New integrals with Barnes function

Denis GALLET
Rectorat de DIJON
2 G, rue Général Delaborde 21000 DIJON (France)
densg71@gmail.com

December 19, 2021

Abstract

In this paper, I study three particular integrals where I give three conjectural formulas in terms of Barnes G-function. There are a article where there are already several similary logarithmics integrals, this is the paper: rediscovery of Malmsten's integrals, their evaluation by contour integration methods and some related results (1). And also, there are a interest for these three integrals because actually softwares as Mathematica or Maple didn't give a correct closed form.

1 Definition

The Barnes function is defined as the following Weierstrass product:

$$G(1+z) = (2\pi)^{\frac{z}{2}} \mathrm{e}^{-\frac{z(1+z)}{2} - \frac{\gamma z^2}{2}} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right)^k \mathrm{e}^{-z + \frac{z^2}{2k}} \qquad (2)$$

where gamma is the Euler-Mascheroni constant.

The following properties of G are well-known.

2 Properties

$$G(1) = 1 \qquad (3)$$

$$G(1+z) = G(z)\,\Gamma(z) \qquad (4)$$

3 First integral: $\int_0^\infty \frac{t \log\left(t^2+z^2\right)}{\mathrm{e}^{a\pi t}-1}\,\mathrm{d}t$

Where a and z are a positive integer number or a positive rational fraction.

I have successively (A is the Glaisher - Kinkelin's constant (5))

$$\int_0^{\infty} \frac{t \log\left(t^2+z^2\right)}{e^{\pi t}-1}\,dt = -4\log\left(G\left(z/2\right)\right) - 4\log\left(\Gamma\left(z/2\right)\right) + \frac{1}{3} - 4\log\left(A\right) - \frac{\log\left(2\right) z^2}{2} + \log\left(2\right) z + \frac{\log\left(2\right)}{3} + \frac{z^2 \log\left(z\right)}{2} - \frac{3 z^2}{4} + z\log\left(\pi\right)$$

Then

$$\int_0^{\infty} \frac{t \log\left(t^2+z^2\right)}{e^{2\pi t}-1}\,dt = -\log\left(G\left(z\right)\right) - \log\left(\Gamma\left(z\right)\right) + \frac{1}{12} - \log\left(A\right) + \frac{\log\left(2\right) z}{2} + \frac{z^2 \log\left(z\right)}{2} - \frac{3 z^2}{4} + \frac{z\log\left(\pi\right)}{2}$$

Then

$$\int_0^{\infty} \frac{t \log\left(t^2+z^2\right)}{e^{3\pi t}-1}\,dt = -\frac{4}{9}\log\left(G\left(\frac{3z}{2}\right)\right) - \frac{4}{9}\log\left(\Gamma\left(\frac{3z}{2}\right)\right) + \frac{1}{27} - \frac{4\log\left(A\right)}{9} - \frac{\log\left(2\right) z^2}{2} + \frac{\log\left(2\right) z}{3} + \frac{\log\left(2\right)}{27} + \frac{z^2 \log\left(z\right)}{2} - \frac{3 z^2}{4} + \frac{z\log\left(\pi\right)}{3} + \log\left(3\right)\left(\frac{z^2}{2} - \frac{1}{27}\right)$$

Then

$$\int_0^{\infty} \frac{t \log\left(t^2+z^2\right)}{e^{4\pi t}-1}\,dt = -\frac{\log\left(G\left(2z\right)\right)}{4} - \frac{\log\left(\Gamma\left(2z\right)\right)}{4} + \frac{1}{48} - \frac{\log\left(A\right)}{4} + \frac{\log\left(2\right) z^2}{2} + \frac{\log\left(2\right) z}{4} - \frac{\log\left(2\right)}{48} + \frac{z^2 \log\left(z\right)}{2} - \frac{3 z^2}{4} + \frac{z\log\left(\pi\right)}{4}$$

And now in general

$$\int_0^{\infty} \frac{t \log\left(t^2+z^2\right)}{e^{a\pi t}-1}\,dt = -4\,\frac{\log\left(G\left(\left(1/2\right) a z\right)\right)}{a^2} - 4\,\frac{\log\left(\Gamma\left(\left(1/2\right) a z\right)\right)}{a^2} + \frac{1}{3a^2} - 4\,\frac{\log\left(A\right)}{a^2} - \frac{\log\left(2\right) z^2}{2} + \frac{\log\left(2\right) z}{a} + \frac{\log\left(2\right)}{3a^2} + \frac{z^2 \log\left(z\right)}{2} - \frac{3 z^2}{4} + \frac{z\log\left(\pi\right)}{a} + \log\left(a\right)\left(\frac{z^2}{2} - \frac{1}{3a^2}\right)$$

4 Second integral: $\int_0^\infty \frac{t\log\left(t^2+z^2\right)}{sinh(a\pi\, t)}\,\mathrm{d}t$

Where a and z are a positive integer number or a positive rational fraction.

