\widetilde{y} := \widetilde{x} - 2 \cdot x = \widetilde{x} - 2 \cdot g^{-1}(\widetilde{x}) =: \widetilde{f}(\widetilde{x}) \square$

Example exponential model

$$f(u_d) := I_s \cdot \left(e^{\frac{u_d}{u_T}} - 1\right)$$

Theorem

1. Using the transformation

$$\widetilde{u}_d := i_d \cdot \widetilde{R} + u_d, \widetilde{i}_d \cdot \widetilde{R} := i_d \cdot \widetilde{R} - u_d$$

it follows

$$\widetilde{i}_d = \widetilde{f}(\widetilde{u}_d)$$

2.

$$\lim_{U_T \to 0} \widetilde{f}(\widetilde{u}_d) = \frac{|\widetilde{u}_d|}{\widetilde{R}}, \text{ for } \widetilde{u}_d > 0$$
$$\lim_{I_s \to 0} \widetilde{f}(\widetilde{u}_d) = \frac{|\widetilde{u}_d|}{\widetilde{R}}, \text{ for } \widetilde{u}_d < 0$$

Proof

1. The expression

$$i_d = f(u_d)$$

defines a monotonically increasing function. Using the previous lemma for

$$x = u_d, y = i_d \cdot \widetilde{R}, \widetilde{x} = \widetilde{u}_d, \widetilde{y} = \widetilde{i}_d \cdot \widetilde{R} \text{ it follows } \widetilde{i}_d = \widetilde{f}(\widetilde{u}_d) \square$$

2. Suppose $\widetilde{u}_d > 0$, then $u_d > 0$ and

$$0 < i_d \cdot \widetilde{R} = I_s \cdot \widetilde{R} \cdot \left(e^{\frac{u_d}{U_T}} - 1\right) = \widetilde{u}_d - u_d < \widetilde{u}_d$$
$$1 < e^{\frac{u_d}{U_T}} < \frac{\widetilde{u}_d}{I_s \cdot \widetilde{R}} + 1$$
$$0 < u_d < U_T \cdot \ln\left(\frac{\widetilde{u}_d}{I_s \cdot \widetilde{R}} + 1\right)$$

Applying the Squeeze Theorem, it follows that

$$\lim_{U_T \to 0} \left(\frac{\widetilde{u}_d}{\widetilde{R}} - \widetilde{f}\left(\widetilde{u}_d\right) \right) = \lim_{U_T \to 0} \left(2 \cdot \frac{u_d}{\widetilde{R}} \right) = \frac{2}{\widetilde{R}} \cdot \lim_{U_T \to 0} u_d = 0 \,\square$$

Suppose $\widetilde{u}_d < 0$, then $u_d < 0$ and

$$\begin{aligned} -I_s &< i_d = I_s \cdot \left(e^{\frac{u_d}{U_T}} - 1 \right) < 0 \\ -I_s &< i_d < 0 \end{aligned}$$

Applying the Squeeze Theorem, it follows that

$$\lim_{I_s \to 0} \left(\frac{\widetilde{u}_d}{\widetilde{R}} - \widetilde{f}\left(\widetilde{u}_d\right) \right) = \lim_{I_s \to 0} \left(-2 \cdot i_d \right) = -2 \cdot \lim_{I_s \to 0} i_d = 0 \,\square$$

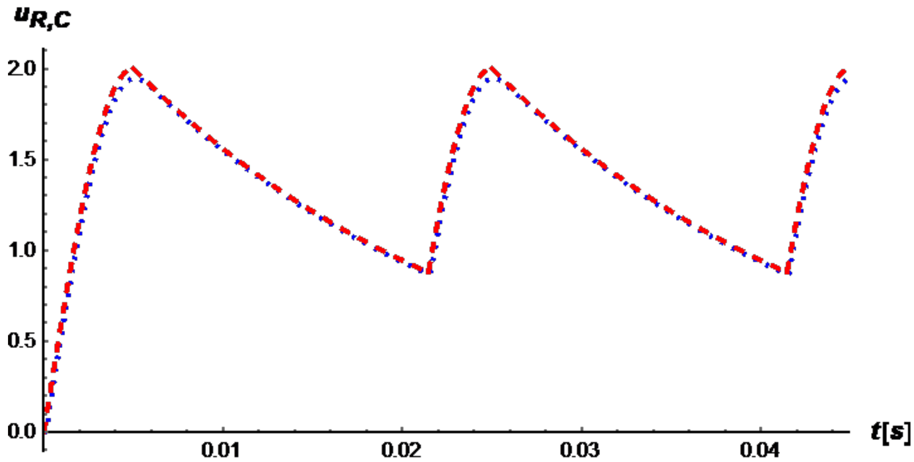

Figure 7: Graph of the exponential diode model, ideal (red), piecewise linear (blue), $I_s \cdot R = 1 \cdot 10^{-2}\,\mathrm{V}$, $U_T = 1 \cdot 10^{-2}\,\mathrm{V}$

2.3 Results and Discussion

In the following table, the results of the calculated and simulated of the magnitude of the ripple voltage are mutually compared. Results were obtained with $f = 50$ (Hz) ,$C = 5$ (μF) $u_0 = 10.0\,(\mathrm{V})$ and $u_k = 0.7\,(\mathrm{V})$.

R ($k\Omega$)	t_{fb} (ms)	$\Delta t = T/4 - t_{fb}$ (ms)	u_{ripple} (V)	
1.0	20.364	15.356	8.778	LTspice® [1]
1.0	20.361	15.361	8.869	Matlab
1.0			37.200	$(u_0 - u_k) \cdot \dfrac{T}{\tau}$
10.0	22.614	17.606	2.717	LTspice®
10.0	22.579	17.579	2.757	Matlab
10.0			3.720	$(u_0 - u_k) \cdot \dfrac{T}{\tau}$
50.0	23.872	18.908	0.619	LTspice®
50.0	23.824	18.824	0.675	Matlab
50.0			0.744	$(u_0 - u_k) \cdot \dfrac{T}{\tau}$
100.0	24.291	18.996	0.318	LTspice®
100.0	24.156	19.156	0.350	Matlab
100.0	24.156	19.156	0.372	$(u_0 - u_k) \cdot \dfrac{T}{\tau}$

Obviously, the deviation in magnitude between the simulation and the calculations mainly results from the resolution of the computer screen.

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