We know this identity:

$$(sinh\,(\pi\, t))^{-1}{=}2\,\left(\mathrm{e}^{\pi t}-1\right)^{-1}-2\,\left(\mathrm{e}^{2\,\pi t}-1\right)^{-1}$$

And so we can write that

$$(sinh\,(a\pi\, t))^{-1}{=}2\,\left(\mathrm{e}^{a\pi t}-1\right)^{-1}-2\,\left(\mathrm{e}^{2\,a\pi t}-1\right)^{-1}$$

Now using the precedent formula we obtain

$$\int_0^\infty \frac{t\log\left(t^2+z^2\right)}{sinh\,(a\pi\, t)}\,\mathrm{d}t=-8\,\frac{\log\left(G\left((1/2)\,az\right)\right)}{a^2}-8\,\frac{\log\left(\Gamma\left((1/2)\,az\right)\right)}{a^2}+2\,\frac{\log\left(G\left(az\right)\right)}{a^2}+2\,\frac{\log\left(\Gamma\left(az\right)\right)}{a^2}+\frac{1}{2\,a^2}-6\,\frac{\log\left(A\right)}{a^2}-\log\left(2\right)z^2+\frac{\log\left(2\right)z}{a}+\frac{2\,\log\left(2\right)}{3\,a^2}+\frac{z\log\left(\pi\right)}{a}-\frac{\log\left(a\right)}{2\,a^2}$$

5 Third integral: $\int_0^\infty \frac{t\log\left(t^2+z^2\right)}{\mathrm{e}^{a\pi\, t}+1}\,\mathrm{d}t$

Where a and z are a positive integer number or a positive rational fraction.

Using the same principe than the first integral and with several closed form of the integral, finally

$$\int_0^\infty \frac{t\log\left(t^2+z^2\right)}{\mathrm{e}^{a\pi\, t}+1}\,\mathrm{d}t=-4\,\frac{\log\left(G\left((1/2)\,az\right)\right)}{a^2}-4\,\frac{\log\left(\Gamma\left((1/2)\,az\right)\right)}{a^2}+2\,\frac{\log\left(G\left(az\right)\right)}{a^2}+2\,\frac{\log\left(\Gamma\left(az\right)\right)}{a^2}+\frac{1}{6\,a^2}-2\,\frac{\log\left(A\right)}{a^2}-\frac{\log\left(2\right)z^2}{2}+\frac{\log\left(2\right)}{3\,a^2}-\frac{z^2\log\left(z\right)}{2}+\frac{3\,z^2}{4}+\log\left(a\right)\left(-\frac{z^2}{2}-\frac{1}{6\,a^2}\right)$$

6 Examples of applications

First example

Consider the integral

$$\int_0^\infty \frac{t\log\left(t^2+2^2\right)}{\mathrm{e}^{(3/2)\pi\, t}-1}\,\mathrm{d}t$$

So we see a=3/2 and z=2. We obtain

$$-\frac{83}{27}+\frac{8\,\log\left(A\right)}{9}+\frac{4\,\log\left(2\right)}{3}+\frac{50\,\log\left(3\right)}{27}$$

Second example

Consider the integral

$$\int_0^\infty \frac{t\log\left(t^2+2^2\right)}{sinh\left((1/3)\pi\,t\right)}\,\mathrm{d}t$$

So we see a=1/3 and z=2. We obtain

$$-\frac{3}{2}+18\,\log\left(A\right)+20\,\log\left(2\right)-\frac{9\,\log\left(3\right)}{4}+18\,\log\left(\pi\right)-\frac{5\,\sqrt{3}\pi}{3}+\frac{5\,\Psi\left(1,1/3\right)\sqrt{3}}{2\,\pi}-36\,\log\left(\Gamma\left(1/3\right)\right)$$

Where $\Psi\left(1,\frac{1}{3}\right)$ is the trigamma function at 1/3 (6).

Third example

Consider the integral

$$\int_0^\infty \frac{t\log\left(t^2+3^2\right)}{\mathrm{e}^{(1/2)\pi\,t}+1}\,\mathrm{d}t$$

So we see a=1/2 and z=3. We obtain

$$\frac{83}{12}-2\,\log\left(A\right)-\frac{35\,\log\left(2\right)}{3}-\frac{9\,\log\left(3\right)}{2}-6\,\log\left(\pi\right)-4\,\frac{G}{\pi}+12\,\log\left(\Gamma\left(1/4\right)\right)$$

Where G is the Catalan's constant (7).

7 References

(1): Iaroslav V. Blagouchine, Rediscovery of Malmsten's integrals, their evaluation by contour integration methods and some related results (2014)

(2): E. W. Barnes. The Theory of the G-function. Quart. J. Pure Appl. Math. 31, pages 264–314, 1899

(3),(4): https://dlmf.nist.gov/5.17

(5): https://mathworld.wolfram.com/Glaisher-KinkelinConstant.html

(6): https://en.wikipedia.org/wiki/Trigamma-function

(7): https://mathworld.wolfram.com/CatalansConstant.html